A GEOMETRIC REALIZATION OF THE CONGRUENCE SUBGROUP KERNEL

JOHN SCHERK

In [1] the fundamental group of the reductive Borel-Serre compactification of a locally symmetric space was computed. The authors noted in an appendix (§7), that there is an intriguing similarity between the results of this paper and computations of the congruence subgroup kernel in [5]. However, no direct connection was demonstrated. The purpose of this note is to explain such a connection. A covering space is constructed from inverse limits of these compactifications. The congruence subgroup kernel then appears as the group of deck transformations of this covering. Notations and definitions are taken from [1].

1. Let k be a number field and let S be a finite set of places of k, which contains the infinite places S_∞. Let $\mathbf{G}$ be a connected, absolutely almost simple, and simply connected algebraic group defined over k.

Fix a faithful representation

$$\rho\colon \mathbf{G} \longrightarrow \mathbf{GL}_N$$

defined over k. Let $\mathcal{O}$ the ring of *S-integers*, and set

$$\mathbf{G}(\mathcal{O}) = \rho^{-1}(\mathbf{GL}_N(\mathcal{O})) \subset \mathbf{G}(k).$$

We recall the definition of the *congruence subgroup kernel* as explained in [2]. Let $\mathfrak{M}_a$, respectively $\mathfrak{M}_c$, be the set of S-arithmetic subgroups, respectively S-congruence subgroups, of $\mathbf{G}$. Taking each of these sets to be a fundamental system of neighbourhoods of 1, we define two topologies, $\mathcal{T}_a$, respectively $\mathcal{T}_c$, on $\mathbf{G}(k)$. Let $\widehat{G}(a)$, respectively $\widehat{G}(c)$, denote the completions of $\mathbf{G}(k)$ in these topologies. The corresponding completions of $\mathbf{G}(\mathcal{O})$ are denoted $\widehat{G}(\mathcal{O}, a)$, respectively $\widehat{G}(\mathcal{O}, c)$.

Denote by $\mathfrak{N}_a$, respectively $\mathfrak{N}_c$, the set of normal subgroups of finite index, respectively principal S-congruence subgroups, of $\mathbf{G}(\mathcal{O})$. These define the topologies $\mathcal{T}_a$, respectively $\mathcal{T}_c$, as well. Then one can regard $\widehat{G}(\mathcal{O}, a)$ and $\widehat{G}(\mathcal{O}, c)$ as inverse limits:

$$\widehat{G}(\mathcal{O}, a) = \varprojlim_{\Gamma \in \mathfrak{N}_a} \mathbf{G}(\mathcal{O})/\Gamma$$

and

$$\widehat{G}(\mathcal{O}, c) = \varprojlim_{\Gamma \in \mathfrak{N}_c} \mathbf{G}(\mathcal{O})/\Gamma\ .$$

Since every S-congruence subgroup is also S-arithmetic, we have homomorphisms $\widehat{G}(a) \to \widehat{G}(c)$ and $\widehat{G}(\mathcal{O}, a) \to \widehat{G}(\mathcal{O}, c)$. They have a common kernel, called the congruence subgroup kernel $C(S, \mathbf{G})$. In particular, we have the exact sequence

$$1 \to C(S, G) \to \widehat{G}(\mathcal{O}, a) \to \widehat{G}(\mathcal{O}, c) \to 1\ . \tag{1}$$

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2. Set $\mathbf{H} = \mathrm{Res}_{k/\mathbb{Q}}\,\mathbf{G}$, and let X_∞ be the symmetric space associated to $\mathbf{H}$. For each $v \in S \setminus S_\infty$, let X_v be the Bruhat-Tits building of $\mathbf{G}(k_v)$. Set $X = X_\infty \times \prod_{v\in S\setminus S_\infty} X_v$, and define the reductive Borel-Serre bordification $\overline{X}^{RBS}$ of X as in [1, 2.4]. Set

$$\overline{X}^{RBS}_a = \varprojlim_{\Gamma\in\mathfrak{M}_a} \Gamma\backslash\overline{X}^{RBS} = \varprojlim_{\Gamma\in\mathfrak{N}_a} \Gamma\backslash\overline{X}^{RBS}$$

and

$$\overline{X}^{RBS}_c = \varprojlim_{\Gamma\in\mathfrak{M}_c} \Gamma\backslash\overline{X}^{RBS} = \varprojlim_{\Gamma\in\mathfrak{N}_c} \Gamma\backslash\overline{X}^{RBS}$$

They are both compact Hausdorff spaces. Let p denote the natural map

$$\overline{X}^{RBS}_a \xrightarrow{p} \overline{X}^{RBS}_c \ .$$

Proposition.

$$\pi_1(\overline{X}^{RBS}_a) \cong \varprojlim_{\substack{\Gamma\in\mathfrak{M}_a\\ \Gamma\ neat}} \Gamma/E\Gamma\ , \quad \pi_1(\overline{X}^{RBS}_c) \cong \varprojlim_{\substack{\Gamma\in\mathfrak{M}_c\\ \Gamma\ neat}} \Gamma/E\Gamma$$

Proof. According to [1, Corollary 5.3], if $\Gamma \in \mathfrak{M}_a$ and Γ is neat, then

$$\pi_1(\Gamma\backslash\overline{X}^{RBS}) \cong \Gamma/E\Gamma\ .$$

Since

$$\pi_1(\overline{X}^{RBS}_a) \cong \varprojlim_{\substack{\Gamma\in\mathfrak{M}_a\\ \Gamma \text{ neat}}} \pi_1(\Gamma\backslash\overline{X}^{RBS})\ ,$$

the result follows for $\overline{X}^{RBS}_a$. The proof for $\overline{X}^{RBS}_c$ is similar. □

For $\Gamma \in \mathfrak{N}_a$ there is a well-defined action on the left of $\mathbf{G}(\mathcal{O})$, and therefore of $\mathbf{G}(\mathcal{O})/\Gamma$, on $\Gamma\backslash\overline{X}^{RBS}$. This determines an action of $\widehat{G}(\mathcal{O},a)$ on $\overline{X}^{RBS}_a$. Similarly, $\widehat{G}(\mathcal{O},c)$ acts on $\overline{X}^{RBS}_c$. Then p is equivariant. It follows from the descriptions of $\widehat{G}(\mathcal{O},a)$ and $\widehat{G}(\mathcal{O},c)$ as inverse limits and from the definitions of $\overline{X}^{RBS}_a$ and $\overline{X}^{RBS}_c$, that $C(S,G)$ acts transitively on each fibre of p.

3. In [5, Theorem A, Corollary 1] and [3, 2.4.6, I], it is shown that if k-rank $\mathbf{G} \geq 1$ and S-rank $\mathbf{G} \geq 2$, then for any $\Gamma \in \mathfrak{M}_a$, $E\Gamma \in \mathfrak{M}_a$. This implies that

$$\widehat{G}(\mathcal{O},a) = \varprojlim_{\Gamma\in\mathfrak{N}_c} \mathbf{G}(\mathcal{O})/E\Gamma\ .$$

It follows then from the exact sequence (1) that

$$C(S,G) = \varprojlim_{\Gamma\in\mathfrak{N}_c} \Gamma/E\Gamma\ . \tag{2}$$

Furthermore,

$$\overline{X}^{RBS}_a = \varprojlim_{\Gamma\in\mathfrak{M}_c} E\Gamma\backslash\overline{X}^{RBS}\ . \tag{3}$$

Under these rank assumptions, we obtain a simple geometric realization of $C(S,G)$.

Theorem. *Assume that k-rank $\mathbf{G} \geq 1$ and S-rank $\mathbf{G} \geq 2$. Then $\overline{X}_a^{RBS}$ is a simply connected covering of $\overline{X}_c^{RBS}$, and $C(S,G)$ is the group of deck transformations.*

Proof. Note that if Γ is neat, then by [1, Corollary 5.2], $\Gamma/E\Gamma$ acts freely on $E\Gamma\backslash\overline{X}^{RBS}$. Then (2) and (3) imply that $C(S,G)$ acts freely on $\overline{X}_a^{RBS}$. Since $C(S,G)$ acts transitively on the fibres of p, it follows that $p : \overline{X}_a^{RBS} \to \overline{X}_c^{RBS}$ is a covering space, with $C(S,G)$ the group of deck transformations. Since the groups $\{E\Gamma \in \mathfrak{M}_a,\ \Gamma \text{ neat}\}$ are cofinal under the rank assumption, the Proposition implies that $\overline{X}_a^{RBS}$ is simply connected. □

Corollary.

$$C(S,\mathbf{G}) \cong \pi_1(\overline{X}_c^{RBS}) \cong \varprojlim_{\Gamma\in\mathfrak{M}_c} \pi_1(\Gamma\backslash\overline{X}^{RBS}) \ .$$

Compare [1, §7 (12)]

References

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Department of Computer and Mathematical Sciences, University of Toronto Scarborough, 1265 Military Trail, Toronto, Ontario, M1C 1A4, CANADA

E-mail address: `scherk@math.toronto.edu`