UNIVERSITY OF TORONTO
Faculty of Arts & Science
MAT344H1 Intro to Combinatorics
Homework Set #10

1. Construct a random graph with n vertices $v_1, \ldots, v_n$ as follows: For each pair $v_i \neq v_j$ flip a coin to decide if the edge $e = (v_i, v_j)$ is in the graph. Write down the probability space and calculate the probability that the resulting graph has m edges.

2. Construct a random graph with n vertices $v_1, \ldots, v_n$ by choosing uniformly among all graphs with these vertices that have exactly m edges. What is the probability for each particular graph in this model? This is called the *Erdos-Renyi Random Graph*. For $n_1 + n_2 = n$ find the probability that the graph is made up of two disconnected parts, one which is a complete graph on n_1 vertices and one which is a complete graph on n_2 vertices.

3. In the Erdos-Renyi random graph above, let W be a subset of the vertex set of size k. It is called a *clique* if for every two vertices v_i, v_j in W the edge $e = (v_i, v_j)$ is present in the graph. Compute the probability that W is a clique. For each such subset W, let $X_W = 1$ if W is a clique and $X_W = 0$ if W is not a clique. Are the random variables X_W and $X_{W'}$ independent if W and W' are two such subsets? Show that

$$N_k = \sum_{\substack{W \subset \{v_1, \ldots, v_n\} \\ |W| = k}} X_W$$

is the number of cliques of size k in the graph. Compute $E[N_k]$.

4. Suppose that in Toronto the rate of Covid is 0.1%. A certain Covid test has a probability 1% to give a negative result if you genuinely have Covid (false negative) and a probability 5% to give a positive result if you genuinely don't have Covid (false positive). You take a test and it gives a positive result. What is the conditional probability that you have Covid?

5. Find

$$\sum_{m=0}^{n} m^2 \binom{n}{m} p^m (1-p)^{n-m}$$

6. Roll a single die until you get two sixes in a row. Let N be the number of rolls it took. For each $m = 2, 3, \ldots$ find the probability that $N = m$. Compute the expectation of N.

7. Consider the following game and betting strategy.

   The game is you bet $\$k$, and a fair coin is flipped. If it comes up heads, you win $2k$ dollars, if it comes up tails, you lose your k dollars (i.e. if you will you gain k, if you lose you gain $-k$). The expected gain is $k(\frac{1}{2}) + (-k)(\frac{1}{2}) = 0$, so it is a fair game.

   Here is your strategy: You bet \$1 on the first round, \$2 on the second round, \$4 on the third round, ..., $\$2^{n-1}$ on the nth round. You keep going until you win, then you walk away. If you win on the first round you walk away with \$1, if you win on the second round you walk away with $-\$1 + \$2 = \$1$. Show that if you win on the nth round you walk away with \$1.

   What is wrong with this? It appears you always get \$1. So if M your net gain, then $E[M] = 1$. But the game was supposed to be fair! It sounds great, all we have to do is find a game sort of like this (they exist) and start playing!

   Of course, we need to have quite a lot of money going in, or it is not going to work: Obviously if we start with \$1, we either go away with \$2 or \$0 dollars and it didn't work the way we hoped. Suppose we start with $\$N$ and N is huge, like a billion dollars (we told Elon Musk our trick and he lent us the money). Of course, there is some tiny probability we just keep not winning and after a certain amount of time we have to leave the game because we just ran out of money. How small is this probability? Calculate the expected net gain with this new constraint (reality).