

Real Analysis

Quest University Canada

Block 1, Spring 2017

Considered from this point of view, mathematical analysis is as extensive as nature itself; it defines all perceptible relations, measures times, spaces, forces, temperatures; this difficult science is formed slowly, but it preserves every principle which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. – Joseph Fourier

There is no rigorous definition of ‘rigour’. – Morris Kline

One of the most important revolutions in mathematical thought occurred between the eighteenth and nineteenth centuries when the efficiency of novel results, powerful techniques, and intuition were rejected in favour of rigour and proof. While this revolution was centred around calculus, it changed the standards of mathematical truth and ushered in a new mathematical era.

For mathematicians of the eighteenth century, it seemed as though calculus could do it all. Their main concern was with getting results and, with the newly developed tools of calculus, problems from physics, astronomy, engineering, and mathematics were solved at an unprecedented rate. However, as the nineteenth century approached, mathematicians such as Cauchy and Bolzano began to give rigorous definitions of limits, derivatives, and integrals as they pointed out the failure of their predecessors’ free-wheeling approach. As this revolution in thought spread, mathematicians increasingly analyzed the foundations of calculus. Modern mathematicians refer to this new way of studying calculus as *analysis*.

In this course we will consider the factors that spurred the revolution from results to rigour and carefully study the concepts that came out of it. Along the way, you will stretch your ability to think logically, understand multi-layered statements, and communicate complicated ideas with precision.

By the end of this course you should:

1. Understand the mathematics that forms the rigorous foundation of calculus
2. Analyze, compare, and use precise definitions
3. Write proofs clearly and with proficiency
4. Construct examples and counterexamples to illustrate, verify, and attack definitions and claims
5. Present and defend mathematical arguments
6. Value the need for rigour in contemporary mathematics, and explain its historical roots the historical and contemporary need for rigour
7. Frame precise and useful questions in response to written and oral arguments
8. Come to see yourself as a capable mathematician

Course Information

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Course Topics

During this course, we will re-examine central ideas from calculus such as limits, continuity, derivatives, and integrals, and how they apply to functions and functions defined by series. The central topics for each week are as follows

- **Week 1: Understanding the power of and problems with infinite series.** Fourier series, infinite sums of numbers, approximating π , the Nested Interval Property, Taylor Series, Lagrange Remainder Theorem
- **Week 2: Introducing rigour to understand continuity, differentiability, and Taylor series.** $\epsilon - \delta$ definitions, Mean Value Theorem, continuity and the Intermediate Value Theorem, proving Lagrange Remainder Theorem
- **Week 3: Operations on series of numbers and functions.** convergence (tests, uniform, pointwise), regrouping and rearranging series, continuity of functions defined by series, term-by-term differentiation and integration
- **Week 4: Convergence of Fourier Series and defining the integral.** constructing Fourier series and proving convergence, the Cauchy integral, the Riemann integral

Course Text & Computer Software

The required text for this class is *A Radical Approach to Real Analysis, 2nd ed.* by David Bressoud, available at the university's bookstore. This text uses the mathematical software package Maple and Mathematica to help develop intuition for complicated concepts. I recommend that you purchase a license for Maple, but you may also develop work-arounds with an open-source software that you are comfortable with.

Weekly Meeting

I will schedule at least one individual meeting with you each week to discuss your progress in the class and your mathematical writing. Please bring your assignments and any questions or concerns. If we run out of time during the meeting, we will schedule a time to continue our discussion.

Our Approach: Inquiry-Based Learning

Don't just read it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special cases? Where does the proof use the hypothesis? – Paul Halmos

We will follow an approach called *inquiry-based learning*, or IBL, in this class. While IBL is often seen as radical and uncomfortable by students encountering it for the first time, it should not be a great departure from what you have seen in your other classes at Quest. Here are some characteristics of an IBL classroom:

- Students look to each other (rather than to the instructor) to validate the ideas presented
- Students are responsible for acquiring their own knowledge
- Problems drive the material and form the basis for discussion
- Students critique the work of others and ask each other for clarification when necessary

IBL is known as a *student-centered* approach, meaning that you will hold most of the responsibility for your learning and understanding. This approach fits particularly nicely in an advanced mathematics class, where the greatest challenge is simply understanding the definitions, lemmas, theorems, problems, and proofs that you encounter.

How will we implement the IBL approach this block? Every day, you will complete a reading and approximately 10-15 problems in the afternoon and evening. Class the following day will be based around this work, and we will discuss the readings and dissect solutions to the problems. If there is time left over, you may work through additional problems in groups or we may look forward to provide motivation for upcoming ideas.

Students learn real analysis at a largely individual pace. The same concepts that seem very natural to you might be quite difficult for another student to understand. Therefore, you should be prepared for the workload to vary from day-to-day throughout the block.

Course Requirements

These approaches cultivate in us the virtue of rigorous thinking: the ability to handle ideas well, and craft clear arguments with those ideas. This virtue serves us well in every area of life. – Francis Su

To pass this course you will need to do the following.

- Complete daily assignments, consisting of reading and problems
- Present solutions to problems in class
- Be an active classroom participant
- Demonstrate understanding of key concepts on daily quizzes
- Independently craft convincing and formal mathematical arguments of Challenge Problems
- Identify themes and unifying concepts of course content on the Final Exam

Daily Assignments

Making the simple complicated is commonplace; making the complicated simple, awesomely simple - that's creativity . – Charles Mingus

Every day, you will be assigned a reading and a list of problems to complete. The reading will usually consist of one two to sections from the textbook. The problems will serve several purposes and be of varying difficulty: some will be easy warm-ups to the reading or simply check details of presented proofs, while others will ask you to write intricate proofs. I expect that you will spend nearly all of your study time on these daily assignments.

Your notes on the assigned problems should contain enough detail to be easily understood by you at a later time. Results in mathematics build on each other, so you will refer to your solutions later in the block. Further, your notes will help you to swiftly present and discuss solutions during class. While you do not need to write your solutions in complete sentences, they should be easy to follow and clear. **If you decide to add to these notes during class before they are collected, you must use a separate colour to clearly distinguish what you prepared before class.**

I will collect, grade, and return these assignments each day. You will receive one point (a ✓) for a question when it appears that you have put adequate thought and work into it (not necessarily if it is right or wrong). On most days I will just glance at your solutions, but I will occasionally read them in detail to ensure that you are actually working through the problem and not just writing lots of impressive-looking words.

In addition to your daily homework, I **strongly recommend** that you keep a journal of key definitions and theorems throughout the course that you review and revise regularly. One of the drawbacks of using a historical approach to study real analysis is that the results in the textbook are not as neatly laid out as a traditional textbook is, and you will find a systematically organized recap very valuable both in this course and in your later studies.

Presenting Solutions¹

The silly question is the first intimation of some totally new development. – Alfred North Whitehead

We have a lot of work to accomplish this block and your contributions in-class are crucial to the success of others. Student presentations of homework problems will be the centrepiece of the class, so you should take them seriously. Here are some things to know about making presentations in real analysis.

- The purpose of class presentations is not to prove to the instructor that you have done the problem. It is to make the ideas of the proof clear to the other students.
- You must use proper English and mathematical grammar during presentations.
- Presenters should explain their reasoning as they go along, not simply write everything down and then turn to explain.

¹from Carol Schumacher

- Fellow students are allowed to ask questions at any point and it is the responsibility of the person making the presentation to answer those questions to the best of his or her ability.
- Presentations are directed at students, so the presenter should look at students to be able to see how well they are following the presentation.

I will seek volunteers to present solutions in-class, but you should expect to present solutions to problems every day. The following scale will be used to grade your presentations.

Mark	Criteria
4	Completely correct and clear proof or solution
3	Solution has minor technical flaws, unclear language, or lacking details. Essentially correct.
2	A partial explanation or proof is provided but a significant gap still exists
1	Minimal progress has been made, including relevant information and could lead to a proof
0	Completely unprepared

Daily Quizzes

We must have perseverance and above all confidence in ourselves. We must believe that we are gifted for something and that this thing must be attained. –Marie Curie

Every day, I will pose 1 to 3 basic questions about the material covered in the class so far. You may be asked to give a definition, a theorem, or an example. While these questions will generally be shorter and less complicated than the ones that you present formally, the same 4-point grading scale will be used to grade these problems.

Participation

The silly question is the first intimation of some totally new development. – Alfred North Whitehead

To learn real analysis and to contribute to the learning of your peers, you must be active and engaged during class. During a presentation you are not off the hook just because someone else volunteered. Even if you are sitting down, you are responsible for contributing questions and comments to help clarify what is being presented.

It is important to note that you should be just as eager to offer questions about concepts that you understand as those that you do not understand. For example, if you have done the problem and understand it well, you can often make suggestions that help clarify the wording or structure of a presentation.

Challenge Problems

A ship in port is safe, but that's not what ships are built for. – Grace Hopper

Throughout the course, I will designate certain problems (one every 1-2 days) as a Challenge Problem. You will submit the solutions to these problems in writing for grading and feedback. I will be paying significant attention to the style and completeness of your writing, as one of the main purposes of these assignments is to develop your ability to write formal mathematical proofs like those you would see in a textbook or a research paper. This means that *every* detail counts and needs to be included. See the handout ‘How to write mathematics’ for more information about writing guidelines. You must neatly *handwrite* or type your solutions using \LaTeX ; please do not attempt to use Microsoft Word or another general-use word processor.

Challenge problems will be completed independently, without consultation with other students or resources other than our course textbook. Of course, you may ask me questions about the Challenge Problems.

All Challenge Problems must be submitted by Monday, February 6 (this is Monday of Week 4). There are no further deadlines for individual Challenge Problems, although you should only use machinery available to you at the time the problem was assigned in your solutions.

When I grade your work, I will consider the following questions.

Mathematical Correctness/ Completeness	Is your proof mathematically correct? Are all sentences/claims legitimate? Is your proof complete or does it contain gaps and/or limitations?
Depth of Understanding	Does your solution demonstrate an understanding of both the problems at hand and your proposed proof? Are there important aspects of the problem that you have neglected to consider?
Justification & Explanation	Have you justified your reasoning? Have you clearly explained your thinking? Does your writing compel the reader to believe that you truly understand the problem and have an appropriate proof?
Coherence & Clarity	Are your solutions coherent and readable? Has your written work clearly expressed the mathematical intent of your proof? Is the identity of objects you refer to clear, and are all variables defined within the problem? Do you include a complete introduction and conclusion?
Neatness, Organization, Grammar, Spelling	Is your solution easily readable? Are the spelling and grammar correct? Are all sentences complete?

Final Exam

You can't win a game you haven't defined. –David Allen

On the final day of the course, you will be given one hour of class time to respond to the following: *Create a concept map that organizes the topics that you have encountered in the course and demonstrates how they are connected. Be sure to explain and define key ideas from throughout the course and clearly indicate connections.*

There are many ways that you could respond to this prompt. I am not looking for one particular approach to organizing course topics, but rather for solid evidence that you have

reflected on topics from the entire course and created your own mental map of topics from the course. While you can discuss this with your classmates ahead of the final exam, I expect each person's map to look quite different and do not want you to create these maps collaboratively.

Academic Integrity

While googling a homework problem or trading solutions with a classmate may seem like good strategies for doing well in this class, these actions will prevent you from learning material, refining your problem-solving skills, and developing self-sufficiency and self-esteem.

The consequences for cheating are severe. *Any* blatant academic dishonesty will result in failure of the course and immediate reporting to the Chief Academic Officer. If you have questions about whether something is considered cheating, please speak with me first.

Grading

While grades are (one) measure of progress, they are not a measure of promise. – Francis Su

Your final grade will be calculated as follows.

- 30% Challenge Problems
- 20% Daily Assignment Completion and Participation
- 20% In-Class Presentations
- 15% Daily Quizzes
- 15% Final Exam

The course grading scale is:

A	93-100%	B	83-86%	C	73-76%
A-	90-92%	B-	80-82%	C-	70-72%
B+	87-89%	C+	77-79%	D	60-69%

Narrative Evaluation

Any student at Quest can request a narrative evaluation (e.g. a written paragraph) in addition to their letter grade in any course. A narrative evaluation will give you more comprehensive feedback that you can learn from and additional information to present employers and graduate schools. If you wish to take advantage of this option, you have until the end of the 6th day of a course to sign up on the Registrar's Office Portal site.

Disability Accommodations

If you have a disability for which you seek accommodation, please make sure to have registered with the Learning Commons, as specified in the Student Accommodation Policy and provide me with your Memorandum by the second day of class.

Additional Course Policies

- Please be on time to class. If you are consistently late you will be asked to leave.

- I expect that you will attend every class session. If you must miss a class for a valid reason (such as illness or a family emergency), please let me know *before* class. I reserve the right to ask for documentation to excuse your absence. For every class that you miss without a valid reason, 5% will be deducted from your *full* course grade.
- Bring pencils, paper, scientific calculator, and laptop to every class.
- Keep your cell phone in your bag during class. If you using or looking at your cell phone, you will be asked to leave.
- Always be respectful in your speaking and actions. Do not use profanity.
- Office hours: If you are not able to make my drop-in office hours, or need to speak with my privately, please e-mail me to arrange an appointment.
- E-mail: During the block I check my e-mail in the morning and at the end of the day, and sporadically throughout the rest of the days and weekends. Please do not e-mail me with questions that may be easily answered by looking at this syllabus, the course website, or asking other members of the class. Be polite and use proper English grammar.
- University policy does not allow food in classrooms. You may bring drinks.