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Renormalization Group Analysis of Spectral Problems in Quantum Field Theory

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In this paper we present a self-contained and detailed exposition of the new renormalization group technique proposed in [1, 2]. Its main feature is that the renormalization group transformation acts directly on a space of operators rather than on objects such as a propagator, the partition function, or correlation functions.

We apply this renormalization transformation to a Hamiltonian describing the physics of an atom interacting with the quantized electromagnetic field, and we prove that excited atomic states turn into resonances when the coupling between electrons and field is nonvanishing.

Key Words: renormalization group; spectrum; resonances; Fock space; QED.

I. INTRODUCTION

In this paper we give a detailed and mathematically self-contained presentation of the new renormalization group technique proposed in [1, 2].

The mathematical problems we address in this paper are encountered in the quantum theory of atoms and molecules coupled to the quantized electromagnetic field. As explained in [1, 2], our task is to explore spectral properties of Hamiltonians that generate the time evolution of systems of

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electrons ("particles") bound to static nuclei and interacting with the quantized radiation field ("bosons"). In paper [2] we justify studying the following simplified quantum mechanical model system.

The Hilbert space of pure state vectors of the system is given by

$$\mathscr{H} = \mathscr{H}_{el} \otimes \mathscr{F}_b[L^2(\mathbb{R}^d)], \tag{I.1}$$

where $\mathscr{H}_{el}$, the "particle Hilbert space", is some separable Hilbert space (for a single electron, e.g., given by $L^2(X) \otimes \mathbb{C}^2$, $X \cong \mathbb{R}^d$, in the Schrödinger representation), and $\mathscr{F}_b \equiv \mathscr{F}_b[L^2(\mathbb{R}^d)]$ is the Fock space of the quantized radiation field, i.e., $\mathscr{F}_b$ is the Hilbert space of symmetric tensors of arbitrary rank ("multi-boson states") over the one-boson Hilbert space $L^2(\mathbb{R}^d)$. Here $\mathbb{R}^d$ is the momentum space of one boson (while $X \cong \mathbb{R}^d$ is the physical configuration space). The Fock space $\mathscr{F}_b$ carries a unique, unitary, irreducible representation of the *canonical commutation relations* between creation- and annihilation operators, $a^\dagger(k)$, $a(k)$,

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0, \qquad [a(k), a^\dagger(k')] = \delta(k - k'), \tag{I.2}$$

for arbitrary $k, k' \in \mathbb{R}^d$, in the sense of operator-valued distributions. The boson Fock space contains a special unit vector, Ω, the *vacuum vector*, which is characterized, up to a phase, by the equations

$$a(k)\, \Omega = 0, \qquad \text{for all} \quad k \in \mathbb{R}^d \tag{I.3}$$

("lowest-weight condition"). The vacuum vector is a cyclic vector for the polynomial algebra generated by creation-and annihilation operators smeared out with test functions; (the representation of the commutation relations (I.2) on $\mathscr{F}_b$ is uniquely characterized by unitarity, the lowest-weight condition (I.3), and the cyclicity of Ω).

The time evolution of the system is generated by a Hamilton operator $H_g(\theta = 0)$, which is a self-adjoint operator defined on a dense domain in $\mathscr{H}$. The family of operators $H_g(\theta)$ is defined by

$$H_g(\theta) := H_{el} \otimes \mathbf{1} + \mathrm{e}^{-\theta} \mathbf{1} \otimes H_f + W_g^{(\theta)}, \tag{I.4}$$

where H_{el} denotes the particle Hamiltonian,

$$H_f := \int \omega(k)\, a^\dagger(k)\, a(k)\, \mathrm{d}k \tag{I.5}$$

is the (free) boson field Hamiltonian, which is self-adjoint on its natural domain $\mathscr{D}(H_f) \subseteq \mathscr{F}_b[L^2(\mathbb{R}^d)]$, where the dispersion ω is given by $\omega(k) := |k|^\gamma$,

$\gamma > 0$ (note that $\gamma = 1$ and $d = 3$ for physical photons). The term $W_g^{(\theta)}$ is of the form

$$W_g^{(\theta)} = \sum_{1 \leqslant M+N \leqslant 2} g^{M+N} W_{M,N}^{(\theta)}, \tag{I.6}$$

$$W_{M,N}^{(\theta)} = \int G_{M,N}(k_1, ..., k_M, \tilde{k}_1, ..., \tilde{k}_N; \theta)$$
$$\otimes a^\dagger(k_1) \cdots a^\dagger(k_M)\, a(\tilde{k}_1) \cdots a(\tilde{k}_N)\, dk_1 \cdots dk_M\, d\tilde{k}_1 \cdots d\tilde{k}_N, \tag{I.7}$$

where $G_{M,N}$ are functions with values in the operators on $\mathscr{H}_{el}$ and g is a coupling constant. It describes the interactions between the particle(s) and the boson field. The parameter θ, a complex number with $|\mathrm{Im}\,\theta|$ small, is introduced, in order to reveal the resonance structure of the *spectrum* of H_g; the physical value of θ is $\theta = 0$. Exploring properties of the spectrum of $H_g(\theta)$, in particular of $H_g(0)$, is the main purpose of the present paper. For a detailed summary of our results, our methods, and background material, as well as references to related work see [2].

Next, we define H_{el}, H_f, and $W_g^{(\theta)}$ more precisely and describe the basic assumptions underlying our analysis.

I.1. *Assumptions on the Model and Main Result*

We assume that H_{el} is a self-adjoint operator on a dense domain of definition $\mathscr{D}(H_{el}) \subseteq \mathscr{H}_{el}$ with a *standard spectrum*, $\sigma(H_{el})$, i.e., $\sigma(H_{el})$ contains isolated eigenvalues $e_0 < e_1 < e_2 < \cdots \leqslant \Sigma$, of finite multiplicity without any accumulation point below Σ and (absolutely) continuous spectrum contained in $[\Sigma, \infty)$ (see Fig. 1). These spectral properties are typical for Schrödinger operators, $-\Delta_x + V(x)$, describing the dynamics of nonrelativistic point particles (see e.g. [3, 8]). We wish to study the spectrum of $H_g(\theta)$ in the vicinity of one of the eigenvalues, e_j, of H_{el}. For simplicity, we assume e_j to be non-degenerate and isolated by $\delta > 0$ from the other spectrum of H_{el}.

The two parameters γ (the power in the dispersion relation ω) and d (the dimension of the photon momentum space) will affect the perturbation theory that we shall develop. For this reason, we want to keep their values unspecified. Of course, we ensure that our results cover the physically relevant

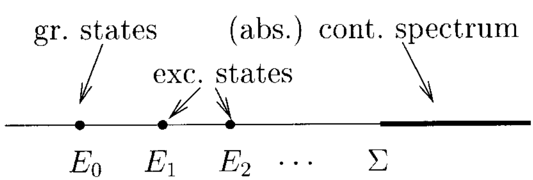

FIG. 1. The spectrum of H_{el}.

case. In contrast, polarization degrees of freedom, which physical photons possess, do not have any effect on the present results and are neglected.

Fixing $\theta := i\vartheta$, $0 < \vartheta < \pi/2$, we observe that $H_0(\theta) = H_{g=0}(\theta)$ in (I.4) is a non-self-adjoint, but normal operator on $\mathscr{H}$ with spectrum $\sigma(H_0(\theta)) = \sigma(H_{el}) + e^{-\theta}\sigma(H_f)$ (see Fig. 2).

As illustrated in Fig. 2, the eigenvalues $e_0, e_1, e_2, ..., e_j, ...$ of H_{el} are eigenvalues of $H_0(\theta)$, as well. They are not isolated from the remaining spectrum anymore, however, and one cannot apply standard perturbation theory to study their fate when the perturbation is switched on, $g > 0$. Indeed, the main achievement in the present work is to construct a perturbation theory for eigenvalues which are not isolated from continuous spectrum.

We remark that $H_0(\theta)$ (trivially) is an analytic family of type A [6, 8] for θ in a sufficiently small neighborhood of zero. Under suitable analyticity assumptions on $W_g^{(\theta)}$, which we state in Hypothesis H-2 below, this extends to $H_g(\theta)$ for small values of g. One may thus obtain the analytic continuation of matrix elements $\langle \Psi | [H_g(\theta = 0) - z]^{-1} \Phi \rangle$ from $z \in \mathbb{C}_+$ across the real axis into $z \in \mathbb{C}_-$, by studying that of $\langle \Psi | [H_g(\theta = i\vartheta) - z]^{-1} \Phi \rangle$ for $0 < \vartheta < \pi/2$ and Ψ, Φ taken from a universal dense set $\mathscr{D} \subseteq \mathscr{H}$. By means of this meromorphic continuation, we can identify spectral properties of $H_g(\theta = i\vartheta)$ such as eigenvalues and continuous spectrum, with poles, branch points and cuts on the domain of analyticity of the meromorphic continuation of matrix elements of the physically relevant resolvent $[H_g(\theta = 0) - z]^{-1}$. This procedure is known in mathematics as complex scaling and is described, e.g., in [3, 5, 8].

We turn to the specification of $W_g \equiv W_g^{(\theta)}$. We assume that, for $k \in \mathbb{R}^d$, $G_{1,0}(k)$, $G_{0,1}(k)$, and their adjoints are quadratic forms on $\mathscr{H}_{el}$ with form domain containing $\mathscr{D}(|H_{el}|^{1/4})$. Likewise, $G_{2,0}(k, k')$, $G_{0,2}(k, k')$, $G_{1,1}(k, k')$, are assumed to be bounded operators on $\mathscr{H}_{el}$, for all $k, k' \in \mathbb{R}^d$.

Before we specify our bounds on the norms of the *coupling functions*, $G_{m,n}(k_1, ..., k_{m+n})$, in a Hypothesis, we introduce natural dimensionless units, in order to rewrite the Hamiltonian of the system in a simpler form: We introduce the distance δ of the eigenvalue e_j to the rest of the spectrum of H_{el} as our unit of energy. This enables us to set $\delta = 1$, throughout the following.

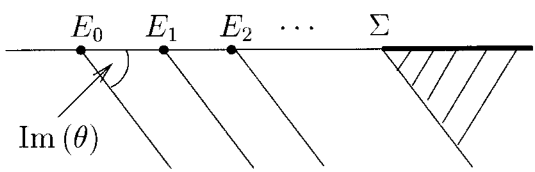

FIG. 2. The spectrum of $H_0(\theta)$.

In order to see that we are not loosing generality in setting $\delta=1$, we apply a unitary scale transformation, Γ_η, to H_g: This transformation unitarily implements the scaling of boson momenta,

$$k \mapsto \eta^{1/\gamma} k, \qquad k \in \mathbb{R}^d, \qquad \eta > 0, \tag{I.8}$$

on the Fock space $\mathscr{F}_b[L^2(\mathbb{R}^d)]$ and is determined by the properties that $\Gamma_\eta \Omega = \Omega$,

$$\begin{aligned} \Gamma_\eta a^\dagger(k)\, \Gamma_\eta^* &= \eta^{-(d/2\gamma)} a^\dagger(n^{-(1/\gamma)} k), \qquad \text{and} \\ \Gamma_\eta a(k)\, \Gamma_\eta^* &= \eta^{-(d/2\gamma)} a(\eta^{-(1/\gamma)} k). \end{aligned} \tag{I.9}$$

Note that Γ_η is obtained by lifting the operator $f(k) \mapsto \eta^{d/2\gamma} f(\eta^{1/\gamma} k)$, defined on the one-particle space, $L^2(\mathbb{R}^d)$, to the Fock space. Thus, using $\omega(k) = |k|^\gamma$ and choosing $\eta := \delta$, we observe that

$$\Gamma_\eta H_f \Gamma_\eta^* = \eta H_f = \delta H_f, \tag{I.10}$$

and, for $1 \leqslant M+N \leqslant 2$,

$$W_{M,N}^{(resc)} \equiv \delta^{-1} (\mathbf{1} \otimes \Gamma_\eta)\, W_{M,N} (\mathbf{1} \otimes \Gamma_\eta^*) \tag{I.11}$$

$$\begin{aligned} &= \int G_{M,N}^{(resc)}(k_1, ..., k_M, \tilde{k}_1, ..., \tilde{k}_N) \\ &\qquad \otimes a^\dagger(k_1) \cdots a^\dagger(k_M)\, a(\tilde{k}_1) \cdots a(\tilde{k}_N)\, \mathrm{d}k_1 \cdots \mathrm{d}k_M\, \mathrm{d}\tilde{k}_1 \cdots \mathrm{d}\tilde{k}_N, \end{aligned} \tag{I.12}$$

where

$$G_{M,N}^{(resc)}(k_1, ..., k_{M+N}) := \delta^{(d(M+N)/2\gamma)-1} G_{M,N}(\delta^{1/\gamma} k_1, ..., \delta^{1/\gamma} k_{M+N}). \tag{I.13}$$

So, defining $H_g^{(resc)} := \delta^{-1} (\mathbf{1} \otimes \Gamma_\eta)\, H_g (\mathbf{1} \otimes \Gamma_\eta^*)$ and $W_g^{(resc)} := \sum_{1 \leqslant M+N \leqslant 2} g^{M+N}\, W_{M,N}^{(resc)}$, we find that

$$H_g^{(resc)} = \delta^{-1} H_{el} \otimes \mathbf{1} + \mathrm{e}^{-\theta} \mathbf{1} \otimes H_f + W_g^{(resc)}. \tag{I.14}$$

The benefit of passing from H_g to $H_g^{(resc)}$ is that the eigenvalue $\delta^{-1} e_j$ of the rescaled particle Hamiltonian $H_{el}^{(resc)} := \delta^{-1} H_{el}$ is now separated by a distance 1 from the rest of its spectrum. Moreover, by a trivial energy shift, we may replace $\delta^{-1} e_j$, by zero.

We formulate our assumptions and henceforth drop the superscript "($resc$)".

HYPOTHESIS 1. *Zero is a non-degenerate isolated eigenvalue of H_{el} with normalized eigenvector φ_{el}, separated by a distance 1 from the rest of its spectrum, i.e.,*

$$\mathrm{Ker}(H_{el}) = \mathrm{span}\{\varphi_{el}\}, \qquad \mathrm{dist}[0, \sigma(H_{el})\backslash\{0\}] = 1. \tag{I.15}$$

Next we give the precise assumptions on the coupling functions $G_{M,N}$, which also depend on $\theta \in \mathbb{C}$ in practical applications.

HYPOTHESIS 2. *There exists a non-negative function $J \geqslant 0$ such that, for all $k, k' \in \mathbb{R}^d$ and $M+N=1$ (i.e., $(M, N) = (0, 1)$ or $(M, N) = (1, 0)$),*

$$\|(|H_{el}|+1)^{-1/4}\, G_{M,N}(k)(|H_{el}|+1)^{-1/4}\|_{\mathscr{H}_{el}} \leqslant J(k), \tag{I.16}$$

and, for $M+N=2$ (i.e., $(M, N) = (2, 0)$, $(M, N) = (1, 1)$, or $(M, N) = (0, 2)$),

$$\|G_{M,N}(k, k')\|_{\mathscr{H}_{el}} \leqslant J(k)\, J(k'). \tag{I.17}$$

If $G_{M,N}$ additionally depend on $\theta \in \mathbb{C}$, then the maps $\theta \mapsto G_{M,N}$ are assumed to be bounded analytic in a complex neighborhood of $\theta = 0$ with respect to the norms specified in (I.16) and (I.17).

Moreover, J is a square integrable function obeying

$$\Lambda_1 := \left\{ \int [1 + \omega(k)^{-1}]\, J^2(k)\, d^dk \right\}^{1/2} < \infty. \tag{I.18}$$

Under the assumptions of Hypothesis H-2 the map $\theta \mapsto H_g(\theta)$ defines an analytic family of type A in a complex neighborhood of $\theta = 0$. Thus we obtain the analytic continuation of the resolvent $(H_g(\theta = 0) - z)^{-1}$ across the real axis from the deformed resolvent $(H_g(\theta = i\vartheta) - z)^{-1}$ for $0 < \vartheta < \pi/2$ sufficiently small. (Note that if φ_{el} is the ground state of the particle Hamiltonian H_{el}, i.e., $j = 0$, then $\theta = 0$.)

Although we do not give a proof here, we shall make it plausible to the reader that Hypothesis H-2 is both necessary and sufficient for the application in atomic physics discussed in [2]. First, every factor of $a^\dagger$ or a in W_g accounts for a factor of $H_f^{1/2}$ in its relative bound with respect to H_0. (This is made precise in Chapter III). So, Hypothesis H-2 guarantees that, for $M+N=1$,

$$|\langle \psi \mid W_{M,N} \psi \rangle| \leqslant \mathrm{const} \|(|H_{el}|+1)^{1/4} \otimes (H_f + 1)^{1/4}\, \psi\|^2, \tag{I.19}$$

and that, for $M+N=2$,

$$|\langle \psi \mid W_2 \psi \rangle| \leqslant \mathrm{const}\ \|\mathbf{1} \otimes (H_f + \mathbf{1})^{1/2}\, \psi\|^2. \tag{I.20}$$

These bounds are sufficient to ensure the boundedness of W_g relative to H_0. Requiring that $G_{0,1}(k)$ and $G_{1,0}(k)$ are bounded would, however, not allow for the main application in atomic physics, where these coefficients are bounded relatively to $\sqrt{-\Delta}$, only. This is exactly what is required in (I.16), because in atomic physics H_{el} is comparable to $-\Delta$ and, hence, $(|H_{el}|+1)^{1/2}$ is comparable to $\sqrt{-\Delta+1}$. For an explicit specification of H_{el} and $G_{M,N}$ derived from a model of nonrelativistic quantum electrodynamics we refer the reader to [2].

It turns out that the properties of J assumed in Hypothesis H-2 are not sufficient for the convergence of the renormalization group developed in Chapter IV. Indeed, one of the central aspects of the present work in the search for the weakest possible condition on J that still admits a convergent renormalization group analysis. We have found the following hypothesis to be sufficient.

HYPOTHESIS 3. *There exists a positive number $\mu>0$ such that the function J in Hypothesis H-2 additionally obeys*

$$\Lambda_5 := \sup_{k\in\mathbb{R}^d} \{\omega(k)^{(d/2\gamma)-1-(\mu/2)}\, J(k)\} < \infty. \tag{I.21}$$

Note that if the coupling functions are non-vanishing and thus $\Lambda_1, \Lambda_5>0$, then we may additionally assume without loss of generality that

$$\Lambda_1 \geqslant 1 \qquad \text{and} \qquad \Lambda_5 \geqslant 1, \tag{I.22}$$

by replacing the coupling constant g by $\min\{\Lambda_1, \Lambda_5\} g$, if necessary.

Next we describe our main result. We assume Hypotheses H-1, H-2, and H-3. In these hypotheses, the parameters

$$\mu>0, \qquad \Lambda_1\geqslant 1, \qquad \Lambda_5\geqslant 1, \qquad 0<\vartheta<\pi/2 \tag{I.23}$$

are specified, and we consider these fixed, henceforth. Moreover, for purely technical reasons, we assume $\mu\leqslant 2$. Using these parameters, we define additional constants C_d, Λ_6, g_0, ρ_0, ρ, ξ, and ε by

$$C_d := d\gamma^{-1}\pi^{d/2}\Gamma[(d/2)+1]^{-1}, \tag{I.24}$$

$$\rho_0 := 2^{-1/2}\sin(\vartheta/2), \tag{I.25}$$

$$\rho := \min\{C_d^{-1}, 2^{-8/\mu}\}, \qquad \varepsilon := \rho^{1/2}/12\,800 \tag{I.26}$$

$$\xi := \min\{C_d^{-1/2}\rho^{(3+\mu)/4}, C_d^{-1}\}, \tag{I.27}$$

$$\Lambda_6 := \frac{\Lambda_1 \Lambda_5 (1 + \sqrt{2C_d})}{\sin(\vartheta/2)}. \tag{I.28}$$

$$g_0 := \frac{\rho^{1/2}}{(1.28)\cdot(10^6)\cdot \Lambda_6} \min\left\{\sqrt{6400}\cdot \rho_0^{1/2}, \frac{\xi}{2}, \frac{\xi C_d \rho^{1/2}}{25}\right\}. \tag{I.29}$$

Our main result is the following theorem.

THEOREM I.1. *Assume Hypotheses* H-1, H-2, H-3, *and Condition* (I.23). *Let* $0 < g \leqslant g_0$. *Then*

(i) *the operator* $H_g(\theta)$ *has a simple eigenvalue,* $E_{(\infty)} = E_{(\infty)}(g, \theta)$ *corresponding to an eigenvector* $\Psi_{(\infty)} = \Psi_{(\infty)}(g, \theta) \in \mathcal{H}$;

(ii) *the eigenvalue* $E_{(\infty)}$ *and the eigenvector* $\Psi_{(\infty)}$ *can be constructed by iterating a renormalization map,* $\mathcal{R}_\rho$, *on a ball in a Banach space of effective Hamiltonians;*

(iii) *if additionally* $g \leqslant \rho^{(3+\mu)/2} g_0$ *then the spectrum,* $\sigma(H_g(\theta))$, *of* $H_g(\theta)$ *is located as follows:*

$$\sigma(H_g(\theta)) \cap D_{\rho_0/2} \subseteq E_{(\infty)} + K_{(\infty)}, \tag{I.30}$$

where $D_{\rho_0/2} = \{z \in \mathbb{C}: |z| \leqslant \rho_0/2\}$ *is the disk of radius* $\rho_0/2$ *about the origin, and* $K_{(\infty)} \subseteq \mathbb{C}$ *is the cuspidal domain defined by*

$$K_{(\infty)} := \{T_{(\infty)}(r) + \zeta \mid 0 \leqslant r < 1, |\zeta| \leqslant 78\varepsilon\rho^{-9/2}\cdot r^{1+(\mu/4)}\}, \tag{I.31}$$

where $T_{(\infty)} \in C^0([0,1], \mathbb{C})$ *is a Lipschitz continuous curve in* $\mathbb{C}$ *defined in* (V.72). *It satisfies* $T_{(\infty)}[0] = 0$ *and* $\arg(T_{(\infty)}) = -\vartheta + O(g)$.

Assertion (i) in Theorem I.1 is proven in Theorem V.10 and Assertion (iii) in Theorem V.7. The proof of Theorem I.1 is based on the renormalization group map $\mathcal{R}_\rho$ that we construct in Chapter IV. We outline some of the key ideas involved.

I.2. *The General Strategy of the Renormalization Group Construction*

I.2.1. *Passing from a single operator to a Banach space of operators.* First we construct an effective operator, $H_{(0)}[z]$, from the Hamiltonian $H_g(\theta) - z$ in (I.4) in several steps. We decimate the degrees of freedom corresponding to the particles and to the photons of energies $\geqslant \rho_0$, where $0 < \rho_0 < 1$ is defined in (I.25). Next, we rescale the photon momenta as $k \mapsto \rho_0^{1/\gamma} k$ and the energies as $E \mapsto \rho_0^{-1} E$. In addition, we change the spectral parameter z as $z \mapsto e^{\theta} \rho_0^{-1} z$. Let

$$P_0 = P_{el} \otimes \chi[H_f < \rho_0], \tag{I.32}$$

where $P_{el} = |\varphi_{el}\rangle\,\langle\varphi_{el}|$ is the orthogonal projection onto the eigenspace of the atomic Hamiltonian H_{el} corresponding to the eigenvalue zero (see Hypothesis H-1), and $\chi[H_f < \rho_0]$ is the spectral projection of H_f onto the subspace of vectors in Fock space with field energy $< \rho_0$. The decimation is done with the help of the Feshbach map, $\mathscr{F}_P$, for a given projection P (see Eq. (II.1)),

$$\mathscr{F}_P(H) = PHP - PH\bar{P}(\bar{P}H\bar{P})^{-1}\,\bar{P}HP, \tag{I.33}$$

where $\bar{P} := \mathbf{1} - P$, and the invertibility of $\bar{P}H\bar{P}$ on Ran $\bar{P}$ is assumed. $\mathscr{F}_P$ maps operators on a given space (on which P is defined) into operators on Ran P. A key point is that the map $\mathscr{F}_P$ is isospectral (in the sense of Theorem II.1 below), i.e.,

$$z \in \sigma_{\#}(H) \Leftrightarrow 0 \in \sigma_{\#}(H - z) \Leftrightarrow 0 \in \sigma_{\#}(\mathscr{F}_P(H - z)) \tag{I.34}$$

where $\sigma_{\#} = \sigma$ or $\sigma_{\#} = \sigma_{\mathrm{pp}}$. Applying the Feshbach map $\mathscr{F}_{P_0}$, we obtain the effective Hamiltonian

$$\mathrm{e}^{-i\vartheta}[\mathscr{F}_{P_0}(H_g(\theta) - z) + zP_0], \tag{I.35}$$

where z belongs to the disk $D_{\rho_0/2}$, i.e., $|z| \leqslant \rho_0/2$. Since P_{el} is one-dimensional, we can view the operator in (I.35) as acting on the Hilbert space $\chi[H_f < \rho_0]\,\mathscr{F}_b[L^2(\mathbb{R}^d)]$.

Next, we rescale the photon momenta as

$$k \mapsto \rho_0^{1/\gamma} k \tag{I.36}$$

by means of the unitary dilation Γ_{ρ_0}, where Γ_η is defined in (I.9), and we rescale the energies as $E \mapsto \rho_0^{-1} E$, passing from the operator in (I.35) to a unitarily equivalent Hamiltonian, $\rho_0 H_{\mathrm{eff}}[\zeta]$, which is defined by

$$P_{el} \otimes H_{\mathrm{eff}}[\zeta] := \frac{\mathrm{e}^{-i\vartheta}}{\rho_0}\,\Gamma_{\rho_0}(\mathscr{F}_{P_0}(H_g(\theta) - \zeta) + \zeta\,\chi[H_f < \rho_0])\,\Gamma_{\rho_0}^*, \tag{I.37}$$

for all $\zeta \in D_{\rho_0/2}$, on the Hilbert space

$$\mathscr{H}_{\mathrm{red}} := \chi[H_f < 1]\,\mathscr{F}_b[L^2(\mathbb{R}^d)] \equiv \mathrm{Ran}\,\chi[H_f < 1]. \tag{I.38}$$

Just like the photon momenta, we rescale the spectral parameter, z. We introduce the bijection

$$Z_{(0)}\colon D_{\rho_0/2} \to D_{1/2}, \qquad \zeta \mapsto \frac{\mathrm{e}^{i\vartheta}}{\rho_0}\zeta \tag{I.39}$$

(see Fig. 3), and we define

$$H_{(0)}[Z_{(0)}(\zeta)] := H_{\mathrm{eff}}[\zeta], \tag{I.40}$$

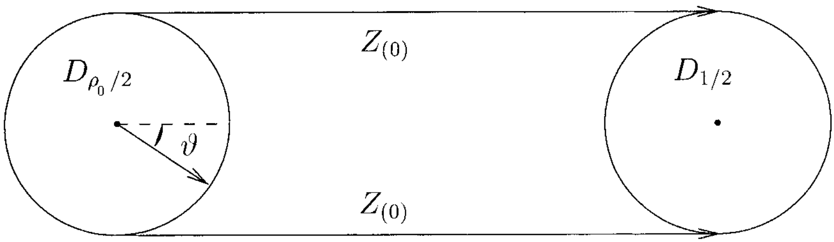

FIG. 3. First rescaling of the spectral parameter.

for all $\zeta \in D_{\rho_0/2}$. Composing these two operations, we explicitly have

$$P_{el} \otimes H_{(0)}[z] = \frac{e^{-i\vartheta}}{\rho_0} \Gamma_{\rho_0}(\mathscr{F}_{P_0}(H_g(\theta) - Z_{(0)}^{-1}[z]) + Z_{(0)}^{-1}[z]\, \chi[H_f < \rho_0])\, \Gamma^*_{\rho_0}, \tag{I.41}$$

on $\mathscr{H}_{\rm red}$, for all $z \in D_{1/2} := \{|z| \leqslant 1/2\}$.

After rescaling of the photon momenta and the spectral parameter, we expand the resolvent $\bar{P}_0(\bar{P}_0 H_g(\theta)\, \bar{P}_0 - \zeta)^{-1}\, \bar{P}_0$, where $\bar{P}_0 = \mathbf{1} - P_0$, in $\mathscr{F}_{P_0}(H_g(\theta) - \zeta)$ of (I.41), in powers of $\bar{P}_0 W_g \bar{P}_0$. Using a straightforward generalization of Wick's theorem, we show that the operator $H_{(0)}[z]$ can be represented in the form

$$H_{(0)}[z] := \chi[H_f < 1](E_{(0)}[z] \cdot \mathbf{1} + T_{(0)}[z; H_f] + W_{(0)}[z])\, \chi[H_f < 1], \tag{I.42}$$

where $E_{(0)}[z] \in \mathbb{C}$ is a number, $T_{(0)}[z; H_f]$ is a function of H_f, and $W_{(0)}[z] = \sum_{M+N \geqslant 1} W^{(0)}_{M,N}[z]$ is a sum of "Wick monomials" of the form

$$W^{(0)}_{M,N}[z] = \int dk^{(M)}\, d\tilde{k}^{(N)} a^\dagger(k^{(M)})\, w^{(0)}_{M,N}[z; H_f; k^{(M)}, \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}), \tag{I.43}$$

for $M + N \geqslant 1$. Here, we use the shorthand notation

$$k^{(M)} := (k_1, ..., k_M) \in \mathbb{R}^{dM}, \qquad dk^{(M)} := \prod_{i=1}^{M} d^d k_i, \tag{I.44}$$

$$a^\dagger(k^{(M)}) := \prod_{i=1}^{M} a^\dagger(k_i), \qquad \omega(k^{(M)}) := \sum_{i=1}^{M} \omega(k_i). \tag{I.45}$$

In Section III.2, we show that, for each $z \in D_{1/2}$, $H_{(0)}[z]$ belongs to a certain *Banach space*, $\mathscr{W}'_\Delta$, of Hamiltonians on $\mathscr{H}_{\rm red} = \chi[H_f < 1]\, \mathscr{F}_b$, which we now define.

The Banach space $\mathscr{W}'_\Delta$, defined as

$$\mathscr{W}'_\Delta := \mathbb{C} \oplus \mathscr{T} \oplus \bigoplus_{M+N \geqslant 1} \mathscr{W}_\Delta(M, N), \tag{I.46}$$

depends on three parameters $0 < \rho < 1/16$, $0 < \xi < 1$, and $\mu > 0$ (the scaling parameter in Hypothesis H-3), which we collect in the triple

$$\Delta = (\mu, \rho, \xi). \tag{I.47}$$

In (I.46),

$$\mathscr{T} := \{ T: [0, 1] \to \mathbb{C} \mid \|T'\|_\infty < \infty,\ T(0) = 0 \} \tag{I.48}$$

denotes the Lipschitz continuous functions on $[0, 1]$ that vanish at the origin. Furthermore, $\mathscr{W}_\Delta(M, N)$ is the Banach space of functions

$$w_{M,N}: [0, 1] \times B_1^M \times B_1^N \to \mathbb{C}, \tag{I.49}$$

Lipschitz continuous in the first variable and such that

$$\|w_{M,N}\|_\Delta < \infty, \tag{I.50}$$

where $B_r := \{k \in \mathbb{R}^2 \mid |k| \leqslant r\}$, B_r^M is the cartesian product of M copies of B_r with B_1^M for $M = 0$ omitted, and $\|w_{M,N}\|_\Delta$ is the norm defined by

$$\|w_{M,N}\|_\Delta := \max\{\xi^{-(M+N)} \|w_{M,N}\|_\Delta^{(\infty)},\ (C_d \xi \rho^{1/2})^{-(M+N)} \|\partial_r w_{M,N}\|_\Delta^{(1)}\}, \tag{I.51}$$

where

$$\|w_{M,N}\|_\Delta^{(\infty)} := \sup_{[0,1] \times B_1^{M+N}} \left\{ |w_{M,N}[r; k^{(M)}, \tilde{k}^{(N)}]| \prod_{i=1}^M \omega(k_i)^{(d/2\gamma) - 1 - (\mu/2)} \prod_{j=1}^N \omega(\tilde{k}_j)^{(d/2\gamma) - 1 - (\mu/2)} \right\} \tag{I.52}$$

and

$$\|w_{M,N}\|_\Delta^{(1)} := \sup_{[0,1]} \left\{ \int |w_{M,N}[r; k^{(M)}, \tilde{k}^{(N)}]| \prod_{i=1}^M \frac{d^d k_i}{\omega(k_i)^{(d/2\gamma) + (\mu/2)}} \prod_{j=1}^N \frac{d^d \tilde{k}_j}{\omega(\tilde{k}_j)^{(d/2\gamma) + (\mu/2)}} \right\}. \tag{I.53}$$

We denote elements of $\bigoplus_{M+N\geqslant 1} \mathscr{W}_\Delta(M, N)$ by $\underline{W} \equiv \{w_{M,N}\}_{M+N\geqslant 1}$. The linear space $\mathscr{W}'_\Delta$ is a Banach space with norm given by

$$\|(E, T, \underline{W})\|'_\Delta := \max\{|E|, \|T'\|_\infty, \|\underline{W}\|_\Delta := \max_{M+N\geqslant 1} \|w_{M,N}\|_\Delta\}, \tag{I.54}$$

for $(E, T, \underline{W}) \in \mathscr{W}'_\Delta$.

To every element $(E, T, \underline{W}) \in \mathscr{W}'_\Delta$ we assign the operator

$$H := \chi[H_f < 1](E \cdot \mathbf{1} + T[H_f] + W)\, \chi[H_f < 1] \tag{I.55}$$

on $\mathscr{H}_{\text{red}}$, where $W = \sum_{M+N\geqslant 1} W_{M,N}$, with

$$W_{M,N} = \int dk^{(M)}\, d\tilde{k}^{(N)} a^\dagger(k^{(M)})\, w_{M,N}[H_f; k^{(M)}, \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}). \tag{I.56}$$

Clearly, H in (I.55) uniquely determines an element $(E, T, \underline{W}) \in \mathscr{W}'_\Delta$, and $H \equiv (E, T, \underline{W}) \in \mathscr{W}'_\Delta$ whenever this appears to be convenient. Furthermore, operators of the form (I.56) are called (M, N)-*monomials* and the functions $w_{M,N}$ entering their definition, *coupling functions* of W_{MN}. Since the correspondence between $W = \sum_{M+N\geqslant 1} W_{M,N}$ and $\underline{W} \in \bigoplus_{M+N\geqslant 1} \mathscr{W}_\Delta(M, N)$ is one-to-one, as well, we also write W instead of $\underline{W}$ whenever this appears to be convenient.

To control the z-dependence of the operators $H[z] \in \mathscr{W}'_\Delta$, we introduce the Banach space, $\mathscr{W}_\Delta$, of analytic families of bounded operators, $H: D_{1/2} \to \mathscr{B}[\mathscr{H}_{\text{red}}]$, parametrized by elements $H[z] \equiv (E[z], T[z], \underline{W}[z]) \in \mathscr{W}'_\Delta$ with the property that

$$\|H[\cdot]\|_\Delta := \sup_{z \in D_1} \|H[z]\|'_\Delta < \infty. \tag{I.57}$$

I.2.2. (*Unprojected*) *renormalization map on* $\mathscr{W}_\Delta$. In order to elucidate general features of the infrared renormalization problem studied in this paper, we first introduce a formal renormalization map, $\hat{\mathscr{R}}_\rho$, defined on a subset of the Banach space $\mathscr{W}_\Delta$ and then sketch some properties of orbits under iterations of the map $\hat{\mathscr{R}}_\rho$ by identifying the fixed points of $\hat{\mathscr{R}}_\rho$ and the stable and unstable manifolds through these fixed points. We define a cylinder $\hat{\mathscr{C}} \subseteq \mathscr{W}'_\Delta$ by

$$\hat{\mathscr{C}} := \{H[z] \in \mathscr{W}'_\Delta \mid |\arg E[z]| < \theta_0, |\partial_r T[z] - \lambda| \leqslant \delta, |\arg \lambda| < \theta_0,$$
$$|\lambda| > 0, \|W[z]\|'_\Delta \leqslant \varepsilon, |\arg z| \geqslant 4\theta_0\}. \tag{I.58}$$

where δ and ε are small constants (depending on λ), and $\theta_0 > 0$ is sufficiently large.

The map $\hat{\mathscr{R}}_\rho$ is defined by

$$\hat{\mathscr{R}}_\rho(H)[z] := \rho^{-1}\Gamma_\rho[\mathscr{F}_{\chi[H_f<\rho]}(H[z]-z) + z\chi[H_f<\rho]]\,\Gamma_\rho^*, \quad (I.59)$$

for $H[z] \in \hat{\mathscr{C}}$. The renormalization map $\hat{\mathscr{R}}_\rho$ has the following properties:

(1) The fixed points of $\hat{\mathscr{R}}_\rho$ are the operators in

$$\mathscr{F}\mathscr{P} := \{\lambda H_f \mid |\arg\lambda| < \theta_0\}. \quad (I.60)$$

(2) At a point $\lambda H_f \in \mathscr{F}\mathscr{P}$ ($\lambda \neq 0$, $|\arg\lambda| < \theta_0$), the space $\mathscr{W}_\Delta$ can be split into a direct sum, $\mathscr{R} \oplus \mathscr{M} \oplus \mathscr{I}$, of a one-dimensional subspace, $\mathscr{R}$, of relevant perturbations, defined by

$$\mathscr{R} := E[z] \cdot \mathbf{1}, \quad (I.61)$$

a one-dimensional subspace of marginal perturbations,

$$\mathscr{M} := \{\mu H_f \mid \mu \in \mathbb{C}\}, \quad (I.62)$$

and a co-dimension-2 subspace, $\mathscr{I}$, of irrelevant perturbations, defined by

$$\mathscr{I} := \{\underline{W} \mid \|\underline{W}\|_\Delta < \infty\}. \quad (I.63)$$

(3) The expansion rate of $\hat{\mathscr{R}}_\rho$ in the direction of $\mathscr{R}$ is given by ρ^{-1}, in the direction of $\mathscr{M}$ it is $=0$, and the contraction rate of $\hat{\mathscr{R}}_\rho$ in the direction of $\mathscr{I}$ is $\geqslant \rho^{\mu/2}$. An orbit of an operator in $\hat{\mathscr{C}}$ is sketched in Fig. 4.

Our interest in the renormalization map $\hat{\mathscr{R}}_\rho$ lies in the circumstance that it is *isospectral* in the sense of Theorem II.1. Of course, the *low-lying*

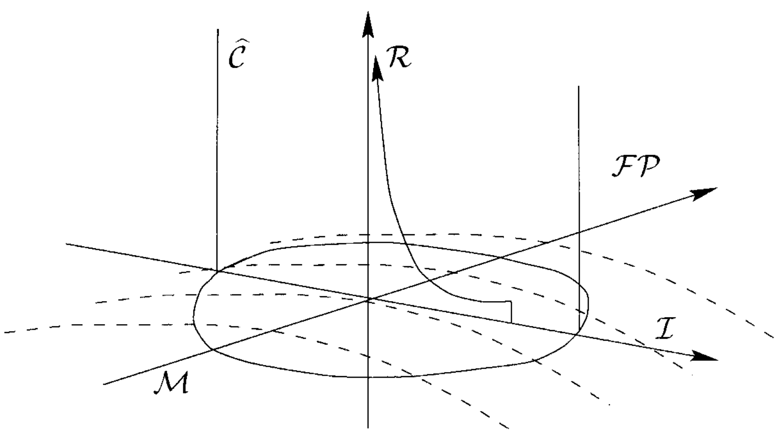

FIG. 4. Orbit under $\hat{\mathscr{R}}$ starting at H_0.

spectrum of $H[z]$ is related to the spectrum of $\hat{\mathcal{R}}_\rho(H)[z]$ by some invertible map, $\mathbb{C} \to \mathbb{C}$. Thus, in order to study, e.g., the resolvent set of a family $H[z] \in \hat{\mathcal{C}}$, we may study the resolvent set of $\hat{\mathcal{R}}^n_\rho(H)[z]$. Since the perturbation $W = \sum_{M+N \geqslant 1} W_{M,N}$ becomes small under iterations of $\hat{\mathcal{R}}_\rho$, the operators $\hat{\mathcal{R}}^n_\rho(H)[z]$ are simpler to analyze than the original operator $H[z]$.

A difficulty in analyzing orbits of families of operators in $\hat{\mathcal{C}}$ under iterations of $\hat{\mathcal{R}}_\rho$ is the divergence of such orbits in the direction of the relevant perturbations, $\mathcal{R}$. This difficulty can be avoided by projecting orbits along $\mathcal{R}$ onto the stable manifold of $\hat{\mathcal{R}}_\rho$ and by successive fine-tuning of the initial value of the spectral parameter, z, in such a way that it approaches an eigenvalue of the operator $H_g(\theta)$. Some details of our construction of such a modified renormalization map, $\mathcal{R}_\rho$, are described in the next subsection.

I.2.3. *Projected Renormalization Map on $\mathcal{W}_\Delta$.* We define a polydisc, $\mathcal{B}(\delta, \varepsilon)$, of operators in $\mathcal{W}_\Delta$ by

$$\mathcal{B}(\delta, \varepsilon) := \{ (E, T, \underline{W}) \in \mathcal{W}_\Delta \mid |\partial_r T - 1| \leqslant \delta,\ \|\underline{W}\|_\Delta \leqslant \varepsilon,\ |E| \leqslant \varepsilon \}. \tag{I.64}$$

Next, we pick $H \in \mathcal{B}(1/8, 1/16)$ and define

$$Z\colon \mathcal{U}^{(\mathrm{in})} \to D_{1/2}, \qquad \zeta \mapsto \frac{1}{\rho}(\zeta - E[\zeta]), \tag{I.65}$$

where

$$\mathcal{U}^{(\mathrm{in})} := \{ \zeta \in D_{1/2} \mid |\zeta - E[\zeta]| \leqslant \rho/2 \} \tag{I.66}$$

(see Fig. 5). We observe that $\zeta \in \mathcal{U}^{(\mathrm{in})}$, $H \in \mathcal{B}(1/8, 1/16)$, and $0 < \rho \leqslant 1/16$ imply that $|\zeta| \leqslant |E[\zeta]| + |\zeta - E[\zeta]| \leqslant 1/4$. Thus,

$$\mathcal{U}^{(\mathrm{in})} \subseteq D_{1/4}. \tag{I.67}$$

Then Cauchy's estimate with contour on $\partial D_{1/2}$ yields that

$$|\partial_\zeta Z(\zeta) - 1| \leqslant 4 \sup_{\zeta \in D_{1/2}} \{ |E[\zeta]| \} < \tfrac{1}{2}. \tag{I.68}$$

This proves that $Z\colon \mathcal{U}^{(\mathrm{in})} \to D_{1/2}$ is a bijection.

We now proceed to defining a projected renormalization map, $\mathcal{R}_\rho$. For $H \in \mathcal{B}(1/8, 1/16)$ and $z \in D_{1/2}$, we set

$$\mathcal{R}_\rho(H)[z] - z := (\mathcal{S}_\rho \circ \mathcal{D}_\rho)[H[Z^{-1}(s)] - Z^{-1}(z)], \tag{I.69}$$

where

$$\mathcal{D}_\rho(X) := \mathcal{F}_{\chi_\rho}(X) \qquad \text{and} \tag{I.70}$$

$$\mathcal{S}_\rho(Y) := \rho^{-1} \Gamma_\rho Y \Gamma_\rho^*, \tag{I.71}$$

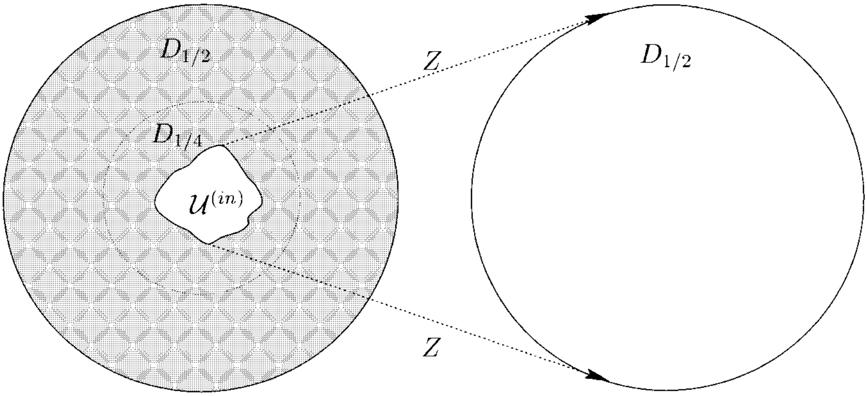

FIG. 5. Rescaling of the spectral parameter.

with $\chi_\rho \equiv \chi[H_f < \rho]$. We note that $\mathscr{D}_\rho$ is a decimation map, projecting out all degrees of freedom corresponding to energies $\geqslant \rho$, while $\mathscr{S}_\rho$ changes the maximal energy scale back to unity. More explicitly, $\mathscr{R}_\rho(H)$ is given by

$$\mathscr{R}_\rho(H)[z] = \frac{1}{\rho} \Gamma_\rho[\mathscr{F}_{\chi_\rho}(H[Z^{-1}(z)] - Z^{-1}(z)) - (E[Z^{-1}(z)] + Z^{-1}(z)) \chi_\rho] \Gamma_\rho^*. \tag{I.72}$$

Our strategy in applying the projected renormalization map, $\mathscr{R}_\rho$, is as follows. First, we demonstrate in Section III.2 that the initial operator family $H_{(0)}[z]$, defined in (I.41), belongs to $\mathscr{B}(1/16, 1/16)$,

$$H_{(0)} \in \mathscr{B}(\tfrac{1}{16}, \tfrac{1}{16}). \tag{I.73}$$

Then we show that $\mathscr{R}_\rho$ maps $\mathscr{B}(\delta, \varepsilon)$ into $\mathscr{B}(\delta + \eta\varepsilon, \eta\varepsilon)$ with $\eta < 1/2$, provided ρ is sufficiently small,

$$\mathscr{R}_\rho : \mathscr{B}(\delta, \varepsilon) \to \mathscr{B}(\delta + \eta\varepsilon, \eta\varepsilon). \tag{I.74}$$

This obviously enables us to iterate $\mathscr{R}_\rho$ on $H_{(0)}[z]$, for suitable values of $z \in D_{1/2}$,

$$H_{(n)}[z] \equiv (E_{(n)}[z], T_{(n)}[z], \underline{W}_{(n)}[z]) := \mathscr{R}_\rho^n(H_{(0)})[z]. \tag{I.75}$$

The isospectral property of $\mathscr{R}_\rho$ guarantees that

$$\mathscr{Z}_{(n)}^{-1}[z] \in \sigma_\#(H_g(\theta)) \Leftrightarrow 0 \in \delta_\#(H_{(n)}[z] - z), \tag{I.76}$$

where

$$\mathscr{Z}_{(n)}^{-1} := Z_{(0)}^{-1} \circ Z_{(1)}^{-1} \circ \cdots \circ Z_{(n)}^{-1} : D_{1/2} \to \mathscr{U}_{(n)}^{(\mathrm{in})}, \tag{I.77}$$

with $Z_{(0)}(\zeta) = e^{i\vartheta} \rho_0^{-1} \zeta$ on $D_{\rho 0/2}$, and, for $n \geqslant 1$,

$$Z_{(n)}: \mathscr{U}_{(n)}^{(\mathrm{in})} \to D_{1/2}, \qquad \zeta \mapsto \frac{1}{\rho} (\zeta - E_{(n-1)}[\zeta]), \tag{I.78}$$

where

$$\mathscr{U}_{(n)}^{(\mathrm{in})} := \{ \zeta \in D_{1/2} \mid |\zeta - E_{(n-1)}[\zeta]| \leqslant \rho/2 \}. \tag{I.79}$$

I.3. *Organization of the Paper*

Our paper is organized as follows. In Chapter II we introduce the Feshbach map which we apply the first time to $H_g(\theta)$ in Chapter III to eliminate the particle and high photon-energy degrees of freedom. The resulting operator, suitably rescaled, is the initial operator $H_{0)}$ (defined in (I.41)) for the iteration of $\mathscr{R}_\rho$. The proof that $H_{(0)}$ actually lies in the ball $\mathscr{B}(1/16, 1/16)$ about H_f is given in Chapter IV. In Chapter IV we show that $\mathscr{R}_\rho$ is a contraction in the sense that it maps $\mathscr{B}(\delta, \varepsilon)$ into $\mathscr{B}(\delta + \eta\varepsilon, \eta\varepsilon)$, with $\eta \leqslant 1/2$. From this contraction property, we derive Assertions (i) and (ii) of Theorem I.1 in Section V.

Our paper has three appendices. In Appendix A, we develop two algebraic tools, the Pull-trough formula and a generalized form of Wick's theorem, which are used to rewrite $H_{(0)}$ in the form prescribed by (I.42)–(I.43). In Appendix B, we derive the basic norm bound for M, N-monomials of the form (I.43), which is one of the building blocks of our analysis. Finally, in Appendix C we compute integrals over simplices with special attain paid to their behaviour as the number of integration variables gets large.

II. THE FESHBACH MAP

In this chapter we develop an important ingredient of our analysis: the *Feshbach Map*, deriving from the *Feshbach projection method.* On a formal level, at least for the perturbation of finitely degenerate, isolated eigenvalues, it can be found in almost every textbook on Quantum Mechanics, e.g., [7]. To meet our aim of developing a perturbation theory for non-isolated eigenvalues, however, we need to formulate the Feshbach projection method in a framework that provides sufficient generality. Before doing so, we remark that this method is known in the physics literature under the name *Feshbach projection method*, while to mathematicians the *Grushin Problem* or *Krein's Formula* may be more familiar, but are essentially identical.

To formulate the method (here, we use the terminology of [6], especially Ch. VI), we require the following: a separable Hilbert space $\mathscr{H}$, a (bounded, but not necessarily orthogonal) projection $P = P^2$ whose complement is denoted by $\bar{P} := \mathbf{1} - P$, a closed operator H_0 which is densely defined on $\mathscr{D} \subseteq \mathscr{H}$ with $\mathscr{D} \supseteq \operatorname{Ran} P$ and which commutes with P, i.e., $\bar{P} H_0 P = P H_0 \bar{P} = 0$, an operator W which is defined on $\mathscr{D}$, and which is relatively H_0-bounded in a sense made precise below. Furthermore, we set $H := H_0 + W$.

With these ingredients, we (formally) define the Feshbach map $\mathscr{F}_P$ by

$$\begin{aligned} \mathscr{F}_P(H - z) &:= P(H - z)\, P - PH\bar{P}(\bar{P}H\bar{P} - z\bar{P})^{-1}\, \bar{P}HP \\ &= P(H_0 - z) + PWP - PW\bar{P}[\bar{P}H_0 + \bar{P}W\bar{P} - z\bar{P}]^{-1}\, \bar{P}WP. \end{aligned} \tag{II.1}$$

To give meaning to the right side of (II.1) and to make the Feshbach projection method work, it is sufficient to assume that $\bar{P}H\bar{P} - z$ is invertible on $\bar{P}\mathscr{H}$ and that the operators

$$\begin{aligned} &(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}, \qquad PW\bar{P}(\bar{P}H\bar{P} - z)^{-1}, \qquad (\bar{P}H\bar{P} - z)^{-1}\, \bar{P}WP, \\ &PW\bar{P}(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}WP, \qquad PWP \end{aligned} \tag{II.2}$$

extend to bounded operators on $\mathscr{H}$. Indeed, (II.2) implies that $\mathscr{F}_P(H - z)$ is a closed operator on $P\mathscr{H}$ because $\mathscr{F}_P(H - z) - PH_0$ is bounded and PH_0 is closed on $P\mathscr{H}$. Thus, we may regard $\mathscr{F}_P$ as a map from the operators on $\mathscr{H}$ into the operators on $P\mathscr{H}$. The virtue of $\mathscr{F}_P$ is that it provides an isospectral map from a class of operators on $\mathscr{H}$ into the operators on $P\mathscr{H}$. More precisely, assuming (II.2), we can reconstruct the full resolvent $(H - z)^{-1}$ on $\mathscr{H}$ from $[\mathscr{F}_P(H - z)]^{-1}$ on $P\mathscr{H}$ and, if neither exists, we can reconstruct the kernel of $H - z$ from the kernel of $\mathscr{F}_P(H - z)$, and vice versa. Yet, $\mathscr{F}_P(H - z)$ poses a potentially simpler spectral problem, as compared to H, since $P\mathscr{H}$ is contained in $\mathscr{H}$. The tradeoff is the non-linear dependence of the new operator $\mathscr{F}_P(H - z)$ on the spectral parameter z.

Theorem II.1 (Feshbach Projection Method). *Assume that $\bar{P}H\bar{P} - z$ is invertible on $\bar{P}\mathscr{H}$ and that*

$$\begin{aligned} &(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}, \qquad PW\bar{P}(\bar{P}H\bar{P} - z)^{-1}, \qquad (\bar{P}H\bar{P} - z)^{-1}\, \bar{P}WP, \\ &PW\bar{P}(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}WP, \qquad PWP \end{aligned} \tag{II.3}$$

all extend to bounded operators on $\mathscr{H}$. Then

(a) *the operator $\mathscr{F}_P(H - z)$ is invertible on $P\mathscr{H}$ if and only if $H - z$ is invertible on $\mathscr{H}$. In this case $[\mathscr{F}_P(H - z)]^{-1} = P(H - z)^{-1}\, P$.*

(b) *if* $H\psi = z\psi$, *for some eigenvector* $0 \neq \psi \in \mathscr{H}$, *then the projected vector* $0 \neq P\psi \in P\mathscr{H}$ *solves* $\mathscr{F}_P(H-z)\, P\psi = 0$,

(c) *if* $\mathscr{F}_P(H-z)\, \varphi = 0$, *for some eigenvector* $0 \neq \varphi = P\varphi \in P\mathscr{H}$, *then the vector* $0 \neq \psi \in \mathscr{H}$, *defined by* $\psi := [P\_(\bar{P}H\bar{P}-z)^{-1}\, \bar{P}WP]\, \varphi$, *solves* $H\psi = z\psi$.

(d) $$\dim \operatorname{Ker}[H-z] = \dim \operatorname{Ker}[\mathscr{F}_P(H-z)]. \tag{II.4}$$

Proof. Passing from H to H_0 to $H+z$ and H_0+z, respectively, we may henceforth assume that $z=0$.

(a) Let us first assume that H is invertible on $\mathscr{H}$. Then

$$\begin{aligned}\mathscr{F}_P(H)\, PH^{-1}P &= [PHP - PH\bar{P}(\bar{P}H\bar{P})^{-1}\, \bar{P}HP]\, PH^{-1}P \\ &= PH(1-\bar{P})\, H^{-1}P - PH\bar{P}(\bar{P}H\bar{P})^{-1}\, \bar{P}H(1-\bar{P})\, H^{-1}P \\ &= P - PH\bar{P}H^{-1}P + PH\bar{P}(\bar{P}H\bar{P})^{-1}\, \bar{P}H\bar{P}H^{-1}P = P.\end{aligned} \tag{II.5}$$

Similarly, $PH^{-1}P\mathscr{F}_P(H) = P$. So we conclude that $\mathscr{F}_P(H)^{-1} = PH^{-1}P$.

Conversely, assume that $\mathscr{F}_P(H)^{-1}$ exists on $P\mathscr{H}$. Then the operator R on $\mathscr{H}$, defined below, is bounded.

$$PRP := \mathscr{F}_P(H)^{-1} \tag{II.6}$$

$$PR\bar{P} := -\mathscr{F}_P(H)^{-1}\, PW\bar{P}(\bar{P}H\bar{P})^{-1}, \tag{II.7}$$

$$\bar{P}RP := -(\bar{P}H\bar{P})^{-1}\, \bar{P}WP\mathscr{F}_P(H)^{-1}, \tag{II.8}$$

$$\bar{P}R\bar{P} := (\bar{P}H\bar{P})^{-1} + (\bar{P}H\bar{P})^{-1}\, \bar{P}WP\mathscr{F}_P(H)^{-1}\, PW\bar{P}(\bar{P}H\bar{P})^{-1}. \tag{II.9}$$

Somewhat involved but straightforward algebra yields $RH = HR = 1$, and thus $r = H^{-1}$, proving (a).

(b) Assume $H\psi = 0$, for some $0 \neq \psi \in \mathscr{H}$. Using $P = \mathbf{1} - \bar{P}$, it is easily checked that this implies $\mathscr{F}_P(H)\, P\psi = 0$. Moreover, $0 = \bar{P}H\psi = \bar{P}H\bar{P}\psi + \bar{P}WP\psi$ which is equivalent to $\bar{P}\psi = -(\bar{P}H\bar{P})^{-1}\, \bar{P}WP\psi$. Thus, we get

$$\|P\psi\| \geqslant (1 + \|(\bar{P}H\bar{P})^{-1}\, \bar{P}WP\|)^{-1}\, \|\psi\| > 0, \tag{II.10}$$

which yields (b).

(c) Assume that $0 \neq \varphi = P\varphi \in P\mathscr{H}$ solves $\mathscr{F}_P(H)\varphi = 0$, and set $\psi := [P - (\bar{P}H\bar{P})^{-1}\, \bar{P}WP]\, \varphi$. A simple computation shows that $H\psi = 0$. We also have $P\psi = P\varphi$, implying $P\varphi = [1 + (\bar{P}H\bar{P})^{-1}\, \bar{P}WP]\, \psi$. Hence,

$$\|\psi\| \geqslant (1 + \|(\bar{P}H\bar{P})^{-1}\, \bar{P}WP\|)^{-1}\, \|\varphi\| > 0, \tag{II.11}$$

which yields (c).

(d) This part follows by linearity from (II.10) and (II.11). ∎

Next, we derive Assumption (II.3) in Theorem II.1 from more explicit conditions on the perturbation W and the unperturbed operator H_0. In fact, in Lemmata II.2 and II.3 below, we give two independent sets of conditions on W to justify (II.3). In Lemma II.2, we assume W to be a relatively bounded perturbation of the operator H_0. In contrast, in Lemma II.3, H_0 is required to be normal and m-sectorial, and W is treated as a relatively bounded perturbation of the closed sectorial form corresponding to H_0 in Lemma II.3. The main example for H_0 and P which meets all these requirements is $H_0 = T(A)$, where T is a function of a positive operator $A \geqslant 0$ which obeys $|T'(r) - 1| \ll 1$, and $P = \chi_\Delta(A)$ is the spectral projection of A onto an interval $\Delta \subseteq \mathbb{R}$. It depends on the context which set of conditions is more convenient to work with. In the present paper, it is more natural to formulate the renormalization scheme in Chapter IV using Lemma II.3.

Lemma II.2 *Assume H_0 to be a closed operator, densely defined on $\mathscr{D} \subseteq \mathscr{H}$ and P to be a bounded projection commuting with H_0 (as explained above). Suppose that $R_0 := (H_0 - z)^{-1}$ exists on $\bar{P}\mathscr{H}$, and assume that*

$$\| W\bar{P}R_0 \| < 1, \qquad \| WP \| < \infty. \tag{II.12}$$

Then H is a closed operator with dense domain $\mathscr{D} \subseteq \mathscr{H}$, and $\mathscr{F}_P(H - z)$ defines a closed operator with dense domain $P\mathscr{D} \subseteq P\mathscr{H}$. Moreover, $\bar{P}H\bar{P} - z$ is invertible on $\bar{P}\mathscr{H}$, and

$$\begin{aligned} &(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}, && PW\bar{P}(\bar{P}H\bar{P} - z)^{-1}, && (\bar{P}H\bar{P} - z)^{-1}\, \bar{P}WP, \\ &PW\bar{P}(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}WP, && PWP && \end{aligned} \tag{II.13}$$

all define bounded operators on $\mathscr{H}$.

Proof. To prove that H is closed, we write $H = H_0 + W\bar{P} + WP$. By assumption, WP is bounded and $W\bar{P}$ is relatively H_0-bounded with bound strictly smaller than one. It follows (see, e.g., [6, Ch. VI]) that H is closed on the same domain $\mathscr{D} \subseteq \mathscr{H}$ as H_0 is.

Next, we construct $(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}$ and $W(\bar{P}H\bar{P} - z)^{-1}\, \bar{P}$ from Neumann series

$$(\bar{P}H\bar{P} - z)^{-1}\, \bar{P} = \sum_{n=0}^{\infty} (-1)^n\, R_0\bar{P}(W\bar{P}R_0)^n, \tag{II.14}$$

$$W(\bar{P}H\bar{P} - z)^{-1}\, \bar{P} = \sum_{n=0}^{\infty} (-1)^n\, W\bar{P}R_0\bar{P}(W\bar{P}R_0)^n, \tag{II.15}$$

which converge to bounded operators because $\|W\bar{P}R_0\| < 1$ and $\|R_0\bar{P}\| < \infty$. We complete the construction of the operators in (II.13) by observing that

$$\|(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}WP\| \leqslant \|(\bar{P}H\bar{P}-z)^{-1}\bar{P}\|\cdot\|WP\| < \infty, \tag{II.16}$$

$$\|PW\bar{P}(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}WP\| \leqslant \|PW\bar{P}(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}\|\cdot\|WP\| < \infty, \tag{II.17}$$

$$\|PWP\| \leqslant \|P\|\cdot\|WP\| < \infty. \tag{II.18}$$

Thus, $\mathscr{F}_P(H-z)-PH_0$ is bounded and $\mathscr{F}_P(H-z)$ is closed on $P\mathscr{D}\subset P\mathscr{H}$ since PH_0 is. ∎

LEMMA II.3. *Assume H_0 to be a normal, m-sectorial operator, densely defined on $\mathscr{D}\subseteq\mathscr{H}$ and P to be a bounded projection commuting with H_0 (as explained above). Suppose that $R_0 := (H_0-z)^{-1}$ and, thus, $R_0^{1/2} := |H_0-z|^{-1/2}$ exists on $\bar{P}\mathscr{H}$, and assume that*

$$\|R_0^{1/2}\bar{P}W\bar{P}R_0^{1/2}\| < 1 \tag{II.19}$$

and

$$\|R_0^{1/2}\bar{P}WP\|, \qquad \|PW\bar{P}R_0^{1/2}\|, \qquad \|PWP\| < \infty. \tag{II.20}$$

Then H is a closed m-sectorial operator with dense domain $\mathscr{D}'\subseteq\mathscr{H}$, and $\mathscr{F}_p(H)$ defines a closed operator with dense domain $P\mathscr{D}'\subseteq P\mathscr{H}$. Moreover, $\bar{P}H\bar{P}-z$ is invertible on $\bar{P}\mathscr{H}$, and

$$(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}, \qquad PW\bar{P}(\bar{P}H\bar{P}-z)^{-1}, \qquad (\bar{P}H\bar{P}-z)^{-1}\,\bar{P}WP,$$
$$PW\bar{P}(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}WP, \qquad PWP \tag{II.21}$$

all define bounded operators on $\mathscr{H}$.

Proof. To prove that H is closed, we first consider the quadratic form associated with H_0-z which is closed and sectorial, since H_0 is closed and m-sectorial. The bounds (II.19) and (II.20) guarantee that W is a relatively (H_0-z)-bounded perturbation with relative form bound strictly smaller than 1. It follows (see e.g. [6, Ch. VI]) that H is a closed sectorial quadratic form with form domain $\mathscr{Q}$ which arises from a unique closed m-sectorial operator H with dense domain $\mathscr{D}'$.

Next, we construct $A\bar{P}(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}B$, for arbitrary bounded A and B, using

$$\begin{aligned} &A\bar{P}(\bar{P}H\bar{P}-z)^{-1}\,\bar{P}B \\ &\quad = A\bar{P}R_0^{1/2}(U+R_0^{1/2}\bar{P}W\bar{P}R_0^{1/2})^{-1}\,\bar{P}R_0^{1/2}\bar{P}B \\ &\quad = A\bar{P}R_0^{1/2}\bar{P}\left[\sum_{n=0}^{\infty} U^*(-R_0^{1/2}\bar{P}W\bar{P}R_0^{1/2}U^*)^n\right]\bar{P}R_0^{1/2}\bar{P}B, \end{aligned} \tag{II.22}$$

where $U^* = U^{-1} := |H_0 - z| \, (H_0 - z)^{-1} \, \bar{P}$ is the unitary on $\bar{P}\mathscr{H}$ resulting from the polar decomposition. Condition (II.19) together with $\|U^*\| = 1$ ensures the norm-convergence of the series in (II.22), and we may estimate

$$\|A\bar{P}(\bar{P}H\bar{P} - z)^{-1} \, \bar{P}B\| \leqslant C \cdot \|A\bar{P}R_0^{1/2}\| \cdot \|R_0^{1/2} \bar{P}B\|, \tag{II.23}$$

for some finite constraint C. Inserting $A := \mathbf{1}$ or $A := PW$, and $B := \mathbf{1}$ or $B := WP$, the assumption (II.20) and $\|R_0 \bar{P}\| < \infty$ yields the boundedness of the operators in (II.21).

Turning to $\mathscr{F}_p(H - z)$, the bounds in (II.21) imply that $\mathscr{F}_p(H - z) - PH_0$ is bounded. Thus, $\mathscr{F}_P(H - z)$ is closed on $P\mathscr{D}$, since PH_0 is. ∎

III. ELIMINATION OF PARTICLE-AND HIGH PHOTON-ENERGY DEGREES OF FREEDOM

In this chapter, we perform the first step of our renormalization group analysis. Recall from Subsection I.2.1 the construction of an effective operator, $H_{(0)}[z]$, defined in (I.41), that we obtain in several steps from the Hamiltonian $H_g - z$ in (I.4). We decimate the degrees of freedom corresponding to the particles and to the photons of energies $\geqslant \rho_0$ by means of the Feshbach map $\mathscr{F}_{P_0}$, introduced in Eqn. (II.1). Here

$$P_0 = P_{el} \otimes \chi[H_f < \rho_0], \tag{III.1}$$

where $P_{el} = |\varphi_{el}\rangle \, \langle \varphi_{el}|$ is the orthogonal projection onto the one-dimensional eigenspace $\mathbb{C}\varphi_{el}$ of the atomic Hamiltonian H_{el} corresponding to the eigenvalue zero (see Hypothesis H-1), and $\chi[H_f < \rho_0]$ is the spectral projection of H_f onto the subspace of vectors in Fock space with field energy $< \rho_0$. Then, we rescale photon momenta as $k \mapsto \rho_0^{1/\gamma} k$, by means of the unitary dilatation Γ_{ρ_0} defined in (I.9), and we rescale the energies as $E \mapsto \rho_0^{-1} E$. In addition, we change the spectral parameter z as $Z_{(0)}: z \mapsto e^{\theta} \rho_0^{-1} z$.

Carrying out these steps as described in Eqns. (I.32)–(I.41), we obtain a family, $z \mapsto H_{(0)}[z]$, of effective Hamiltonian operators,

$$P_{el} \otimes H_{(0)}[z] = \frac{e^{-i\vartheta}}{\rho_0} \, \Gamma_{\rho_0}[\mathscr{F}_{P_0}(H_g - Z_{(0)}^{-1}[z]) + Z_{(0)}^{-1}[z] \, \chi[H_f < \rho_0]] \, \Gamma_{\rho_0}^*, \tag{III.2}$$

on $\mathscr{H}_{\mathrm{red}} = \mathrm{Ran} \, \chi[H_f < 1]$. Since the Feshbach map is isospectral, in the sense of Theorem II.1, the following crucial property guarantees that the analysis of the spectrum of H_g is equivalent to the analysis of the spectrum of $H_{(0)}$. Indeed, Theorem II.1 implies that

$$z \in \sigma_{\#}(H_g) \Leftrightarrow 0 \in \sigma_{\#}(H_g - z)$$
$$\Leftrightarrow 0 \in \sigma_{\#}(H_{(0)}[Z_{(0)}(z)] - Z_{(0)}(z)), \tag{III.3}$$

where $\sigma_{\#} = \sigma$ or $\sigma_{\#} = \sigma_{\mathrm{pp}}$, under the assumption that $\bar{P}_0(H_g - z)\,\bar{P}_0$ is invertible on Ran $\bar{P}_0$. In Section III.1, we verify this condition for all $z \in D_{\rho_0/2} := \{|z| \leqslant \rho_0/2\}$.

Next, we expand the resolvent $\bar{P}_0(\bar{P}_0 H_g(\theta)\,\bar{P}_0 - \zeta)^{-1}\,\bar{P}_0$, where $\bar{P}_0 = \mathbf{1} - P_0$, entering $\mathscr{F}_{P_0}(H_g(\theta) - \zeta)$ in (III.3), in powers of $\bar{P}_0 W_g \bar{P}_0$. Using a straightforward generalization of Wick's theorem (which is derived in Appendix A) and the estimates from Section III.1, we prove the following theorem (which is our main result of this chapter).

Theorem III.1. *Let* $\theta := i\vartheta$ *for some* $0 < \vartheta < \pi/2$. *Pick* $0 < \rho_0 \leqslant \rho^{(\mathrm{out})} = 2^{-1/2} \sin(\vartheta/2)$ *and* $z \in D_{1/2}$. *Assume Hypotheses* H-1, H-2, H-3, *and*

$$\frac{g\Lambda_6}{\rho_0^{1/2}} \leqslant \frac{1}{100}, \tag{III.4}$$

where Λ_1, Λ_5, Λ_6, *and* C_d *are defined in* (I.18), (I.21), *and* (I.22)–(I.28). *Then, for* $z \in D_{1/2}$, $H_{(0)}[z]$ *is well-defined and representable in the form*

$$H_{(0)}[z] = \chi_1(E_{(0)}[z] + T_{(0)}[z] + W_{(0)}[z])\,\chi_1 \in \mathscr{W}'_{\Delta}, \tag{III.5}$$

where $\chi_1 \equiv \chi[H_f < 1]$, *and the coupling functions of* $W_{(0)}[z]$,

$$w^{(0)}_{M,N}\colon D_{1/2} \times [0, 1) \times B_1^M \times B_1^N \to \mathbb{C}, \tag{III.6}$$

obey the following estimates, for $\Delta \equiv (\mu, \rho, 100 g\Lambda_6)$.

$$\|w^{(0)}_{M,N}[z]\|^{(\infty)}_{\Delta} \leqslant 2, \tag{III.7}$$

$$\|\partial_r w^{(0)}_{M,N}[z]\|^{(1)}_{\Delta} \leqslant 25, \tag{III.8}$$

for $M + N \geqslant 1$, *and the components* $E_{(0)}[z]$ *and* $T_{(0)}[z;\,\cdot\,]$ *obey the estimate*

$$|E_{(0)}[z]| \leqslant 2\left(\frac{100 g\Lambda_6}{\rho_0^{1/2}}\right)^2, \tag{III.9}$$

$$|T_{(0)}[z; r] - r| \leqslant 4\left(\frac{100 g\Lambda_6}{\rho_0^{1/2}}\right)^2, \tag{III.10}$$

$$|\partial_r T_{(0)}[z; r] - 1| \leqslant 25\left(\frac{100 g\Lambda_6}{\rho_0^{1/2}}\right)^2. \tag{III.11}$$

Theorem III.1 has the following important consequence.

COROLLARY III.2 *Under the same hypotheses of Theorem* III.1, *and for* $g > 0$ *so small that*

$$\delta := 25(100 g \Lambda_6 \rho_0^{-1/2}) \leqslant \tfrac{1}{8}, \tag{III.12}$$

$$\varepsilon := 100 g \Lambda_6 \max\{200 g \Lambda_6 \rho_0^{-1}, 2\xi^{-2}, 25 C_d^{-1} \xi^{-1} \rho^{-1/2}\} \leqslant \tfrac{1}{16}, \tag{III.13}$$

one has that

$$H_{(0)} \in \mathscr{B}(\delta, \varepsilon). \tag{III.14}$$

III.1. *Domain of the Feshbach Map* $\mathscr{F}_{P_0}$

In this section, we investigate the Feshbach map $\mathscr{F}_{P_0}$, which is the only non-trivial ingredient entering the definition of $H_{(0)}$. In particular, we verify the assumptions of Theorem II.1, thus establishing that $H_{(0)}[z]$ is well-defined, for all $z \in D_{1/2}$.

Recall from the introduction that the eigenvalue $e_j = 0$ was assumed to be non-degenerate and isolated by a distance 1 from the rest of the spectrum of H_{el}, and that the continua in $\sigma(H_0)$ generated by H_f are branching off the real axis at an angle $0 > -\vartheta > -\pi/2$ into the lower half-plane. We investigate spectral parameters contained in $D_{\rho^{(\mathrm{out})}}$, where $D_\rho := \{z \in \mathbb{C} \mid \rho \geqslant |z|\}$ and

$$\rho^{(\mathrm{out})} := \frac{1}{\sqrt{2}} \sin\left(\frac{\vartheta}{2}\right) < 1. \tag{III.15}$$

We pick $0 < \rho_0 \leqslant \rho^{(\mathrm{out})}$ and $0 < \delta' < \vartheta/2$ to specify two subsets, $\mathscr{U}_{(0)}^{(\mathrm{in})}$ and $\mathscr{U}_{(0)}^{(\mathrm{out})}$, of $D_{\rho^{(\mathrm{out})}}$ (see Fig. 3),

$$\mathscr{U}_{(0)}^{(\mathrm{out})}(\delta') := \{e^{-i(\vartheta+\gamma)} r \in \mathbb{C} \mid \rho_0/2 \leqslant r \leqslant \rho^{(\mathrm{out})}, \delta' \leqslant |\gamma| \leqslant \pi\} \tag{III.16}$$

$$\mathscr{U}_{(0)}^{(\mathrm{in})} := \{\zeta \in \mathbb{C} \mid |\xi| \leqslant \rho_0/2\} = D_{\rho_0/2}. \tag{III.17}$$

The two regions $\mathscr{U}_{(0)}^{(\mathrm{out})}(\delta')$ and $\mathscr{U}_{(0)}^{(\mathrm{in})}$ serve different purposes: First, for all $z \in \mathscr{U}_{(0)}^{(\mathrm{out})}(\delta')$ and sufficiently large $\delta' > 0$, we show directly that $H_g - z$ has a bounded inverse. Secondly, for all $z \in \mathscr{U}_{(0)}^{(\mathrm{in})} = D_{\rho_0/2}$, we show that the spectral properties of $H_g - z$ can be determined from those of $H_{(0)}[Z_{(0)}(z)]$.

The validity of Assumption (II.12) of Theorem II.1, which ensures the convergence of the Neumann series, crucially depends on the parameter $\Lambda_1 \geqslant 1$ that we defined in Hypothesis H-2 as

$$\Lambda_1 := \left\{ \int [1 + \omega(k)^{-1}]\, J^2(k)\, d^d k \right\}^{1/2} < \infty. \tag{III.18}$$

The first main result of this section is

THEOREM III.3. *Let* $\theta := i\vartheta$, *for some* $0 < \vartheta < \pi/2$, *and pick* $0 < \rho_0 \leqslant \rho^{(\mathrm{out})} = 2^{-1/2}\sin(\vartheta/2)$ *and* $0 < \delta' < \vartheta/2$. *Define* P_0, $\mathcal{U}^{(\mathrm{out})}_{(0)}$ *and* $\mathcal{U}^{(\mathrm{in})}_{(0)}$ *as in* (III.16)–(III.17) *above. Assume Hypotheses* H-1, H-2, *and*

$$g < \frac{\rho_0^{1/2}\sin(\vartheta/2)}{24\Lambda_1}. \tag{III.19}$$

Then, for $1 \leqslant M+N \leqslant 2$,

$$\|R_0^{1/2}\bar{P}_0 W_{M,N}\bar{P}_0 R_0^{1/2}\| \leqslant \frac{\Lambda_1^{M+N}Y(z)}{\rho_0^{1/2}\sin(\vartheta/2)}, \tag{III.20}$$

$$\|R_0^{1/2}\bar{P}_0 W_{M,N} P_0\| \leqslant \frac{\Lambda_1^{M+N}Y(z)^{1/2}}{\sin(\vartheta/2)}, \tag{III.21}$$

$$\|P_0 W_{M,N}\bar{P}_0 R_0^{1/2}\| \leqslant \frac{\Lambda_1^{M+N}Y(z)^{1/2}}{\sin(\vartheta/2)}, \tag{III.22}$$

$$\|P_0 W_{M,N} P_0\| \leqslant \Lambda_1^{M+N}\rho_0^{1/2}, \tag{III.23}$$

where

$$Y(z) := \begin{cases} \dfrac{\sin(\vartheta/2)}{\sin(\delta'/2)} & \text{for } \quad z \in \mathcal{U}_0^{(\mathrm{out})}(\delta'), \\ 1 & \text{for } \quad z \in \mathcal{U}_0^{(\mathrm{in})}, \end{cases} \tag{III.24}$$

and $R_0^{1/2} := |H_0 - z|^{-1/2}$. *Furthermore,*

(a) *if, in addition,* $0 < \delta' < \vartheta$ *is chosen large enough, so that* $\sin(\delta') > 24\Lambda_1 g\rho^{-1/2}\sin(\vartheta/2)$, *then* $H_g - z$ *has a bounded inverse for all* $z \in \mathcal{U}^{(\mathrm{out})}_{(0)}(\delta')$.

(b) *for all* $z \in \mathcal{U}^{(\mathrm{in})}_{(0)}$, $H_g - z$ *is isospectral to* $\mathcal{F}_{P_0}(H_g - z)$ *in the sense of Theorem II.*1. *Here,* $\mathcal{F}_{P_0}(H_g - z)$ *is the operator on* $\chi[H_f < \rho_0]\,\mathcal{F}_b[L^2(\mathbb{R}^d)]$ *that results from the Feshbach projection method with projection* P_0.

Proof. First, we derive Estimates (III.20), (III.21), and (III.23) from Lemma III.4 and Lemma III.5. We denote

$$A := |H_{el}| \otimes \mathbf{1} + \mathbf{1} \otimes H_f + \rho_0. \tag{III.25}$$

Then we estimate

$$\|R_0^{1/2}\bar{P}_0 W_{M,N}\bar{P}_0 R_0^{1/2}\| \leqslant \|A^{-1/2}W_{M,N}A^{-1/2}\| \cdot \|R_0^{1/2}\bar{P}_0 A^{1/2}\|^2, \tag{III.26}$$

$$\|R_0^{1/2}\bar{P}_0 W_{M,N}P_0\| \leqslant \|A^{-1/2}W_{M,N}A^{-1/2}\| \cdot \|R_0^{1/2}\bar{P}_0 A^{1/2}\| \cdot \|P_0 A^{1/2}\|, \tag{III.27}$$

$$\|P_0 W_{M,N}\bar{P}_0 R_0^{1/2}\| \leqslant \|A^{-1/2}W_{M,N}A^{-1/2}\| \cdot \|R_0^{1/2}\bar{P}_0 A^{1/2}\| \cdot \|P_0 A^{1/2}\|, \tag{III.28}$$

$$\|P_0 W_{M,N}P_0\| \leqslant \|A^{-1/2}W_{M,N}A^{-1/2}\| \cdot \|P_0 A\|. \tag{III.29}$$

Next, we invoke Lemma III.5 below. Note that $\|A^{-1/2}W_{M,N}A^{-1/2}\|$ refers to the case $M=p$, $N=q$ (hence $p+q\geqslant 1$), and $m=n=0$ in Lemma III.5. Thus (III.42) below applies and yields

$$\|A^{-1/2}W_{M,N}A^{-1/2}\| \leqslant \Lambda_1^{M+N}\rho_0^{-1/2}. \tag{III.30}$$

Bounds (III.20), (III.21), and (III.23) are obtained by inserting Lemma III.4 and (III.30) into (III.26)–(III.29).

Second, we establish (b), assuming that $z\in\mathscr{U}_{(0)}^{(\mathrm{in})}$. Inequalities (III.21) and (III.23) clearly imply that $R_0^{1/2}\bar{P}_0 W_g P_0$, $P_0 W_g \bar{P}_0 R_0^{1/2}$ and $P_0 W_g P_0$ are bounded operators. To demonstrate (b) it thus remains to show that the norm $\|R_0^{1/2}\bar{P}_0 W_g \bar{P}_0 R_0^{1/2}\|$ is strictly smaller than one, appealing to Lemma II.3 and Theorem II.1. We remark that by Assumption (III.19) we have

$$g\Lambda_1 \leqslant (24)^{-1}\rho_0^{1/2}\sin(\vartheta/2) \leqslant \Lambda \leqslant 1/24, \tag{III.31}$$

and thus $\max_{M+N\geqslant 1}\{g^{M+N}\Lambda_1^{M+N}\} = g\Lambda_1$. This, in turn, implies

$$\|R_0^{1/2}\bar{P}_0 W_g \bar{P}_0 R_0^{1/2}\| \leqslant \frac{6\max_{l\geqslant 1}\{g^l\Lambda_1^l\}}{\rho_0^{1/2}\sin(\vartheta/2)} \leqslant 1/4, \tag{III.32}$$

and (b) is proven.

Finally, we prove (a): fixing $z\in\mathscr{U}_{(0)}^{(\mathrm{out})}(\delta')$, we attempt to construct the inverse of $H_g - z$ by a Neumann series as in (II.22)

$$(H_g - z)^{-1} = R_0^{1/2}\left[\sum_{n=0}^{\infty} U^*(-R_0^{1/2}W_g R_0^{1/2}U^*)^n\right] R_0^{1/2}, \tag{III.33}$$

whose convergence is guaranteed by $\|R_0^{1/2}W_g R_0^{1/2}\| < 1$. But this last bound is a simple matter using (III.20), (III.21), (III.23) and

$$\|R_0 P_0\| = \sup_{r>0} |\mathrm{e}^{-i\theta}r - z|^{-1} \leqslant \frac{\sqrt{2}}{\rho_0 \sin(\delta'/2)}, \tag{III.34}$$

which follows from $z = e^{-\theta + i\gamma} t$ with $t > \rho_0/2$ and $|\gamma| \geqslant \delta'$. Thus

$$\begin{aligned}\|R_0^{1/2} W_g R_0^{1/2}\| \leqslant 6 \max_{1 \leqslant M+N \leqslant 2} \{ & \|R_0^{1/2} \bar{P}_0 g^{M+N} W_{M,N} \bar{P}_0 R_0^{1/2}\| \\ & + \|R_0 P_0\|^{1/2} \cdot \|P_0 g^{M+N} W_{M,N} \bar{P}_0 R_0^{1/2}\| \\ & + \|R_0 P_0\|^{1/2} \cdot \|R_0^{1/2} \bar{P}_0 g^{M+N} W_{M,N} P_0\| \\ & + \|R_0 P_0\| \cdot \|P_0 g^{M+N} W_{M,N} P_0\| \} \\ \leqslant \frac{24}{\rho_0^{1/2} \sqrt{1-\cos\delta'}} & \max_{1 \leqslant M+N \leqslant 2} \{ g^{M+N} \Lambda_1^{M+N} \} < 1. \quad \blacksquare \end{aligned} \tag{III.35}$$

The proof of Theorem III.3 above is based on the estimates in Lemmata III.4 and III.5, which we state and prove next.

Lemma III.4. *Assume Hypothesis* H-1, H-2, *and* $\theta := i\vartheta$, $0 < \vartheta < \pi/2$. *Pick* $0 < \rho_0 \leqslant \rho^{(\mathrm{out})}$ *and* $0 < \delta' < \vartheta/2$, *and define* $\mathscr{U}_{(0)}^{(\mathrm{out})}(\delta')$, $\mathscr{U}_{(0)}^{(\mathrm{in})}$ *as in* (III.16), (III.17). *Denote* $A := |H_{el}| \otimes \mathbf{1} + \mathbf{1} \otimes H_f + \rho_0$, *as in* (III.25), *and recall* $R_0 := (H_0 - z)^{-1}$. *Then* $\|P_0 A\| \leqslant 2\rho_0$ *and*

$$\|R_0 \bar{P}_0 A\| \leqslant \frac{6\, Y(z)}{\sqrt{1-\cos\vartheta}} = \begin{cases} 3(\sin(\delta'/2))^{-1} & \text{for } z \in \mathscr{U}_{(0)}^{(\mathrm{out})}(\delta') \\ 3(\sin(\vartheta/2))^{-1} & \text{for } z \in \mathscr{U}_{(0)}^{(\mathrm{in})} \end{cases} \tag{III.36}$$

Proof. The inequality before (III.36) is trivial. To prove (III.36), we introduce the orthogonal decomposition $\bar{P}_0 = \bar{P}_0^{(1)} + \bar{P}_0^{(2)}$, where

$$\bar{P}_0^{(1)} := P_{el} \otimes \chi[H_f \geqslant \rho_0] \qquad \text{and} \qquad \bar{P}_0^{(2)} := \bar{P}_{el} \otimes \mathbf{1}. \tag{III.37}$$

Note that both R_0 and A commute with $\bar{P}_0^{(1)}$ and $\bar{P}_0^{(2)}$. Thus

$$\|R_0 \bar{P}_0 A\| = \max\{ \|R_0 \bar{P}_0^{(1)} A\|, \|R_0 \bar{P}_0^{(2)} A\| \}. \tag{III.38}$$

First, we consider $\mathrm{Ran}\{\bar{P}_0^{(2)}\}$ and $z \in \mathscr{U}_{(0)}^{\mathrm{out})}(\delta') \cup \mathscr{U}_{(0)}^{(\mathrm{in})} \subseteq D_{\rho^{(\mathrm{out})}}$. Then $|z| \leqslant 1/\sqrt{2} \sin(\vartheta/2)$, which yields

$$\|R_0 \bar{P}_0^{(2)} A\| \leqslant \sup_{\substack{r>0 \\ |t| \geqslant 1}} \left| \frac{|t| + r + \rho_0}{t + e^{-i\vartheta} r - z} \right| \leqslant \frac{3\sqrt{2}}{\sqrt{1-\cos\vartheta}} \leqslant \frac{3}{\sin(\vartheta/2)}. \tag{III.39}$$

Second, we consider $\mathrm{Ran}\{\bar{P}_0^{(1)}\}$. Fix $z \in \mathscr{U}_{(0)}^{(\mathrm{out})}(\delta')$. Then $z = \mathrm{e}^{-\vartheta + \gamma} t$ with $|\gamma| \geqslant \delta'$ and $t \geqslant 0$. Hence,

$$\|R_0 \bar{P}_0^{(1)} A\| \leqslant \sup_{r \geqslant \rho_0} \left| \frac{r + \rho_0}{\mathrm{e}^{-i\vartheta} r - z} \right| \leqslant \sup_{r > 0} \left| \frac{2r}{r + \mathrm{e}^{-i\gamma} t} \right| \leqslant \frac{\sqrt{2}}{\sin(\delta'/2)}. \tag{III.40}$$

Finally, on $\mathrm{Ran}\{\bar{P}_0^{(1)}\}$ and for $z \in \mathscr{U}_{(0)}^{(\mathrm{in})}$, we have that

$$\|R_0 \bar{P}_0^{(1)} A\| \leqslant \sup_{r \geqslant \rho_0} \left| \frac{r + \rho_0}{\mathrm{e}^{-i\vartheta} r - z} \right| \leqslant \sup_{r \geqslant \rho_0} \left| \frac{r + \rho_0}{r - \rho_0/2} \right| \leqslant 4 \leqslant \frac{3}{\sin(\vartheta/2)}. \quad \blacksquare \tag{III.41}$$

Next, we impose suitable conditions on J so that every factor of $a^\dagger$ or a in W_g accounts for a factor of $H_f^{1/2}$ in the relative bound.

LEMMA III.5. *Assume Hypotheses* H-1 *and* H-2 *with* $\rho_0 \leqslant 1$. *Pick two reals* $\omega, \tilde{\omega} \geqslant 0$, *and non-negative integers* $m, n, p, q \geqslant 0$, *such that* $m+n+p+q = 1$ *or* $m+n+p+q = 2$. *Furthermore, fix* $k^{(m)} := (k_1, ..., k_m) \in (\mathbb{R}^d)^m$ *and* $\tilde{k}^{(n)} := (\tilde{k}_1, ..., \tilde{k}_n) \in (\mathbb{R}^d)^n$. *Let* $A := |H_{el}| \otimes \mathbf{1} + \mathbf{1} \otimes H_f + \rho_0$ *as in* (*III*.25). *Denote* $dx^{(p)} := \prod_{j=1}^p d^d x_j$, *and* $a^\dagger(x^{(p)}) := \prod_{j=1}^p a^\dagger(x_j)$. *Then*,

$$\begin{aligned} &\left\| (A+\omega)^{-1/2} \left(\int dx^{(p)} d\tilde{x}^{(q)} G_{m+p,n+q}(k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}) \right.\right. \\ &\qquad \left.\left. \otimes\, a^\dagger(x^{(p)})\, a(\tilde{x}^{(q)}) \right) (A+\tilde{\omega})^{-1/2} \right\| \\ &\qquad \leqslant \Lambda_1^{m+n+p+q} \rho_0^{-(1+\delta_{p+q,0})/2} \prod_{j=1}^m J(k_j) \prod_{j=1}^n J(\tilde{k}_j). \end{aligned} \tag{III.42}$$

Proof. We first remark that it suffices to prove (III.42) for $\omega = \tilde{\omega} = 0$. For convenience, we temporarily write $k := k^{(m)}$, $\tilde{k} := \tilde{k}^{(n)}$, $x := x^{(p)}$, $\tilde{x} := \tilde{x}^{(q)}$, and $G_{k,\tilde{k}}(x; \tilde{x}) := G_{m+p,n+q}(k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)})$. We pick a pair $\phi, \psi \in \mathscr{H}$ and study

$$\begin{aligned} A_{\phi,\psi} &= \left| \left\langle \phi \,\middle|\, A^{-1/2} \left(\int dx\, d\tilde{x}\, G_{k,\tilde{k}}(x; \tilde{x}) \otimes a^\dagger(x)\, a(\tilde{x}) \right) A^{-1/2} \psi \right\rangle \right| \\ &\leqslant \int dx\, d\tilde{x}\, \|\mathbf{1} \otimes [(H_f + \omega_x)^{-p/2} a(x)]\, \phi\| \\ &\quad \times \|\mathbf{1} \otimes [(H_f + \tilde{\omega}_x)^{-q/2} a(\tilde{x})]\, \psi\| \\ &\quad \times \|[\mathbf{1} \otimes (H_f + \omega_x)]^{p/2} (A + \omega_x)^{-1/2} [G_{k,\tilde{k}}(x; \tilde{x}) \otimes \mathbf{1}] \\ &\quad \times (A + \tilde{\omega}_x)^{-1/2} [\mathbf{1} \otimes (H_f + \tilde{\omega}_x)]^{q/2}\|, \end{aligned} \tag{III.43}$$

where $\omega_x := \sum_{j=1}^{p} \omega(x_j)$, $\tilde{\omega}_x := \sum_{j=1}^{q} \omega(\tilde{x}_j)$ and, furthermore, we use the fact that $A^{-1/2} a^\dagger(x) = a^\dagger(x)(A+\omega_x)^{-1/2}$ and $a(\tilde{x})\, A^{-1/2} = (A+\tilde{\omega}_x)^{-1/2}\, a(\tilde{x})$. We refer to these identities later as *Pull-Through Formulae*, established in Lemma A.1. By Schwarz' inequality, we further estimate

$$A_{\phi,\psi} \leqslant B_p^{1/2}(\phi)\, B_q^{1/2}(\psi) \times \left\{ \int C^2_{k,\tilde{k}}(x,\tilde{x}) \prod_{j=1}^{p} \omega(x_j)^{-1}\, d^d x_j \prod_{j=1}^{q} \omega(\tilde{x}_j)^{-1}\, d^d \tilde{x}_j \right\}^{1/2}, \tag{III.44}$$

where

$$C_{k,\tilde{k}}(x,\tilde{x}) := \| [\mathbf{1} \otimes (H_f + \omega_x)]^{p/2}\, (A+\omega_x)^{-1/2} \times [G_{k,\tilde{k}}(x;\tilde{x}) \otimes \mathbf{1}] (A+\tilde{\omega}_x)^{-1/2}\, [\mathbf{1} \otimes (H_f + \tilde{\omega}_x)]^{q/2} \|, \tag{III.45}$$

and (remember $a(x) = a(x^{(p)}) = \prod_{j=1}^{p} a(x_j)$)

$$\begin{aligned} B_p(\phi) &:= \int \| \mathbf{1} \otimes (H_f + \omega_x)^{-p/2}\, a(x^{(p)})\, \phi \|^2 \prod_{j=1}^{p} \omega(x_j)\, dx_j \\ &= \int \left\langle a(x^{(p-1)})\, \phi \,\middle|\, \mathbf{1} \otimes \frac{H_f}{[H_f + \omega(x_1) + \cdots + \omega(x_{p-1})]^p} \right. \\ &\qquad \left. \times\, a(x^{(p-1)})\, \phi \right\rangle \prod_{j=1}^{p-1} \omega(x_j)\, d^d x_j \\ &\leqslant \int \langle a(x^{(p-1)})\, \phi \mid \mathbf{1} \otimes [H_f + \omega(x_1) + \cdots + \omega(x_{p-1})]^{1-p} \\ &\qquad \times\, a(x^{(p-1)})\, \phi \rangle \prod_{j=1}^{p-1} \omega(x_j)\, d^d x_j \\ &= B_{p-1}(\phi) \leqslant \cdots \leqslant B_0(\phi) = \|\phi\|^2. \end{aligned} \tag{III.46}$$

As an intermediate result we thus have that, for $p+q \geqslant 1$, or for $p=q=0$,

$$\left\| A^{-1/2} \int dx^{(p)}\, d\tilde{x}^{(q)} G_{m+p,n+q}(k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}) \otimes a^\dagger(x^{(p)})\, a(\tilde{x}^{(q)})\, A^{-1/2} \right\| \leqslant \left\{ \int C^2_{k,\tilde{k}}(x,\tilde{x}) \prod_{j=1}^{p} \omega(x_j)^{-1}\, d^d x_j \prod_{j=1}^{q} \omega(\tilde{x}_j)^{-1}\, d^d \tilde{x}_j \right\}^{1/2}. \tag{III.47}$$

So, it suffices to show that

$$C_{k,\tilde{k}}(x,\tilde{x}) \leqslant \rho_0^{-(1/2)-(1/2)\,\delta_{p+q,0}} (1+\omega_x)^{1/2}\, (1+\tilde{\omega}_x)^{1/2} \cdot \prod J, \tag{III.48}$$

where we abbreviated $\prod J := \prod_{j=1}^{m} J(k_j) \prod_{j=1}^{p} J(x_j) \prod_{j=1}^{n} J(\tilde{k}_j) \prod_{j=1}^{q} J(\tilde{x}_j)$. Indeed, using $1+\omega_x \leqslant \prod_{j=1}^{p} [1+\omega(x_j)]$, $1+\tilde{\omega}_x \leqslant \prod_{j=1}^{q} [1+\omega(\tilde{x}_j)]$, and $\Lambda_1 \geqslant 1$, (III.42) derives from (III.48).

To verify (III.48), we first notice that the operator estimated in (III.45) may be represented as a direct integral over the spectrum of H_f. We may thus regard H_f as a positive number r, say, and the norm, $C_{k,\tilde{k}}(x,\tilde{x})$, of this operator is given by the supremum over $r \geqslant 0$. Thus, we obtain

$$C_{k,\tilde{k}}(x,\tilde{x}) = \sup_{r\geqslant 0}\{(r+\omega_x)^{p/2}(r+\tilde{\omega}_x)^{q/2} \|(|H_{el}|+\rho_0+r+\omega_x)^{-1/2} \times G_{k,\tilde{k}}(x;\tilde{x})(|H_{el}|+\rho_0+r+\tilde{\omega}_x)^{-1/2}\|_{\mathscr{H}_{el}}\}. \tag{III.49}$$

We distinguish two cases: First, we consider $m+p+n+q=1$. Then Hypothesis H-2 gives

$$\|(|H_{el}|+1)^{-1/4} G_{k,\tilde{k}}(x;\tilde{x})(|H_{el}|+1)^{-1/4}\|_{\mathscr{H}_{el}} \leqslant \prod J. \tag{III.50}$$

We insert (III.50) in (III.49) and get

$$C_{k,\tilde{k}}(x,\tilde{x}) \leqslant \prod J \cdot \sup_{t,\tilde{t},r\geqslant 0} \left\{\frac{(r+\omega_x)^{p/2}(r+\tilde{\omega}_x)^{q/2}(t+1)^{1/4}(\tilde{t}+1)^{1/4}}{(t+\rho_0+r+\omega_x)^{1/2}(\tilde{t}+\rho_0+r+\tilde{\omega}_x)^{1/2}}\right\}. \tag{III.51}$$

Since $\rho_0 \leqslant 1$, we immediately obtain, for $p=q=0$,

$$C_{k,\tilde{k}}(x,\tilde{x}) \leqslant \prod J \cdot \left(\sup_{t\geqslant 0}\left\{\frac{(t+1)^{1/4}}{(t+\rho_0)^{1/2}}\right\}\right)^2 \leqslant \rho_0^{-1} \prod J. \tag{III.52}$$

For $p=1$ and $q=0$, we distinguish $r \geqslant \rho_0$ from $r<\rho_0$. If $r\geqslant\rho_0$ then $r+\omega_x \leqslant \rho_0^{-1} r(1+\omega_x)$ yields

$$\frac{(r+\omega_x)^{1/2}(t+1)^{1/4}(\tilde{t}+1)^{1/4}}{(t+\rho_0+r+\omega_x)^{1/2}(\tilde{t}+\rho_0+r)^{1/2}} \leqslant \frac{\rho_0^{-1/2} r^{1/2}(1+\omega_x)^{1/2}}{(t+\rho_0+r+\omega_x)^{1/4}(\tilde{t}+\rho_0+r)^{1/4}} \leqslant \rho_0^{-1/2}\cdot(1+\omega_x)^{1/2}, \tag{III.53}$$

while, for $r<\rho_0$, we obtain the same result by minimizing the following expression over $t, \tilde{t} \geqslant 0$:

$$\frac{(r+\omega_x)^{1/2}\,(t+1)^{1/4}\,(\tilde{t}+1)^{1/4}}{(t+\rho_0+r+\omega_x)^{1/2}\,(\tilde{t}+\rho_0+r)^{1/2}}$$
$$\leqslant \frac{(\rho_0+\omega_x)^{1/2}\,(t+1)^{1/4}\,(\tilde{t}+1)^{1/4}}{(t+\rho_0+\omega_x)^{1/2}\,(\tilde{t}+\rho_0)^{1/2}}$$
$$\leqslant \frac{(\rho_0+\omega_x)^{1/2}\max\{1,(\rho_0+\omega_x)\}^{1/4}\max\{1,\rho_0\}^{1/4}}{(\rho_0+\omega_x)^{1/2}\,\rho_0^{1/2}}$$
$$=\rho_0^{-1/2}\cdot\max\{1,(\rho_0+\omega_x)\}^{1/4}\leqslant \rho_0^{-1/2}\cdot(1+\omega_x)^{1/2}. \tag{III.54}$$

Of course, the estimate for the case $p=0$, $q=1$ is similar.

Second, we consider the case $m+p+n+q=2$, for which Hypothesis H-2 gives

$$\|G_{k,\tilde{k}}(x;\tilde{x})\|_{\mathscr{H}_{el}}\leqslant \prod J. \tag{III.55}$$

Thus, we obtain

$$G_{k,\tilde{k}}(x,\tilde{x})\leqslant \prod J\cdot\sup_{r\geqslant 0}\left\{\frac{(r+\omega_x)^{p/2}\,(r+\tilde{\omega}_x)^{q/2}}{(\rho_0+r+\omega_x)^{1/2}\,(\rho_0+r+\tilde{\omega}_x)^{1/2}}\right\}. \tag{III.56}$$

For $p+q\leqslant 1$, we observe that the right side of (III.56) is smaller than the right side of (III.51), which we bound in (III.52)–(III.54). It remains to consider $p+q=2$. The case $p=q=1$ is trivial, and we only consider $p=2$, $q=0$, since the case $p=0$, $q=2$ is similar. So, assuming $p=2$ and $q=0$, we estimate

$$\frac{r+\omega_x}{(\rho_0+r+\omega_x)^{1/2}\,(\rho_0+r)^{1/2}}\leqslant\left(\frac{r+\omega_x}{r+\rho_0}\right)^{1/2}\leqslant\max\left\{1,\frac{\omega_x}{\rho_0}\right\}^{1/2}, \tag{III.57}$$

which is obviously smaller than $\rho_0^{-1/2}\cdot(1+\omega_x)^{1/2}$. Collecting all these estimates, we observe that we have established (III.48) in all possible cases. ∎

III.2. *The Operator* $H_{(0)}[z]$

In this section, we transform the operator $H_{(0)}[z]$ to the generalized form and estimate its coupling functions.

First, we use a Neumann series expansion to convert $\mathscr{F}_{P_0}(H_g-z)$ into a Wick-ordered form, moving all creation operators $a^\dagger(k_{l,j})$ to the left of functions of H_f and moving all annihilation operators $a(k_{l',j'})$ to the right of functions of H_f. Counting the number M of creation operators to their left and the number N of annihilation operators to their right, the sum of all these functions of H_f form a new (but not yet rescaled) interaction coefficient $\tilde{w}_{M,N}$.

In the second step we derive certain L^p-bounds on these coefficients and their derivatives w.r.t. H_f or z. These bounds enable us to control the *interaction part*, the terms with $M+N \geqslant 1$, as a relatively bounded perturbation of the *free part* of $\mathscr{F}_{P_0}(H_g - z)$, the sum of all terms with $M=N=0$, i.e., pure functions of H_f. At this point, we must make use of Hypothesis H-3, (besides H-1 and H-2). We recall that Hypothesis H-3 demand that there exist a positive number $\mu > 0$ such that the smallest number $\Lambda_5 \geqslant 1$ which satisfies, for all $k \in \mathbb{R}^d$,

$$J(k) \leqslant \Lambda_5 \cdot \omega(k)^{1 + (\mu/2) - (d/2\gamma)} \tag{III.58}$$

is finite. It is convenient to introduce two other parameters, α and β, given by

$$\alpha := \frac{1}{2}(1+\mu) - \frac{1}{2}\left(\frac{d}{\gamma} - 1\right) = 1 + \frac{\mu}{2} - \frac{d}{2\gamma}, \tag{III.59}$$

$$\beta := \frac{1}{2}(1+\mu) + \frac{1}{2}\left(\frac{d}{\gamma} - 1\right) = \frac{\mu}{2} + \frac{d}{2\gamma}. \tag{III.60}$$

Note that (III.58) now reads

$$J(k) \leqslant \Lambda_5 \cdot \omega(k)^{\alpha}. \tag{III.61}$$

Third, we eliminate the dependence on ρ_0 of the domains $\chi[H_f < \rho_0]\, \mathscr{F}$ and $\mathscr{U}^{(\mathrm{in})}_{(0)} = D_{\rho_0/2}$ by rescaling and shifting $\mathscr{F}_{P_0}[H_g - z)$ (and thus $\tilde{w}_{M,N}$). Surpressing its dependence on g and ρ_0, the resulting operator $H_{(0)}[z]$ acts on $\mathscr{H}_{\mathrm{red}} := \chi[H_f < 1]\, \mathscr{F}_b$ with $z \in D_{1/2}$. Indeed, in Appendix B we prove that

$$H_{(0)} \colon D_{1/2} = \{|z| \leqslant 1/2\} \to \mathscr{B}[\mathscr{H}_{\mathrm{red}}] \tag{III.62}$$

defines an analytic family of bounded operators on $\mathscr{H}_{\mathrm{red}}$.

We start working through this program by converting $\mathscr{F}_{P_0}(H_g - z)$ into a Wick-ordered form, moving all creation operators $a^\dagger(k_{l,j})$ to the left of functions of H_f and moving all annihilation operators $a(k_{l',j'})$ to the right of functions of H_f. This amounts to determining the coupling functions,

$$\tilde{w}^{(0)}_{M,N} \colon \mathscr{U}^{(\mathrm{in})}_{(0)} \times \mathbb{R}^+ \times B^M_{\rho_0} \times B^N_{\rho_0} \to \mathbb{C} \tag{III.63}$$

(recall that $B_\rho := \{k \in \mathbb{R}^d \mid \omega(k) \leqslant \rho\}$), of the operator $\tilde{H}_{\mathrm{eff}}[z]$, defined by

$$\mathscr{F}_{P_0}(H_g - z) =: P_{el} \otimes (\tilde{H}_{\mathrm{eff}}[z] - z), \tag{III.64}$$

for all $M+N \geqslant 0$ and $z \in \mathscr{U}^{(\mathrm{in})}_{(0)}$. Later, $\tilde{w}_{0,0}[z; r]$ is shown to be Lipschitz continuous with respect to r on $\mathscr{U}^{(\mathrm{in})}_{(0)} \times [0, \rho_0)$, and, for $M+N \geqslant 1$, $\tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]$ turns out to be bounded and Lipschitz continuous with respect to r, i.e., it is differentiable w.r.t. r up to finite jump discontinuities in the derivative on $\mathcal{U}^{(\text{in})}_{(0)} \times [0, \rho_0) \times \mathbb{R}^{Md} \times \mathbb{R}^{Nd}$. Then, $\tilde{w}_{M,N}[z; H_f; k^{(M)}; \tilde{k}^{(N)}]$ is an operator on $\mathcal{H}_{\text{red}}$ defined via the functional calculus for H_f.

LEMMA III.6. *Let $\theta := i\vartheta$ for some $0 < \vartheta < \pi/2$ and assume Hypothesis* H-1, H-2, *and* (III.19). *Pick $0 < \rho_0 \leqslant \rho^{(\text{out})}$ and $z \in \mathcal{U}^{\text{in})}_{(0)}$. Let $P_0 := P_{el} \otimes \chi_{\rho_0}[H_f]$, $\bar{P}_0 := \mathbf{1} - P_0$ and $\bar{R}_0[H_f] := \bar{P}_0(H_0 - z)$. Then, for all $M + N \geqslant 0$, $\tilde{w}_{M,N}$, defined by* (III.63), *is given by*

$$\tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}]$$

$$= \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{\substack{m_l + p_l + n_l + q_l = 1, 2; \\ l = 1, \ldots, L}} \delta_{M, \sum_{l=1}^{L} m_l} \cdot \delta_{N, \sum_{l=1}^{L} n_l}$$

$$\times \prod_{l=1}^{L} \left\{ \binom{m_l + p_l}{p_l} \binom{n_l + q_l}{q_l} \right\}$$

$$\times \left\{ \tilde{D}_L[z; H_f; \{ W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}; \{ \bar{R}_0 \}_{l=1}^{L-1}] \right\}^{symm}_{M,N}, \qquad \text{(III.65)}$$

where

$$\tilde{D}_L[z; r; \{ W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}; \{ \bar{R}_0 \}_{l=1}^{L-1}]$$

$$:= \prod_{l=1}^{L} (-g)^{m_l + p_l + n_l + q_l}$$

$$\times \langle \varphi_{el} \otimes \Omega \mid W^{m_1, n_1}_{p_1, q_1}[k_1^{(m_1)}; \tilde{k}_1^{(n_1)}] \, \bar{R}_0[H_f + r + \mu_1]$$

$$\times \cdots \bar{R}_0[H_f + r + \mu_{L-1}] \, W^{m_L, n_L}_{p_L, q_L}[k_L^{(m_L)}; \tilde{k}_L^{(n_L)}] \, \varphi_{el} \otimes \Omega \rangle, \qquad \text{(III.66)}$$

$$W^{m_l, n_l}_{p_l, q_l}[k_l^{(m_l)}; \tilde{k}_l^{(n_l)}]$$

$$:= \int dx_l^{(p_l)} \, d\tilde{x}_l^{(q_l)} \, G_{m_l + p_l, n_l + q_l}[k_l^{(m_l)}, x_l^{(p_l)}; \tilde{k}_l^{(n_l)}, \tilde{x}_l^{(q_l)}] \otimes a^\dagger(x_l^{(p_l)}) \, a(\tilde{x}_l^{(q_l)}), \qquad \text{(III.67)}$$

and

$$\mu_l := \sum_{j=1}^{l} \omega(\tilde{k}_j^{(n_j)}) + \sum_{j=l+1}^{L} \omega(k_j^{(m_j)}). \qquad \text{(III.68)}$$

Proof. We expand $\mathcal{F}_{P_0}(H_g - z)$ in a Neumann series whose convergence is guaranteed by (III.20) and (III.19).

$$\mathscr{F}_{P_0}(H_g - z) = P_0(H_0 - z) + P_0 W P_0 - P_0 W \bar{P}_0[\bar{P}_0(H_0 - z) + \bar{P}_0 W \bar{P}_0]^{-1} \bar{P}_0 W P_0 - z P_0$$
$$= P_0(H_0 - z) - \sum_{L=1}^{\infty} (-L)^L P_0 W(\bar{P}_0(H_0 - z)^{-1} W P_0)^{L-1}$$
$$= P_0(H_0 - z) - \sum_{L=1}^{\infty} (-1)^L \sum_{\substack{M_l + N_l = 1, 2; \\ l = 1, ..., L}} g^{\sum_{l=1}^{L} M_L + N_L} \times P_0 W_{M_1+N_1} \bar{R}_0 \cdots \bar{R}_0 W_{M_L+N_L} P_0. \tag{III.69}$$

The assertion follows directly from an application of Theorem A.4 to every single term in the last sum on the right side of (III.69). Note that in this present application, the coefficients $w_{M,N}$ in Theorem A.4 have their values in the operators on $\mathscr{H}_{el}$—this does not affect the (purely algebraic) assertion, since we carefully avoided commutations of these coefficients. ∎

We come to address our second task: the proof of bounds on the interaction coefficients $\tilde{w}_{M,N}$. We learn from Lemma III.6 that estimates for $\tilde{w}_{M,N}$ can be derived by summing up similar estimates for $\tilde{D}_L$, as is done below.

Lemma III.7. *Require the same hypotheses as in Lemma III.6 and assume that $m_l + n_l + p_l + q_l \in \{1, 2\}$ for all $1 \leqslant l \leqslant L$. Write $\sum_{l=1}^{L} m_l := M$ and $\sum_{l=1}^{L} n_l := N$. Then $\tilde{D}_L$, defined as in (III.66), obeys the following bounds:*

$$|\tilde{D}_L[r; z; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{\bar{R}_0\}_{l=1}^{L-1}]| \leqslant \prod_{l=1}^{L} \left(\frac{3g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)}\right)^{m_l + n_l + p_l + q_l} \rho_0^{1-(M+N)/2} \Lambda_5^{M+N} \times \prod_{j=1}^{M} \omega(k_j)^{\alpha} \prod_{j=1}^{N} \omega(\tilde{k}_j)^{\alpha}, \tag{III.70}$$

$$|\partial_z \tilde{D}_L[r; z; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{\bar{R}_0\}_{l=1}^{L-1}]| \leqslant \left(\frac{L}{\rho_0}\right) \prod_{l=1}^{L} \left(\frac{3g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)}\right)^{m_l + n_l + p_l + q_l} \rho_0^{1-(M+N)/2} \Lambda_5^{M+N} \times \prod_{j=1}^{m} \omega(k_j)^{\alpha} \prod_{j=1}^{N} \omega(\tilde{k}_j)^{\alpha}, \tag{III.71}$$

Proof. To make the argument more transparent, we abbreviate

$$\tilde{D}_L[\cdots] := \tilde{D}_L[r; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{\bar{R}_0\}_{l=1}^{L-1}] \tag{III.72}$$
$$W_l := W^{m_l, n_l}_{p_l, q_l}[k_l^{(m_l)}; \tilde{k}_l^{(n_l)}]. \tag{III.73}$$

Using the fact that $\varphi_{el} \otimes \Omega = \rho_0^{1/2} A^{-1/2} \varphi_{el} \otimes \Omega$, we estimate

$$\begin{aligned} |\tilde{D}_L[\cdots]| &= \rho_0 \, |\langle \varphi_{el} \otimes \Omega \,|\, [A^{-1/2} W_1 (A+\mu_1)^{-1/2}][\bar{R}_0[\mu_1](A+\mu_1)] \\ &\quad \times \cdots [\bar{R}_0[\mu_{L-1}](A+\mu_{L-1})] \\ &\quad \times \cdots [(A+\mu_{L-1})^{-1/2} \, W_L A^{-1/2}](\varphi_{el} \otimes \Omega) \rangle| \\ &\leqslant \rho_0 \cdot \prod_{l=1}^{L} \|(A+\mu_{l-1})^{-1/2} \, W_l (A+\mu_l)^{-1/2}\| \\ &\quad \times \prod_{l=1}^{L-1} \|\bar{R}_0[\mu_l](A+\mu_l)\|, \end{aligned} \tag{III.74}$$

where $\mu_0 := \mu_L := 0$. Note that $\|\bar{R}_0[H_f + \mu](A+\mu)\| \leqslant \|\bar{R}_0[H_f] \, A\|$. Thus, an application of Lemma III.4 and III.5 yields

$$|\tilde{D}_L[\cdots]| \leqslant \rho_0 \cdot \prod_{l=1}^{L} \frac{3(g\Lambda_1)^{m_l+n_l+p_l+q_l}}{\sin(\vartheta/2) \, \rho_0^{(1+\delta_{p_l+q_l,0})/2}} \cdot \prod_{j=1}^{M} J(k_j) \prod_{j=1}^{N} J(\tilde{k}_j). \tag{III.75}$$

Now, we observe that $m_l + p_l + n_l + q_l \geqslant 1$ implies that $1 + \delta_{p_l+q_l,0} \leqslant 2(m_l+n_l) + p_l + q_l$ and, hence,

$$\frac{(g\Lambda_1)^{m_l+n_l+p_l+q_l}}{\rho_0^{(1/2)+(1/2)\,\delta_{p_l+q_l,0}}} \leqslant \left(\frac{g\Lambda_1}{\rho_0^{1/2}}\right)^{m_l+n_l+p_l+q_l} \cdot \rho_0^{-(1/2)(m_l+n_l)}. \tag{III.76}$$

To prove (III.71), we note that $\partial_z \bar{R}_0[H_p + \mu_l] = -\bar{R}_0^2[H_p + \mu_l]$. Thus we obtain from Leibniz' rule

$$\begin{aligned} \partial_z \tilde{D}_L[\cdots] = -\sum_{l=1}^{L-1} \tilde{D}_L[r; z; \{W_j, k_j^{(m_j)}; \tilde{k}_j^{(n_j)}\}_{j=1}^{L}; \\ \{(\bar{R}_0)_{j=1}^{l-1}, \bar{R}_0^2, (\bar{R}_0)_{j=l+1}^{L-1}\}]. \end{aligned} \tag{III.77}$$

Then (III.71) follows from estimating each term in the sum in (III.77) as in (III.74), taking into account, in addition, that

$$\|\bar{R}_0^2[H_p + \mu](A+\mu)\| \leqslant \rho_0^{-1} \cdot \|\bar{R}_0[H_f] \, A\|^2. \quad \blacksquare \tag{III.78}$$

We recall the definition of the constraint C_d, which is the volume of the d-dimensional unit ball times d/γ, where γ is the exponent in $\omega(k) = |k|^\gamma$.

$$C_d := \frac{d\pi^{d/2}}{\gamma \Gamma[(d/2)+1]}, \tag{III.79}$$

Note that

$$\int_{\mathbb{R}^d} f[\omega(k)] \, dk = C_d \cdot \int_0^\infty f[\omega]^{(d/\gamma)-1} \, d\omega. \tag{III.80}$$

LEMMA III.8 *Require the hypotheses of Lemma* III.6, *and assume, in addition, that Hypothesis* H-3 *holds and that* $m_l+n_l+p_l+q_l \geqslant 1$, *for all* $1 \leqslant l \leqslant L$. *Write* $\sum_{l=1}^{L} m_l := M$ *and* $\sum_{l=1}^{L} n_l := N$. *For any* $0<r<\rho_0$, $\widetilde{D}_L$ *(defined as in* (III.66)) *obeys the following bound.*

$$\int_{B_{\rho_0}^{M+N}} |\partial_r \widetilde{D}_L[r; z; \{W_{p_l, q_l}^{m_l, n_l}; k_l^{(m_l)}; \widetilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{\bar{R}_0\}_{l=1}^{L-1}]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\widetilde{k}_j}{\omega(\widetilde{k}_j)^\beta}$$
$$\leqslant 4L\rho_0^{(M+N)/2} \prod_{l=1}^{L} \left(\frac{6g\Lambda_1 \Lambda_5 (1+\sqrt{2C_d})}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{m_l+p_l+n_l+q_l}, \tag{III.81}$$

where $B_r := \{k \mid \omega(k) \leqslant r\}$.

Proof. First, we recall that

$$\bar{R}_0[H_f] = [\bar{P}_{el}(H_{el}-z) \otimes 1 + \mathbf{e}^{-i\vartheta} \bar{P}_{el} \otimes H_f]^{-1} + P_{el} \otimes \frac{\chi[H_f \geqslant \rho_0]}{\mathrm{e}^{-i\vartheta} H_f - z}. \tag{III.82}$$

Thus, its ($\mathscr{B}[\mathscr{H}_{\mathrm{red}}]$-valued) distributional derivative is given by

$$\partial_{H_f} \bar{R}_0[H_f] = \mathrm{e}^{-i\vartheta} \bar{R}_0^2[H_f] + (\mathrm{e}^{-i\vartheta}\rho_0 - z)^{-1} P_{el} \otimes \delta[H_f - \rho_0]. \tag{III.83}$$

Using the abbreviation (III.72) and (III.73) again, Leibniz' rule in conjunction with (III.82) gives

$$\begin{aligned}
&\partial \widetilde{D}_L[\cdots] \\
&= \sum_{l=1}^{L-1} \widetilde{D}_L[r; z; \{W_j; k_j^{(m_j)}; \widetilde{k}_j^{(n_j)}\}_{j=1}^{L}; \{(\bar{R}_0)_{j=1}^{l-1}, \partial_{H_f} \bar{R}_0, (\bar{R}_0)_{j=l+1}^{L-1}\}] \\
&= -\mathrm{e}^{-i\vartheta} \partial_z \widetilde{D}[\cdots] + (\mathrm{e}^{-i\vartheta}\rho_0 - z)^{-1} \\
&\quad \times \sum_{l=1}^{L-1} \widetilde{D}_L[r; z; \{W_j; k_j^{(m_j)}; \widetilde{k}_j^{(n_j)}\}_{j=1}^{L}; \{(\bar{R}_0)_{j=1}^{l-1}, P_{el} \otimes \delta[\,\cdot - \rho_0], (\bar{R}_0)_{j=l+1}^{L-1}\}] \\
&= -\mathrm{e}^{-i\vartheta} \partial_z \widetilde{D}_L[\cdots] + \frac{1}{(\mathrm{e}^{-i\vartheta}\rho_0 - z)} \\
&\quad \times \sum_{l=1}^{L-1} \widetilde{D}_L[\cdots (P_{el} \otimes \delta[\,\cdot - \rho_0])_l \cdots],
\end{aligned} \tag{III.84}$$

using the shorthand notation

$$\begin{aligned}
&\widetilde{D}_L[\cdots (P_{el} \otimes \delta[\,\cdot - \rho_0])_l \cdots] \\
&\quad = \widetilde{D}_L[r; z; \{W_j; k_j^{(m_j)}; \widetilde{k}_j^{(n_j)}\}_{j=1}^{L}; \{(\bar{R}_0)_{j=1}^{l-1}, P_{el} \otimes \delta[\,\cdot - \rho_0], (\bar{R})_{j=l+1}^{L-1}\}].
\end{aligned} \tag{III.85}$$

The δ-function in the last sum of (III.84) is too singular, to be estimated directly. We rather make the dependence of $\widetilde{D}_L[\cdots(P_{el}\otimes\delta[\cdot-\rho_0]_l\cdots]$ on δ explicit by means of the recurrence relation (A.47). For fixed $1\leqslant l\leqslant L-1$ and $V:=\sum_{j=1}^{l} v_j$, $U:=\sum_{j=l+1}^{L} u_j$, this recurrence relation gives

$$
\begin{aligned}
&\widetilde{D}_L[\cdots(P_{el}\otimes\delta[\cdot-\rho_0])_l\cdots]\\
&\quad=\sum_{\substack{v_j=0,\\ j=1,\dots,l}}^{q_j}\ \sum_{\substack{u_j=0,\\ j=l+1,\dots,L}}^{p_j}\int\prod_{j=1}^{l}\left\{d\tilde{y}_j^{(v_j)}\binom{q_j}{v_j}\right\}\prod_{j=l+1}^{L}\left\{dy_j^{(u_j)}\binom{q_j}{u_j}\right\}\\
&\qquad\times\widetilde{D}_l[r+\omega_{\geqslant l}(k^{(m)});z;\{W_j;k_j^{(m_j)};\tilde{k}_j^{(n_j)},\tilde{y}_j^{(v_j)}\}_{j=1}^{l};\{\bar{R}_0\}_{j=1}^{l-1}]\\
&\qquad\times\delta\left[r+\sum_{j=1}^{l}\omega(\tilde{k}_j^{(n_j)})+\sum_{j=l+1}^{L}\omega(k_j^{(m_j)})+\omega(y^{(U)})-\rho_0\right]\\
&\qquad\times\langle a(\tilde{y}^{(V)})\,a^{\dagger}(y^{(U)})\rangle\\
&\qquad\times\widetilde{D}_{L-1}[r+\omega_{\leqslant l}(\tilde{k}^{(n)});z;\{W_j;k_j^{(m_j)},y_j^{(u_j)};\tilde{k}_j^{(n_j)}\}_{j=l+1}^{L};\{\bar{R}_0\}_{j=l+1}^{L-1}].
\end{aligned}
\tag{III.86}
$$

Now the assumption $r<\rho_0$ becomes important because it implies

$$N_{\leqslant l}+M_{>l}+U:=\sum_{j=1}^{l}n_j+\sum_{j=l+1}^{L}m_j+U\geqslant 1 \tag{III.87}$$

in (III.86). Formally, the term in (III.86) with $N_{\leqslant l}+M_{>l}+U=0$ vanishes since we would have $\delta[r-\rho_0]$ in (III.86) which is supported at $r=\rho_0$. Rigorously, (III.87) follows from first expanding $\widetilde{D}_L[\cdots]$ by means of the recurrence relation (A.47) (before differentiating it). Then we get a similar formula as (III.86) for $\widetilde{D}_L[\cdots]$ except that

$$\delta\left[r+\sum_{j=1}^{l}\omega(\tilde{k}_j^{(n_j)})+\sum_{j=l+1}^{L}\omega(k_j^{(m_j)})+\omega(y^{(U)})-\rho_0\right]$$

is replaced by

$$\chi\left[r+\sum_{j=1}^{l}\omega(\tilde{k}_j^{(n_j)})+\sum_{j=l+1}^{L}\omega(k_j^{(m_j)})+\omega(y^{(U)})\geqslant\rho_0\right].$$

For $N_{\leqslant l}+M_{>l}+U=0$ this gives $\chi[r\geqslant\rho_0]$ which vanishes for $r<\rho_0$, and there is nothing to differentiate, to begin with.

Next, we integrate (III.86) against $\prod_{j=1}^{M}dk_j/\omega(k_j)^{\beta}\prod_{j=1}^{N}d\tilde{k}_j/\omega(\tilde{k}_j)^{\beta}$. We insert for $\widetilde{D}_l[\cdots]$ and $\widetilde{D}_{L-l}[\cdots]$ the estimate (III.70) and get

$$\int_{B_{\rho_0}^{M+N}} |\tilde{D}_L[\cdots(P_{el}\otimes\delta[\cdot-\rho_0])_l\cdots]| \prod_{j=1}^{M} \frac{dk_j}{\partial(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta}$$

$$\leqslant \sum_{\substack{v_j=0,\\ j=1,...,l}}^{q_j} \sum_{\substack{u_j=0,\\ j=l+1,...,L}}^{p_j} \prod_{j=1}^{l} \binom{q_j}{v_j} \prod_{j=l+1}^{L} \binom{p_j}{u_j}$$

$$\times \prod_{l=1}^{L} \left(\frac{3g\Lambda_1}{\rho_0^{1/2}\sin(\vartheta/2)}\right)^{m_l+n_l+p_l+q_l}$$

$$\times \chi[N_{\leqslant l}+M_{>l}+U\geqslant 1]\,\rho_0^{2-(1/2)(M+N+U+V)}\Lambda_5^{M+N+U+V}$$

$$\times \int_{B_{\rho_0}^{M+N+U+V}} \langle a(\tilde{y}^{(V)})\,a^\dagger(y^{(U)})\rangle$$

$$\times \delta\left[r+\sum_{j=1}^{l}\omega(\tilde{k}_j^{(n_j)})+\sum_{j=l+1}^{L}\omega(k_j^{(m_j)})+\omega(y^{(U)})-\rho_0\right]$$

$$\times \prod_{j=1}^{M}\{\omega(k_j)^{\alpha-\beta}\,dk_j\} \prod_{j=1}^{N}\{\omega(\tilde{k}_j)^{\alpha-\beta}\,d\tilde{k}_j\}$$

$$\times \prod_{j=1}^{U}\{\omega(y_j)^\alpha\,dy_j\} \prod_{j=1}^{V}\{\omega(\tilde{y}_j)^\alpha\,d\tilde{y}_j\}. \tag{III.88}$$

Now, $\langle a(\tilde{y}^{(V)})\,a^\dagger(y^{(U)})\rangle=0$ unless $U=V$, and in this case we obtain

$$\langle a(\tilde{y}^{(V)})\,a^\dagger(y^{(U)})\rangle=\sum_{\pi}\prod_{j=1}^{U}\delta[y_j-\tilde{y}_{\pi(j)}], \tag{III.89}$$

where π are the permutations of $\{1, ..., U\}$. Moreover, $\alpha-\beta=1-(d/\gamma)$ and $2\alpha=(1+\mu)+1-(d/\gamma)$. Thus, using (III.80), the integral on the right side of (III.88) is given by

$$\mathscr{I} := (U!)\int_{B_{\rho_0}^{M+N+U}} \delta\left[r+\sum_{j=1}^{l}\omega(\tilde{k}_j^{(n_j)})+\sum_{j=l+1}^{L}\omega(k_j^{(m_j)})+\omega(y^{(U)})-\rho_0\right]$$

$$\times \prod_{j=1}^{M}\{\omega(k_j)^{\alpha-\beta}\,dk_j\} \prod_{j=1}^{N}\{\omega(\tilde{k}_j)^{\alpha-\beta}\,d\tilde{k}_j\} \prod_{j=1}^{U}\{\omega(y_j)^{2\alpha}\,dy_j\}$$

$$=(U!)\,C_d^{M+N+U}\cdot\int_{B_{\rho_0}^{M+N+U}} \delta\left[r+\sum_{j=1}^{N_{\leqslant l}+M_{>l}}\omega_j+\sum_{j=1}^{U}\hat{\omega}_j-\rho_0\right]$$

$$\times \prod_{j=1}^{M+N} d\omega_j \prod_{j=1}^{U} \hat{\omega}_j^{1+\mu}\,d\hat{\omega}_j. \tag{III.90}$$

We distinguish the case $U=0$ and $U \geqslant 1$. If $U=0$ then the constraint (III.87) reads $N_{\leqslant l}+M_{>l} \geqslant 1$ and we estimate

$$\mathscr{I} \leqslant C_d^{M+N} \prod_{j=1}^{M+N-1} \int_{\omega_j \leqslant \rho_0} d\omega_j = C_d^{M+N} \rho_0^{M+N-1}. \tag{III.91}$$

Conversely, if $U \geqslant 1$ then we compensate for the δ-function by first integrating against $d\hat{\omega}_U$ and obtain

$$\begin{aligned} \mathscr{I} &\leqslant (U!)\, C_d^{M+N+U} \rho_0^{(1+\mu)+M+N} \\ &\quad \times \int \chi[\hat{\omega}_1 + \cdots + \hat{\omega}_{U-1} \leqslant \rho_0] \prod_{j=1}^{U-1} \hat{\omega}_j^{1+\mu}\, d\hat{\omega}_j \\ &\leqslant (U!)\, C_d^{M+N+U} \rho_0^{(1+\mu)\, U+M+N} \\ &\quad \times \int \chi[\hat{\omega}_1 + \cdots + \hat{\omega}_{U-1} \leqslant \rho_0] \prod_{j=1}^{U-1} d\hat{\omega}_j \\ &\leqslant \frac{U!}{(U-1)!}\, C_d^{M+N+U} \rho_0^{(2+\mu)\, U+M+N-1} \\ &\leqslant U C_d^{M+N+U} \rho_0^{M+N+U-1}. \end{aligned} \tag{III.92}$$

Here we make use of the fact that the integral over the $U-1$-dimensional simplex, $\{(\omega_1, ..., \omega_{U-1}) \in \mathbb{R}_+^{U-1} \mid \omega_1 + \cdots + \omega_{U-1} \leqslant \rho_0\}$, is given by $\rho_0^{U-1}/(U-1)!$. Moreover, we used $\rho_0 \leqslant 1$ in the last inequality. Combining (III.91) with (III.92), we observe that the following estimate is valid for all $U \geqslant 0$:

$$\mathscr{I} \leqslant \max\{U, 1\}\; C_d^{M+N+U} \rho_0^{M+N+U-1}. \tag{III.93}$$

Furthermore, we observe that $2U \leqslant \sum_{l=1}^{L} p_l + q_l$ and that

$$\sum_{v_j=0}^{q_j} \binom{q_j}{v_j} \cdot \sum_{u_j=0}^{p_j} \binom{p_j}{u_j} = 2^{p_j+q_j}, \tag{III.94}$$

Inserting (III.93) and the last two observations into (III.88), we arrive at

$$\begin{aligned} &\int |\tilde{D}_L[\cdots (P_{el} \otimes \delta[\,\cdot - \rho_0])_l \cdots]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta} \\ &\quad \leqslant \max\{U, 1\}\; \Lambda_5^{M+N+2U} C_d^{M+N+U} \rho_0^{1+(M+N)/2} \\ &\qquad \times \prod_{l=1}^{L} \left(\frac{6g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{m_l+p_l+n_l+q_l} \\ &\quad \leqslant \rho_0^{1+(M+N)/2} \prod_{l=1}^{L} \left(\frac{9g\Lambda_1 \Lambda_5 (1+\sqrt{C_d})}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{m_l+p_l+n_l+q_l} \end{aligned} \tag{III.95}$$

Here, we make use of the fact that, for all $U \geqslant 0$,

$$\max\{U, 1\}(C_d \Lambda_5^2)^U \leqslant (\max\{1, U\}\, 2^{-U}) \cdot ((1+\sqrt{C_d})\, \Lambda_5)^{\sum_{l=1}^L p_l + q_l}$$
$$\leqslant ((1+\sqrt{C_d})\, \Lambda_5)^{\sum_{l=1}^L p_l + q_l}. \tag{III.96}$$

Next, we insert the estimate (III.71) from Lemma III.7 to estimate the integral over $\partial_z \tilde{D}[\cdots]$ and obtain

$$\int |\partial_z \tilde{D}_L[\cdots]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta}$$
$$\leqslant L(C_d \Lambda_5 \rho_0^{1/2})^{M+N} \prod_{l=1}^{L} \left(\frac{3g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{m_l + p_l + n_l + q_l}. \tag{III.97}$$

Finally, we note that $|e^{-i\vartheta}\rho_0 - z| \geqslant \rho_0/2$. Thus, in view of (III.84), we obtain the claim (III.81) by adding (III.97) and $3(L-1)/\rho_0$ times (III.95). ∎

Having estimated the $\tilde{D}_L[\cdots]$'s, we can now easily derive the corresponding estimates for $\tilde{w}_L[\cdots]$ by summing up (III.65). This yields the following lemma.

LEMMA III.9. *Let $\theta := i\vartheta$ for some $0 < \vartheta < \pi/2$. Pick $0 < \rho_0 \leqslant \rho^{(\text{out})} = 2^{-1/2} \sin(\vartheta/2)$ and $z \in \mathscr{U}_{(0)}^{(\text{in})}$. Assume Hypotheses* H-1, H-2, H-3 *and*

$$\frac{g\Lambda_6}{\rho_0^{1/2}} \leqslant \frac{1}{100}, \qquad \textit{where} \quad \Lambda_6 := \frac{\Lambda_1 \Lambda_5 (1+\sqrt{2C_d})}{\sin(\vartheta/2)}. \tag{III.98}$$

Let $P_0 := P_{el} \otimes \chi_{\rho_0}[H_f]$, $\bar{P}_0 := \mathbf{1} - P_0$ and $\bar{R}_0[H_f] := \bar{P}_0(H_0 - z)$. Then $\tilde{w}_{M,N}$, defined by (III.64), *obeys the following estimates for all $M+N \geqslant 0$ and $0 < r < \rho_0$:*

$$|\tilde{w}_{M,N}[r; z; k^{(M)}; \tilde{k}^{(N)}]|$$
$$\leqslant 2\rho_0 \left(\frac{100 g \Lambda_6}{\rho_0} \right)^{M+N} \cdot \left(\frac{100 g \Lambda_6}{\rho_0^{1/2}} \right)^{2\delta_{M+N,0}} \prod_{j=1}^{M} \omega(k_j)^\alpha \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha, \tag{III.99}$$

and

$$\int_{B_{\rho_0}^{M+N}} |\partial_r \tilde{w}_{M,N}[r; z; k^{(M)}; \tilde{k}^{(N)}]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta}$$
$$\leqslant 25 \left(\frac{100 g \Lambda_6}{\rho_0} \right)^{M+N} \cdot \left(\frac{100 g \Lambda_6}{\rho_0^{1/2}} \right)^{2\delta_{M+N,0}}. \tag{III.100}$$

Proof. We first prove (III.99) by inserting (III.70) into (III.65) which yields

$$
\begin{aligned}
&|\tilde{w}_{M,N}[r; z; k^{(M)}; \tilde{k}^{(N)}]| \\
&\quad \leqslant \sum_{L=L_0}^{\infty} \sum_{\substack{m_l+p_l+n_l+q_l=1,2; \\ l=1,\dots,L}} \delta_{M, \sum_{l=1}^{L} m_l} \cdot \delta_{N, \sum_{l=1}^{L} n_l} \cdot \rho_0^{1-(M+N)/2} \Lambda_5^{M+N} \\
&\qquad \times \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \left(\frac{3g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{m_l+n_l+p_l+q_l} \right\} \\
&\qquad \times \prod_{j=1}^{M} \omega(k_j)^{\alpha} \prod_{j=1}^{N} \omega(\tilde{k}_j)^{\alpha}, \qquad \text{(III.101)}
\end{aligned}
$$

where $L_0 := 1$ for $M+N \geqslant 1$ and $L_0 := 2$ in case that $M = N = 0$. Next,

$$
\binom{m+p}{p} \binom{n+q}{q} \leqslant 2^{m+p+n+q} \qquad \text{and} \qquad \frac{100 g \Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \leqslant \frac{100 g \Lambda_6}{\rho_0^{1/2}} \leqslant 1. \qquad \text{(III.102)}
$$

Thus

$$
\begin{aligned}
&\sum_{m+p+n+q=1,2} \binom{m+p}{p} \binom{n+q}{q} \left(\frac{3g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{m+n+p+q} \\
&\quad \leqslant \left(\frac{100 g \Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{\max\{m+n, 1\}} \sum_{m+n+p+q \geqslant 1} \left(\frac{3}{50} \right)^{m+n+p+q} \\
&\quad = \left(\frac{100 g \Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{\max\{m+n, 1\}} \cdot \left[\left(\frac{1}{1-\frac{3}{50}} \right)^4 - 1 \right]. \qquad \text{(III.103)}
\end{aligned}
$$

Thus, (III.99) results from $\sum_{l=1}^{L} \max\{m+n, 1\} \geqslant \max\{M+N, L\}$ which implies

$$
\begin{aligned}
&|\tilde{w}_{M,N}[r; z; k^{(M)}; \tilde{k}^{(N)}]| \\
&\quad \leqslant 2\rho_0^{1-(M+N)/2} \Lambda_5^{M+N} \cdot \left(\frac{100 g \Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^{M+N+2\delta_{M+N,0}} \\
&\qquad \times \prod_{j=1}^{M} \omega(k_j)^{\alpha} \prod_{j=1}^{N} \omega(\tilde{k}_j)^{\alpha}. \qquad \text{(III.104)}
\end{aligned}
$$

The proof of (III.100) is analogous except that it makes use of $\sum_{L=1}^{\infty} L x^L = x(1-x)^2$. ∎

We come to the third step indicated at the beginning of this chapter: Elimination of the dependence on ρ_0 of the underlying Hilbert space by rescaling and changing the spectral parameter. First, we observe that the family $\widetilde{H}_{\mathrm{eff}}[z]$ is related to the desired one, $H_{(0)}[z]$, by

$$H_{(0)}[z] := \frac{e^{i\vartheta}}{\rho_0} \Gamma_{\rho_0} \widetilde{H}_{\mathrm{eff}}[Z_{(0)}^{-1}(z)]\, \Gamma_{\rho_0}^*, \tag{III.105}$$

where, recall,

$$Z_{(0)}: \mathscr{U}_{(0)}^{(\mathrm{in})} \to D_{1/2}, \qquad \zeta \mapsto e^{i\vartheta} \rho_0^{-1} \zeta. \tag{III.106}$$

Thus, for any $z \in D_{1/2}$,

$$H_{(0)}[z] = \chi_1 (E_{(0)}[z] + T_{(0)}[z; H_f] + W_{(0)}[z])\, \chi_1, \tag{III.107}$$

where $\chi_1 \equiv \chi[H_f < 1]$ and

$$E_{(0)}[z] = \rho_0^{-1} \cdot \tilde{w}_{0,0}[Z_{(0)}^{-1}(z); 0], \tag{III.108}$$

$$T_{(0)}[z; H_f] = H_f + \rho_0^{-1} \{\tilde{w}_{0,0}[Z_{(0)}^{-1}(z); \rho_0 r]; -\tilde{w}_{0,0}[Z_{(0)}^{-1}(z); 0]\}, \tag{III.109}$$

$$\tilde{w}_{0,0}[z; s] := e^{i\vartheta} \langle \mathscr{F}_{P_0}(H_g + e^{-i\vartheta} s - z) + z \rangle_{\varphi_{el} \otimes \Omega}, \tag{III.110}$$

and

$$w_{M,N}^{(0)}[z; H_f; k^{(M)}, \tilde{k}^{(N)}] := \rho_0^{(d/2\gamma)(M+N)-1} \tilde{w}_{M,N}[Z_{(0)}^{-1}(z); \rho_0 H_f; \rho_0^{1/\gamma} k^{(M)}; \rho_0^{1/\gamma} \tilde{k}^{(N)}]. \tag{III.111}$$

The last equations and Lemma III.9 imply that Theorem III.1.

IV. THE RENORMALIZATION MAP $\mathscr{R}_\rho$

In this chapter, we present the key tool of our analysis: an inductive renormalization group construction. The idea underlying our construction is that by lowering the photon energy scale, thereby decimating the degrees of freedom of the system, the dynamics of the remaining degrees of freedom is approximated ever better by the free photon dynamics H_f. Our construction is realized in terms of a renormalization map, $\mathscr{R}_\rho$, consisting of a decimation of degrees of freedom, followed by rescaling by a factor of ρ^{-1}. The renormalization map associates to every operator H on the subspace $\mathscr{H}_{\mathrm{red}} \equiv \mathrm{Ran}\, \{\chi[H_f < 1]\} \subseteq \mathscr{F}$ an operator $\mathscr{R}_\rho[H]$ on $\mathscr{H}_{\mathrm{red}}$ with the property that $H - z$ is invertible if and only if $\mathscr{R}_\rho[H] - z$ is, for z belonging to a small neighborhood of a certain part of $\sigma(H)$ described below.

IV.1. *Construction of the Renormalization Map* $\mathscr{R}_\rho$

The renormalization map $\mathscr{R}_\rho$ depends on the scale parameters $0<\rho<1$ and $\mu, \xi>0$, which we collect in the triple

$$\Delta := (\mu, \rho, \xi). \tag{IV.1}$$

It is defined on a certain subset of a linear space, $\mathscr{W}_\Delta$, of Hamiltonians on $\mathscr{H}_{\text{red}}$, constructed in (I.46)–(I.57). Here $\mu>0$ is the same parameter as in Hypothesis H-3, and we recall that d is the spatial dimension, $\omega(k)=|k|^\gamma$, and the parameters α and β are defined in (III.59)-(III.60) as

$$\alpha = \frac{1}{2}(1+\mu) - \frac{1}{2}\left(\frac{d}{\gamma} - 1\right) = 1 + \frac{\mu}{2} - \frac{d}{2\gamma},$$

$$\beta := \frac{1}{2}(1+\mu) + \frac{1}{2}\left(\frac{d}{\gamma} - 1\right) = \frac{\mu}{2} + \frac{d}{2\gamma}.$$

Furthermore, we recall from (I.64) that the polydisk $\mathscr{B}(\delta, \varepsilon) \subseteq \mathscr{W}_\Delta$, on which the renormalization map $\mathscr{R}_\rho$ is defined, is given by

$$\mathscr{B}(\delta, \varepsilon) := \Big\{ (E, T, \underline{W}) \in \mathscr{W}_\Delta \mid \sup_{z \in D_{1/2}} \|T'[z, \cdot\,] - 1\|_\infty \leqslant \delta,$$

$$\sup_{z \in D_{1/2}} |E[z]| \leqslant \varepsilon, \ \sup_{z \in D_{1/2}} \|\underline{W}[z]\|_\Delta \leqslant \varepsilon \Big\}, \tag{IV.2}$$

assuming that $\delta \leqslant 1/8$ and $\varepsilon \leqslant 1/16$. In analogy to (III.16)–(III.17), we define

$$\mathscr{U}^{(\text{in})} := \{ z \in D_{1/2} \mid |z - E[z]| \leqslant \rho/2 \}, \tag{IV.3}$$

$$\mathscr{U}^{(\text{out})}(v) := \{ z \in D_{1/2} \backslash \mathscr{U}^{(\text{in})} \mid |\arg\{z - E[z]\}| \geqslant v \}, \tag{IV.4}$$

We note that $H[\cdot\,] \in \mathscr{B}(\delta, \varepsilon)$ and $z \in \mathscr{U}^{(\text{in})}$ implies that $|z| \leqslant \rho/2 + |E[z]|$ and thus

$$\mathscr{U}^{(\text{in})} \subseteq D_{1/4}, \tag{IV.5}$$

provided that $\varepsilon, \rho \leqslant 1/8$. Another simple consequence of $H \in \mathscr{B}(\delta, \varepsilon)$ is the following lemma.

LEMMA IV.1 *Let* $T[z; \cdot\,] \in \mathscr{T}$ *with* $\|T'[z; \cdot\,] - 1\|_\infty \leqslant \varepsilon \leqslant 1/6$, $z \in D_{1/2}$, *and* $0 \leqslant r \leqslant 1$. *Then*

(a) *for any* $z \in D_{1/2}$,

$$(1-\delta)\, r \leqslant |T[z;r]| \leqslant (1+\delta)\, r \qquad \textit{and} \qquad |\arg\{T[z;r]\}| \leqslant \delta, \tag{IV.6}$$

(b) *for any* $z \in \mathscr{U}^{(\mathrm{out})}(\delta')$ *with* $\delta \leqslant \delta' \leqslant \pi/2$,

$$|T[z;r]+E[z]-z| \geqslant \left(\frac{\delta'-\delta}{3\sqrt{2\pi}}\right)(r+\rho), \tag{IV.7}$$

(c) *for any* $z \in \mathscr{U}^{(\mathrm{in})}$ *and* $r \geqslant \rho$,

$$|T[z;r]+E[z]-z| \geqslant (\tfrac{1}{6})(r+\rho). \tag{IV.8}$$

Proof. Part (a) follows trivially from

$$\frac{|\mathrm{Im}\, T[z;r]|}{\mathrm{Re}\, T[z;r]} \leqslant \frac{\|\partial_r T[z;r]-1\|_\infty}{1-\|\partial_r T[z;r]-1\|_\infty} \leqslant \frac{\delta}{1-\delta} \leqslant \tan\delta. \tag{IV.9}$$

To prove (b), we pick $z \in \mathscr{U}^{(\mathrm{out})}(\delta')$ and write $s \cdot \mathrm{e}^{i\varphi} := z - E[z]$ with $s \geqslant \rho/2$ and $|\varphi| \geqslant \delta'$. Similarly, we write $t \cdot \mathrm{e}^{i\psi} := T[z;r]$ with $t \geqslant \frac{1}{2}(1-\delta) \times (r+\rho)$ and $|\psi| \leqslant \delta$. Thus, using $1-\delta \geqslant 1/3$,

$$\begin{aligned} |T[z;r]+E[z]-z| &= |t\mathrm{e}^{i\psi} - s\mathrm{e}^{i\varphi}| \\ &= \sqrt{t^2+s^2-2ts\cos(\varphi-\psi)} \\ &\geqslant (2)^{-1/2}\sqrt{1-\cos(\varphi-\psi)}\,(|t|+|s|) \\ &\geqslant (18)^{-1/2}\sqrt{1-\cos(\varphi-\psi)}\,(r+\rho). \end{aligned} \tag{IV.10}$$

Now (b) follows from $1-\cos x \geqslant x^2/\pi$ for $|x| \leqslant \pi/2$.

Finally, we note that $|z-E[z]| \leqslant \rho/2$, for $z \in \mathscr{U}^{(\mathrm{in})}$, which, together with $r \geqslant \rho$, implies that

$$|T[z;r]+E[z]-z| \geqslant (1-\delta)\, r - \frac{\rho}{2} \geqslant \left(\frac{1-\delta}{2} - \frac{1}{4}\right)(r+\rho), \tag{IV.11}$$

and hence (c). ∎

The previous lemma has two important consequences. On $\mathscr{U}^{(\mathrm{out})}(\delta')$, it guarantees the invertibility of $H[z]-z$, thus identifying z as a point in the resolvent set $\rho(H[z])$ of $H[z]$. On $\mathscr{U}^{(\mathrm{in})}$, it justifies the application of the Feshbach map $\mathscr{F}_{\chi_\rho}$. We make this precise in the following theorem.

THEOREM IV.2. *Let* $H \equiv (E, T, \underline{W}) \in \mathscr{B}(\delta, \varepsilon)$ *with* $\rho, \delta, \varepsilon \leqslant 1/8$, *and assume that* $\varepsilon\rho^{-1/2} \leqslant 1/90$ *and* $\xi C_d \leqslant 1$.

(a) *$H[z]-z$ is invertible, for all $z \in \mathcal{U}^{(\mathrm{out})}(120\varepsilon\rho^{-1/2}+\delta)$.*

(b) *For any $z \in \mathcal{U}^{(\mathrm{in})}$, the operator $\bar{\chi}_\rho H[z]\bar{\chi}_\rho - z$ is invertible on $\chi[H_f \geqslant \rho]\mathcal{H}_{\mathrm{red}}$. Moreover, writing $\chi_\rho \equiv \chi_\rho[H_f] \equiv \chi[H_f < \rho]$, the Feshbach map $\mathcal{F}_{\chi_\rho}(H[z]-z)$ is defined, and it is invertible on $\chi[H_f<\rho]\mathcal{H}_{\mathrm{red}}$ if and only if $H[z]-z$ is invertible on $\mathcal{H}_{\mathrm{red}}$.*

Proof. The argument is similar to the proof of Lemma II.3 and Theorem III.3(a), (b). First we remark that $T[z; H_f]+E[z]-z$ is a normal, bounded operator because it is defined as a spectral function of H_f. Moreover, Lemma IV.1(a) guarantees that $T[z; H_f]+E[z]-z$ is m-sectorial and bounded invertible if $z \in \mathcal{U}^{(\mathrm{out})}(120\varepsilon\rho^{-1/2}+\delta)$ or if $T[z; H_f]+E[z]-z$ is restricted to $\chi[H_f \geqslant \rho]\mathcal{H}_{\mathrm{red}}$ and $z \in \mathcal{U}^{(\mathrm{in})}$. We denote by U the unitary that results from its polar decomposition, i.e.,

$$T[z; r]+E[z]-z = U\,|T[z; r]+E[z]-z|. \tag{IV.12}$$

To prove (a), we construct the inverse of $H[z]-z$ by a Neumann series,

$$(H[z]-z)^{-1} = R_0^{1/2}\left[\sum_{n=0}^{\infty} U^*(-R_0^{1/2}\chi_1 W[z]\chi_1 R_0^{1/2}U^*)^n\right], \tag{IV.13}$$

for $z \in \mathcal{U}^{(\mathrm{out})}(120\varepsilon\rho^{-1/2})$, where $R_0^{1/2} := |T[z; r]+E[z]-z|^{-1/2}$. We apply Theorem B.2 and Lemma IV.1(b) and obtain

$$\begin{aligned}
&\|R_0^{1/2}\chi_1 W[z]\chi_1 R_0^{1/2}\| \\
&\quad\leqslant \|(H_f+\rho)^{-1/2}\chi_1 W[z]\chi_1(H_f+\rho)^{-1/2}\| \cdot \left\|\frac{H_f+\rho}{T[z; H_f]+E[z]-z}\right\| \\
&\quad\leqslant \frac{2e^2\varepsilon\, 3\sqrt{2\pi}}{\rho^{1/2}(120\varepsilon\rho^{-1/2}-\varepsilon)} < 1,
\end{aligned} \tag{IV.14}$$

since $\xi C_d \leqslant 1$. Thus the series in (IV.13) is norm convergent.

Part (b) follows similarly from

$$\begin{aligned}
&(\bar{\chi}_\rho H[z]\bar{\chi}_\rho - z)^{-1} \\
&\quad= R_0^{1/2}\left[\sum_{n=0}^{\infty} U^*(-R_0^{1/2}\chi_1\bar{\chi}_\rho W[z]\chi_1\bar{\chi}_\rho R_0^{1/2}U^*)^n\right],
\end{aligned} \tag{IV.15}$$

and the estimate

$$\|R_0^{1/2}\chi_1\bar{\chi}_\rho W[z]\bar{\chi}_\rho\chi_1 R_0^{1/2}\| \leqslant \frac{12e^2\varepsilon}{\rho^{1/2}} < 1. \quad ▮ \tag{IV.16}$$

Next, we rewrite $\mathscr{F}_{\chi_\rho}(H[z]-z)$, for $z \in \mathscr{U}^{(\mathrm{in})}$, in a Wick-ordered form

$$\mathscr{F}_{\chi_\rho}(H[z]-z) =: \tilde{H}[z] + E[z] - z \tag{IV.17}$$

where $\tilde{H}[z] = \chi_\rho(\tilde{E}[z] + \tilde{T}[z; H_f] + \tilde{W}[z])\,\chi_\rho$ and $\tilde{T}$, $\tilde{E}$, $\tilde{W}$ and $\tilde{W}_{M,N}$, with corresponding coupling functions $\tilde{w}_{M,N}$, are defined similarly to (III.63). The coupling functions $\tilde{w}_{M,N}$ are given by the same formulae as (III.65)–(III.68) in Lemma III.6, except that in (IV.18), below, the summation runs over all $m_l, n_l, p_l, q_l \geqslant 0$ such that their sum is $\geqslant 1$. More precisely,

$$\begin{aligned}
&\tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}] \\
&\quad = \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{\substack{m_l+p_l+n_l+q_l \geqslant 1; \\ l=1, ..., L}} \delta_{M, \sum_{l=1}^{L} m_l} \cdot \delta_{N, \sum_{l=1}^{L} n_l} \\
&\quad \times \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \right\} \\
&\quad \times \{ \tilde{D}_L[H_f; \{ W_{p_l, q_l}^{m_l, n_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}; \{ \bar{R}_0 \}_{l=1}^{L-1}] \}_{M,N}^{symm}.
\end{aligned} \tag{IV.18}$$

The definition of $\tilde{D}_L$ is identical to (III.66) except that the expectation value $\langle \varphi_{el} \otimes \Omega \mid (\cdot)\, \varphi_{el} \otimes \Omega \rangle$ is replaced by $\langle \Omega \mid (\cdot)\, \Omega \rangle$.

Our next step is to rescale $\tilde{H}[z]$ by Γ_ρ and to use the map

$$Z: \mathscr{U}^{(\mathrm{in})} \to D_{1/2}, \qquad Z(\zeta) := \rho^{-1}(\zeta - E[\zeta]). \tag{IV.19}$$

We remark that Z is bijective provided that $H \in \mathscr{B}(\delta, \varepsilon)$ and $\varepsilon \leqslant 1/8$. To see this, we differentiate $Z(\zeta)$ which yields

$$\frac{\rho}{2} \cdot \partial_\zeta Z(\zeta) = 1 - \partial_\zeta E[\zeta]. \tag{IV.20}$$

Since $\mathscr{U}^{(\mathrm{in})} \subseteq D_{1/4}$, by (IV.5), *Cauchy's estimate* with a contour on $\partial D_{1/2} = \{|z| = 1/2\}$ yields that, for all $\zeta \in \mathscr{U}^{(\mathrm{in})}$,

$$|\partial_\zeta E[\zeta]| \leqslant 4 \sup_{D_{1/2}} |E| \leqslant 4\varepsilon \leqslant \tfrac{1}{2}, \tag{IV.21}$$

and hence that $Z: \mathscr{U}^{(\mathrm{in})} \to D_{1/2}$ is bijective, indeed.

We define the renormalization map $\mathscr{R}_\rho$ by

$$\mathscr{R}_\rho(H)[z] - z := \rho^{-1} \Gamma_\rho \mathscr{F}_{\chi_\rho}(H[Z^{-1}(z)] - Z^{-1}(z))\, \Gamma_\rho^*, \tag{IV.22}$$

for $z \in D_{1/2}$.

IV.2. *The Contraction Property of the Renormalization Map $\mathscr{R}_\rho$*

We come to the heart of the matter: the contraction property of $\mathscr{R}_\rho$. Concretely, we propose to show that

$$\mathscr{R}_\rho : \mathscr{B}(\delta, \varepsilon) \to \mathscr{B}(\delta + \eta\varepsilon, \eta\varepsilon), \tag{IV.23}$$

where $\delta \leqslant 1/8$, $\varepsilon \leqslant 1/16$, $\eta \leqslant 1/2$, and $\mathscr{B}(\delta, \varepsilon)$ is defined in (IV.2). Obviously, this property enables us to iterate the renormalization map, yielding a sequence of analytic families in $\mathscr{W}_\Delta$ which converges to a (trivial) fixpoint of $\mathscr{R}_\rho$,

$$(0, \lambda r, \underline{0}) = \mathscr{R}_\rho(0, \lambda r, \underline{0}), \tag{IV.24}$$

where $\lambda \in \mathbb{C} \backslash \{0\}$. By a trivial change of scales, we may assume $\lambda = 1$ without loss of generality. Note that $(0, r, \underline{0})$ corresponds to $H[z] \equiv H_f$, the free photon Hamiltonian.

The proof of this contraction property is split into a series of lemmata. We start with estimating the coupling functions $\tilde{w}_{M,N}$, defined in (IV.18). To make our argument more transparent, we write

$$\tilde{w}_{M,N} = \tilde{w}^T_{M,N} + \Delta\tilde{w}_{M,N}, \tag{IV.25}$$

where

$$\tilde{w}^T_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}] := w_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}], \tag{IV.26}$$

and

$$\begin{aligned} &\Delta\tilde{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}] \\ &\quad = \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{m_l + p_l + n_l + q_l \geqslant 1; \\ l = 1, ..., L}} \delta_{M, \sum_{l=1}^L m_l} \cdot \delta_{N, \sum_{l=1}^L n_l} \\ &\qquad \times \prod_{l=1}^L \left\{ \binom{m_l + p_l}{p_l} \binom{n_l + q_l}{q_l} \right\} \\ &\qquad \times \{ \tilde{D}_L[H_f; \{ W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^L; \{ \bar{R}_0 \}_{l=1}^{L-1}] \}^{symm}_{M,N}. \end{aligned} \tag{IV.27}$$

Then we have the following lemma

Lemma IV.3. *Let $(E, T, \underline{W}) \in \mathscr{B}(\delta, \varepsilon)$ with $\rho, \delta \leqslant 1/8$, $\varepsilon \leqslant 1/16$, and assume that $\xi C_d \leqslant 1$, $\xi^2 C_d \leqslant 1/16$, and $\varepsilon\rho^{-1/2} \leqslant 1/320$. Then*

$$|\widetilde{D}_L[\cdots]| \leqslant \xi^{M+N} \cdot \left(\frac{\rho}{6}\right) \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{l=1}^{L} \left\{\frac{(\xi C_d^{1/2})^{p_l+q_l}}{(p_l!)^{1/2}\,(q_l!)^{1/2}}\right\}$$
$$\times \prod_{j=1}^{M} \max\{\rho, \omega(k_j)\}^{-1/2} \cdot \prod_{j=1}^{N} \max\{\rho, \omega(\tilde{k}_j)\}^{-1/2} \qquad \text{(IV.28)}$$

and

$$\|\widetilde{D}_L[\cdots]\|^{(\infty)}_{\Delta,\rho} \leqslant \left(\frac{\rho}{6}\right) \cdot \left(\frac{\xi}{\rho^{1/2}}\right)^{M+N} \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{l=1}^{L} \left\{\frac{(\xi C_d^{1/2})^{p_l+q_l}}{(p_l!)^{1/2}\,(q_l!)^{1/2}}\right\} \qquad \text{(IV.29)}$$

where $\widetilde{D}_L[\cdots] := \widetilde{D}_L[z; r; \{W^{m_l,n_l}_{p_l,q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}: \{\bar{R}_0\}_{l=1}^{L-1}]$.

Moreover, for $M+N \geqslant 1$,

$$\|\tilde{w}^T_{M,N}\|^{(\infty)}_{\Delta,\rho} \leqslant \varepsilon \cdot \xi^{M+N}, \qquad \text{(IV.30)}$$

$$\|\Delta\tilde{w}_{M,N}\|^{(\infty)}_{\Delta,\rho} \leqslant 2\rho \left(\frac{80\varepsilon}{\rho^{1/2}}\right)^2 \left(\frac{4\xi}{\rho^{1/2}}\right)^{M+N}, \qquad \text{(IV.31)}$$

$$\|\Delta\tilde{w}_{0,0}\|^{(\infty)}_{\Delta,\rho} \leqslant 2\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^2 \cdot (4\xi C_d^{1/2})^2. \qquad \text{(IV.32)}$$

Proof. Inequality (IV.30) is trivial. To prove (IV.28), we apply (IV.8) and the fact that $\mu_l \geqslant \omega(\tilde{k}_1^{(n_l)}) + \omega(k_{l+1}^{(m_{l+1})})$ to the estimate analogous to (III.74) and obtain

$$|\widetilde{D}_L[\cdots]| \leqslant 6^{L-1}\rho \prod_{l=1}^{L} \|(H_f + \mu_{l-1} + \rho)^{-1/2}\,\chi_1\,W^{m_l,n_l}_{p_l,q_l}\chi_1$$
$$\times (H_f + \mu_l + \rho)^{-1/2}\|$$
$$\leqslant 6^{L-1}\rho \prod_{l=1}^{L} \|(H_f + \omega(k_l^{(m_l)}) + \rho)^{-1/2}\,\chi_1\,W^{m_l,n_l}_{p_l,q_l}\chi_1$$
$$\times (H_f + \omega(\tilde{k}_l^{(n_l)}) + \rho)^{-1/2}\|. \qquad \text{(IV.33)}$$

Next, we apply Theorem B.1 (with $M := p_l$ and $N := q_l$). This yields

$$|\widetilde{D}_L[\cdots]| \leqslant 6^{L-1}\rho\varepsilon^L\xi^{M+N} \prod_{j=1}^{M} \omega(k_j)^\alpha \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha$$
$$\times \prod_{l=1}^{L} (\min\{\rho + \omega(k_l^{(m_l)}), 1\}^{-\delta_{p_l,0}/2}$$
$$\times \min\{\rho + \omega(\tilde{k}_l^{(n_l)}), 1\}^{-\delta_{q_l,0}/2})$$
$$\times \prod_{l=1}^{L} \left\{\frac{\rho^{-(\delta_{p_l,0}+\delta_{q_l,0})/2}(\xi C_d^{1/2}\,\Gamma[\mu+1]^{1/2})}{\Gamma[(1+\mu)\,p_l+1]^{1/2}\,\Gamma[(1+\mu)\,q_l+1]^{1/2}}\right\}. \qquad \text{(IV.34)}$$

Now, we claim that

$$\min\{\rho+\omega(k_l^{(m_l)}),1\}^{\delta_{p_l,0}}\cdot\min\{\rho+\omega(\tilde{k}_l^{(n_l)}),1\}^{\delta_{q_l,0}} \geqslant \rho\sum_{j=1}^{m_l}\max\{\rho,\omega(k_{l,j})\}\cdot\prod_{j=1}^{n_l}\max\{\rho,\omega(\tilde{k}_{l,j})\}. \qquad \text{(IV.35)}$$

We check (IV.35) by inspection: For $p_l+q_l\geqslant 1$, the left side in (IV.35) is bounded from below by $\rho\leqslant 1$, while ρ is an upper bound for its right side. It remains to consider the case $p_l=q_l=0$, in which case we have that $m_l+n_l\geqslant 1$. We estimate

$$\min\{\rho+\omega(k_l^{(m_l)}),1\}^{\delta_{p_l,0}}\cdot\min\{\rho+\omega(\tilde{k}_l^{(n_l)}),1\}^{\delta_{q_l,0}} \geqslant \max\{\omega,\omega(k_{l,1}),...,\omega(k_{l,m_l})\}\cdot\max\{\rho,\omega(\tilde{k}_{l,1}),...,\omega(\tilde{k}_{l,n_l})\}, \qquad \text{(IV.36)}$$

using $\rho,\omega(k_{l,j}),\omega(\tilde{k}_{l,j})\leqslant 1$. Estimate (IV.35) now directly follows from (IV.36) and $\rho\leqslant 1$.

We obtain (IV.28) from inserting (IV.35) and

$$\frac{\Gamma[1+\mu]^p}{\Gamma[(1+\mu)\,p+1]}\leqslant\frac{\Gamma[1]^p}{\Gamma[p+1]}=\frac{1}{p!} \qquad \text{(IV.37)}$$

into (IV.34). In turn, (IV.29) follows from (IV.28) and $\max\{\rho,\omega(k_j)\}\geqslant\rho$.

To prove (IV.31), we insert (IV.29) into (IV.27) and obtain

$$\|\Delta\tilde{\omega}_{M,N}\|_{\Delta,\rho}^{(\infty)}\leqslant\sum_{L=2}^{\infty}\sum_{\substack{m_l+n_l+p_l+q_l\geqslant 1,\\ l=1,...,L}}\delta_{M,\sum_{l=1}^L m_l}\cdot\delta_{N,\sum_{l=1}^L n_l} \times\rho\left(\frac{\xi}{\rho^{1/2}}\right)^{M+N}\cdot\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \times\prod_{l=1}^{L}\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\frac{(\xi C_d^{1/2})^{p_l+q_l}}{(p_l!)^{1/2}(q_l!)^{1/2}}\right\}. \qquad \text{(IV.38)}$$

Since $(m+p)!\leqslant 2^{m+p}m!\,p!$, Eq. (IV.38) is seen to imply that

$$\|\Delta\tilde{w}_{M,N}\|_{\Delta,\rho}^{(\infty)}\leqslant\rho\cdot\left(\frac{4\xi}{\rho^{1/2}}\right)^{M+N}\cdot\sum_{L=2}^{\infty}A^L, \qquad \text{(IV.39)}$$

where

$$A:=\frac{6\varepsilon}{\rho^{1/2}}\left[\left(\sum_{m=0}^{\infty}2^{-m}\right)^2\left(\sum_{p=0}^{\infty}\frac{(2\xi C_d^{1/2})^p}{(p!)^{1/2}}\right)^2-1\right]. \qquad \text{(IV.40)}$$

An application of Schwarz' inequality gives

$$\left(\sum_{p=0}^{\infty} \frac{(2\xi C_d^{1/2})^p}{(p!)^{1/2}} \right)^2 \leqslant \left(\sum_{p=0}^{\infty} 2^{-p} \right) \cdot \left(\sum_{p=0}^{\infty} \frac{(8\xi^2 C_d)^p}{p!} \right) = 2 \exp[8\xi^2 C_d], \quad \text{(IV.41)}$$

and hence

$$A \leqslant \frac{48\varepsilon}{\rho^{1/2}} \cdot \exp[8\xi^2 C_d] \leqslant \frac{80\varepsilon}{\rho^{1/2}} \leqslant \frac{1}{2}. \quad \text{(IV.42)}$$

Therefore, $\sum_{L=2}^{\infty} A^L = A^2(1-A)^{-1} \leqslant 2A^2$ and we arrive at (IV.31).

In order to prove (IV.32), we examine (IV.38) again for the special case that $M = N = 0$. Then we estimate

$$\begin{aligned} \|\Delta \tilde{w}_{0,0}\|_{\delta,\rho}^{(\infty)} &\leqslant \sum_{L=2}^{\infty} \sum_{\substack{p_l+q_l \geqslant 1, \\ l=1,\dots,L}} \left(\frac{6\varepsilon}{\rho^{1/2}} \right)^L \cdot \prod_{l=1}^{L} \left\{ \frac{(\xi C_d^{1/2})^{p_l+q_l}}{(p_l!)^{1/2}\,(q_l!)^{1/2}} \right\} \\ &\leqslant \sum_{L=2}^{\infty} \left(\frac{6\varepsilon}{\rho^{1/2}} \right)^L \left[\left(\sum_{p=0}^{\infty} (\xi C_d^{1/2})^p \right)^2 - 1 \right]^L \\ &\leqslant \sum_{L=2}^{\infty} \left(\frac{6\varepsilon}{\rho^{1/2}} \right)^L (4\xi C_d^{1/2})^L \leqslant 2 \left(\frac{6\varepsilon}{\rho^{1/2}} \right)^2 (4\xi C_d^{1/2})^2, \end{aligned} \quad \text{(IV.43)}$$

using again that $\xi C_d^{1/2} \leqslant 1/4$ and $6\varepsilon\rho^{-1/2} \leqslant 1/2$. ∎

Lemma IV.4. *Let $(E, T, \underline{W}) \in \mathscr{B}(\delta, \varepsilon)$ with $\delta \leqslant 1/8$ and $\varepsilon < 1/16$, and assume that $\xi C_d \leqslant 1$, $\xi^2 C_d \leqslant 1/144$, and $\varepsilon\rho^{-1/2} \leqslant 1/320$. Then, for all $M + N \geqslant 1$,*

$$\|\partial_r \tilde{w}_{M,N}^T\|_{\Delta,\rho}^{(1)} \leqslant \frac{24\varepsilon}{\rho^{1/2}} \cdot (\xi C_d \rho^{1/2})^{M+N}, \quad \text{(IV.44)}$$

$$\|\partial_r \Delta \tilde{w}_{M,N}\|_{\Delta,\rho}^{(1)} \leqslant 12 \left(\frac{90\varepsilon}{\rho^{1/2}} \right)^2 \cdot (4\xi C_d \rho^{1/2})^{M+N}, \quad \text{(IV.45)}$$

$$\|\partial_r \Delta \tilde{w}_{0,0}\|_{\Delta,\rho}^{(1)} \leqslant 48 \left(\frac{6\varepsilon}{\rho^{1/2}} \right)^2 \cdot (4\xi C_d)^2, \quad \text{(IV.46)}$$

Proof. As in the proof of Lemma III.8, we first recall that

$$\bar{R}_0[H_f] \equiv \bar{R}_0[z; H_f] \equiv \frac{\bar{\chi}_\rho}{T+E-z} := \frac{\chi[H_f \geqslant \rho]}{T[z; H_F] + E[z] - z}, \quad \text{(IV.47)}$$

which implies that

$$\partial_{H_f} \bar{R}_0[H_f] = (T[\rho] + E - z)^{-1} \, \delta(H_f - \rho) - \bar{R}_0^2[H_f] \cdot \partial_{H_f} T. \tag{IV.48}$$

By Leibniz' rule, we thus obtain

$$\partial_r \tilde{D}_L[\cdots] = \sum_{l=1}^{L} \tilde{D}_L[\cdots W_l \to \partial_r W_l \cdots]$$

$$+ (T[\rho] + E - z)^{-1} \tag{IV.49}$$

$$\sum_{l=1}^{L-1} \tilde{D}_L[\cdots \{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, \delta(r-\rho), (\bar{R}_0)_{j=l+1}^{L-1}\} \cdots] \tag{IV.50}$$

$$+ \sum_{l=1}^{L-1} \tilde{D}_L[\cdots \{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, \bar{R}_0^2 \partial_r T, (\bar{R}_0)_{j=l+1}^{L-1}\} \cdots], \tag{IV.51}$$

where we abbreviated by

$$\tilde{D}_L[\cdots W_l \to \partial_r W_l \cdots]$$
$$:= \tilde{D}_L[z; H_f; \{(W_{p_j, q_j}^{m_j, n_j}; k_j^{(m_j)}; \tilde{k}_j^{(n_j)})_{j=1}^{l-1},$$
$$(\partial_r W_{p_l, q_l}^{m_l, n_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}), (W_{p_j, q_j}^{m_j, n_j}; k_j^{(m_j)}; \tilde{k}_j^{(n_j)})_{j=l+1}^{L}\}; \{\bar{R}_0\}_{j=1}^{L-1}], \tag{IV.52}$$

$$\tilde{D}_L[\cdots \{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, f_l(r), (\bar{R}_0)_{j=l+1}^{L-1}\} \cdots]$$
$$:= \tilde{D}_L[z; H_f; \{W_{p_j, q_j}^{m_j, n_j}; k_j^{(m_j)}; \tilde{k}_j^{(n_j)}\}_{j=1}^{L}; \{(\bar{R}_0)_{j=1}^{l-1}, f_l(r), (\bar{R}_0)_{j=l+1}^{L-1}\}], \tag{IV.53}$$

with $f_l(r) := \delta(r - \rho)$ and $f_l(r) := \bar{R}_0^2(r) \cdot \partial_r T$, respectively.

We begin with estimating Term (IV.51). Since

$$\|(H_f + \rho) \, \bar{R}_0^2[H_f] \, \partial_{H_f} T[z; H_f]\|$$
$$\leqslant \|(H_f + \rho) \, \bar{R}[H_f]\|^2 \cdot \left\| \frac{\chi[H_f \geqslant \rho]}{H_f + \rho} \right\| \cdot \|\partial_{H_f} T[z; H_f]\|$$
$$\leqslant \frac{18(1+\varepsilon)}{\rho} \leqslant \frac{21}{\rho}, \tag{IV.54}$$

we obtain as in (IV.34)

$$|\tilde{D}_L[\cdots\{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, \bar{R}_0^2 \partial_r T, (\bar{R}_0)_{j=l+1}^{L-1}\}\cdots]| \leqslant \left(\frac{\xi}{\rho^{1/2}}\right)^{M+N} \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{l=1}^{L} \frac{(\xi C_d^{1/2})^{p_l+q_l}}{(p_l!)^{1/2}\,(q_l!)^{1/2}} \prod_{j=1}^{M} \omega(k_j)^\alpha \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha. \tag{IV.55}$$

Inserting $\int_{\omega(k)\leqslant\rho} \omega(k)^{\alpha-\beta}\,dk = C_d\rho$, we observe that (IV.55) yields

$$\|\tilde{D}_L[\cdots\{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, \bar{R}_0^2 \partial_r T, (\bar{R}_0)_{j=l+1}^{L-1}\}\cdots]\|_{\Delta,\rho}^{(1)} \leqslant (\xi C_d \rho^{1/2})^{M+N} \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{l=1}^{L} \frac{(\xi C_d^{1/2})^{p_l+q_l}}{(p_l!)^{1/2}\,(q_l!)^{1/2}}. \tag{IV.56}$$

Second, we estimate the term in (IV.50). Similarly to (III.86), we again use the recurrence relation (A.47) to rewrite this term as

$$\begin{aligned}
&\tilde{D}_L[\cdots\{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, \delta(r-\rho), (\bar{R}_0)_{j=l+1}^{L-1}\}\cdots] \\
&\quad = \sum_{\substack{v_j=0,\\ j=1,\ldots,l}}^{q_j} \sum_{\substack{u_j=0,\\ j=l+1,\ldots,L}}^{p_j} \int \prod_{j=1}^{l} \left\{ d\tilde{y}_j^{(v_j)} \binom{q_j}{v_j} \right\} \prod_{j=l+1}^{L} \left\{ dy_j^{(u_j)} \binom{p_j}{u_j} \right\} \\
&\qquad \times \tilde{D}_l[r+\omega_{\geqslant l}(k^{(m)}); z; \{W_j; k_j^{(m_j)}; \tilde{k}_j^{(n_j)}, \tilde{y}_j^{(v_j)}\}_{j=1}^{l}; \{\bar{R}_0\}_{j=1}^{l-1}] \\
&\qquad \times \delta\left[r + \sum_{j=1}^{l} \omega(\tilde{k}_j^{(n_j)}) + \sum_{j=l+1}^{L} \omega(k_j^{(m_j)}) + \omega(y^{(U)}) - \rho_0\right] \\
&\qquad \times \langle a(\tilde{y}^{(V)})\, a^\dagger(y^{(U)}) \rangle \\
&\qquad \times \tilde{D}_{L-l}[r+\omega_{\leqslant l}(\tilde{k}^{(n)}); z; \{W_j; k_j^{(m_j)}, y_j^{(u_j)}; \tilde{k}_j^{(n_j)}\}_{j=l+1}^{L}; \{\bar{R}_0\}_{j=l+1}^{L-1}].
\end{aligned} \tag{IV.57}$$

where $U := \sum_{j=1+l}^{L} u_j$, $V := \sum_{j=1}^{l} v_j$, and we have that

$$M_{\leqslant l} + M_{>l} + U := \sum_{j=1}^{l} n_j + \sum_{j=l+1}^{L} m_j + U \geqslant 1. \tag{IV.58}$$

To estimate this term, we closely follow the argument given after (III.86). We first insert (IV.29) into (IV.57), which yields

$$\|\tilde{D}_L[\cdots\{\bar{R}_0\}_{j=1}^{L-1} \to \{(\bar{R}_0)_{j=1}^{l-1}, \delta(r-\rho), (\bar{R}_0)_{j=l+1}^{L-1}\}\cdots]\|_{\Delta,\rho}^{(1)}$$

$$\leqslant \sum_{\substack{v_j=0,\\ j=1,\ldots,l}}^{q_j} \sum_{\substack{u_j=0,\\ j=l+1,\ldots,L}}^{p_j} \prod_{j=1}^{l} \binom{q_j}{v_j} \prod_{j=l+1}^{L} \binom{p_j}{u_j} \left(\frac{\rho}{6}\right)^2 \cdot \left(\frac{\xi}{\rho^{1/2}}\right)^{M+N+U+V}$$

$$\times \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{j=1}^{l} \frac{(\xi C_d^{1/2})^{m_j+q_j-v_j}}{(p_j!)^{1/2}\,((q_j-v_j)!)^{1/2}} \prod_{j=l+1}^{L} \frac{(\xi C_d^{1/2})^{p_j+q_j-u_j}}{((p_j-u_j)!)^{1/2}\,(q_j!)^{1/2}}$$

$$\times \chi[N_{\leqslant l}+M_{>l}+U \geqslant 1]\cdot \mathscr{I} \tag{IV.59}$$

where $\mathscr{I}$ denotes the following integral.

$$\mathscr{I} := (U!)\cdot C_d^{M+N+U} \int_0^\infty \delta\left[r + \sum_{j=1}^{N_{\leqslant l}+M_{>l}} \omega_j + \sum_{j=1}^{U} \hat{\omega}_j - \rho\right]$$

$$\times \prod_{j=1}^{M+N} d\omega_j \prod_{j=1}^{U} \hat{\omega}_j^{1+\mu}\, d\hat{\omega}_j, \tag{IV.60}$$

as in (III.88)–(III.90). If $U=0$ then $M_{>l}+N_{\leqslant l}=1$ and

$$\mathscr{I} \leqslant C_d^{M+N} \rho^{M+N-1}. \tag{IV.61}$$

Conversely, if $U \geqslant 1$ then

$$\mathscr{I} \geqslant U!\cdot C_d^{M+N+U} \rho^{(1+\mu)+M+N}$$

$$\times \int_0^\infty \chi[\hat{\omega}_1 + \cdots + \hat{\omega}_{w-1} \leqslant \rho] \prod_{j=1}^{u-1} \hat{\omega}_j^{1+\mu}\, d\hat{\omega}_j$$

$$= \frac{U!}{\Gamma[(2+\mu)(U-1)+1]} C_d^{M+N+U}\cdot \rho^{(2+\mu)\,U+M+N-1}$$

$$\leqslant \frac{U!\, C_d^{M+N+U} \rho^{(2+\mu)\,U+M+N-1}}{(2U-2)!} \tag{IV.62}$$

Now we use that $(U!)^2 \leqslant 2(2U-2)!$, which one easily verifies, for $U=1$ and $U=2$, by inspection and, for $U \geqslant 3$, by induction. This inequality implies that, for $U \geqslant 1$,

$$\mathscr{I} \leqslant \frac{2}{U!\,\rho} (C_d \rho)^{M+N} (C_d \rho^{2+\mu})^U. \tag{IV.63}$$

Comparing to (IV.61), we observe that Estimate (IV.63) is valid for $U=0$, as well, and thus for all $U \geqslant 0$. Inserting (IV.63) into (IV.59) and using that $\sum_{n=0}^{N} \binom{N}{n} = 2^N$, $\rho \leqslant 1$, and $2\xi C_d^{1/2} \leqslant 1/2$, we arrive at

$$\begin{aligned}
&\| \tilde{D}_L[\cdots \{\bar{R}_0\}_{j=1}^{L-1} \rightarrow \{(\bar{R}_0)_{j=1}^{l-1}, \delta(r-\rho), (\bar{R}_0)_{j=l+1}^{L-1}\} \cdots] \|_{\Delta,\rho}^{(1)} \\
&\quad \leqslant \sum_{\substack{v_j=0, \\ j=1,\dots,l}}^{q_j} \sum_{\substack{u_j=0, \\ j=l+1,\dots,L}}^{p_j} \prod_{j=1}^{l} \binom{q_j}{v_j} \prod_{j=l+1}^{L} \binom{p_j}{u_j} \binom{\rho}{18} \cdot (\xi C_d \rho^{1/2})^{M+N} \\
&\qquad \times \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{j=1}^{l} \frac{2^{-v_j} \cdot (\xi C_d^{1/2})^{p_j+q_j-v_j}}{(p_j!)^{1/2} ((q_j-v_j)!)^{1/2}} \prod_{j=l+1}^{L} \frac{2^{-u_j} \cdot (\xi C_d^{1/2})^{p_j+q_j-u_j}}{((p_j-u_j)!)^{1/2} (q_j!)^{1/2}} \\
&\qquad \times \max_{U \in \mathbb{N}_0} \{(U!)^{-1} \cdot (4\xi^2 C_d \rho^{1+\mu})^U\} \\
&\quad \leqslant \binom{\rho}{18} \cdot (\xi C_d \rho^{1/2})^{M+N} \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \exp[4\xi^2 C_d \rho^{1+\mu}] \cdot \prod_{l=1}^{L} (2\xi C_d^{1/2})^{p_l+q_l} \\
&\quad \leqslant \binom{\rho}{9} \cdot (\xi C_d \rho^{1/2})^{M+N} \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{l=1}^{L} (2\xi C_d^{1/2})^{p_l+q_l}.
\end{aligned} \tag{IV.64}$$

We come to the most difficult part, the estimate on (IV.49). Up to now, we did not make full use of the inverse factorials that Theorem B.1 yields, but to estimate (IV.49), we compensate with these inverse factorials for the many terms that the contractions of creation- and annihilation operators generate. To this end, we apply (A.46) and obtain

$$\begin{aligned}
&\tilde{D}_L[\cdots W_l \rightarrow \partial_r W_l \cdots] \\
&\quad = \sum_{\substack{v_j=0, \\ j=1,\dots,l-1}}^{q_j} \sum_{\substack{u_j=0, \\ j=l+1,\dots,L}}^{p_j} \int \prod_{j=1}^{l-1} \left\{ d\tilde{y}_j^{(v_j)} \binom{q_j}{v_j} \right\} \prod_{j=l+1}^{L} \left\{ dy_j^{(u_j)} \binom{p_j}{u_j} \right\} \\
&\qquad \times \tilde{D}_{l-1}[z; r+\omega_{\geqslant l}(k^{(m)}); \{W_j; k_j^{(m_j)}; \tilde{k}_j^{(n_j)}, \tilde{y}_j^{(v_j)}\}_{j=1}^{l-1}; \{\bar{R}_0\}_{j=1}^{l-1}] \\
&\qquad \times \bar{R}_0[\mu_{l-1}] \langle a(\tilde{y}^{(V)}) \, \partial_r W_l[\lambda_l; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}] \, a^\dagger(y^{(U)}) \rangle_\Omega \, \bar{R}_0[\mu_l] \\
&\qquad \times \tilde{D}_{L-l}[z; r+\omega_{\leqslant l}(\tilde{k}^{(n)}); \{W_j; k_j^{(m_j)}, y_j^{(u_j)}; \tilde{k}_j^{(n_j)}\}_{j=l+1}^{L}; \{\bar{R}_0\}_{j=l+1}^{L-1}].
\end{aligned} \tag{IV.65}$$

Now, we fully exploit Estimate (IV.28) which yields

$$|\tilde{D}_{l-1}[z; r+\omega_{\geqslant l}(k^{(m)}); \{W_j; k_j^{(m_j)}; \tilde{k}_j^{(n_j)}, \tilde{y}_j^{(v_j)}\}_{j=1}^{l-1}; \{\bar{R}_0\}_{j=1}^{l-2}]|$$

$$\leqslant \left(\frac{\rho}{6}\right)\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^{l-1} \prod_{j=1}^{l-1} \left\{\frac{(\xi C_d^{1/2})^{p_j+q_j-v_j} (\xi\rho^{-1/2})^{m_j+n_j}}{(p_j!)^{1/2} ((q_j-v_j)!)^{1/2}}\right\}$$

$$\times \prod_{j=1}^{l-1} \left\{\prod_{i=1}^{m_j} \omega(k_{j,i})^\alpha \prod_{i=1}^{n_j} \omega(\tilde{k}_{j,i})^\alpha\right\} \prod_{j=1}^{V} \left\{\frac{\xi\omega(\tilde{y}_j)^\alpha}{\max\{\rho, \omega(\tilde{y}_j)\}^{1/2}}\right\} \tag{IV.66}$$

and

$$|\tilde{D}_{L-l}[z; r+\omega_{\leqslant l}(\tilde{k}^{(n)}); \{W_j; k_j^{(m_j)}, y_j^{(u_j)}; \tilde{k}_j^{(n_j)}\}_{j=l+1}^{L}; \{\bar{R}_0\}_{j=l+1}^{L-1}]|$$

$$\leqslant \left(\frac{\rho}{6}\right)\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^{L-l} \prod_{j=l+1}^{L} \left\{\frac{(\xi C_d^{1/2})^{p_j+q_j-u_j} (\xi\rho^{-1/2})^{m_j+n_j}}{((p_j-u_j)!)^{1/2} (q_j!)^{1/2}}\right\}$$

$$\times \prod_{j=l+1}^{L} \left\{\prod_{i=1}^{m_j} \omega(k_{j,i})^\alpha \prod_{i=1}^{n_j} \omega(\tilde{k}_{j,i})^\alpha\right\} \cdot \prod_{j=1}^{U} \left\{\frac{\xi\omega(y_j)^\alpha}{\max\{\rho, \omega(y_j)\}^{1/2}}\right\}. \tag{IV.67}$$

Upon inserting (IV.66) and (IV.67) into (IV.65), we obtain, for $(k^{(M)}, \tilde{k}^{(N)}) \in B_\rho^M \times B_\rho^N$,

$$|\tilde{D}_L[\cdots W_l \to \partial_r W_l \cdots]|$$

$$\leqslant \sum_{\substack{v_j=0,\\ j=1,\ldots,l-1}}^{q_j} \sum_{\substack{u_j=0,\\ j=l+1,\ldots,L}}^{p_j} \prod_{j=1}^{l-1} \binom{q_j}{v_j} \prod_{j=l+1}^{L} \binom{p_j}{u_j}$$

$$\times \frac{1}{4}\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^{L-1} (\xi\rho^{-1/2})^{M+N-m_l-n_l}$$

$$\times \prod_{j=l+1}^{L} \left\{\frac{(\xi C_d^{1/2})^{p_j+q_j-u_j}}{((p_j-u_l)!)^{1/2} (q_j!)^{1/2}}\right\} \prod_{j=1}^{l-1} \left\{\frac{(\xi C_d^{1/2})^{p_j+q_j-v_j}}{(p_j!)^{1/2} ((q_j-v_j)!)^{1/2}}\right\}$$

$$\times \prod_{j=1}^{M} \omega(k_j)^\alpha \cdot \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha \cdot \mathscr{I}(k_l^{(m_l)}, \tilde{k}_l^{(n_l)}), \tag{IV.68}$$

where

$$
\begin{aligned}
&\mathscr{I}(k_l^{(m_l)}, \tilde{k}_l^{(n_l)}) \\
&\quad := \int |\langle a(\tilde{y}^{(V)})\, \chi_1\, a^\dagger(x^{(p_l)}) \\
&\qquad \times \partial_r w_{m_l+p_l, n_l+q_l}[z; \lambda_l; k_l^{(m_l)}, x^{(p_l)}; \tilde{k}_l^{(n_l)}, \tilde{x}^{(q_l)}]\, a(\tilde{x}^{(q_l)})\, \chi_1\, a^\dagger(y^{(U)})\rangle_\Omega| \\
&\qquad \times \prod_{j=1}^{U} \left\{\frac{\omega(y_j)^\alpha\, dy_j}{\max\{\rho, \omega(y_j)\}^{1/2}}\right\} \cdot \prod_{j=1}^{V} \left\{\frac{\omega(\tilde{y}_j)^\alpha\, d\tilde{y}_j}{\max\{\rho, \omega(\tilde{y}_j)\}^{1/2}}\right\} dx^{(p_l)}\, d\tilde{x}^{(q_l)} \\
&\quad \leqslant \chi[V - p_l = U - q_l \geqslant 1] \cdot \left(\frac{V!}{(V-p_l)!}\right)\left(\frac{U!}{(U-q_l)!}\right) \cdot (U-q_l)! \\
&\qquad \times \rho^{-(p_l+q_l)/2} \int \chi_1[\omega(x^{(p_l)})]\, \chi_1[\omega(\tilde{x}^{(q_l)})] \\
&\qquad \times \sup_r |\partial_r w_{m_l+p_l, n_l+q_l}[z; r; k_l^{(m_l)}, x^{(p_l)}; \tilde{k}_l^{(n_l)}, \tilde{x}^{(q_l)}]| \\
&\qquad \times \prod_{j=1}^{p_l} \{\omega(x_j)^\alpha\, dx_j\} \prod_{j=1}^{q_l} \{\omega(\tilde{x}_j)^\alpha\, d\tilde{x}_j\} \\
&\qquad \times \int \chi_1[\omega(y^{(U-q_l)})] \prod_{j=1}^{U-q_l} \{\omega(y_j)^{2\alpha-1}\, dy_j\}. \qquad \text{(IV.69)}
\end{aligned}
$$

Here, Wick-ordering of $a(\tilde{y}^{(V)})\, a^\dagger(x^{(p_l)})$ gives rise to the factor $V!\,((V-p_l)!)^{-1}$, Wick-ordering of $a(\tilde{x}^{(q_l)})\, a^\dagger(y^{(U)})$ gives rise to the factor $U!\,((U-q_l)!)^{-1}$, and Wick-ordering of $a(\tilde{y}^{(V-p_l)})\, a^\dagger(y^{(U-q_l)})$ accounts for the factor $(U-q_l)!$ on the right side of (IV.69).

Next, we apply Lemma C.3 with $\eta := \alpha + \beta = 1 + \mu$, and we use (B.12). This yields

$$
\begin{aligned}
\mathscr{I}(k_l^{(m_l)}, \tilde{k}_l^{(n_l)}) \leqslant{}& C_d^{U-q_l} \rho^{-(p_l+q_l)/2} \cdot \chi[V - p_l = U - q_l \geqslant 1] \\
&\times \left(\frac{V!}{(V-p_l)!\; p_l^{(1+\mu)\, p_l}}\right) \cdot \left(\frac{U!}{(U-q_l)!\; q_l^{(1+\mu)\, q_l}}\right) \\
&\times \int \sup_r |\partial_r w_{m_l+p_l, n_l+q_l}[z; r; k_l^{(m_l)}, x^{(p_l)}; \tilde{k}_l^{(n_l)}, \tilde{x}^{(q_l)}]| \\
&\times \prod_{j=1}^{p_l} \frac{dx_j}{\omega(x_j)^\beta} \prod_{j=1}^{q_l} \frac{d\tilde{x}_j}{\omega(\tilde{x}_j)^\beta}. \qquad \text{(IV.70)}
\end{aligned}
$$

We insert (IV.70) into (IV.68) and integrate against the measure $\prod_{j=1}^{M} dk_j/\omega(k_j)^\beta \prod_{j=1}^{N} d\tilde{k}_j/\omega(\tilde{k}_j)^\beta$ over the region $B^M_{\rho^{1/\gamma}} \times B^N_{\rho^{1/\gamma}}$. We arrive at

$$\|\tilde{D}_L[\cdots W_l \to \partial_r W_l \cdots]\|^{(1)}_{\Delta,\rho} \leqslant \frac{\rho^{1/2}}{24\varepsilon} \sum_{\substack{v_j=0,\\ j=1,\ldots,l-1}}^{q_j} \sum_{\substack{u_j=0,\\ j=l+1,\ldots,L}}^{p_j} \prod_{j=1}^{l-1} \left\{2^{v_j}\binom{q_j}{v_j}\right\} \prod_{j=l+1}^{L} \left\{2^{u_j}\binom{p_j}{u_j}\right\}\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \times (\xi C_d \rho^{-1/2})^{M+N-m_l-n_l-p_l-q_l} \cdot (\xi C_d^{1/2})^{P+Q} \|\partial_r w_l\|^{(1)}_{\Delta,\rho}, \tag{IV.71}$$

denoting $Q := \sum_{j=1}^{l-1} q_j$ and $P := \sum_{j=l+1}^{L} p_j$. Note that

$$\sum_{\substack{v_j=0,\\ j=1,\ldots,l-1}}^{q_j} \prod_{j=1}^{l-1} \left\{2^{v_j}\binom{q_j}{v_j}\right\} = \prod_{j=1}^{l-1} 3^{q_j} \leqslant 3^Q, \tag{IV.72}$$

$$\sum_{\substack{u_j=0,\\ j=l+1,\ldots,L}}^{p_j} \prod_{j=l+1}^{L} \left\{2^{u_j}\binom{p_j}{u_j}\right\} = \prod_{j=l+1}^{L} 3^{p_j} \leqslant 3^P. \tag{IV.73}$$

Thus

$$\|\tilde{D}_L[\cdots W_l \to \partial_r W_l \cdots]\|^{(1)}_{\Delta,\rho} \leqslant \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot (\xi C_d \rho^{-1/2})^{M+N} \cdot (3\xi C_d^{1/2})^{P+Q}, \tag{IV.74}$$

using that $\|\partial_r w_l\|^{(1)}_{\Delta} \leqslant 24\varepsilon\rho^{-1/2} \cdot (\xi C_d \rho^{1/2})^{-m_l-n_l-p_l-q_l}$ and $P+Q \geqslant U+V$.

Adding up (IV.74), (IV.64), and (IV.56) according to (IV.49), we obtain the bound

$$\|\partial_r \tilde{D}_L[\cdots]\|^{(1)}_{\Delta,\rho} \leqslant 3L \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot (\xi C_d \rho^{1/2})^{M+N} \cdot \prod_{l=1}^{L} (3C_d^{1/2}\xi)^{p_l+q_l}. \tag{IV.75}$$

We insert this estimate and $3\xi C_d^{1/2} \leqslant 1/4$ into the series (IV.18), which yields

$$\begin{aligned}\|\partial_r \Delta \tilde{w}_{M,N}\|^{(1)}_{\Delta,\rho} &\leqslant \sum_{L=2}^{\infty} \sum_{\substack{m_l+n_l+p_l+q_l \geqslant 1\\ l=1,\ldots,L}} \delta_{M,\sum_{l=1}^{L} m_l} \cdot \delta_{N,\sum_{l=1}^{L} n_l} \cdot 3L \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \\ &\quad \times (\xi C_d \rho^{1/2})^{M+N} \cdot \prod_{l=1}^{L} \left\{4^{-p_l-q_l}\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\right\} \\ &\leqslant (4\xi C_d \rho^{1/2})^{M+N} \cdot \sum_{L=2}^{\infty} 3L \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \left[\left(\sum_{p=0}^{\infty} 2^{-p}\right)^4 - 1\right]^L \\ &\leqslant 12\left(\frac{90\varepsilon}{\rho^{1/2}}\right)^2 \cdot (4\xi C_d \rho^{1/2})^{M+N},\end{aligned} \tag{IV.76}$$

since $\hat{A} := 90\varepsilon\rho^{-1/2} \leqslant 1/2$ and thus $\sum_{L=2}^{\infty} L\hat{A}^L = 2\hat{A}^2(1-\hat{A})^{-1} \leqslant 4\hat{A}^2$.

Finally, for $M=N=0$, Equation (IV.75) and the series expansion (IV.18) yield

$$\|\partial_r \Delta \tilde{w}_{0,0}\|^{(1)}_{\Delta,\rho} \leqslant \sum_{L=2}^{\infty} \sum_{\substack{p_l+q_l \geqslant 1 \\ l=1,\ldots,L}} 3L \cdot \left(\frac{6\varepsilon}{\rho^{1/2}}\right)^L \cdot \prod_{l=1}^{L} (2\xi C_d^{1/2})^{p_l+q_l}$$

$$\leqslant \sum_{L=2}^{\infty} 3L \cdot \left(\frac{48\varepsilon\xi C_d^{1/2}}{\rho^{1/2}}\right)^L$$

$$\leqslant 48\left(\frac{6\varepsilon}{\rho^{1/2}}\right)^2 \cdot (4\xi C_d^{1/2})^2, \tag{IV.77}$$

proving (IV.45). ∎

We remark that in our derivation of (IV.71), we used the inequality

$$p! \leqslant p^{(1+\mu)p}. \tag{IV.78}$$

This is one point in our proof where the condition $\mu \geqslant 0$ (here, including $\mu=0$) inevitably enters, namely to compensate large factorials.

Theorem IV.5. *Fix $\mu, \rho, \xi > 0$ such that $\rho C_d \leqslant 1$, $C_d^{1/2}\xi \leqslant \rho^{(3+\mu)/4}$, and $\rho^{\mu/2} < 1/16$. Assume that $\delta \leqslant 1/8$ and $\varepsilon\rho^{-1/2} \leqslant 1/12800$. Then*

$$\mathscr{R}_\rho\colon \mathscr{B}(\delta,\varepsilon) \to \mathscr{B}(\delta+\eta\varepsilon, \eta\varepsilon), \tag{IV.79}$$

where

$$\eta := 4\rho^{\mu/2}\left(1+\frac{12800\varepsilon}{\rho^{1/2}}\right) \leqslant 8\rho^{\mu/2} < \frac{1}{2}. \tag{IV.80}$$

Proof. We assume that $(E, T, \underline{W}) \in \mathscr{B}(\delta,\varepsilon)$, and we write $\hat{H} \equiv (\hat{E}, \hat{T}, \underline{\hat{W}}) := \mathscr{R}_\rho[(E, T, \underline{W})]$. Equations (IV.17)–(IV.22) imply that

$$\hat{H}[Z(\zeta)] - Z(\zeta) := \rho^{-1}\Gamma_\rho(\tilde{H}[\zeta] + E[\zeta] - \zeta)\,\Gamma_\rho^*, \tag{IV.81}$$

for any $\zeta \in \mathscr{U}^{(\mathrm{in})}$, where, as before, we identify $\hat{H} \mapsto (\hat{E}, \hat{T}, \underline{\hat{W}}) \in \mathscr{W}_\Delta$ by

$$\hat{H}[z] =: \chi_1(\hat{T}[z; H_f] + \hat{E}[z] + \hat{W}[z])\,\chi_1, \tag{IV.82}$$

$$\hat{T}[z; H_f] := \rho^{-1} T[Z^{-1}(z); \rho H_f] + \rho^{-1}(\tilde{w}_{0,0}[Z^{-1}(z); \rho H_f] - \tilde{w}_{0,0}[Z^{-1}(z); 0]), \tag{IV.83}$$

$$\hat{E}[z] := \rho^{-1}\tilde{w}_{0,0}[Z^{-1}(z); 0], \tag{IV.84}$$

$$W[z] := \sum_{M+N \geqslant 1} W_{M,N}[z], \tag{IV.85}$$

$$W_{M,N}[z] := \int dk^{(M)}\, d\tilde{k}^{(N)} a^\dagger(k^{(M)})\, \hat{w}_{M,N}[z; H_f; k^{(M)}; \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}), \tag{IV.86}$$

$$\begin{aligned} &\hat{w}_{M,N}[z; r; k^{(M)}; \tilde{k}^{(N)}] \\ &\quad := \rho^{(d/2\gamma)(M+N)-1}\, \tilde{w}_{M,N}[\rho H_f; Z^{-1}(z); \rho^{1/\gamma} k^{(M)}; \rho^{1/\gamma}\tilde{k}^{(N)}]. \end{aligned} \tag{IV.87}$$

We use Lemma IV.3 and Lemma IV.4 and the relations (IV.82)–(IV.87). First, we examine $\hat{w}_{M,N}$, for $M+N \geqslant 1$. By (IV.87) and (IV.30), we have that

$$\begin{aligned} &|\hat{w}^T_{M,N}[Z(z); r; k^{(M)}, \tilde{k}^{(N)}]| \\ &\quad = \rho^{(d/2\gamma)(M+N)-1}\, |\tilde{w}^T_{M,N}[z; \rho r; \rho^{1/\gamma} k^{(M)}; \rho^{1/\gamma}\tilde{k}^{(N)}]| \\ &\quad \leqslant \rho^{(d/2\gamma)(M+N)-1} \cdot \|\tilde{w}^T_{MN}\|^{(\infty)}_{\Delta,\rho} \cdot \prod_{j=1}^{M} \omega(\rho^{1/\gamma} k_j)^\alpha \prod_{j=1}^{N} \omega(\rho^{1/\gamma}\tilde{k}_j)^\alpha \\ &\quad \leqslant \rho^{(\alpha+(d/2\gamma))(M+N)-1} \cdot \|\tilde{w}^T_{MN}\|^{(\infty)}_{\Delta,\rho} \cdot \prod_{j=1}^{M} \omega(k_j)^\alpha \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha. \end{aligned} \tag{IV.88}$$

Thus, since $\alpha + (d/2\gamma) = 1 + (\mu/2)$ and $M+N \geqslant 1$,

$$\|\hat{w}^T_{M,N}\|^{(\infty)}_{\Delta} \leqslant \rho^{(1+(\mu/2))(M+N)-1}\, \|w_{M,N}\|^{(\infty)}_{\Delta} \leqslant \varepsilon \rho^{\mu/2} \xi^{M+N}. \tag{IV.89}$$

Similarly, we get from (IV.31) that

$$\begin{aligned} \|\Delta \hat{w}_{M,N}\|^{(\infty)}_{\Delta} &\leqslant 2(4\rho^{(1+\mu)/2})^{M+N}\, (80\varepsilon)^2\, \rho^{-1} \xi^{M+N} \\ &\leqslant \rho^{\mu/2} \left(\frac{(2^9)\cdot(10^2)\cdot\varepsilon}{\rho^{1/2}} \right) \varepsilon \xi^{M+N}. \end{aligned} \tag{IV.90}$$

Adding up these terms, we obtain

$$\xi^{-(M+N)}\, \|\hat{w}_{M,N}\|^{(\infty)}_{\Delta} \leqslant 4\rho^{\mu/2} \left(1 + \frac{12800\varepsilon}{\rho^{1/2}} \right) \cdot \varepsilon. \tag{IV.91}$$

Next we compute from (IV.87) that

$$\int |\partial_r \hat{w}_{M,N}[Z(z); r; k^{(M)}; \tilde{k}^{(N)}]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta}$$

$$= \rho^{(d/2\gamma)(M+N)} \int |\partial_r \tilde{w}_{M,N}[z; \rho r; \rho^{1/\gamma} k^{(M)}; \rho^{1/\gamma} \tilde{k}^{(N)}]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta}$$

$$\leqslant \rho^{(\beta - (d/2\gamma))(M+N)}$$

$$\times \int_{B_\rho^M \times B_\rho^N} |\partial_r \tilde{w}_{M,N}[z; \rho r; k^{(M)}; \tilde{k}^{(N)}]| \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\beta} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^\beta}$$

$$\leqslant \rho^{(\mu/2)(M+N)} \cdot \|\partial_r \tilde{w}_{M,N}\|_{\Delta,\rho}^{(1)}. \tag{IV.92}$$

Thus, we obtain from (IV.43)–(IV.44) the following estimate.

$$\frac{\rho^{1/2}}{24} (\xi C_d \rho^{1/2})^{-(M+N)} \|\hat{w}_{M,N}\|_\Delta^{(1)} \leqslant 4\rho^{\mu/2} \cdot \left(1 + \frac{4050\varepsilon}{\rho^{1/2}}\right) \cdot \varepsilon. \tag{IV.93}$$

Putting together (IV.93) and (IV.90), we see that

$$\max_{M+N \geqslant 1} \|\hat{w}_{M,N}\|_\Delta \leqslant 4\rho^{\mu/2} \cdot \left(1 + \frac{12800\varepsilon}{\rho^{1/2}}\right) \cdot \varepsilon. \tag{IV.94}$$

Second, we use (IV.84), (IV.32), and $\xi C_d^{1/2} \leqslant \rho^{(3+\mu)/4}$ to estimate

$$|\hat{E}[z]| \leqslant \rho^{-1} \|\Delta \tilde{w}_{0,0}\|_\Delta^{(\infty)} \leqslant 12 \left(\frac{\varepsilon \xi C_d^{1/2}}{\rho}\right)^2$$

$$\leqslant 4\rho^{\mu/2} \left(\frac{12800\varepsilon}{\rho^{1/2}}\right) \varepsilon. \tag{IV.95}$$

Since $\eta = 4\rho^{\mu/2}(1 + 12800\varepsilon\rho^{-1/2})$, we obtain

$$\sup_{z \in D_{1/2}} |\hat{E}[z]| \leqslant \eta\varepsilon \qquad \text{and} \qquad \sup_{z \in D_{1/2}} |\underline{\hat{W}}[z]|'_\Delta \leqslant \eta\varepsilon. \tag{IV.96}$$

Third, Equation (IV.83) implies that

$$\partial_r \hat{T}[Z(\zeta); r] = \partial_r T[\zeta; \rho r] + \partial_r \tilde{w}_{0,0}[\zeta; \rho r], \tag{IV.97}$$

for any $\zeta \in \mathscr{U}^{(\text{in})}$, and thus

$$\sup_{z \in D_{1/2}} \|\partial_r \hat{T}[z;\,\cdot\,] - 1\|_\infty$$
$$= \sup_{\zeta \in \mathscr{U}^{(\text{in})}} \|\partial_r T[\zeta;\,\cdot\,] - 1\|_\infty + \sup_{\zeta \in \mathscr{U}^{(\text{in})}} \|\partial_r \tilde{w}_{0,0}[\zeta;\,\cdot\,]\|_\infty$$
$$= \delta + \sup_{z \in D_{1/2}} \|\partial_r \Delta \tilde{w}_{0,0}[z]\|^{(1)}_{\Delta,\rho} = \delta + \eta\varepsilon, \tag{IV.98}$$

using (IV.46) and $H \in \mathscr{B}(\delta, \varepsilon)$. By (IV.97) and (IV.98), we arrive that the assertion that

$$\mathscr{R}_\rho((E, T, \underline{W})) = (\hat{E}, \hat{T}, \underline{\hat{W}}) \in \mathscr{B}(\delta + \eta\varepsilon, \eta\varepsilon). \quad \blacksquare \tag{IV.99}$$

V. ANALYSIS OF THE FLOW GENERATED BY $\mathscr{R}_\rho$

Our next goal is to apply Theorem IV.5 to the starting operator, $H_{(0)}$, defined in (III.105). We show that $H_{(0)}$ fulfills its hypotheses and we may thus apply $\mathscr{R}_\rho$ to generate $H_{(1)} := \mathscr{R}_\rho[H_{(0)}]$, for which we again verify the hypotheses of Theorem IV.5 and apply $\mathscr{R}_\rho$, and so on. In the vein, we produce a sequence

$$H_{(n)} := \mathscr{R}_\rho^n[H_{(0)}], \tag{V.1}$$

$n \in \mathbb{N}$, of "isospectral" Hamiltonians, and we locate the spectrum of $H_{(n)}$ on the energy scale

$$\rho_n := \rho_0 \cdot \rho^n, \tag{V.2}$$

for all n. By the isospectral property, we can then put these pieces of spectral information together to locate the spectrum on $H_{(0)}$ on arbitrarily small energy scales.

V.1. *Adjustment of the Initial Condition*

Recall from Corollary III.2 that

$$H_{(0)} \in \mathscr{B}(\delta, \varepsilon), \tag{V.3}$$

provided that $g > 0$ is sufficiently small such that

$$\delta := 25(100 g \Lambda_6 \rho_0^{-1/2})^2 \leqslant \tfrac{1}{8}, \tag{V.4}$$
$$\varepsilon := 100 g \Lambda_6 \max\{200 g \Lambda_6 \rho_0^{-1}, 2\xi^{-1}, 25 C_d^{-1} \xi^{-1} \rho^{-1/2}\} \leqslant \tfrac{1}{16}. \tag{V.5}$$

In the following, we require Hypotheses H-1, H-2, H-3, and $0 < \vartheta < \pi/2$, $0 < \rho_0 \leqslant \rho^{(\text{out})} = 2^{-1/2} \sin(\vartheta/2)$, $0 < \mu \leqslant 2$, $\Lambda_1 \geqslant 1$, $\Lambda_5 \geqslant 1$, and we define

$$C_d = d\gamma^{-1}\pi^{d/2}\Gamma[(d/2)+1]^{-1}, \tag{V.6}$$

$$\Lambda_6 = \Lambda_1 \Lambda_5 (1 + \sqrt{2C_d})(\sin(\vartheta/2))^{-1}. \tag{V.7}$$

The following lemma shows for how large a choice of the coupling constant $g > 0$ the requirements of Theorem IV.5 and of (V.4)–(V.5) are still met.

LEMMA V.1. *Given* $0 < \mu \leqslant 2$, $\Lambda_1 \geqslant 1$, $\Lambda_5 \geqslant 1$, $0 < \vartheta \leqslant \pi/2$, *we define* C_d, Λ_6, δ, *and* ε *by* (*V*.4)–(*V*.5), *and we set*

$$\rho_0 := (2^{-1/2}) \sin(\vartheta/2), \qquad \rho := \min\{C_d^{-1}, 2^{-8/\mu}\}, \tag{V.8}$$

$$\xi := \min\{C_d^{-1/2}\rho^{(3+\mu)/4}, C_d^{-1}\}, \qquad \varepsilon := \rho^{1/2}/12\,800 \tag{V.9}$$

$$g_0 := \frac{\rho^{1/2}}{12800 \cdot 100 \cdot \Lambda_6} \min\left\{\sqrt{6400} \cdot \rho_0^{1/2}, \frac{1}{2}\xi, \frac{1}{25} C_d \xi \rho^{1/2}\right\}. \tag{V.10}$$

Then, for any $0 < g \leqslant g_0$, *the following relations hold true*:

$$\rho_0 \leqslant \sqrt{2}^{-1} \sin(\vartheta/2), \qquad \frac{100 g \Lambda_6}{\rho_0^{1/2}} \leqslant 1, \tag{V.11}$$

$$\rho C_d \leqslant 1, \qquad C_d \xi \leqslant 1, \qquad C_d^{1/2}\xi \leqslant \rho^{(3+\mu)/2}, \qquad \rho^{\mu/2} \leqslant 1/16, \tag{V.12}$$

$$\delta \leqslant 1/16, \qquad \varepsilon \leqslant 1/16, \qquad \varepsilon\rho^{-1/2} \leqslant 1/12800. \tag{V.13}$$

Proof. First, we recall that $\rho_0 \leqslant \sqrt{2}^{-1} \sin(\vartheta/2)$, $\rho C_d \leqslant 1$, $C_d \xi \leqslant 1$, $C_d^{1/2}\xi \leqslant \rho^{(3+\mu)/4}$, and $\rho^{\mu/2} \leqslant 1/16$, by definition. Second, we remark that

$$\frac{100 g_0 \Lambda_6}{\rho_0^{1/2}} \leqslant \frac{\sqrt{6400} \cdot \rho^{1/2}}{12800} \leqslant \frac{1}{2\sqrt{6400}} \leqslant 1. \tag{V.14}$$

This implies that

$$\begin{aligned}
\varepsilon\rho^{-1/2} &\leqslant \frac{1}{12800} \min\left\{\sqrt{6400} \cdot \rho_0^{1/2}, \frac{\xi}{2}, \frac{C_d \xi \rho^{1/2}}{25}\right\} \max\left\{\frac{2}{\rho_0^{1/2}} \frac{100 g \Lambda_6}{\rho_0^{1/2}}, \frac{2}{\xi}, \frac{25}{C_d \xi \rho^{1/2}}\right\} \\
&\leqslant \frac{1}{12800} \min\left\{\sqrt{6400} \cdot \rho_0^{1/2}, \frac{\xi}{2}, \frac{C_d \xi \rho^{1/2}}{25}\right\} \max\left\{\frac{1}{\sqrt{6400} \cdot \rho_0^{1/2}}, \frac{2}{\xi}, \frac{25}{C_d \xi \rho^{1/2}}\right\} \\
&\leqslant \frac{1}{12800}.
\end{aligned} \tag{V.15}$$

Finally, we observe that

$$\varepsilon\rho^{-1/2} \geqslant \varepsilon \geqslant \tfrac{2}{25}\delta. \tag{V.16}$$

So, if $\varepsilon\rho^{-1/2} \leqslant 1/12800$ then $\delta \leqslant 1/16$ and $\varepsilon \leqslant 1/16$. ∎

The choice of ρ_0, ρ, and ξ in (V.8)–(V.9) was made so as to meet the conditions imposed in (V.11)–(V.13) for the largest possible values of the coupling constant g. As a consequence of Lemma V.1, we have the following theorem.

THEOREM V.2. *Assume Hypothesis* H-1, H-2, *and* H-3, *with* $0 < \mu \leqslant 2$, $\Lambda_1 \geqslant 1$, $\Lambda_5 \geqslant 1$, *and* $0 < \vartheta < \pi/2$. *Define* C_d, ρ_0, ρ, ξ, ε, δ, *and* g_0 *by* (V.4)–(V.10), *and assume that* $0 < g \leqslant g_0$. *Then* $H_{(0)} \in \mathscr{B}(\delta, \varepsilon)$. *Furthermore, defining*

$$H_{(n)} \equiv (E_{(n)}, T_{(n)}, \underline{W}_{(n)}) := \mathscr{R}_\rho^n[(E_{(0)}, T_{(0)}, \underline{W}_{(0)})], \tag{V.17}$$

$$Z_{(n)}: \mathscr{U}_{(n)}^{(\mathrm{in})} \to D_{1/2}, \qquad z \mapsto \rho^{-1}(z - E_{(n-1)}[z]), \tag{V.18}$$

and

$$\mathscr{U}_{(n)}^{(\mathrm{in})} := \{z \in d_{1/2} \mid |z - E_{(n-1)}[z]| \leqslant \rho/2\}, \tag{V.19}$$

we have that

$$H_{(n)} \in \mathscr{B}(\delta + \varepsilon, \eta^n \varepsilon), \tag{V.20}$$

for all $n = 1, 2, \ldots$.

Proof. Equation (V.20) follows from (V.3) and Theorem IV.5, which is applicable thanks to Lemma V.1, by iterating

$$\mathscr{R}_\rho: \mathscr{B}(\delta, \varepsilon) \to \mathscr{B}(\delta + \eta\varepsilon, \eta\varepsilon) \to \mathscr{B}(\delta + \eta\varepsilon + \eta^2\varepsilon, \eta^2\varepsilon) \to \cdots, \tag{V.21}$$

additionally taking into account that $\sum_{n=1}^\infty \eta^n \leqslant 1$ since $\eta \leqslant 1/2$. ∎

V.2. *Cuspidal Domains of (Possible) Spectrum*

Next, we investigate the convergence of $\sum_{n=0}^\infty E_{(n)}$. Recall from (V.18)–(V.19) that, for $n = 1, 2, \ldots$,

$$\mathscr{U}_{(n)}^{(\mathrm{in})} = \{z \in D_{1/2} \mid |z - E_{(n-1)}[z]| \leqslant \rho/2\}, \tag{V.22}$$

and

$$Z_{(n)}: \mathscr{U}_{(n)}^{(\mathrm{in})} \to D_{1/2}, \qquad z \mapsto \rho^{-1}(z - E_{(n-1)}[z]). \tag{V.23}$$

We use $Z_{(0)}$ from (III.106) to define

$$\mathscr{Z}_{(n)}^{-1}: D_{1/2} \to \mathscr{U}_{(0)}^{(\mathrm{in})}, \qquad \mathscr{Z}_{(n)}^{-1} := Z_{(0)}^{-1} \circ Z_{(1)}^{-1} \circ \cdots \circ Z_{(n)}^{-1}, \tag{V.24}$$

and

$$\mathscr{S}_{(n)} := \mathscr{Z}_{(n)}^{-1}(D_{1/4}). \tag{V.25}$$

LEMMA V.3. *For all* $n = 1, 2, ...$

$$\mathscr{S}_{(0)} \supseteq \mathscr{S}_{(1)} \supseteq \mathscr{S}_{(2)} \supseteq \cdots \supseteq \mathscr{S}_{(n)}, \tag{V.26}$$

$$\rho_0(\tfrac{2}{5})^n \leqslant \text{inner rad}(\mathscr{S}_{(n)}) \leqslant \text{outer rad}(\mathscr{S}_{(n)}) \leqslant \rho_0(\tfrac{4}{3}\rho)^n, \tag{V.27}$$

where, for $A \subseteq \mathbb{C}$, *the inner and outer radius of* A *are defined by* inner rad$(A) := \sup_{z,r}\{r \mid z + D_r \subseteq A\}$ *and* outer rad$(A) := \inf_{z,r}\{r \mid z + D_r \supseteq A\}$. *Moreover, the number* $E_{(\infty)} \in \mathbb{C}$, *defined by* $\{E_{(\infty)}\} = \bigcap_{n \in \mathbb{N}} \mathscr{S}_{(n)}$, *is uniquely determined by the sequence*

$$E_{(\infty)} = \lim_{n \to \infty} \mathscr{Z}_{(n)}^{-1}(0). \tag{V.28}$$

Proof. Since, $\mathscr{U}_{(n)}^{(\mathrm{in})} \subseteq D_{1/4}$, for $n \in \mathbb{N}_0$, we clearly have

$$\begin{aligned}\mathscr{S}_{(n+1)} &= \mathscr{Z}_{(n+1)}^{-1}(D_{1/4}) \subseteq \mathscr{Z}_{(n+1)}^{-1}(D_{1/2}) \\ &= \mathscr{Z}_{(n)}^{-1}(\mathscr{U}_{(n)}^{(\mathrm{in})}) \subseteq \mathscr{Z}_{(n)}^{-1}(D_{1/4}) = \mathscr{S}_{(n)},\end{aligned} \tag{V.29}$$

and thus (V.26). By the same argument, $\mathscr{Z}_{(n)}^{-1}(\mathscr{U}_{(n)}^{(\mathrm{in})}) \subseteq \mathscr{Z}_{(n-1)}^{-1}(\mathscr{U}_{(n-1)}^{(\mathrm{in})}) \subseteq \cdots \subseteq \mathscr{Z}_{(0)}^{-1}(\mathscr{U}_{(0)}^{(\mathrm{in})})$, and $|\partial_z E_{(n)}| \leqslant 4\eta\varepsilon \leqslant 1/4$, by Theorem V.2. Thus, for $\zeta \in \mathscr{U}_{(n)}^{(\mathrm{in})}$,

$$\frac{3}{4\rho}\,|\zeta| \leqslant |Z_{(n)}(\zeta)| = \rho^{-1}\,|\zeta - E[\zeta]| \leqslant \frac{5}{4\rho}\,|\zeta|, \tag{V.30}$$

which implies that

$$\frac{4\rho}{5}\,|z| \leqslant |Z_{(n)}^{-1}(z)| \leqslant \frac{4\rho}{3}\,|z|, \tag{V.31}$$

for $z \in D_{1/2}$. Iterating this estimate, we obtain

$$\text{outer rad}\{\mathscr{Z}_{(n)}^{-1}(D_{1/4})\} \leqslant \frac{\rho_0}{4} \cdot \left(\frac{4\rho}{3}\right)^n, \tag{V.32}$$

proving the right inequality in (V.27). Next, using again that $|\partial_z E_{(n)}| \leqslant 1/4$, we infer that, for $z \in D \setminus \{0\}$,

$$|\arg[\mathscr{Z}_{(n)}^{-1}(z)] - \arg[z]| \leqslant \pi/4, \tag{V.33}$$

and hence

$$\text{inner rad}\{\mathscr{Z}_{(n)}^{-1}(D_{1/4})\} \geqslant \frac{\rho_0}{4}\left(\frac{2\rho}{5}\right)^n. \quad \blacksquare \tag{V.34}$$

Having found $E_{(\infty)}$, the number in $\mathbb{C}_-$ that we later identify to be the resonance energy we sought for, we also wish to determine a deformed curve, i.e., a function

$$T_{(\infty)}: [0, 1] \to \mathbb{C}_- \tag{V.35}$$

that represents the "continuous spectrum" for the perturbed operator $H_g(\theta)$. We put "continuous spectrum" in quotation marks because we do not prove the existence of continuous spectrum for $H_g(\theta)$, but we show that any spectrum of $H_g(\theta)$ in $\mathscr{U}_{(0)}^{(\text{in})} = D_{\rho_0/2}$ is contained in a cuspidal domain about $E_{(\infty)} + \{T_{(\infty)}(r) \mid r \in [0, 1]\}$.

Before going into mathematical detail, we motivate and outline the construction of $T_{(\infty)}$. The first difficulty we encounter when trying to define $T_{(\infty)}$ is that the functions $T_{(n)}$ depend on both $r \in [0, 1]$ and $z \in D_{1/2}$. We thus need to impose a sensible condition that determines $z = \zeta_{(n)}(r)$ as a function of r so that on the nth energy scale, $\rho^{n+1} \leqslant r < \rho^n$, the curve $T_{(\infty)}[r]$ is essentially given by $\tilde{T}_{(\infty)}$, defined by

$$E_{(\infty)} + \tilde{T}_{(\infty)}[r] = \mathscr{Z}_{(n)}^{-1}(T_{(n)}[\zeta_{(n)}(\rho^{-n}r); \rho^{-n}r]). \tag{V.36}$$

The condition that yields z as a function of r is expressed in terms of functions

$$\zeta_{(n)}: [0, 5/16] \to D_{3/8}, \tag{V.37}$$

for each $n \in \mathbb{N}_0$, by the requirement that $|\zeta_{(n)}(r) - r| \leqslant 1/16$ and by the fix point equation

$$\zeta_{(n)}(r) = T_{(n)}[\zeta_{(n)}(r), r], \tag{V.38}$$

where $T_{(n)}$ is defined in (V.17). We postpone the discussion of Eqn. (V.38), and we temporarily ignore the fact that, according to (V.37), $\zeta_{(n)}$ is only defined on $[0, 5/16]$ rather than $[0, 1]$. Instead, we insert (V.38) into (V.36) and obtain

$$E_{(\infty)} + \tilde{T}_{(\infty)}[r] = \mathscr{Z}_{(n)}^{-1}(\zeta_{(n)}(\rho^{-n}r)), \tag{V.39}$$

for $\rho^{n+1} \leqslant r < \rho^n$. Equation (V.36) is still not satisfactory because $\tilde{T}_{(\infty)}$ may have jump discontinuities at $r = \rho^n$. We solve this problem by smoothly interpolating between $\mathscr{Z}_{(n)}^{-1}(\zeta_{(n)}(\rho^{-n} r))$ and $\mathscr{Z}_{(n-1)}^{-1}(\zeta_{(n-1)}(\rho^{-n+1} r))$. Namely, we pick $\phi \in C_0^\infty(\mathbb{R}_0^+)$ such that $\phi' \leqslant 0$, $\phi \equiv 1$ on $[0, 1/4]$, and $\phi \equiv 0$ on $[5/16, \infty]$. We use ϕ to define, for $n = 0, 1, 2, 3, \ldots$

$$\zeta_{(n)}^{(av)}(r) := \phi(r) \cdot \zeta_{(n)}(r) + (1 - \phi(r)) \cdot Z_{(n)}(\zeta_{(n-1)}(\rho r)), \tag{V.40}$$

where $\zeta_{(-1)}(r) := r$ and $0 \leqslant r < 1$. We emphasize that $\zeta_{(n)}^{(av)}$ is defined on the full interval $[0, 1]$. Moreover, the curve $T_{(\infty)}$ defined by

$$E_{(\infty)} + T_{(\infty)}[r] = \mathscr{Z}_{(n)}^{-1}(\zeta_{(n)}^{(av)}(\rho^{-n} r)), \tag{V.41}$$

for $\rho^{n+1} \leqslant r < \rho^n$, is Lipschitz continuous since $\zeta_{(n)}^{(av)}$ has that property and because

$$\lim_{r \nearrow 1} \{\zeta_{(n)}^{(av)}(r)\} = Z_{(n)}(\zeta_{(n-1)}(\rho)) = Z_{(n)}(\zeta_{(n-1)}^{(av)}(\rho)). \tag{V.42}$$

The key property of $\zeta_{(n)}^{(av)}$, defined by (V.39) and (V.38), is that, for all $z \in D_{1/2}/\mathscr{U}_{(n)}^{(\mathrm{in})}$ and $\rho \leqslant r < 1$, we have that

$$|T_{(n)}[z; r] + E_{(n)}[z] - z| \geqslant \frac{1}{3} |z - \zeta_{(n)}^{(av)}(r)| - \left(\frac{9\varepsilon}{\rho}\right) \eta^{n-1}, \tag{V.43}$$

as is proved in Lemma V.5 below. Thus, if z is sufficiently far away from the graph of $\zeta_{(n)}^{(av)}$, then (V.43) yields

$$|T_{(n)}[z; r] + E_{(n)}[z] - z| \geqslant c \cdot (r + \rho), \tag{V.44}$$

for some constant $c > 0$, and we obtain the invertibility of $H_{(n)}[z] - z$ by a norm-convergent Neumann series expansion.

We now come to the precise mathematical discussion of $T_{(\infty)}$. Recall that the condition that yields z as a function of r is expressed in terms of functions

$$\zeta_{(n)} \colon [0, 5/16] \to D_{3/8}, \tag{V.45}$$

for each $n \in \mathbb{N}_0$, by the requirement that $|\zeta_{(n)}(r) - r| \leqslant 1/16$ and the equation

$$\zeta_{(n)}(r) = T_{(n)}[\zeta_{(n)}(r), r], \tag{V.46}$$

where $T_{(n)}$ is defined in (V.17). To see that Equation (V.46) has a unique solution for every $r \in [0, 5/16]$, we set $\zeta_{(n)}(r) := r + \delta$ and $\Delta T(\tau) := T_{(n)}[r + \tau; r] - r$. Then, (V.46) reads

$$\delta = \Delta T(\delta). \tag{V.47}$$

By Cauchy's estimate and $\varepsilon \leqslant 1/16$, we have that

$$|\partial_z \Delta T(\zeta)| \leqslant \left(\frac{1}{2} - \frac{5}{16}|\zeta|\right)^{-1} \cdot \varepsilon \leqslant \frac{1}{2}, \tag{V.48}$$

for $|\zeta| \leqslant 1/16$, and the existence of δ in (V.47) follows from a fix point argument. Note that $\zeta_{(n)}$ is uniformly Lipschitz-continuous, since $T_{(n)}$ is.

We cast our investigation in a series of small lemmata.

LEMMA V.4. *For $0 \leqslant r \leqslant 5/16$ and $\varepsilon, \rho \leqslant 1/16$,*

$$|Z_{(n)}(\zeta_{(n-1)}(\rho r)) - \zeta_{(n)}(r)| \leqslant \left(\frac{3\varepsilon}{\rho}\right) \cdot \eta^{n-1}. \tag{V.49}$$

Proof. Using (V.46), we observe that

$$\begin{aligned}
&Z_{(n)}(\zeta_{(n-1)}(\rho r)) - \zeta_{(n)}(r) \\
&\quad = Z_{(n)}(T_{(n-1)}[\zeta_{(n-1)}(\rho r), \rho r]) - T_{(n)}[\zeta_{(n)}(r), r] \\
&\quad = \rho^{-1} T_{(n-1)}[\zeta_{(n-1)}(\rho r), \rho r] - \rho^{-1} E_{(n-1)}[\zeta_{(n-1)}(\rho r)] \\
&\qquad - T_{(n)}[\zeta_{(n)}(r), r] \\
&\quad = \{T_{0,(n)}[\zeta_{(n)}(r); r] - T_{(n)}[\zeta_{(n)}(r), r]\} - \rho^{-1} E_{(n-1)}[\zeta_{(n-1)}(\rho r)] \\
&\qquad + \rho^{-1}\{T_{(n-1)}[Z_{(n)}^{-1}(Z_{(n)}(\zeta_{(n-1)}(\rho r))), \rho r] \\
&\qquad - T_{(n-1)}[Z_{(n)}^{-1}(\zeta_{(n)}(r)); \rho r]\},
\end{aligned} \tag{V.50}$$

where $T_{0,(n)}[z; r] := \rho^{-1} T_{(n-1)}[Z_{(n)}^{-1}(z); \rho r]$. From (IV.97) and (IV.46), we obtain

$$|T_{0,(n)}[r; \zeta_{(n)}(r)] - T_{(n)}[r; \zeta_{(n)}(r)]| \leqslant \eta^n \varepsilon, \tag{V.51}$$

$$|\rho^{-1} E_{(n-1)}[\zeta_{(n-1)}(\rho r)]| \leqslant \eta^{n-1} \varepsilon \rho^{-1}. \tag{V.52}$$

Since $\zeta_{(n-1)}(\rho r) \in D_{1/4}$ and $Z_{(n)}^{-1}(\zeta_{(n)}(r)) \in D_{1/4}$, we may apply Cauchy's estimate (with contour on $\partial D_{1/2}$) and get

$$\begin{aligned}
&|T_{(n-1)}[Z_{(n)}^{-1}(Z_{(n)}(\zeta_{(n-1)}(\rho r))), \rho r] - T_{(n-1)}[Z_{(n)}^{-1}(\zeta_{(n)}(r)), \rho r]| \\
&\quad \leqslant |Z_{(n)}(\zeta_{(n-1)}(\rho r)) - \zeta_{(n)}(r)| \cdot \sup_{D_{1/4}} |\partial_z Z_{(n)}^{-1}(z)| \cdot \sup_{D_{1/4}} |\partial_z T_{(n-1)}[z; \rho r]| \\
&\quad \leqslant \frac{4\rho}{3} \cdot |Z_{(n)}(\zeta_{(n-1)}(\rho r)) - \zeta_{(n)}(r)| \cdot 4 \sup_{D_{1/2}} |T_{(n-1)}[z; \rho r] - \rho r| \\
&\quad \leqslant \frac{16\varepsilon\rho}{3} \cdot |Z_{(n)}(\zeta_{(n-1)}(\rho r)) - \zeta_{(n)}(r)|.
\end{aligned} \tag{V.53}$$

Now, (V.49) follows from inserting (V.51)–(V.53) into (V.50), taking into account that $\varepsilon \leqslant 1/16$. ∎

Next, we pick $\phi \in C_0^\infty(\mathbb{R}_0^+)$ such that $\phi' \leqslant 0$, $\phi \equiv 1$ on $[0, 1/4]$, and $\phi \equiv 0$ on $[5/16, \infty]$. We use ϕ to define, for $n = 0, 1, 2, 3, ...$

$$\zeta_{(n)}^{(av)}(r) := \phi(r) \cdot \zeta_{(n)}(r) + (1 - \phi(r)) \cdot Z_{(n)}(\zeta_{(n-1)}(\rho r)), \tag{V.54}$$

where $\zeta_{(-1)}(r) := r$ and $0 \leqslant r < 1$.

LEMMA V.5. *Let $z \in D_{1/2} \backslash \mathscr{U}_{(n)}^{(\mathrm{in})}$ and $\rho \leqslant r < 1$. Then*

$$|T_{(n)}[z; r] + E_{(n)}[z] - z| \geqslant \frac{1}{3} |z - \zeta_{(n)}^{(av)}(r)| - \left(\frac{9\varepsilon}{\rho}\right) \eta^{n-1}. \tag{V.55}$$

Proof. We define a function $Q_r \colon \mathscr{U}_{(n)}^{\mathrm{in})} \to D_{1/2}$, for fixed $\rho \leqslant r < 1$, by

$$\begin{aligned} Q_r(z) &:= \rho^{-1} T_{(n-1)}[z; \rho r] - Z_{(n)}(z) \\ &= \rho^{-1} \{ T_{(n-1)}[z; \rho r] + E_{(n-1)}[z] - z \}. \end{aligned} \tag{V.56}$$

Then, by Cauchy's estimate and $\mathscr{U}_{(n)}^{(\mathrm{in})} \subseteq D_{1/4}$,

$$|\partial_z Q_r(z) + \rho^{-1}| \leqslant \frac{8\varepsilon}{\rho} \leqslant \frac{1}{2\rho}. \tag{V.57}$$

Next, we observe that

$$\begin{aligned} &|\{ T_{(n)}[z; r] + E_{(n)}[z] - z \} - Q_r(Z_{(n)}^{-1}[z])| \\ &\quad = |T_{(n)}[z; r] - T_{0,(n)}[z; r] + E_{(n)}[z]| \leqslant 2\varepsilon\eta^n. \end{aligned} \tag{V.58}$$

We come to our analysis of $Q_r(Z_{(n)}^{-1}(\zeta_{(n)}^{(av)}(r)))$: Note that $Q_r(\zeta_{(n-1)}(\rho r)) = \rho^{-1} E_{(n-1)}(\zeta_{(n-1)}(r))$ and hence, by (V.57), (V.31), and Lemma V.4, we obtain the bound

$$\begin{aligned} &|Q_r(Z_{(n)}^{-1}(\zeta_{(n)}^{(av)}(r)))| \\ &\quad \leqslant |Q_r(Z_{(n)}^{-1}(\zeta_{(n)}^{(av)}(r))) - Q_r(\zeta_{(n-1)}(\rho r))| + \varepsilon \rho^{-1} \eta^{n-1} \\ &\quad \leqslant \left(\frac{3}{2\rho}\right) \cdot \left(\frac{4\rho}{3}\right) \cdot |\zeta_{(n)}^{(av)}(r) - Z_{(n)}(\zeta_{(n-1)}(\rho r))| + \left(\frac{\varepsilon}{\rho}\right) \eta^{n-1} \\ &\quad \leqslant \left(\frac{7\varepsilon}{\rho}\right) \eta^{n-1}. \end{aligned} \tag{V.59}$$

Inserting (V.59), (V.58) and using again (V.57) and (V.31), we thus obtain

$$\begin{aligned} &|T_{(n)}[z; r] + E_{(n)}[z] - z| \\ &\quad \geqslant |Q_r(Z_{(n)}^{-1}(z))| - 2\varepsilon\eta^n \\ &\quad \geqslant |Q_r(Z_{(n)}^{-1}(z)) - Q_r(Z_{(n)}^{-1}(\zeta_{(n)}^{(av)}(r)))| - \left(\frac{9\varepsilon}{\rho}\right)\eta^{n-1} \\ &\quad \geqslant \left(\frac{1}{2\rho}\right)\cdot\left(\frac{2\rho}{3}\right)\cdot|z - \zeta_{(n)}^{(av)}(r)| - \left(\frac{9\varepsilon}{\rho}\right)\eta^{n-1}. \quad \blacksquare \end{aligned} \tag{V.60}$$

We define the analogue $\mathscr{U}_{(0)}^{(\mathrm{out})}(\delta')$ as follows.

$$\mathscr{U}_{(n)}^{(\mathrm{out})}(\delta') := \{z \in D_{1/2} \backslash \mathscr{U}_{(n)}^{(\mathrm{in})} \mid \forall r: \\ |z - \zeta_{(n)}^{(av)}(r)| \geqslant \delta' \, |z - E_{(n-1)}[z]|\}. \tag{V.61}$$

THEOREM V.6. *Assume that* $108\varepsilon\rho^{-2}\eta^{n-1} \leqslant 1$. *Then* $H_{(n)}[z] - z$ *is invertible for all* $z \in \mathscr{U}_{(n)}^{(\mathrm{out})}(108\varepsilon\rho^{-2}\eta^{n-1})$.

Proof. We use again the polar decomposition

$$T_{(n)}[z; r] + E_{(n)}[z] - z = U \cdot |T_{(n)}[z; r] + E_{(n)}[z] - z| \tag{V.62}$$

to construct the inverse of $H_{(n)}[z] - z$ by a Neumann series:

$$(H_{(n)}[z] - z)^{-1} = R_0^{1/2}\left\{\sum_{n=0}^{\infty} U^*(-R_0^{1/2}\chi_1 W_{(n)}\chi_1 R_0^{1/2} U^*)^n\right\}, \tag{V.63}$$

where $R_0 := |T_{(n)}[z; r] + E_{(n)}[z] - z|^{-1}$. Now, from Theorem V.2 and Theorem B.2, and using that $C_d\xi \leqslant 1$, we obtain

$$\|(H_f + \rho)^{-1/2}\chi_1 W_{(n)}\chi_1(H_f + \rho)^{-1/2}\| \leqslant \frac{2e^2\varepsilon}{\rho^{1/2}}\eta^n. \tag{V.64}$$

Thus, to establish convergence of the series in (V.63), it remains to be shown that

$$\left\|\frac{(H_f + \rho)\chi_1[H_f]}{T_{(n)}[z; H_f] + E_{(n)}[z] - z}\right\| = \sup_{0 \leqslant r < 1}\left\{\frac{r + \rho}{|T_{(n)}[z; r] + E_{(n)}[z] - z|}\right\} \\ < \frac{\rho^{1/2}}{2e^2\varepsilon}\eta^{-n}. \tag{V.65}$$

By the definition of $\mathscr{U}^{(\text{out})}_{(n)}(\delta')$ and Lemma V.5, and with $\delta' := (108)\,\varepsilon\rho^{-2}\eta^{n-1}$, we have that

$$|T_{(n)}[z;r]+E_{(n)}[z]-z| \geqslant \frac{\delta'}{3}\,|z-E_{(n-1)}[z]| - \left(\frac{9\varepsilon}{\rho}\right)\eta^{n-1}$$
$$\geqslant \frac{\delta'}{6}\,|z-E_{(n-1)}[z]|, \tag{V.66}$$

also using that $|z-E_{(n-1)}[z]| \geqslant \rho/2 \geqslant (6/\delta)' \cdot (9\varepsilon/\rho)\,\eta^{n-1}$. In case that $|z-E_{(n-1)}[z]| \geqslant \frac{1}{2}r$, we observe that

$$|z-E_{(n-1)}[z]| \geqslant \tfrac{1}{2}\max\{r,\rho\} \geqslant \tfrac{1}{4}(r+\rho), \tag{V.67}$$

and thus

$$|T_{(n)}[z;r]+E_{(n)}[z]-z| \geqslant \frac{\delta'}{24}\,(r+\rho). \tag{V.68}$$

Conversely, if $|z-E_{(n-1)}[z]| < \frac{1}{2}r$ (but $|z-E_{(n-1)}[z]| \geqslant \rho/2$), we estimate

$$|T_{(n)}[z;r]+E_n[z]-z|$$
$$\geqslant \frac{1}{2}\left(|T_{(n)}[z;r]-z| - \frac{1}{2}r\right) + \frac{1}{2}\cdot\frac{\delta'}{6}\cdot\frac{\rho}{2}$$
$$\geqslant \frac{1}{2}\left(1-\frac{1}{16}-\frac{1}{2}\right)r + \frac{\delta'\rho}{24} \geqslant \frac{\delta'}{24}\,(r+\rho). \tag{V.69}$$

Thus, (V.65) follows from $\rho \leqslant \rho^{\mu/2} \leqslant 1/16$ and

$$\frac{24}{\delta'} < \frac{\rho^{1/2}}{2e^2\varepsilon}\,\eta^{-n}. \quad \blacksquare \tag{V.70}$$

Using the isospectral property of the Feshbach map, we conclude from Theorem V.6 that the resolvent set, $\rho(H_g(\theta))$, of the dilated Hamiltonian contains

$$\mathscr{U}^{(\text{in})}_{(0)} \cap \rho(H_g(\theta)) \supseteq \bigcup_{n=0}^{\infty} \mathscr{Z}^{-1}_{(n)}(\mathscr{U}^{(\text{out})}_{(n+1)}(108\varepsilon\rho^{-2}\eta^n)). \tag{V.71}$$

We now return to the definition of $T_{(\infty)}$ mentioned in (V.35). For all $r \in [0,1]$, we set

$$E_{(\infty)} + T_{(\infty)}(r) := \sum_{n=0}^{\infty} \chi[\rho^{n+1} \leqslant r < \rho^n] \cdot \mathscr{Z}^{-1}_{(n)}(\zeta^{(av)}_{(n)}(\rho^{-n}r)). \tag{V.72}$$

Note that $T_{(\infty)}$ is uniformly Lipschitz-continuous, since $\zeta_{(n)}$ has this property because $\rho \leqslant 1/4$ and

$$\lim_{r \nearrow \rho^n} T_{(\infty)}(r) = \mathscr{Z}_{(n)}^{-1}(\zeta_{(n)}^{(av)}(1)) = \mathscr{Z}_{(n)}^{-1} \circ Z_{(n)}(\zeta_{(n-1)}(\rho))$$
$$= \mathscr{Z}_{(n-1)}^{-1}(\zeta_{(n-1)}^{(av)}(\rho)) = T_{(\infty)}(\rho^n). \tag{V.73}$$

Next, we define a cuspidal domain,

$$K_{(\infty)}(\delta) := \{ T_{(\infty)}(r) + \zeta \mid 0 \leqslant r < 1, |\zeta| \leqslant \delta \cdot r^{1+(\mu/4)} \}, \tag{V.74}$$

for $\delta > 0$. We claim that the following inclusion holds true.

THEOREM V.7. *Let $\sigma(H_g) := \mathbb{C} \backslash \rho(H_g)$ be the spectrum of the Hamiltonian $H_g = H_g(i\vartheta)$ where $\theta = i\vartheta$, for some $0 < \vartheta < \pi/2$. Assume Hypotheses* H-1, H-2, *and* H-3, *with $0 \leqslant \mu \leqslant 2$, $\Lambda_1 \geqslant 1$, $\Lambda_5 \geqslant 1$. Define C_d, ρ_0, ρ, ξ, ε, δ, and g_0 by* (V.4)–(V.10), *and assume that $0 < g \leqslant \rho^{(3+\mu)/2} g_0$. Then*

$$\sigma(H_g(\theta)) \cap \mathscr{U}_{(0)}^{(\text{in})} \subseteq E_{(\infty)} + K_{(\infty)}(78 \varepsilon \rho^{-9/2}). \tag{V.75}$$

Proof. First, we remark that the choices (V.8)–(V.20) of ρ_0, ρ, and μ ensure that $\rho^{\mu/4} \leqslant 3/32$ and that $\delta'_n := 108 \varepsilon \rho^{-2} \eta^{n-1} \leqslant 108 \varepsilon \rho^{-1/2} \cdot \rho^{(3+\mu)/2} \leqslant 1/4$. Using the last inequality, we obtain from (V.71) that

$$\begin{aligned} &\sigma(H_g(\theta)) \cap \mathscr{U}_{(0)}^{(\text{in})} \\ &\quad \subseteq \mathscr{U}_{(0)}^{(\text{in})} \Big\backslash \bigcup_{n=0}^{\infty} \mathscr{Z}_{(n)}^{-1}(\mathscr{U}_{(n+1)}^{(\text{out})}(\delta'_{n-1})) \\ &\quad = \left(\bigcup_{n=0}^{\infty} \mathscr{Z}_{(n)}^{-1}(D_{1/2} \backslash \mathscr{U}_{(n+1)}^{(\text{in})}) \right) \Big\backslash \left(\bigcup_{n=0}^{\infty} \mathscr{Z}_{(n)}^{-1}(\mathscr{U}_{(n+1)}^{(\text{out})}(\delta'_{n+1})) \right) \\ &\quad = \bigcup_{n=1}^{\infty} \mathscr{Z}_{(n-1)}^{-1}(D_{1/2} \backslash (\mathscr{U}_{(n)}^{(\text{in})} \cup \mathscr{U}_{(n)}^{(\text{out})}(\delta'_n))), \end{aligned} \tag{V.76}$$

using the fact that $\mathscr{U}_{(n)}^{(\text{out})}(\delta'_n) \subseteq D_{1/2} \backslash \mathscr{U}_{(n)}^{(\text{in})}$ and that $\mathscr{U}_{(0)}^{(\text{in})}$ is the disjoint union of $\mathscr{Z}_{(n)}^{-1}(D_{1/2} \backslash \mathscr{U}_{(n+1)}^{(\text{in})})$, for $n \in \mathbb{N}_0$. By (V.76), it suffices to show that, for every $n \in \mathbb{N}$,

$$\mathscr{Z}_{(n-1)}^{-1}(D_{1/2} \backslash (\mathscr{U}_{(n)}^{(\text{in})} \cup \mathscr{U}_{(n)}^{(\text{out})}(\delta'_n))) \subseteq E_{(\infty)} + E_{(\infty)}(78 \varepsilon \rho^{-9/2}), \tag{V.77}$$

in order to prove (V.75). To this end, we pick $z \in D_{1/2} \backslash (\mathscr{U}_{(n)}^{(\text{in})} \cup \mathscr{U}_{(n)}^{(\text{out})}(\delta'_n))$. There then exists $0 \leqslant r_0 < 1$ such that

$$|z - \zeta_{(n)}^{(av)}(r_0)| < \delta'_n \, |z - E_{(n)}[z]|. \tag{V.78}$$

Note that $|\zeta^{(av)}_{(n)}(r_0)| \leqslant (1+\delta)\, r_0 \leqslant (9/8)\, r_0$, because $|\partial_r T_{(n)} - 1| \leqslant \delta$. Moreover, $|z - E_{(n)}[z]| \geqslant \rho/2$ and $|E_n[z]| \leqslant \varepsilon \cdot \eta^n$, and we obtain from (V.78) and $\delta'_n \leqslant 1/4$ that

$$
\begin{aligned}
r_0 &\geqslant \left(\frac{8}{9}\right) \cdot (|z - E_{(n)}[z]| - |E_{(n)}[z]| - |z - \zeta^{(av)}_{(n)}(r_0)|) \\
&\geqslant \left(\frac{8}{9}\right) \cdot \left((1-\delta'_n)\, \frac{\rho}{2} - \varepsilon \eta^n \right) \\
&= \left(\frac{8}{9}\right) \cdot \left((1-\delta'_n)\, \frac{\rho}{2} - \delta'_n \frac{\eta \rho^2}{108} \right) \\
&\geqslant \left(\frac{16}{17}\right) \cdot \left[1 - \left(1 + \frac{1}{54}\right) \delta'_n \right] \left(\frac{\rho}{2}\right) > \frac{5\rho}{16}.
\end{aligned}
\tag{V.79}
$$

The importance of (V.79) lies in the definition (V.54) of $\zeta^{(av)}_{(n)}$ and $\zeta^{(av)}_{(n+1)}$ and the definition (V.72) of $T_{(\infty)}$. To see this, we first consider the case that $\rho \leqslant r_0 < 1$. Then

$$
E_{(\infty)} + T_{(\infty)}(\rho^n r_0) = \mathscr{Z}^{-1}_{(n)}(\zeta^{(av)}_{(n)}(r_0)). \tag{V.80}
$$

In case that $(5/16)\, \rho < r_0 < \rho$, we have that $\rho^{n+2} \leqslant \rho^n r_0 < \rho^{n+1}$ and thus

$$
\begin{aligned}
E_{(\infty)}(\rho^n r_0) &= \mathscr{Z}^{-1}_{(n+1)}(\zeta^{(av)}_{(n+1)}(\rho^{-1} r_0)) \\
&= \mathscr{Z}^{-1}_{(n)}(\zeta_{(n)}(r_0)) = \mathscr{Z}^{-1}_{(n)}(\zeta^{(av)}_{(n)}(r_0)),
\end{aligned}
\tag{V.81}
$$

since $\phi(\rho^{-1}) = 0$. Inserting (V.80)–(V.81) into (V.78) and using that $\eta \leqslant 8\rho^{\mu/2}$, we get

$$
\begin{aligned}
&|\mathscr{Z}^{-1}_{(n)}(z) - E_{(\infty)} - T_{(\infty)}(\rho^n r_0)| \\
&\quad = |\mathscr{Z}^{-1}_{(n)}(z) - \mathscr{Z}^{-1}_{(n)}[\zeta^{(av)}_{(n)}(r_0)]| \\
&\quad \leqslant \|\partial_z \mathscr{Z}^{-1}_{(n)}(z)\|_{(\infty)} \cdot |z - \zeta^{(av)}_{(n)}(r_0)| \leqslant \left(\frac{4\rho}{3}\right)^n \cdot \delta'_n \\
&\quad \leqslant \left(\frac{4\rho}{3}\right)^n \cdot \left(\frac{108\varepsilon}{\rho^2}\right) \cdot \eta^{n-1} \\
&\quad \leqslant \left(\frac{27}{2}\right) \left(\frac{16}{5}\right)^{1+(\mu/4)} \cdot \varepsilon \cdot \rho^{-3-(3\mu/4)} \cdot \left(\frac{32}{3}\, \rho^{\mu/4}\right)^n \cdot (\rho^n r_0)^{1+(\mu/4)} \\
&\quad \leqslant \left[\frac{27}{2} \left(\frac{16}{5}\right)^{3/2} \varepsilon \rho^{-9/2} \right] \cdot (\rho^n r_0)^{1+(\mu/4)} < 78 \varepsilon \rho^{-9/2} \cdot (\rho^n r_0)^{1+(\mu/4)}.
\end{aligned}
\tag{V.82}
$$

Thus, $\mathscr{Z}^{-1}_{(n)}(z) \in K_{(\infty)}(78 \varepsilon \rho^{-9/2})$. ∎

V.3. *Existence of Resonances*

Our last topic is the proof of the existence of a resonance at $E_{(\infty)}$. Our approach here is more direct and constructive than the one in [2]. To this end, we introduce a sequence of complex numbers, $\{v_{(n)}\}_{n \in \mathbb{N}_0}$, by setting $v_{(0)} := E_{(\infty)}$ and

$$v_{(n)} := \mathscr{Z}_{(n-1)}(E_{(\infty)}) = \lim_{m \to \infty} \{Z_{(n)}^{-1} \circ Z_{(n+1)}^{-1} \circ \cdots \circ Z_{(m)}^{-1}(0)\} \in \mathscr{U}_{(n)}^{(\text{in})}, \tag{V.83}$$

compare to (V.28). Note that, according to (V.29), we have that $v_{(n)} \in \mathscr{S}_{(n)}$, and thus

$$|v_{(n)}| \leqslant \left(\frac{4\rho}{3}\right)^n. \tag{V.84}$$

Furthermore, we observe that

$$v_{(n+1)} = Z_{(n)}(\eta_{(n)}). \tag{V.85}$$

Using $\{v_{(n)}\}$, we define a sequence of operators, $\{S_{(n)}\}_{n \in \mathbb{N}_0}$, by

$$S_{(0)} := P_0 - \bar{P}_0(\bar{P}_0 H_g \bar{P}_0 - E_{(\infty)})^{-1} \bar{P}_0 W_g P_0, \tag{V.86}$$

and for $n \in \mathbb{N}$,

$$S_{(n)} := \chi_\rho - \bar{\chi}_\rho(\bar{\chi}_\rho H_{(n-1)}[v_{(n)}] \bar{\chi}_\rho - v_{(n)})^{-1} \bar{\chi}_\rho W_{(n-1)}[v_{(n)}] \chi_\rho. \tag{V.87}$$

The crucial properties of $\{S_{(n)}\}$ are collected in the following lemma.

Lemma V.8. *Assume the same hypotheses as in Theorem* V.2. *Then, for* $n \in \mathbb{N}$,

$$\|S_{(0)}\| \leqslant \frac{17}{16}, \qquad \|S_{(0)} - P_0\| \leqslant \frac{1}{16}, \tag{V.88}$$

$$\|S_{(n)}\| \leqslant 1 + \left(\frac{220\varepsilon}{\rho^{1/2}}\right) \eta^{n-1}, \qquad \|S_{(n)} - \chi_\rho\| \leqslant \left(\frac{220\varepsilon}{\rho^{1/2}}\right) \eta^{n-1}, \tag{V.89}$$

and,

$$\frac{e^{i\vartheta}}{\rho_0} \Gamma_{\rho_0}(H_g - E_{(\infty)}) S_{(0)} \Gamma_{\rho_0}^* = H_{(0)}[v_{(1)}] - v_{(1)}, \tag{V.90}$$

$$\frac{1}{\rho} \Gamma_\rho(H_{(n+1)}[v_{(n)}] - v_{(n)}) S_{(n)} \Gamma_\rho^* = H_{(n)}[v_{(n+1)}] - v_{(n+1)}, \tag{V.91}$$

where Γ_{ρ_0} and Γ_ρ are the unitary dilatations defined in (I.9).

Proof. To derive (V.88), we expand $S_{(0)}-P_0$ in a Neumann series, just as in Theorem III.3. We obtain

$$\|S_{(0)}-P_0\| = \left\| \sum_{L=1}^{\infty} \left\{ \left(\frac{\bar{P}_0}{H_0 - E_{(\infty)}} \right) (-W_g) \right\}^L P_0 \right\|$$

$$\leqslant \|R_0^{1/2} \bar{P}_0\| \left(\sum_{L=1}^{\infty} \|R_0^{1/2} \bar{P}_0 W_g \bar{P}_0 R_0^{1/2}\|^{L-1} \right) \cdot \|R_0^{1/2} \bar{P}_0 W_g P_0\|$$

$$\leqslant \left(\frac{2}{\rho_0} \right)^{1/2} \left\{ \sum_{L=0}^{\infty} \left(\frac{6g\Lambda_1}{\rho_0^{1/2} \sin(\vartheta/2)} \right)^L \right\} \left(\frac{6g\Lambda_1}{\sin(\vartheta/2)^{1/2}} \right)$$

$$\leqslant \frac{\sqrt{2} \cdot (4/3) \cdot 6 \cdot g\Lambda_1}{\sin(\vartheta/2)^{1/2} \cdot \rho_0^{1/2}} \leqslant \frac{1}{16}. \tag{V.92}$$

using (III.19)–(III.24), $E_{(\infty)} \in \mathscr{U}_{(0)}^{(\mathrm{in})}$, and (V.11). To prove (V.89), we also expand $S_{(n)} - \chi_\rho$ in a Neumann series and obtain

$$\|S_{(n)} - \chi_\rho\| = \left\| \sum_{L=1}^{\infty} \left\{ \left(\frac{\bar{\chi}_\rho}{T_{(n-1)}[H_f; v_{(n)}] + E_{(n-1)}[v_{(n)}] - v_{(n)}} \right) \right.\right.$$

$$\left.\left. \times (-W_{(n-1)}[v_{(n)}]) \right\}^L \chi_\rho \right\|$$

$$\leqslant \sum_{L=1}^{\infty} \left\{ \left\| \frac{(H_f + \rho)\, \bar{\chi}_\rho[H_f]}{T_{(n-1)}[H_f; v_{(n)}] + E_{n-1}[v_{(n)}] - v_{(n)}} \right\|^{L+1/2} \right.$$

$$\left. \times \|(H_f+\rho)^{-1/2}\, \bar{\chi}_\rho W_{(n-1)}[v_{(n)}]\, \bar{\chi}_\rho \cdot (H_f+\rho)^{-1/2}\|^L \right\}$$

$$\leqslant \sqrt{6} \sum_{L=1}^{\infty} \left(\frac{12e^2 \cdot \varepsilon \eta^{n-1}}{\rho^{1/2}} \right)^L \leqslant \left(\frac{220\varepsilon}{\rho^{1/2}} \right) \cdot \eta^{n-1}, \tag{V.93}$$

using Theorem B.2, $C_d \xi \leqslant 1$, (IV.8), and Theorem V.2.

To prove (V.90), we observe that

$$\bar{P}_0(H_g - E_{(\infty)})\, S_{(0)} = 0, \tag{V.94}$$

$$\frac{e^{i\vartheta}}{\rho_0} \Gamma_{\rho_0} P_0 (H_g - E_{(\infty)})\, S_{(0)} \Gamma_{\rho_0}^* = \frac{e^i}{\rho_0} \Gamma_{\rho_0} \mathscr{F}_{P_0}(H_g - E_{(\infty)})\, \Gamma_{\rho_0}^*$$

$$= H_{(0)}[Z_{(0)}(E_{(\infty)})] - Z_{(0)}(E_{(\infty)})$$

$$= H_{(0)}[v_{(1)}] - v_{(1)}. \tag{V.95}$$

The proof of (V.91) is similar. ∎

The virtue of the operators $S_{(n)}$ is explicit in (V.91): If $(H_{(n)}[v_{(n+1)}] - v_{(n+1)})\,\psi = 0$ then also $(H_{(n-1)}[v_{(n)}] - v_{(n)})(S_{(n)} \Gamma^*_{\rho^{1/\gamma}} \psi) = 0$. We already Used this property in our description of the Feshbach map in Chapter II. So using (V.90) and (V.91), we construct a sequence of vectors, $\{\Psi_{(n)}\}_{n \in \mathbb{N}_0}$, which converges to the desired eigenvector of H_g, by setting

$$\Psi_{(n)} := S_{(0)} \Gamma^*_0 S_{(1)} \Gamma^*_{\rho^{1/\gamma}} S_{(2)} \cdots S_{(n)} \Gamma^*_{\rho^{1/\gamma}} \Omega, \tag{V.96}$$

where Ω is the Fock vacuum.

Lemma V.9. *For all* $n \in \mathbb{N}$,

$$\| \Psi_{(n)} \| \leqslant 2, \tag{V.97}$$

$$\| \Psi_{(n+1)} - \Psi_{(n)} \| \leqslant \frac{250\varepsilon}{\rho^{1/2}}\, \eta^n, \tag{V.98}$$

$$\| (H_g - E_{(\infty)})\, \Psi_{(n)} \| \leqslant 16 \rho_0 \rho^{1/2} \cdot \eta^n. \tag{V.99}$$

Proof. To prove (V.97), we observe that because $\rho^{\mu/2} \leqslant 1/16$, (V.89), and because of the unitarity of Γ_ρ,

$$\begin{aligned} \| \Psi_{(n)} \| &\leqslant \left(\frac{17}{16} \right) \cdot \prod_{j=1}^{n} \left(1 + \left(\frac{220\varepsilon}{\rho^{1/2}} \right) \eta^{j-1} \right) \\ &\leqslant \left(\frac{17}{16} \right) \cdot \exp\left[\left(\frac{220\varepsilon}{\rho^{1/2}} \right) \cdot \sum_{j=1}^{n} \eta^{j-1} \right] \\ &\leqslant \left(\frac{17}{16} \right) \cdot \exp\left[\frac{440\varepsilon}{\rho^{1/2}} \right]. \end{aligned} \tag{V.100}$$

Similarly, we observe that

$$\begin{aligned} \| \Psi_{(n+1)} - \Psi_{(n)} \| &\leqslant \left(\frac{17}{16} \cdot \frac{220\varepsilon}{\rho^{1/2}} \right) \eta^n \cdot \prod_{j=1}^{n} \left(1 + \left(\frac{220\varepsilon}{\rho^{1/2}} \right) \eta^{j-1} \right) \\ &\leqslant \left(\frac{17 \cdot 220 \cdot \varepsilon}{16 \cdot \rho^{1/2}} \right) \cdot \exp\left[\frac{440\varepsilon}{\rho^{1/2}} \right] \cdot \eta^n, \end{aligned} \tag{V.101}$$

using in addition that

$$\Psi_{(n+1)} - \Psi_{(n)} = S_0 \Gamma^*_0 S_{(1)} \cdots S_{(n)} \Gamma^*_{\rho^{1/\gamma}} (S_{(n+1)} - \chi_\rho)\, \Gamma^*_{\rho^{1/\gamma}} \Omega. \tag{V.102}$$

Then, (V.97) and (V.98) follow from $\varepsilon \rho^{-1/2} \leqslant 1/12800$.

It remains to prove (V.99). From (V.90) and (V.91), we obtain

$$
\begin{aligned}
&(H_g - E_{(\infty)})\, \Psi_{(n)} \\
&\quad = (H_g - E_{(\infty)})\, S_{(0)} \Gamma^*_{(0)} S_{(1)} \Gamma^*_{\rho^{1/\gamma}} \Omega \\
&\quad = e^{-i\vartheta} \rho_0 \Gamma^*_0 (H_{(0)}(v_{(1)}) - v_{(1)})\, S_{(1)} \Gamma^*_{\rho^{1/\gamma}} S_{(2)} \Gamma^*_{\rho^{1/\gamma}} \cdots S_{(n)} \Gamma^*_{\rho^{1/\gamma}} \Omega \\
&\quad = e^{-i\vartheta} \rho_0 \rho \Gamma^*_0 \Gamma^*_{\rho^{1/\gamma}} (H_{(1)}(v_{(2)}) - v_{(2)})\, S_{(2)} \Gamma^*_{\rho^{1/\gamma}} \cdots S_{(n)} \Gamma^*_{\rho^{1/\gamma}} \Omega \\
&\quad = \cdots = e^{-i\vartheta} \rho_0 \rho^n \Gamma^*_0 (\Gamma^*_{\rho^{1/\gamma}})^n\, (H_{(n)}(v_{(n+1)}) - v_{(n+1)})\, \Omega. \qquad \text{(V.103)}
\end{aligned}
$$

Since

$$
\begin{aligned}
&(H_{(n)}(v_{(n+1)}) - v_{(n+1)})\, \Omega \\
&\quad = [\, W_{(n)}(v_{(n+1)}) - (v_{(n+1)} - E_{(n)}(v_{(n+1)}))]\, \Omega \\
&\quad = W_{(n)}(v_{(n+1)})\, \Omega - \rho^{-1} v_{(n+1)} \Omega, \qquad \text{(V.104)}
\end{aligned}
$$

we have that

$$
\begin{aligned}
\|(H_{(n)}(v_{(n+1)}) - v_{(n+1)})\, \Omega\| &\leqslant 2e^2\varepsilon \cdot \eta^n + \rho_0 \cdot \left(\frac{4\rho}{3}\right)^{n+1} \\
&\leqslant 2(e^2\varepsilon + \rho_0 \rho) \cdot \eta^n, \qquad \text{(V.105)}
\end{aligned}
$$

and hence (V.99) follows. ∎

THEOREM V.10. *Assume the same hypotheses as in Theorem* V.2. *Then*

$$\Psi_{(\infty)} := \lim_{n \to \infty} \Psi_{(n)} \text{ exists and is contained in } \mathscr{D}(H_g); \qquad \text{(V.106)}$$

$$|\langle \Psi_{(\infty)} \,|\, \Omega \rangle| \geqslant \tfrac{2}{3} \|\Psi_{(\infty)}\| > 0, \qquad \text{(V.107)}$$

$$H_g \Psi_{(\infty)} = E_{(\infty)} \Psi_{(\infty)}. \qquad \text{(V.108)}$$

Proof. By (V.98), $\{\Psi_{(n)}\}_{n \in \mathbb{N}}$ converges and by (V.99),

$$\|H_g \Psi_{(n)}\| \leqslant 2\, |E_{(\infty)}| + 16, \qquad \text{(V.109)}$$

implying that $\Psi_{(n)} \in \mathscr{D}(H_g)$, for every $n \in \mathbb{N}$. Since H_g is closed, this implies that $\Psi_{(\infty)} \in \mathscr{D}(H_g)$, s well. This proves (V.106). Eqn. (V.108) then follows directly from (V.99). Finally,

$$\|\Omega - \Psi_{(\infty)}\| \leqslant \|(S_{(0)} - P_0)\, \Gamma_0^* \Omega\| + \sum_{n=0}^{\infty} \|\Psi_{(n+1)} - \Psi_{(n)}\|$$
$$\leqslant \frac{1}{16} + \frac{500\varepsilon}{\rho^{1/2}}, \tag{V.110}$$

and hence (V.107) is proven. ∎

APPENDIX A. PULL-THROUGH FORMULA AND WICK'S THEOREM

In this chapter we systematize an algebraic technique that allows us to convert an arbitrary product of creation operators, annihilation operators, and functions of H_f into a sum of Wick-ordered products of such operators. Here, Wick-order means that creation operators stand to the left of the functions of H_f, and the annihilation operators are to their right. A close look at $\mathscr{F}_{P_0}(H_g - z)$ in part (b) of Theorem III.3 reveals that we have to deal with these arbitrary (unordered) products, if we seek for more precise information about its spectral properties.

We turn to the Pull-Through formula, recalling from Chapter III the definition of P_0, where we used $\chi[H_f < \rho_0]$, the spectral projection of H_f onto the interval $[0, \rho_0)$, and $\chi[r < \rho] \equiv \chi_\rho[r]$ denoted the characteristic function of the interval $[0, \rho)$. In general, $f[H_f]$ is defined on $\mathscr{D}(H_f)$ by the functional calculus for any measurable function $f: \mathbb{R}^d \to \mathbb{C}$ with $f[r] = O(r)$. More explicitly, $f[H_f]$ acts in the n-particle sector of $\mathscr{F}_b[L^2(\mathbb{R}^d)]$ as a multiplication operator, multiplying

$$f[H_f]\, a^\dagger(k_1) \cdots a^\dagger(k_n) \Omega = f[\omega(k_1) + \cdots + \omega(k_n)]\, a^\dagger(k_1) \cdots a^\dagger(k_n)\, \Omega. \tag{A.1}$$

Using, for example, this representation of $f[H_f]$ as a multiplication operator, it is easy to check the following

Lemma A.1 (Pull-Through Formula). *Let $f: \mathbb{R}^d \to \mathbb{C}$ be measurable, obeying $f[r] = O(r)$. Then $f[H_f]$ is defined on $\mathscr{D}(H_f)$ and, for all $k \in \mathbb{R}^d$,*

$$f[H_f]\, a^\dagger(k) = a^\dagger(k)\, f[H_f + \omega(k)], \tag{A.2}$$
$$a(k)\, f[H_f] = f[H_f + \omega(k)]\, a(k), \tag{A.3}$$

in the sense of operator-valued distributions. Moreover, extending f to the whole real line by setting $f \equiv 0$ on $\mathbb{R}_0^-$, (A.2) and (A.3) extend to $a^\dagger(k)\, f[H_f] = f[H_f - \omega(k)]\, a^\dagger(k)$ and $f[H_f]\, a(k) = a(k)\, f[H_f - \omega(k)]$.

Next, we present Wick's theorem. To this end, we introduce some notation. Pick any $(\sigma_1, \sigma_2, ..., \sigma_N) \in \{+1, -1\}^N$. For any subset $\mathcal{Q} \subset \mathcal{N} := \{1, 2, ..., N\}$, $N \in \mathbb{N}$, we denote $\mathcal{Q}_\pm := \{ j \in \mathcal{Q} \mid \sigma_j = \pm 1\}$. Then, writing $a^+(k) := a^\dagger(k)$ and $a^-(k) := a(k)$, we define the Wick-ordered product $:\cdot:$ by

$$:\prod_{j\in\mathcal{Q}} a^{\sigma_j}(k_j): := \prod_{j\in\mathcal{Q}_+} a^{\sigma_j}(k_j) \prod_{j\in\mathcal{Q}_-} a^{\sigma_j}(k_j). \tag{A.4}$$

Furthermore, we denote

$$\langle A \rangle := \langle \Omega \mid A\Omega \rangle, \tag{A.5}$$

for any operator A which has the vacuum Ω in its domain. Wick's theorem is a simple formula that converts arbitrary products of annihilation-and creation operators into a sum of Wick-ordered products.

LEMMA A.2 (Wick's Theorem). *Denote $\mathcal{N} := \{1, 2, ..., N\}$ and $\prod_{j\in\mathcal{A}} \equiv \prod_{j=1}^N \chi[j \in \mathcal{A}]$ for any $\mathcal{A} \subseteq \mathcal{N}$. Then, for any $(\sigma_1, \sigma_2, ..., \sigma_N) \in \{+1, -1\}^N$*

$$\prod_{j\in\mathcal{N}} a^{\sigma_j}(k_j) = \sum_{\mathcal{Q}\subseteq\mathcal{N}} \left\langle \prod_{j\in\mathcal{N}\setminus\mathcal{Q}} a^{\sigma_j}(k_j) \right\rangle :\prod_{j\in\mathcal{Q}} a^{\sigma_j}(k_j):. \tag{A.6}$$

Proof. We use an induction in the number of factors, N, observing that (A.6) is trivial for $N=1$. Assume that (A.6) holds for all products with up to N factors, for some $N \geqslant 1$ and consider the left side of (A.6) with $N+1$ factors. The case $\sigma_{N+1} = -1$ is also simple because the last factor annihilates the vacuum $a^{\sigma_{N+1}}(k_{N+1})\Omega = 0$. Thus, denoting $a_j^{\sigma_j} := a^{\sigma_j}(k_j)$, the induction hypothesis yields

$$\begin{aligned}
\prod_{j\in\mathcal{N}+1} a_j^{\sigma_j} &= \sum_{\mathcal{Q}\subseteq\mathcal{N}} \left\langle \prod_{j\in\mathcal{N}\setminus\mathcal{Q}} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j}: a_{N+1}^- \\
&= \sum_{\mathcal{Q}\subseteq\mathcal{N}} \left\langle \prod_{j\in\mathcal{N}\setminus\mathcal{Q}} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j} a_{N+1}^-: \\
&= \sum_{\mathcal{Q}\subseteq\mathcal{N}+1} \left\langle \prod_{j\in\mathcal{N}+1\setminus\mathcal{Q}} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j}:.
\end{aligned} \tag{A.7}$$

To handle the case $\sigma_{N+1} = +1$, we use the following convenient representation of the canonical commutation relation

$$[a_i^{\sigma_i}, a_j^{\sigma_j}] = \langle a_i^{\sigma_i} a_j^{\sigma_j} \rangle, \tag{A.8}$$

which yields

$$\left\langle \prod_{j\in\mathcal{N}\backslash\mathcal{Q}} a_j^{\sigma_j} a_{N+1}^{+} \right\rangle = \sum_{k\in\mathcal{N}\backslash\mathcal{Q}} \langle a_k^{\sigma_k} a_{N+1}^{+}\rangle \left\langle \prod_{j\in\mathcal{N}\backslash(\mathcal{Q}\cup\{k\})} a_j^{\sigma_j} \right\rangle. \tag{A.9}$$

Using the induction hypothesis and again (A.8), we obtain

$$\begin{aligned}\prod_{j\in\mathcal{N}+1} a_j^{\sigma_j} &= a_{N+1}^{+} \prod_{j\in\mathcal{N}} a_j^{\sigma_j} + \sum_{k=1}^{N} \langle a_k^{\sigma_k} a_{N+1}^{\sigma_{N+1}}\rangle \prod_{j\in\mathcal{N}\backslash\{k\}} a_j^{\sigma_j} \\ &= \sum_{\mathcal{Q}\subseteq\mathcal{N}} \left\langle \prod_{j\in\mathcal{N}\backslash\mathcal{Q}} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j} a_{N+1}^{+}: \\ &\quad + \sum_{k=1}^{N} \sum_{\mathcal{Q}\subseteq\mathcal{N}\backslash\{k\}} \langle a_k^{\sigma_k} a_{N+1}^{\sigma_{N+1}}\rangle \left\langle \prod_{j\in\mathcal{N}\backslash(\mathcal{Q}\cup\{k\})} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j}:.\end{aligned} \tag{A.10}$$

Now, we observe that for any function $F(k, \mathcal{Q})$

$$\begin{aligned}\sum_{k=1}^{N} \sum_{\mathcal{Q}\subseteq\mathcal{N}\backslash\{k\}} F(k, \mathcal{Q}) &= \sum_{k=1}^{N} \sum_{\mathcal{Q}\subseteq\mathcal{N}} \chi[k\notin\mathcal{Q}]\, F(k, \mathcal{Q}) \\ &= \sum_{\mathcal{Q}\subseteq\mathcal{N}} \sum_{k\in\mathcal{N}\backslash\mathcal{Q}} F(k, \mathcal{Q}).\end{aligned} \tag{A.11}$$

Inserting (A.11) and (A.9) into the last line of (A.10), we arrive at the claim:

$$\begin{aligned}\prod_{j\in\mathcal{N}+1} a_j^{\sigma_j} &= \sum_{\mathcal{Q}\subseteq\mathcal{N}} \left\langle \prod_{j\in\mathcal{N}\backslash\mathcal{Q}} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j} a_{N+1}^{+}: \\ &\quad + \sum_{\mathcal{Q}\subseteq\mathcal{N}} \left\langle \prod_{j\in\mathcal{N}\backslash\mathcal{Q}} a_j^{\sigma_j} a_{N+1}^{+} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j}: \\ &= \sum_{\mathcal{Q}\subseteq\mathcal{N}+1} \left\langle \prod_{j\in\mathcal{N}+1\backslash\mathcal{Q}} a_j^{\sigma_j} \right\rangle :\prod_{j\in\mathcal{Q}} a_j^{\sigma_j}:. \quad \blacksquare\end{aligned} \tag{A.12}$$

Now we combine the Pull-Through formula (Lemma A.1) and Wick's theorem (Lemma A.2) to derive the following identity.

LEMMA A.3. *Denote* $\mathcal{N} := \{1, 2, ..., N\}$ *and* $\prod_{j\in\mathcal{A}} \equiv \prod_{j=1}^{N} \chi[j\in\mathcal{A}]$ *for any* $\mathcal{A}\subseteq\mathcal{N}$. *Suppose that* $\sigma_j \in \{+1, -1\}$ *and* $f_j[r] = O(r+1)$ *is a measurable function on* $\mathbb{R}^+$, *for any* $j = 1, 2, ..., N$. *Then*

$$\prod_{j=1}^{N} \{a^{\sigma_j}(k_j)\, f_j[H_f]\}$$

$$= \sum_{\mathcal{Q} \subseteq \mathcal{N}} \prod_{j \in \mathcal{Q}_+} a^+(k_j) \left\langle \prod_{j=1}^{N} \left\{ [a^{\sigma_j}(k_j)]^{\chi[j \notin \mathcal{Q}]} \right.\right.$$

$$\left.\left. \times f_j \left[H_f + r + \sum_{\substack{i=1,\\ i \in \mathcal{Q}_-}}^{j} \omega(k_i) + \sum_{\substack{i=j+1,\\ i \in \mathcal{Q}_+}}^{N} \omega(k_i) \right] \right\} \right\rangle \Bigg|_{r=H_f} \prod_{j \in \mathcal{Q}_-} a^-(k_j), \tag{A.13}$$

where $[a^{\sigma_j}(k_j)]^{\chi[j \notin \mathcal{Q}]} = a^{\sigma_j}(K-j)$ *for* $j \notin \mathcal{Q}$ *and* $[a^{\sigma_j}(k_j)]^{\chi[j \notin \mathcal{Q}]} = 1$ *for* $j \in \mathcal{Q}$.

Proof. The proof of (A.13) is a lengthy computation using the Pull-Through formula and Wick's Theorem. For this computation we need to extend $f_j(r) := 0$ for any $r < 0$ and any $j = 1, 2, ..., N$, so that we may use the Pull-Through formula backwards, as is indicated in the second part of Lemma A.1. Again, we denote $a_j^{\sigma_j} := a^{\sigma_j}(k_j)$ and, furthermore, $\omega_j := \omega(k_j)$. Then, we get

$$\prod_{j=1}^{N} \{a_j^{\sigma_j} f_j[H_f]\} = \prod_{j=1}^{N} a_j^{\sigma_j} \prod_{j=1}^{N} f_j \left[H_f + \sum_{i=j+1}^{N} \sigma_i \omega_i \right]$$

$$= \sum_{\mathcal{Q} \subseteq \mathcal{N}} \left\langle \prod_{j \in \mathcal{N} \setminus \mathcal{Q}} a_j^{\sigma_j} \right\rangle : \prod_{j \in \mathcal{Q}} a_j^{\sigma_j} : \prod_{j=1}^{N} f_j \left[H_f + \sum_{i=j+1}^{N} \sigma_i \omega_i \right]$$

$$= \sum_{\mathcal{Q} \subseteq \mathcal{N}} \prod_{j \in \mathcal{Q}_+} a_j^+ \left\langle \prod_{j \in \mathcal{N} \setminus \mathcal{Q}} a_j^{\sigma_j} \right\rangle$$

$$\times \prod_{j=1}^{N} f_j \left[H_f + \sum_{i=j+1}^{N} \sigma_i \omega_i + \sum_{i \in \mathcal{Q}_-} \omega_i \right] \prod_{j \in \mathcal{Q}_-} a_j^- . \tag{A.14}$$

Next, we use $\langle f[H_f] \rangle = f[0]$ which implies $f[H_f] = \langle f[H_f + r] \rangle|_{r=H_f}$ to move the f_j's into the vacuum expectation value:

$$\left\langle \prod_{j \in \mathcal{N} \setminus \mathcal{Q}} a_j^{\sigma_j} \right\rangle \prod_{j=1}^{N} f_j \left[H_f + \sum_{i=j+1}^{N} \sigma_i \omega_i + \sum_{i \in \mathcal{Q}_-} \omega_i \right]$$

$$= \left\langle \prod_{j \in \mathcal{N} \setminus \mathcal{Q}} a_j^{\sigma_j} \prod_{j=1}^{N} f_j \left[r + H_f + \sum_{i=j+1}^{N} \sigma_i \omega_i + \sum_{i \in \mathcal{Q}_-} \omega_i \right] \right\rangle \Bigg|_{r=H_f}$$

$$= \left\langle \prod_{j=1}^{N} \left\{ [a_j^{\sigma_j}]^{\chi[j \notin \mathcal{Q}]} \cdot f_j \left[r + H_f + \sum_{\substack{i=j+1\\ i \in \mathcal{Q}}}^{N} \sigma_i \omega_i + \sum_{i \in \mathcal{Q}_-} \omega_i \right] \right\} \right\rangle \Bigg|_{r=H_f} . \tag{A.15}$$

We finish the proof with the remark that

$$\sum_{i \in \mathcal{Q}_-} \omega_i + \sum_{\substack{i=j+1 \\ i \in \mathcal{Q}}}^{N} \sigma_i \omega_i = \sum_{i \in \mathcal{Q}_-} \omega_i - \sum_{\substack{i=j+1 \\ i \in \mathcal{Q}_-}}^{N} \omega_i + \sum_{\substack{i=j+1 \\ i \in \mathcal{Q}_+}}^{N} \omega_i$$

$$= \sum_{\substack{i=1 \\ i \in \mathcal{Q}_-}}^{j} \omega_i + \sum_{\substack{i=j+1 \\ i \in \mathcal{Q}_+}}^{N} \omega_i. \quad \blacksquare \tag{A.16}$$

We are now ready to formulate the main consequence of the Pull-Through formula and Wick's Theorem, after having introduced some more notation: Besides the family $f_j(r)$, $j \in \mathbb{N}$ of measurable functions, we assume below to be given a family of measurable functions

$$w_{M,N}\colon \mathbb{R}^+ \times (\mathbb{R}^d)^M \times (\mathbb{R}^d)^N \to \mathbb{C} \tag{A.17}$$

for all $M, N \in \mathbb{N}_0$. We assume that $w_{M,N}[r; k^{(M)}; \tilde{k}^{(N)}] = O(r)$, a.e. $\mathbb{R}^{(M+N)d}$, and hence can be defined via the functional calculus as an operator on $\mathcal{D}(H_f)$ for a.e. $k^{(M)}, \tilde{k}^{(N)}$ by the replacement $r \to H_f$. Here, we abbreviated

$$k^{(M)} := (k_1, k_2, ..., k_M) \in (\mathbb{R}^d)^M, \qquad dk^{(M)} := \sum_{j=1}^{M} d^d k_j. \tag{A.18}$$

Another assumption on $w_{M,N}$ is their symmetry in $k^{(M)}$ and $\tilde{k}^{(N)}$. More precisely, we demand that

$$w_{M,N}[r; k^{(M)}; \tilde{k}^{(N)}] = \{ w_{M,N}[r; k^{(M)}; \tilde{k}^{(N)}] \}_{M,N}^{symm}$$

$$:= \frac{1}{M!\, N!} \sum_{\pi, \tilde{\pi}} w_{M,N}[r; k_{\pi}^{(M)}; \tilde{k}_{\tilde{\pi}}^{(N)}], \tag{A.19}$$

where the sum runs over all permutations $\pi \in \mathcal{S}_M$ and $\tilde{\pi} \in \mathcal{S}_N$, and

$$k_{\pi}^{(M)} := (k_{\pi(1)}, k_{\pi(2)}, ..., k_{\pi(M)}). \tag{A.20}$$

We generalize this notations as follows. Given

$$k^{(M_l)} = (k_{l,1}, ..., k_{l,M_l}) \in (\mathbb{R}^d)^{M_l}, \tag{A.21}$$

$$\tilde{k}_l^{(N_l)} = (\tilde{k}_{l,1}, ..., \tilde{k}_{l,N_l}) \in (\mathbb{R}^d)^{N_l}, \tag{A.22}$$

for all $l \in \{1, 2, ..., L\}$, we write

$$k^{(M)} := (k_l^{(M_l)})_{l=1}^L = (k_1^{(M_1)}, ..., k_L^{(M_L)})$$
$$= (k_{1,1}, ..., k_{1,M}, k_{2,1}, ..., k_{2,M_2}, k_{L,1}, ..., k_{L,M_l}), \qquad \text{(A.23)}$$
$$\tilde{k}^{(N)} := (\tilde{k}_l^{(N_l)})_{l=1}^L = (\tilde{k}_1^{(N_1)}, ..., k_L^{(N_L)})$$
$$= (\tilde{k}_{1,1}, ..., \tilde{k}_{1,N_1}, \tilde{k}_{2,1}, ..., \tilde{k}_{2,N_2}, \tilde{k}_{L,1}, ..., \tilde{k}_{L,N_L}), \qquad \text{(A.24)}$$

where we have set

$$M := \sum_{l=1}^{L} M_l \qquad \text{and} \qquad N := \sum_{l=1}^{L} N_l, \qquad \text{(A.25)}$$

provided no confusion arises. Note that, for a function $f(k_1^{(M_1)}, ..., k_1^{(M_L)}, \tilde{k}_1^{(N_1)}, ..., \tilde{k}_L^{(N_L)})$, the symmetrization symbol $\{\cdot\}_{M,N}^{symm}$ indicates the summation over all permutations of the M pairs $\{(l, j) \mid 1 \leqslant l \leqslant L, 1 \leqslant j \leqslant M_l\}$ and all permutations of the N pairs $\{(l, j) \mid 1 \leqslant l \leqslant L, 1 \leqslant j \leqslant N_l\}$, consistent with our notation (A.23) and (A.24). Furthermore, we henceforth write

$$a^\dagger(k^{(M)}) := \prod_{j=1}^{M} a^\dagger(k_j), \qquad a(k^{(M)}) := \prod_{j=1}^{M} a(k_j), \qquad \text{(A.26)}$$

$$\omega(k^{(M)}) := \sum_{j=1}^{M} \omega(k_j). \qquad \text{(A.27)}$$

Next, assume that $m + p + n + q \geqslant 1$ with m, n, p, q nonnegative integers. For a given $w_{m+p,n+q}$ described above, we define

$$W_{p,q}^{m,n}[H_f; k^{(m)}; \tilde{k}^{(n)}]$$
$$:= \int dx^{(p)}\, d\tilde{x}^{(q)}\, a^\dagger(x^{(p)})\, w_{m+p,n+q}[H_f; k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}]\, a(\tilde{x}^{(q)}). \qquad \text{(A.28)}$$

In particular, we abbreviate $W_{m,n} := W_{m,n}^{0,0}[H_f]$, i.e.,

$$W_{m,n} := \int dk^{(m)}\, d\tilde{k}^{(n)}\, a^\dagger(k^{(m)})\, w_{m,n}[H_f; k^{(m)}; \tilde{k}^{(n)}]\, a(\tilde{k}^{(n)}). \qquad \text{(A.29)}$$

Now, we are ready to formulate our next result.

THEOREM A.4. *Assume that* $w_{M,N}$ *are functions as in* (A.17), *obeying* (A.19), *for all* $M + N \geqslant 1$ *and that* $f_j[r] = O(r)$ *are measurable functions on* $\mathbb{R}^+$, *for any* $j \in \mathbb{N}$. *Suppose that* $M_l + N_l \geqslant 1$, *for all* $l = 1, 2, ..., L$. *Then*

$$
\begin{aligned}
&W_{M_1, N_1} f_1[H_f]\, W_{M_2, N_2} f_2[H_f] \cdots f_{L-1}[H_f]\, W_{M_L, N_L} \\
&\quad = \sum_{m_1=0}^{M_1} \sum_{n_1=0}^{N_1} \cdots \sum_{m_L=0}^{M_L} \sum_{n_L=0}^{N_L} \int \prod_{L=1}^{L} \left\{ dk_l^{(M_l-m_l)}\, d\tilde{k}_l^{(N_l-n_l)} \binom{M_l}{m_l} \binom{N_l}{n_l} \right\} \\
&\qquad \times a^\dagger(k^{(m)}) \{ D_L[H_f; \{ W^{m_l, n_l}_{M_l-m_l, N_l-n_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}; \{f_l\}_{l=1}^{L-1}] \}^{symm}_{m,n} \\
&\qquad \times a(\tilde{k}^{(n)}),
\end{aligned} \tag{A.30}
$$

where

$$
\begin{aligned}
&D_L[r; \{ W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)} \}_{l=1}^{L}; \{f_l\}_{l=1}^{L-1}] \\
&\quad := \langle W^{m_1, n_1}_{p_1, q_1}[H_f + r + \lambda_1; k_1^{(m_1)}; \tilde{k}_1^{(n_1)}]\, f_1[H_f + r + \mu_1] \cdots \\
&\qquad \cdots \times f_{L-1}[H_f + r + \mu_{L-1}]\, W^{m_L, n_L}_{p_L, q_L}[H_f + r + \lambda_L; k_L^{(m_L)}; \tilde{k}_L^{(n_L)}] \rangle,
\end{aligned} \tag{A.31}
$$

and

$$
\lambda_l := \sum_{j=1}^{l-1} \omega(\tilde{k}_j^{(n_j)}) + \sum_{j=l+1}^{L} \omega(k_j^{(m_j)}), \tag{(A.32)}
$$

$\mu_l := \lambda_l + \omega(\tilde{k}_l^{(n_l)})$ *and, furthermore,* $m = \sum_{l=1}^{L} m_l$, $n = \sum_{l=1}^{L} n_l$.

Before giving the proof of Theorem A.4, we remark that its assertion is valid without change if we suppose that f_j and $w_{M,N}$ have their values in the bounded operators on $\mathcal{H}_{el}$, rather than $\mathbb{C}$. This is necessary in one application.

Proof of Theorem A.4. Inserting $W_{M,N}$ into (A.29), we observe that

$$
\begin{aligned}
&W_{M_1, N_1} f_1[H_f]\, W_{M_2, N_2} f_2[H_f] \cdots f_{L-1}[H_f]\, W_{M_L, N_L} \\
&\quad = \int \prod_{l=1}^{L} \{ dk_l^{(M_l)}\, d\tilde{k}_l^{(N_l)} \}\, a^\dagger(k_1^{(M_1)})\, w_{M_1, N_1}[H_f; k_1^{(M_1)}; \tilde{k}_1^{(N_1)}] \\
&\qquad \times a(\tilde{k}_1^{(N_1)})\, f_1[H_f] \cdots f_{L-1}[H_f] \\
&\qquad \times a^\dagger(k_L^{(M_L)})\, w_{M_L, N_L}[H_f; k_L^{(M_L)}; \tilde{k}_L^{(N_L)}]\, a(\tilde{k}_L^{(N_L)}).
\end{aligned} \tag{A.33}
$$

We apply Lemma A.3 to the product in (A.33) which contains $K = \sum_{l=1}^{L} (M_l + N_l)$ creation- and annihilation operators. This requires a summation over the subsets $\mathcal{Q} \subseteq \mathcal{K} = \{1, 2, ..., K\}$, distinguishing those creation- and annihilation operators that are Wick-ordered from those that go into the vacuum expectation value. For our present purpose, it is more convenient to represent $\mathcal{K}$ and $\mathcal{Q}$ by

$$\mathscr{K} = \bigcup_{\mu=M,N} \bigcup_{l=1}^{L} \mathscr{K}_{\mu,l}, \qquad \mathscr{K}_{\mu,l} := \{(\mu, l, j) \mid j = 1, ..., \mu_l\}, \tag{A.34}$$

$$\mathscr{Q} = \bigcup_{\mu=M,N} \bigcup_{l=1}^{L} \mathscr{Q}_{\mu,l}, \qquad \mathscr{Q}_{\mu,l} := \mathscr{Q} \cap \mathscr{K}_{\mu,l}. \tag{A.35}$$

Using this representation, the summation over $\mathscr{Q} \subseteq \mathscr{K}$ is replaced by the multiple summation over $\mathscr{Q}_{\mu,l} \subseteq \mathscr{K}_{\mu,l}$:

$$\sum_{\mathscr{Q} \subseteq \mathscr{K}} \to \sum_{\mathscr{Q}_{M,1} \subseteq \mathscr{K}_{M,1}} \sum_{\mathscr{Q}_{N,1} \subseteq \mathscr{K}_{N,1}} \cdots \sum_{\mathscr{Q}_{M,L} \subseteq \mathscr{K}_{M,L}} \sum_{\mathscr{Q}_{N,L} \subseteq \mathscr{K}_{N,L}} . \tag{A.36}$$

Each subset $\mathscr{Q}_{M,l} \subseteq \mathscr{K}_{M,l}$ specifies those $m_l := |\mathscr{Q}_{M,l}| \leqslant M_l$ variables $\{k_{l,j} \mid j \in \mathscr{Q}_{M,l}\}$ among $k_l^{(M_l)} = (k_{l,1}, ..., k_{l,M_l})$ that are Wick-ordered outside the vacuum expectation value, and those $M_l - m_l = |\mathscr{K}_{M,l} \backslash \mathscr{Q}_{M,l}|$ variables $\{k_{l,j} \mid j \in \mathscr{K}_{M,l} \backslash \mathscr{Q}_{M,l}\}$ that appear in the vacuum expectation value in (A.13). For example, one term contributing to the sum over $\mathscr{Q} \subseteq \mathscr{K}$ is given by

$$\begin{aligned} \mathscr{Q}^0_{M,l} &= \{1, 2, ..., m_l\}, \\ \mathscr{K}_{M,l} \backslash \mathscr{Q}^0_{M,l} &= \{m_l + 1, m_l + 2, ..., M_l\}, \end{aligned} \tag{A.37}$$

$$\begin{aligned} \mathscr{Q}^0_{N,l} &= \{1, 2, ..., n_l\}, \\ \mathscr{K}_{N,l} \backslash \mathscr{Q}^0_{N,l} &= \{n_l + 1, n_l + 2, ..., N_l\}. \end{aligned} \tag{A.38}$$

We write the contribution this term generates according to (A.13) and (A.33) in full detail:

$$\begin{aligned}
&\int \prod_{l=1}^{L} \left\{ \prod_{j=1}^{m_l} dk_{l,j} \prod_{j=1}^{n_l} d\tilde{k}_{l,j} \right\} \prod_{l=1}^{L} \prod_{j=1}^{m_l} a^\dagger(k_{l,j}) \\
&\quad \times \Bigg\langle \Bigg(\Bigg(\int \prod_{j=m_1+1}^{M_1} dk_{1,j} \prod_{j=n_1+1}^{N_1} d\tilde{k}_{1,j} \prod_{j=m_1+1}^{M_1} a^\dagger(k_{1,j}) \\
&\quad \times w_{M_1,N_1}[H_f + r + \lambda_1; k_1^{(M_1)}; \tilde{k}_1^{(N_1)}] \prod_{j=n_1+1}^{N_1} a(\tilde{k}_{1,j}) \Bigg) \\
&\quad \times f_1 \left[H_f + r + \lambda_1 + \sum_{j=1}^{n_1} \omega(\tilde{k}_{1,j}) \right] \cdots \\
&\quad \times f_{L-1} \left[H_f + r + \lambda_{L-1} + \sum_{j=1}^{n_{L-1}} \omega(\tilde{k}_{L-1,j}) \right] \\
&\quad \times \Bigg(\int \prod_{j=m_L+1}^{M_L} dk_{L,j} \prod_{j=n_L+1}^{N_L} d\tilde{k}_{L,j} \prod_{j=m_L+1}^{M_L} a^\dagger(k_{L,j}) \\
&\quad \times w_{M_L,N_L}[H_f + r + \lambda_L; k_L^{(m_L)}; \tilde{k}_L^{(N_L)}] \prod_{j=n_L+1}^{N_L} a(\tilde{k}_{L,j}) \Bigg) \Bigg\rangle \prod_{l=1}^{L} \prod_{j=1}^{n_l} a(\tilde{k}_{l,j}),
\end{aligned} \tag{A.39}$$

using

$$\lambda_l = \sum_{\tilde{l}=1}^{l-1} \sum_{j=1}^{n_{\tilde{l}}} \omega(\tilde{k}_{\tilde{l},j}) + \sum_{\tilde{l}=l+1}^{L} \sum_{j=1}^{m_{\tilde{l}}} \omega(k_{\tilde{l},j}). \tag{A.40}$$

In Eq. (A.39) we rename the integration variables for each $l = 1, 2, ..., L$ as follows:

$$\left.\begin{array}{l} k_l^{(M_l)} \mapsto (k_l^{(m_l)}, x_l^{M_l - m_l}), \\ \tilde{k}_l^{(N_l)} \mapsto (\tilde{k}_l^{(n_l)}, \tilde{x}_l^{N_l - n_l}), \end{array}\right\}:$$

$$\Leftrightarrow \begin{cases} k_{l,1} \mapsto k_{l,1}, ..., k_{l,m_l} \mapsto k_{l,m_l}, \\ \quad k_{l,m_l+1} \mapsto x_{l,1}, ..., k_{l,M_l} \mapsto x_{l,M_l-m_l}, \\ \tilde{k}_{l,1} \mapsto \tilde{k}_{l,1}, ..., \tilde{k}_{l,n_l} \mapsto \tilde{k}_{l,n_l}, \\ \quad \tilde{k}_{l,n_l+1} \mapsto \tilde{x}_{l,1}, ..., \tilde{k}_{l,N_l} \mapsto \tilde{x}_{l,N_l-n_l}. \end{cases} \tag{A.41}$$

Observe that the definitions for λ_l given in (A.32) and (A.40) agree, upon this change of variables, and the term (A.39), which now appears as

$$\begin{aligned} &A[M_l, m_l; N_l, n_l] \\ &= \int \prod_{l=1}^{L} \{dk_l^{(m_l)}\, d\tilde{k}_l^{(n_l)}\} \prod_{l=1}^{L} a^\dagger(k_l^{(m_l)}) \\ &\quad \times \langle W^{m_1,n_1}_{M_1-m_1,N_1-n_1}[H_f + r + \lambda_1; k_1^{(m_1)}; \tilde{k}_1^{(n_1)}] \\ &\quad \times f_1[H_f + r + \lambda_1 + \omega(\tilde{k}_1^{(n_1)})] \cdots \\ &\quad \times \cdots f_{L-1}[H_f + r + \lambda_{L-1} + \omega(\tilde{k}_{L-1}^{(n_{L-1})})] \\ &\quad \times W^{m_L,n_L}_{M_L-m_L,N_L-n_L}[H_f + r + \lambda_L; k_L^{(m_L)}; \tilde{k}_L^{(n_L)}]\rangle|_{r=H_f} \prod_{l=1}^{L} a(\tilde{k}_l^{(n_l)}), \end{aligned} \tag{A.42}$$

resembles the desired result. Now, we go back to (A.39) and observe that any subset $\mathcal{Q} \subseteq \mathcal{K}$ with $|\mathcal{Q}_{M,l}| = |\mathcal{Q}^0_{M,l}| = m_l$ and $|\mathcal{Q}_{N,l}| = |\mathcal{Q}^0_{N,l}| = n_l$ generates the same contribution to the sum as (A.39) does, because we assumed in (A.19) the functions $w_{M_l,N_l}[r; k_l^{(M_l)}: \tilde{k}_l^{(N_l)}]$ to be invariant under internal permutations of $k_l^{(M_l)} = (k_{l,1}, ..., k_{l,M_l})$ and $\tilde{k}_l^{(N_l)} = (\tilde{k}_{l,1}, ..., \tilde{k}_{l,N_l})$. For any given $1 \leqslant m_l \leqslant M_l$, there are $M_l!(M_l - m_l!)^{-1} (m_l!)^{-1}$ subsets $\mathcal{Q}_{M,l} \subset \mathcal{K}_{M,l}$ such that $|\mathcal{Q}_{M,l}| = m_l$. Thus, we obtain

$$\begin{aligned} &W_{M_1,N_1} f_1[H_f]\, W_{M_2,N_2} f_2[H_f] \cdots f_{L-1}[H_f]\, W_{M_L,N_L} \\ &= \sum_{m_1=0}^{M_1} \sum_{n_1=0}^{N_1} \cdots \sum_{m_L=0}^{M_L} \sum_{n_L=0}^{N_L} \prod_{l=1}^{L} \left\{ \binom{M_l}{m_l} \binom{N_l}{n_l} \right\} A[M_l, m_l; N_l, n_l], \end{aligned} \tag{A.43}$$

where $A[M_l, m_l: N_l, n_l]$ is given in (A.42). We have verified (A.30) except for one small detail: The symmetrization $\{\cdot\}^{symm}_{m,n}$ of $A[M_l, m_l; N_l, n_l]$. But this again just a change of variables in $A[M_l, m_l; N_l, n_l]$. Namely, we set $m := \sum_{l=1}^{L} m_l$ and $n := \sum_{l=1}^{L} n_l$, $k^{(m)} := (k_1^{(m_1)}, ..., k_L^{(m_L)})$ and $\tilde{k}^{(n)} := (\tilde{k}_1^{(n_1)}, ..., \tilde{k}_L^{(n_L)})$, perform a permutation of variables $(k_1, ..., k_m) \mapsto (k_{\pi(1)}, ..., k_{\pi(m)})$, $(\tilde{k}_1, ..., \tilde{k}_n) \mapsto (\tilde{k}_{\tilde{\pi}(1)}, ..., \tilde{k}_{\tilde{\pi}(n)})$, and sum up the single contributions from every pair of permutations $(\pi, \tilde{\pi})$. This way we get $m!\, n!$ copies of the same integral, thanks to $\prod_{j=1}^{m} a^\dagger(k_j) = \prod_{j=1}^{m} a^\dagger(k_{\pi(j)})$, $\prod_{j=1}^{n} a^\dagger(\tilde{k}_j) = \prod_{j=1}^{n} a^\dagger(\tilde{k}_{\tilde{\pi}(j)})$. ∎

We come to the algebraically most involved expression, the recurrence relation for D_L.

THEOREM A.5. *Assume that $w_{M,N}$ are functions defined in* (A.17), *obeying* (A.19), *for all $M+N \geqslant 1$ and that $f_j[r] = O(r)$ are measurable functions on $\mathbb{R}^+$, for any $j \in \mathbb{N}$. Suppose that $m_l + p_l + n_l + q_l \geqslant 1$, for all $l = 1, 2, ..., L$. As in Theorem* A.4, *we use* (A.28) *to define*

$$\begin{aligned} D_L[r; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{f_l\}_{l=1}^{L-1}] &= \langle W^{m_1, n_1}_{p_1, q_1}[H_f + r + \lambda_1; k_1^{(m_1)}; \tilde{k}_1^{(n_1)}]\, f_1[H_f + r + \mu_1] \cdots \\ &\quad \cdots \times f_{L-1}[H_f + r + \mu_{L-1}]\, W^{m_L, n_L}_{p_L, q_L}[H_f + r + \lambda_L; k_L^{(m_L)}; \tilde{k}_L^{(n_l)}] \rangle. \end{aligned} \tag{A.44}$$

where $\mu_l := \lambda_l + \omega(\tilde{k}_l^{(n_l)})$ and

$$\lambda_l := \sum_{j=1}^{l-1} \omega(\tilde{k}_j^{(n_j)}) + \sum_{j=l+1}^{L} \omega(k_j^{(m_j)}). \tag{A.45}$$

Then, for any $1 \leqslant l \leqslant L$.

$$\begin{aligned} &D_L[r; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{f_l\}_{l=1}^{L-1}] \\ &= \sum_{\substack{v_j = 0, \\ j=1, ..., l-1}}^{q_j} \sum_{\substack{u_j = 0, \\ j=l+1, ..., L}}^{p_j} \int \prod_{j=1}^{l-1} \left\{ d\tilde{y}_j^{(v_j)} \binom{q_j}{v_j} \right\} \prod_{j=l+1}^{L} \left\{ dy_j^{(u_j)} \binom{p_j}{u_j} \right\} \\ &\quad \times D_{l-1}[r + \omega_{\geqslant l}(k^{(m)}); \{W^{m_j, n_j + v_j}_{p_j, q_j - v_j}; k_j^{(m_j)}; \tilde{k}_j^{(n_j)}, \tilde{y}_j^{(v_j)}\}_{j=1}^{l-1}; \{f_i\}_{j=1}^{l-2}] \\ &\quad \times f_{l-1}[r + \mu_{l-1} + \omega_{\leqslant l-1}(\tilde{y}^{(V)})] \cdot f_l[r + \mu_l + \omega_{\geqslant l+1}(y^{(U)})] \\ &\quad \times \left\langle \prod_{j=1}^{l-1} a(\tilde{y}_j^{(v_j)})\, W^{m_l, n_l}_{p_l, q_l}[r + H_f + \lambda_l; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}] \prod_{j=1}^{l-1} a^\dagger(y_j^{(u_j)}) \right\rangle \\ &\quad \times D_{L-1}[r + \omega_{\leqslant l}(\tilde{k}^{(n)}); \{W^{m_j + u_j, n_j}_{p_j - u_j, q_j}; k_j^{(m_j)}, y_j^{(u_j)}; \tilde{k}_j^{(n_j)}\}_{j=l+1}^{L}; \{f_j\}_{j=l+1}^{L-1}], \end{aligned} \tag{A.46}$$

and, for any $1 \leqslant l \leqslant L-1$

$$
\begin{aligned}
& D_L[r; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{f_l\}_{l=1}^{L-1}] \\
&\quad = \sum_{\substack{v_j=0, \\ j=1, ..., l}}^{q_j} \sum_{\substack{u_j=0, \\ j=l+1, ..., L}}^{p_j} \int \prod_{j=1}^{l} \left\{ d\tilde{y}_j^{(v_j)} \binom{q_j}{v_j} \right\} \prod_{j=l+1}^{L} \left\{ dy_j^{(u_j)} \binom{p_j}{u_j} \right\} \\
&\qquad \times D_l[r+\omega_{\geqslant l}(k^{(m)}); \{W^{m_j, n_j+v_j}_{p_j, q_j-v_j}; k_j^{(m_j)}; \tilde{k}_j^{(n_j)}, \tilde{y}_j^{(v_j)}\}_{j=1}^{l}; \{f_j\}_{j=1}^{l-1}] \\
&\qquad \times f_l[r+\mu_l+\omega_{\geqslant l+1}(y^{(U)})] \left\langle \prod_{j=1}^{l} a(\tilde{y}_j^{(v_j)}) \prod_{j=l+1}^{L} a^\dagger(y_j^{(u_j)}) \right\rangle \\
&\qquad \times D_{L-1}[r+\omega_{\leqslant l}(\tilde{k}^{(n)}); \{W^{m_j+u_j, n_j}_{p_j-u_j, q_j}; k_j^{(m_j)}, y_j^{(u_j)}; \tilde{k}_j^{(n_j)}\}_{j=l+1}^{L}; \\
&\qquad \times \{f_j\}_{j=l+1}^{L-1}],
\end{aligned} \tag{A.47}
$$

where we wrote $a := \sum_{l=1}^{L} a_l$ *for* $a = M, N, m, n, V, U$ *and*

$$
\omega_{\leqslant l}(x^{(a)}) := \sum_{j=1}^{l} \omega(x_j^{(a_j)}) = \sum_{j=1}^{l} \sum_{i=1}^{a_j} \omega(x_{j,i}), \tag{A.48}
$$

$$
\omega_{\geqslant l}(x^{(a)}) := \sum_{j=l}^{L} \omega(x_j^{(a_j)}) = \sum_{j=l}^{L} \sum_{i=1}^{a_j} \omega(x_{j,i}). \tag{A.49}
$$

Proof. First, we rewrite (A.44) in two different ways as

$$
\begin{aligned}
& D_L[r; \{W^{m_l, n_l}_{p_l, q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}; \{f_l\}_{l=1}^{L-1}] \\
&\quad = \langle \Phi_{l-1} | f_{l-1}[H_f + r + \mu_{l-1}] \\
&\qquad \times W^{m_l, n_l}_{p_l, q_l}[H_f + r + \lambda_l; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}] \, f_l[H_f + r + \mu_l] \, \Psi_{l+1} \rangle \\
&\quad = \langle \Phi_l | f_l[H_f + r + \mu_l] \, \Psi_{l+1} \rangle,
\end{aligned} \tag{A.50}
$$

with

$$
\begin{aligned}
\Phi_l := {} & W^{m_l, n_l}_{p_l, q_l}[H_f + r + \lambda_l; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}]^* \, f^*_{l-1}[H_f + r + \mu_{l-1}] \cdots \\
& \cdots \times f_1^*[H_f + r + \mu_1] \, W^{m_1, n_1}_{p_1, q_1}[H_f + r + \lambda_1; k_1^{(m_1)}; \tilde{k}_1^{(n_1)}]^* \, \Omega,
\end{aligned} \tag{A.51}
$$

and

$$
\begin{aligned}
\Psi_l := {} & W^{m_l, n_l}_{p_l, q_l}[H_f + r + \lambda_l; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}] \, f_l[H_f + r + \mu_l] \cdots \\
& \cdots \times f_{L-1}[H_f + r + \mu_{L-1}] \, W^{m_L, n_L}_{p_L, q_L}[H_f + r + \lambda_L; k_L^{(m_l)}; \tilde{k}_L^{(n_L)}] \, \Omega.
\end{aligned} \tag{A.52}
$$

Next, according to Theorem A.4, we can expand

$$
\begin{aligned}
&W^{m_l, n_l}_{p_l, q_l}[r+\lambda_l; k^{(m_l)}_l; \tilde{k}^{(n_l)}_l]\, f[r+\mu_l] \cdots \\
&\qquad \cdots \times f_{L-1}[r+\mu_{L-1}]\; W^{m_L, n_L}_{p_L, q_L}[r+\lambda_L; k^{(m_L)}_L; \tilde{k}^{(n_L)}_L] \\
&\qquad = \sum_{\substack{u_j=0 \\ j=l,\dots,L}}^{p_j} \sum_{\substack{v_j=0 \\ j=l,\dots,L}}^{q_j} \int \prod_{j=l}^{L} \left\{ dy^{(u_j)}_j\, d\tilde{y}^{(v_j)}_j \binom{p_j}{u_j}\binom{q_j}{v_j} \right\} \prod_{j=l}^{L} a^\dagger(y^{(u_j)}_j) \\
&\qquad\quad \times \big\langle W^{m_l+u_l, n_l+v_l}_{p_l-u_l, q_l-v_l}[H_f+r+\tilde{\lambda}_l; k^{(m_l)}_l, y^{(u_l)}_l; \tilde{k}^{(n_l)}_l, \tilde{y}^{(v_l)}_l] \\
&\qquad\quad \times f_l[H_f+r+\tilde{\mu}_l] \cdots f_{L-1}[H_f+r+\tilde{\mu}_{L-1}] \\
&\qquad\quad \times W^{m_L+u_L, n_L+v_L}_{p_L-u_L, q_L-v_L}[H_f+r+\tilde{\lambda}_L; k^{(m_L)}_L, y^{(u_L)}_L; \tilde{k}^{(n_L)}_L, \tilde{y}^{(v_L)}_L] \big\rangle \\
&\qquad\quad \times \prod_{j=l}^{L} a(\tilde{y}^{(v_j)}_j),
\end{aligned} \tag{A.53}
$$

where $\tilde{\mu}_l := \tilde{\lambda}_l + \omega(\tilde{k}^{(n_l)}_l) + \omega(\tilde{y}^{(v_l)}_l)$ and

$$
\begin{aligned}
\tilde{\lambda}_j &:= \sum_{i=1}^{j-1} \omega(\tilde{k}^{(n_i)}_i) + \sum_{i=l}^{j-1} \omega(\tilde{y}^{(v_i)}_i) + \sum_{i=j+1}^{L} \omega(k^{(m_i)}_i) + \sum_{i=j+1}^{L} \omega(y^{(u_i)}_i) \\
&= \sum_{i=l}^{j-1} [\omega(\tilde{k}^{(n_i)}_i) + \omega(\tilde{y}^{(v_i)}_i)] + \sum_{i=j+1}^{L} [\omega(k^{(m_i)}_i) + \omega(y^{(u_i)}_i)] + \omega_{\leqslant l-1}(\tilde{k}^{(n)}).
\end{aligned} \tag{A.54}
$$

The second identity in (A.54) enables us to rewrite the operator in (A.53) as

$$
\begin{aligned}
&\sum_{\substack{u_j=0 \\ j=l,\dots,L}}^{p_j} \sum_{\substack{v_j=0 \\ j=l,\dots,L}}^{q_j} \int \prod_{j=l}^{L} \left\{ dy^{(u_j)}_j\, d\tilde{y}^{(v_j)}_j \binom{p_j}{u_j}\binom{q_j}{v_j} \right\} \prod_{j=l}^{L} a^\dagger(y^{(u_j)}_j) \\
&\times D_{L-l+1}[r+\omega_{\leqslant l-1}(\tilde{k}^{(n)}); \{W^{m_j+u_j, n_j+v_j}_{p_j, q_j}; k^{(m_j)}_j, y^{(u_j)}_j; \tilde{k}^{(n_j)}_j, \tilde{y}^{(v_j)}_j\}_{j=l}^{L}; \{f_l\}_{l=1}^{L}] \\
&\times \prod_{j=l}^{L} a(\tilde{y}^{(v_j)}_j).
\end{aligned} \tag{A.55}
$$

When applying this operator on the right of (A.55) to the Fock vacuum Ω, only the terms with $v_1 = \cdots = v_L = 0$ are nonvanishing and we obtain

$$
\begin{aligned}
\Psi_l = &\sum_{\substack{u_j=0 \\ j=l,\dots,L}}^{p_j} \prod_{j=l}^{L} \binom{p_j}{u_j} \int dy^{(U)}\, a^\dagger(y^{(U)})\, \Omega \\
&\times D_{L-l+1}[r+\omega_{\leqslant l-1}(\tilde{k}^{(n)}); \{W^{m_j+u_j, n_j}_{p_j-u_j, q_j}; k^{(m_j)}_j, y^{(m_j)}_j; \tilde{k}^{(n_j)}_j\}_{j=l}^{L}; \{f_l\}_{l=1}^{L}].
\end{aligned} \tag{A.56}
$$

A similar representation holds for Φ_l. Inserting this into (A.50), we arrive at (A.46) and (A.47). ▮

APPENDIX B. BOUNDS ON THE INTERACTION

We recall from Equation (III.64) that the original Hamiltonian $H_g - z$, defined on $\mathscr{H}_{el} \otimes \mathscr{F}$ and shifted by $z \in \mathscr{U}^{(\mathrm{in})}_{(0)}$, is isospectral to $H_{(0)}[Z_{(0)}(z)] - Z_{(0)}(z)$, defined on $\mathscr{H}_{\mathrm{red}} = \mathrm{Ran}\, \chi[H_f < 1] = \chi[H_f < 1]\, \mathscr{F}$. Here, $Z_{(0)}(z) = e^{i\vartheta} \rho_0^{-1} z$ is a bijection from $\mathscr{U}^{(\mathrm{in})}_{(0)} \to D_{1/2} = \{z \colon |z| \leqslant 1/2\}$. In fact, we claimed that $z_0 \mapsto H_{(0)}[z_0]$ is an analytic family of bounded operators on $\mathscr{H}_{\mathrm{red}}$. In the present appendix, we justify this claim. More precisely, we demonstrate how the bounds of the form (III.7) on the interaction coefficients $w^{(0)}_{M,N}$ turn into bounds for $W^{(0)}_{M,N}$, for $M+N \geqslant 1$.

Being slightly more general, we fix $M+N \geqslant 1$ and consider a measurable function

$$w_{M,N} \colon [0, 1) \times \mathbb{R}^{Md} \times \mathbb{R}^{Nd} \to \mathbb{C}. \tag{B.1}$$

Under suitable assumptions on $w_{M,N}$ we show below that

$$W_{M,N} := \int dk^{(M)}\, d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\, w_{M,N}[H_f; k^{(M)}; \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)}) \tag{B.2}$$

defines a bounded operator on $\mathscr{H}_{\mathrm{red}}$, and we give the following bound, denoting $\chi_1 \equiv \chi_1[H_f] := \chi[H_f < 1]$.

THEOREM B.1. *Let $M+N \geqslant 1$, and assume that*

$$|w_{M,N}[H_f; k^{(M)}; \tilde{k}^{(N)}]| \leqslant \varepsilon \prod_{j=1}^{M} \omega(k_j)^\alpha \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha, \tag{B.3}$$

where $\alpha = \frac{1}{2}(1+\mu) - \frac{1}{2}(d/\gamma - 1)$ and $\mu > 0$. Then, for any $0 < \rho, \tilde{\rho} \leqslant 1$,

$$\begin{aligned} &\|(H_f + \rho)^{-1/2} \chi_1[H_f]\, W_{M,N} \chi_1[H_f] (H_f + \tilde{\rho})^{-1/2}\| \\ &\quad \leqslant \frac{\varepsilon \rho^{-\delta_{M,0}/2} \tilde{\rho}^{-\delta_{N,0}/2} (C_d \Gamma[1+\mu])^{(M+N)/2}}{\Gamma[(1+\mu)\, M+1]^{1/2}\, \Gamma[(1+\mu)\, M+1]^{1/2}}, \end{aligned} \tag{B.4}$$

where $\Gamma[x]$ is the Gamma function (see (C.1)) for all $x > 0$.

Before proceeding to the proof of Theorem B.1, we remark that $\Gamma[(1+\mu)\, M+1] \geqslant \Gamma[M+1] = M!$ and the denominator in (B.4) can weigh out factorials that are generated from Wick contractions as in (III.92).

Proof. We pick $\phi = \chi_1[H_f]\,\phi$, $\psi = \chi_1[H_f]\,\psi \in \mathscr{H}_{\text{red}}$ and consider

$$A^2(\phi, \psi) := |\langle \phi \,|\, \chi_1[H_f]\, W_{M,N} \chi_1[H_f]\, \psi \rangle|^2$$
$$= \left| \int dk^{(M)}\, d\tilde{k}^{(N)} \, \langle a(k^{(M)})\, \phi \,|\, w_{M,N}[H_f; k^{(M)}; \tilde{k}^{(N)}]\, a(\tilde{k}^{(N)})\, \psi \rangle \right|. \tag{B.5}$$

Remembering $a(k^{(M)}) \equiv \prod_{j=1}^{M} a(k_j)$ and $\omega(k^{(M)}) \equiv \sum_{j=1}^{M} \omega(k_j)$, the pull-through formula (A.3) implies that

$$a(k^{(M)})\, \chi_1[H_f] = \chi[H_f + \omega(k^{(M)}) < 1]\, a(k^{(M)})\, \chi_1[H_f]$$
$$= \chi[\omega(k^{(M)}) < 1]\, a(k^{(M)})\, \chi_1[H_f], \tag{B.6}$$

which, together with Schwarz' inequality, yields

$$A^2(\phi, \psi) \leqslant B^{(M)}(\phi) \cdot B^{(N)}(\psi) \int \chi_1[\omega(k^{(M)})]$$
$$\times \chi_1[\omega(\tilde{k}^{(N)})] \left\{ \sup_{0<r<1} |w_{M,N}[r; k^{(M)}; \tilde{k}^{(N)}]| \right\} \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)} \prod_{j=1}^{N} \frac{d\tilde{k}_j}{\tilde{\omega}(k_j)}, \tag{B.7}$$

where $B^{(0)}(\phi) := \|\phi\|^2$ and

$$B^{(M)}(\phi) := \int \|a(k^{(M)})\, \chi_1 \phi\|^2 \prod_{j=1}^{M} \omega(k_j)\, dk_j. \tag{B.8}$$

Another application of the pull-through formula then gives

$$B^{(M)}(\phi) = \int \left\langle \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \,\middle|\, H_f \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \right\rangle \prod_{j=1}^{M} \omega(k_j)\, dk_j$$
$$\leqslant \int \left\langle \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \,\middle|\, [H_f + \omega(k^{(M-1)})] \prod_{j=1}^{M-1} a(k_j)\, \chi_1 \phi \right\rangle$$
$$\times \prod_{j=1}^{M} \omega(k_j)\, dk_j$$
$$= \int \|a(k^{(M-1)})\, \chi_1 H_f^{1/2} \phi\|^2 \prod_{j=1}^{M-1} \omega(k_j)\, dk_j$$
$$= B^{(M-1)}(H_f^{1/2} \chi_1 \phi)$$
$$\leqslant B^{(M-2)}(H_f \chi_1 \phi) \leqslant \cdots \leqslant B^{(0)}(H_f^{M/2} \chi[H_f]\, \phi)$$
$$= \|H_f^{M/2} \chi_1[H_f]\, \phi\|^2. \tag{B.9}$$

Additionally note that

$$\|\chi_1[H_f]\, H_f^{M/2}\phi\| \leqslant \|\chi_1[H_f]\, H_f^{1/2}\phi\| \leqslant \|\chi_1[H_f](H_f+\rho)^{1/2}\,\phi\|, \tag{B.10}$$

for $M \geqslant 1$ and hence

$$B^{(M)}(\phi) \leqslant \|\chi_1[H_f](H_f+\rho)^{\min\{M,\,1\}/2}\,\phi\|^2. \tag{B.11}$$

Inserting the assumption on $w_{M,N}$, we estimate the integral on the right side of (B.7) by $\varepsilon^2 \cdot I_M \cdot I_n$, where

$$\begin{aligned} I_M &:= \int \chi_1[\omega(k^{(M)})] \prod_{j=1}^{M} \{\omega(k_j)^{1\alpha-1}\, dk_j\} \\ &= C_d^M \left(\int \chi[\omega_1 + \cdots + \omega_M] \prod_{j=1}^{M} \omega_j^{\mu}\, d\omega_j \right) \\ &= \frac{C_d^M \Gamma[1+\mu]^M}{\Gamma[(1+\mu)\, M+1]}. \end{aligned} \tag{B.12}$$

Here, $\Gamma[x]$ denotes the Gamma function for $x>0$ and the last equality is derived in Lemma C.2. Putting together (B.7), (B.11), and (B.12), we obtain

$$\begin{aligned} &\|(H_f+\rho)^{-\min\{M,\,1\}/2}\, \chi_1[H_f]\, W_{M,N} \chi_1[H_f](H_f+\tilde{\rho})^{-\min\{N,\,1\}/2}\| \\ &\quad \leqslant \frac{\varepsilon (C_d \Gamma[1+\mu])^{(M+N)/2}}{\Gamma[(1+\mu)\, M+1]^{1/2}\, \Gamma[(1+\mu)\, M+1]^{1/2}}. \end{aligned} \tag{B.13}$$

Now, the claim follows from

$$\left\| \frac{(H_f+\rho)^{\min\{M,\,1\}/2}}{(H_f+\rho)^{1/2}} \chi_1[H_f] \right\| = \rho^{-\delta_{M,0}/2}. \quad \blacksquare \tag{B.14}$$

Next, we introduce

$$W := \sum_{M+N \geqslant 1} W_{M,N}, \tag{B.15}$$

and assert

THEOREM B.2. *Assume that*

$$|w_{M,N}[H_f;\, k^{(M)};\, \tilde{k}^{(0)}]| \leqslant \varepsilon \xi^{M+N} \prod_{j=1}^{M} \omega(k_j)^{\alpha} \prod_{j=1}^{N} \omega(\tilde{k}_j)^{\alpha}, \tag{B.16}$$

where $\alpha = \frac{1}{2}(1+\mu) = \frac{1}{2}(d/\gamma - 1)$ *and* $\mu \geqslant 0$. *Then, for any* $0 < \rho \leqslant 1$,

$$\|(H_f + \rho)^{-1/2} \chi_1 W \chi_1 (H_f + \rho)^{-1/2}\| \leqslant \frac{2\varepsilon \cdot \min\{1, C_d \xi\}}{\rho^{1/2}} \cdot \exp[2\max\{1, C_d \xi\}^2]. \tag{B.17}$$

Proof. We use $\mu \geqslant 0$ and Lemma C.2 backwards to estimate

$$\frac{\Gamma[1+\mu]^M}{\Gamma[(1+\mu)\, M+1]} \leqslant \frac{\Gamma[1]^M}{\Gamma[M+1]} = (M!)^{-1} \tag{B.18}$$

Thus, Theorem B.1 yields

$$\begin{aligned} &\|(H_f + \rho)^{-1/2} \chi_1[H_f]\, W \chi_1[H_f](H_f + \rho)^{-1/2}\| \\ &\leqslant \frac{\varepsilon}{\rho^{1/2}} \cdot \sum_{M+N \geqslant 1} (M!\, N!)^{-1/2} \, (C_d^{1/2} \xi)^{M+N} \\ &\leqslant \frac{\varepsilon \min\{1, C_d^{1/2} \xi\}}{\rho^{1/2}} \cdot \sum_{M+N \geqslant 1} (M!\, N!)^{-1/2} \, (\max\{1, C_d^{1/2} \xi\})^{M+N}. \end{aligned} \tag{B.19}$$

By Schwarz' inequality, we have

$$\sum_{M=0}^{\infty} \frac{x^M}{(M!)^{1/2}} \leqslant \sqrt{2} \left(\sum_{M=0}^{\infty} \frac{(2x^2)^M}{M!} \right)^{1/2} = \sqrt{2} \cdot \exp[x^2], \tag{B.20}$$

and, substituting for $x := \max\{1, C_d \xi\}$, we arrive at the assertion. ∎

APPENDIX C. INTEGRALS OVER SIMPLICES OF LARGE DIMENSION

In the preceding paragraph we expressed integrals over M-dimensional simplices in terms of the Gamma function, given by

$$\Gamma[x] := \int_0^\infty e^{-t} t^{x-1} \, dt, \tag{C.1}$$

for $x > 0$. The purpose of this paragraph is to state a few elementary or well-known facts about the Gamma function. For example, it is well-known that

$$x \cdot \Gamma[x] := \Gamma[x+1] \qquad \text{and} \qquad \Gamma[1] = 1, \tag{C.2}$$

which implies $\Gamma[n+1]=n!$ for $n\in\mathbb{N}_0$, in particular. The following estimate is, perhaps, less well-known but quite useful for us.

LEMMA C.1. *For any* $x, y\geqslant 0$,

$$\Gamma[x+1]\cdot\Gamma[y+1]\leqslant\Gamma[x+y+1]. \tag{C.3}$$

Proof. First, we introduce the Beta function (see 8.384 in [4]), given by

$$B[x, y]:=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma[x]\,\Gamma[y]}{\Gamma[x+y]}, \tag{C.4}$$

for $x, y>0$. Thus,

$$\begin{aligned}\frac{\Gamma[x+1]\,\Gamma[y+1]}{\Gamma[x+y+1]}&=(x+y+1)\,B[x+1, y+1]\\ &=(x+y+1)\int_0^1 t^x(1-t)^y\,dt.\end{aligned} \tag{C.5}$$

Now, we define $f_r(s):=\int_0^1 t^{r-s}(1-t)^s\,dt$, noticing that $\int_0^1 t^x(1-t)^y\,dt=f_{x+y}(y)$. One easily checks that $f_r''(s)>0$ on $(0, r)$ and hence $f_r(s)\leqslant\max\{f_r(0), f_r(r)\}=(r+1)^{-1}$. In particular, $(x+y+1)\cdot f_{x+y}(y)\leqslant 1$. ∎

The importance of the Gamma function in our context comes from the inverse factorials it yields for integrals over simplices of large dimension.

LEMMA C.2. *For any* $M\in\mathbb{N}_0$, $\eta>0$ *and* $\rho\geqslant 0$,

$$\int\chi[\omega_1+\cdots+\omega_M<\rho]\prod_{j=1}^{M}\frac{d\omega_j}{\omega_j^{1-\eta}}=\frac{\rho^{\eta M}\Gamma[\eta]^M}{\Gamma[\eta M+1]} \tag{C.6}$$

Proof. By scaling, the left side of (C.6) equals $X_M\cdot\rho^{\eta M}$ where $X_0:=1$ and

$$X_M:=\int\chi[\omega_1+\cdots+\omega_M<1]\prod_{j=1}^{M}\frac{d\omega_j}{\omega_j^{1-\eta}}, \tag{C.7}$$

for $M\geqslant 1$. Also for $M\geqslant 1$, we observe the following recursion relation

$$\begin{aligned}X_M&=\int_0^1\left\{\int\chi[\omega_1+\cdots+\omega_{M-1}<1-\omega_M]\prod_{j=1}^{M-1}\frac{d\omega_j}{\omega_j^{1-\eta}}\right\}\frac{d\omega_M}{\omega_M^{1-\eta}}\\ &=X_{M-1}\cdot\int_0^1(1-\omega)^{\eta(M-1)}\,\omega^{\eta-1}\,d\omega\\ &=X_{M-1}\cdot B[\eta(M-1)+1, \eta].\end{aligned} \tag{C.8}$$

Using (C.4), we thus obtain

$$X_M = \prod_{j=1}^{M} B[\eta(j-1)+1, \eta]$$
$$= \prod_{j=1}^{M} \frac{\Gamma[\eta(j-1)+1]\,\Gamma[\eta]}{\Gamma[\eta j+1]} = \frac{\Gamma[\eta]^M}{\Gamma[\eta M+1]}. \quad \blacksquare \tag{C.9}$$

The following lemma shows that even in the case of a general integral, the restriction to $\{\omega(k_1)+\cdots+\omega(k_M)<\rho\}$ can be turned into a bound with inverse factorials.

Lemma C.3. *Let* $\eta>0$ *and assume* $F\colon \mathbb{R}^{dM}\to\mathbb{R}_+$ *to be such that*

$$A := \int F(k_1, ..., k_M) \prod_{j=1}^{M} \frac{dk_j}{\omega(k_j)^\eta} < \infty. \tag{C.10}$$

Then, with $\omega(k^{(M)}) = \omega(k_1)+\cdots+\omega(k_M)$,

$$\int \chi[\omega(k^{(M)})<\rho]\, F(k_1, ..., k_M) \prod_{j=1}^{M} dk_j \leqslant \frac{\rho^{\eta M} A}{M^{\eta M}}. \tag{C.11}$$

Proof. We use an induction in $M\geqslant 1$. For $M=1$, (C.11) is trivial. Assume that (C.11) holds for $M-1\geqslant 1$ and define $f\colon \mathbb{R}^d\to\mathbb{R}_+$ by

$$f(k) := \int F(k_1, ..., k_{M-1}, k) \prod_{j=1}^{M-1} \frac{dk_j}{\omega(k_j)^\eta}. \tag{C.12}$$

Observe that $\int f(k)\,\omega(k)^{-\eta}\,dk = A$. By induction and by the fundamental theorem of calculus, we have

$$\int \chi[\omega(k_{(M)})<\rho]\, F(k_1, ..., k_M) \prod_{j=1}^{M} dk_j$$
$$= \int \chi[\omega(k_1)+\cdots+\omega(k_{M-1})<\rho-\omega(k_M)]\, F(k_1, ..., k_M) \prod_{j=1}^{M} dk_j$$
$$\leqslant (M-1)^{\eta(M-1)} \int (\rho-\omega(k))^{\eta(M-1)}\, f(k)\, dk$$
$$= (M-1)^{\eta(M-1)} \int_0^\rho (\rho-\omega)^{\eta(M-1)}\, \omega^\eta \left(\frac{d}{d\omega}\left\{\int_{\omega(k)<\omega} \frac{f(k)\,dk}{\omega(k)^\eta}\right\}\right) d\omega. \tag{C.13}$$

Now, we integrate by parts. We set $g(\omega) := (\rho - \omega)^{\eta(M-1)} \omega^{\beta}$ such that $g(\rho) = g(0) = 0$ and

$$-\frac{dg}{d\omega} = \beta(\rho-\omega)^{\eta(M-1)-1} \omega^{\eta-1}(M\omega - \rho) \geqslant 0 \tag{C.14}$$

if and only if $M^{-1}\rho \leqslant \omega \leqslant \rho$. Thus, the right side of (C.13) equals

$$\begin{aligned}(M-1)^{\eta(M-1)} \int_0^{\rho} \left(-\frac{dg}{d\omega}\right)\left(\int_{\omega(k)<\omega} \frac{f(k)\,dk}{\omega(k)^{\eta}}\right) d\omega \\ \leqslant (M-1)^{\eta(M-1)} A \cdot \int_{\rho/M}^{\rho} \left(-\frac{dg}{d\omega}\right) d\omega \\ = (M-1)^{\eta(M-1)} A \cdot g(\rho/M) = \frac{\rho^{\eta M} A}{M^{\eta M}}. \quad \blacksquare \end{aligned} \tag{C.15}$$

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