MAT344 HW 2 Solutions

Solution to Ex. 1: Let $n \geq 2$. There are three types of valid record identifiers (VRIs) of length n. A VRI of length n of the first type begins with any upper-case letter other than D and is followed by any valid record identifier of length $n-1$. Thus, there are $25 \cdot r(n-1)$ VRIs of length n of the first type. A VRI of length n of the second type begins with $1C, 2K$, or $7J$ and is followed by any valid record identifier of length $n-2$. Thus, there are $3 \cdot r(n-2)$ VRIs of length n of the second type. A VRI of length n of the third type begins with D and is followed by any string of $n-1$ decimal digits. Thus there are 10^{n-1} VRIs of length n of the third type. In total, we get

$$r(n) = 25 \cdot r(n-1) + 3 \cdot r(n-2) + 10^{n-1}.$$

Since $r(0) = 1$ and $r(1) = 26$, we have

$$\begin{aligned}
r(2) &= 25 \cdot r(1) + 3 \cdot r(0) + 10^1 = 25 \cdot 26 + 3 \cdot 1 + 10 = 663 \\
r(3) &= 25 \cdot r(2) + 3 \cdot r(1) + 10^2 = 25 \cdot 663 + 3 \cdot 26 + 100 = 16\,753 \\
r(4) &= 25 \cdot r(3) + 3 \cdot r(2) + 10^3 = 25 \cdot 16\,753 + 3 \cdot 663 + 1\,000 = 421\,814 \\
r(5) &= 25 \cdot r(4) + 3 \cdot r(3) + 10^4 = 25 \cdot 421\,814 + 3 \cdot 16\,753 + 10\,000 = 10\,605\,609
\end{aligned}$$

Solution to Ex. 3: For each non-negative integer n, let $S(n)$ denote the set of all ternary strings of length n that do not contain 102 as a substring. Note that $g(n) = |S(n)|$. All ternary strings of length at most 2 do not contain 102 as a substring. Therefore $g(0) = 1$, $g(1) = 3$, and $g(2) = 3^2 = 9$.

Let $n \geq 3$. For $i = 0, 1, 2$, define $S_i(n) = \{x_1 \dots x_n \in S(n) : x_n = i\}$. Then $S(n)$ is the disjoint union of $S_0(n), S_1(n)$, and $S_2(n)$, so $g(n) = |S(n)| = |S_0(n)| + |S_1(n)| + |S_2(n)|$.

Let $i = 0, 1$. If $x_1 \dots x_{n-1}i$ is a ternary string of length n that ends in i, then $x_1 \dots x_{n-1}i$ does not contain 102 as a substring if and only if $x_1 \dots x_{n-1}$ does not contain 102 as a substring. Therefore $S_i(n) = S(n-1) \times \{i\}$, and consequently $|S_i(n)| = |S(n-1)| = g(n-1)$.

Now we will show that $S_2(n) = S(n-3) \times (\{0,1,2\}^2 \setminus \{10\}) \times \{2\}$, and consequently $|S_2(n)| = |S(n-3)| \cdot (3^2 - 1) = 8g(n-3)$. In words, we will show that $S_2(n)$ is the set of ternary strings $x_1 \dots x_{n-1}2$ such that $x_1 \dots x_{n-3}$ does not contain 102 as a substring and $x_{n-2}x_{n-2} \neq 10$. To show this, consider a ternary string $x_1 \dots x_{n-1}2$ of length n that ends in 2. The string $x_1 \dots x_{n-1}2$ does not contain 102 as a substring if and only if $x_1 \dots x_{n-1}$ does not contain 102 as a substring and $x_{n-2}x_{n-1} \neq 10$. Now, when $x_{n-2}x_{n-1} \neq 10$, the string $x_1 \dots x_{n-1}$ does not contain 102 as a substring if and only if $x_1 \dots x_{n-3}$ does not contain 102 as a substring. Therefore $x_1 \dots x_{n-1}2$ does not contain 102 as a substring if and only if $x_1 \dots x_{n-3}$ does not contain 102 as a substring and $x_{n-2}x_{n-2} \neq 10$, as required.

Putting everything together, we have $g(n) = 2g(n-1) + 8g(n-3)$.

Solution to Ex. 5: Recall that S is the set of quaternary strings (strings on the alphabet $\{0, 1, 2, 3\}$) that do not contain 12 or 20 as a substring and we wish to find a recursion for the number $h(n)$ of strings in S of length $n \geq 0$. All quaternary strings of length at most 1 are in S, so $h(0) = 1$ and $h(1) = 4$. Let $n \geq 2$. Consider a quaternary string $x_1 \dots x_n$ of length n.

Suppose that $x_n = 1$ or $x_n = 3$. Then $x_1 \dots x_n$ is in S if and only if $x_1 \dots x_{n-1}$ is in S. Therefore the number of strings in S of length n that end in 1 or 3 is $2h(n-1)$.

Suppose that $x_n = 0$. Then $x_1 \dots x_n$ is in S if and only if $x_1 \dots x_{n-1}$ is in S and $x_{n-1} = 0, 1, 3$, and this happens if and only if $x_1 \dots x_{n-2}$ is in S and $x_{n-1} = 0, 1, 3$. Therefore the number of strings in S of length n that end in 0 is $3h(n-2)$.

Suppose that $x_n = 2$. Then $x_1 \dots x_n$ is in S if and only if $x_1 \dots x_{n-1}$ is in S and $x_{n-1} = 0, 2, 3$, and this happens if and only if $x_1 \dots, x_{n-2}$ is in S and $x_{n-1} = 0, 2, 3$. Therefore the number of strings in S of length n that end in 2 is $3h(n-2)$.

In total, the number of strings in S of length n is thus $h(n) = 2h(n-1) + 6h(n-2)$.

Solution to Ex. 7: We apply the Euclid's algorithm:

$$\begin{aligned}827 &= 3 \cdot 249 + 80\\ 249 &= 3 \cdot 80 + 9\\ 80 &= 8 \cdot 9 + 8\\ 9 &= 1 \cdot 8 + 1\\ 8 &= 8 \cdot 1 + 0.\end{aligned}$$

Therefore $gcd(827, 249) = 1$ and

$$\begin{aligned}1 &= 9 - 1 \cdot 8\\ &= (249 - 3 \cdot 80) - 1 \cdot (80 - 8 \cdot 9)\\ &= 249 - 4 \cdot 80 + 8 \cdot 9\\ &= 249 - 4 \cdot (827 - 3 \cdot 249) + 8 \cdot (249 - 3 \cdot 80)\\ &= -4 \cdot 827 + 21 \cdot 249 - 24 \cdot 80\\ &= -4 \cdot 827 + 21 \cdot 249 - 24 \cdot (827 - 3 \cdot 249)\\ &= -28 \cdot 827 + 93 \cdot 249.\end{aligned}$$

Thus, if we take $a = 6 \cdot (-28) = -168$ and $b = 6 \cdot 93 = 558$, then $827a + 249b = 6$.

Solution to Ex. 9.a: First, we prove the identity by induction. If $n = 1$, then

$$\frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1^2 = \sum_{i=1}^{n} i^2,$$

so the identity holds. Let $n \geq 1$ and assume that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Then

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2\\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2\\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}\\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}\\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6}\\ &= \frac{(n+1)(n+2)(2n+3)}{6}\\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.\end{aligned}$$

Therefore, by the principle of induction the identity holds for all positive integers n.

Now we give a combinatorial proof of the identity. Let n be a positive integer. We will prove that

$$3\sum_{i=1}^{n} i^2 = \frac{n(n+1)}{2} \cdot (2n+1)$$

by counting the number of elements in a set in two different ways. The proof will make clear why the triangular number $\frac{n(n+1)}{2}$ appears. Afterwards we will give a second way of concluding the combinatorial proof, which proves the identity in the form

$$6\sum_{i=1}^{n} i^2 = n(n+1)(2n+1).$$

Consider the following three "step-pyramids":

$$P_1 = \bigcup_{i=0}^{n-1} [i, n-i] \times [i, n-i] \times [i, i+1],$$
$$P_2 = \bigcup_{i=0}^{n-1} [0, n-i] \times [n-i, n-i+1] \times [i, n],$$
$$P_3 = \bigcup_{i=0}^{n-1} [n-i-1, n-i] \times [0, n-i] \times [i+1, n+1].$$

Let us call a unit cube of the form $[i, i+1] \times [j, j+1] \times [k, k+1]$ with $i, j, k \in \mathbb{Z}$ an *integral unit cube*. Note that each of P_1, P_2, P_3 is a union of $\sum_{i=1}^{n} i^2$ distinct integral unit cubes. (For example, consider P_1. For each $i = 0, \ldots, n-1$, the solid box $[i, n-i] \times [i, n-i] \times [i, i+1]$ contains precisely $(n-i)^2$ integral unit cubes, namely $[j, j+1] \times [k, k+1] \times [i, i+1]$ for $j, k = 0, \ldots, n-i-1$.) Moreover, no two of P_1, P_2, P_3 share an integral unit cube. Therefore the union $S = P_1 \cup P_2 \cup P_3$ contains precisely $3\sum_{i=1}^{n} i^2$ integral unit cubes.

We will now count the number of integral unit cubes contained in S in a different way and show that it is $\frac{n(n+1)}{2} \cdot (2n+1)$. Consider the following two "step-triangular prisms":

$$T_1 = \bigcup_{i=0}^{n-1} [i, n] \times [i, i+1] \times [0, 1]$$
$$T_2 = \bigcup_{i=0}^{n-1} [0, n-i] \times [n-i, n-i+1] \times [0, 1]$$

Each of T_1 and T_2 is a union of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ distinct integral unit cubes, and T_1 and T_2 do not share any integral unit cubes. Now, S is the union of the $n+1$ vertical translates of T_1 by the vectors $(0,0,1), \ldots, (0,0,n+1)$ and n vertical translates of T_2 by the vectors $(0,0,1), \ldots, (0,0,n)$. Therefore the number of integral unit cubes contained in S is $(2n+1) \cdot \frac{n(n+1)}{2}$ as required.

Alternatively, one can conclude the proof by observing that the solid box $B = [0, n] \times [0, n+1] \times [0, 2n+1]$ is the union of S and a solid S' congruent to S, which also contains $3\sum_{i=1}^{n} i^2$ integral unit cubes that are all distinct from those contained in S. Therefore the number of integral unit cubes in B is the number contained in S plus the number contained in S', in other words $3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i^2 = 6\sum_{i=1}^{n} i^2$. Since the number of integral unit cubes in B is clearly $n(n+1)(2n+1)$, we obtain $6\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)$.

Solution to Ex. 9.b:

If $n = 0$, then the left hand side of the identity is $\binom{n}{0}2^0 = 1$ and the right hand side is $3^0 = 1$, so the identity holds. Let $n \geq 0$ and suppose that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$. Then

$$\begin{aligned}
\sum_{i=0}^{n+1} \binom{n+1}{i} 2^i &= \binom{n+1}{0} 2^0 + \sum_{i=1}^{n+1} (\binom{n}{i-1} \binom{n}{i}) 2^i \\
&= 1 + \sum_{i=1}^{n+1} \binom{n}{i-1} 2^i + \sum_{i=1}^{n+1} \binom{n}{i} 2^i \\
&= 1 + \sum_{i=0}^{n} \binom{n}{i} 2^{i+1} + \sum_{i=1}^{n+1} \binom{n}{i} 2^i \\
&= \sum_{i=0}^{n} \binom{n}{i} 2^{i+1} + \sum_{i=0}^{n} \binom{n}{i} 2^i \\
&= 2\sum_{i=0}^{n} \binom{n}{i} 2^i + \sum_{i=0}^{n} \binom{n}{i} 2^i
\end{aligned}$$

Since $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$, we have

$$\sum_{i=0}^{n+1} \binom{n+1}{i} 2^i = 2 \cdot 3^n + 3^n = 3^{n+1}.$$

By the principle of induction, the identity holds for all non-negative integers n.

Now we give a combinatorial proof. Let n be a non-negative integer. Let $B(n)$ (resp. $T(n)$) denote the set of binary (resp. ternary) strings of length n. For each $i = 0, \dots, n$, let $T_i(n)$ denote the set of ternary strings of length n with i 0s and 1s (and $n - i$ 2s). Then we have the disjoint union $T(n) = \coprod_{i=0}^{n} T_i(n)$. Therefore $3^n = |T(n)| = \sum_{i=0}^{n} |T_i(n)|$.

We will show that for each $i = 0, \dots, n$ we have $|T_i(n)| = \binom{n}{i} 2^i$. To show this, it suffices to construct a bijection $f_i : T_i(n) \to \binom{\{1,\dots,n\}}{i} \times B(i)$, as the codomain has cardinality $\binom{n}{i} 2^i$. Let $x = x_1 \dots x_n \in T_i(n)$. Define $k_1 < \dots < k_i$ so that $x_{k_1}, \dots, x_{k_i} \in \{0, 1\}$. Then $\{k_1, \dots, k_n\} \in \binom{\{1,\dots,n\}}{i}$ and $x_{k_1} \dots x_{k_i} \in B(i)$. We define $f_i(x) = (\{k_1, \dots, k_i\}, x_{k_1} \dots x_{k_i})$. This gives a well-defined map $f_i : T_i(n) \to \binom{\{1,\dots,n\}}{i} \times B(i)$. The map f_i is a bijection since we can write down its inverse. Indeed, for $(\{k_1, \dots, k_i\}, x_{k_1} \dots x_{k_i}) \in \binom{\{1,\dots,n\}}{i} \times B(i)$ define by $g_i(\{k_1, \dots, k_i\}, y_{k_1} \dots y_{k_i})$ to be the ternary string $x_1 \dots x_n$ with $x_{k_i} = y_{k_i}$ for all $i = 0, \dots, i$ and $x_k = 2$ for all $k \in \{1, \dots, n\} \setminus \{k_1, \dots, k_i\}$. Then g_i is the inverse of f_i.

Solution to Ex. 11: It is straightforward to prove the identity by induction. We give a combintorial proof. Let n be a non-negative integer. The set $2^{[n+1]} \setminus \{\varnothing\}$ of non-empty subsets of $[n+1] = \{1, \dots, n+1\}$ is the disjoint union

$$2^{[n+1]} \setminus \{\varnothing\} = \coprod_{i=0}^{n} S_i,$$

where S_i is the set of non-empty subsets of $[n+1]$ whose largest element is $i+1$. For each $i = 0, \dots, n$, the the map $f_i : S_i \to 2^{[i]}$ defined by $f_i(A) = A \setminus \{i+1\}$ is a bijection, so $|S_i| = 2^i$. Therefore

$$2^{n+1} - 1 = |2^{[n+1]} \setminus \{\varnothing\}| = |\coprod_{i=0}^{n} S_i| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} 2^i$$

as required.

Solution to Ex. 13: We proceed by induction on n. If $n = 1$, then $9^n - 5^n = 9 - 5 = 4$, which is divisible by 4. Let n be a positive integer and assume that $9^n - 5^n$ is divisible by 4. Let q be the integer such that $9^n - 5^n = 4q$. Then

$$9^{n+1} - 5^{n+1} = 9 \cdot 9^n - 5 \cdot 5^n = 4 \cdot 9^n + 5 \cdot (9^n - 5^n) = 4 \cdot (9^n + 5q).$$

Therefore 4 divides $9^{n+1} - 5^{n+1}$. By the principle of induction, we have that 4 divides $9^n - 5^n$ for all positive integers n.

Solution to Ex. 15: If $n = 0$, then $n^3 + (n+1)^3 + (n+2)^3 = 0 + 1 + 8 = 9$, which is divisible by 9. Let n be a non-negative integer and assume that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9. Let q be the integer such that $n^3 + (n+1)^3 + (n+2)^3 = 9q$. Then

$$\begin{aligned}
(n+1)^3 + ((n+1)+1)^3 + ((n+1)+2)^3 &= (n+1)^3 + (n+2)^3 + (n+3)^3 \\
&= (n+1)^3 + (n+2)^3 + n^3 + \binom{3}{1} 3n^2 + \binom{3}{2} 3^2 n + \binom{3}{3} 3^3 \\
&= 9q + 9n^2 + 27n + 27 \\
&= 9(q + n^2 + 3n + 3).
\end{aligned}$$

Thus, $(n+1)^3 + ((n+1)+1)^3 + ((n+1)+2)^3$ is divisible by 9. By the principle of induction, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all non-negative integers n.

Solution to Ex. 17: If $n = 0$, then $3n^2 - n + 2 = 2 = f(0)$. If $n = 1$, then $3n^2 - n + 2 = 3 - 1 + 2 = 4 = f(1)$. Let $n \geq 1$ be a integer and assume that $f(k) = 3k^2 - k + 2$ for all non-negative integers $k \leq n$. Then

$$\begin{aligned} f(n+1) &= 2f(n) - f(n-1) + 6 \\ &= 2(3n^2 - n + 2) - (3(n-1)^2 - (n-1) + 2) + 6 \\ &= 6n^2 - 2n + 4 - (3(n^2 - 2n + 1) - n + 3) + 6 \\ &= 6n^2 - 2n + 4 - (3n^2 - 7n + 6) + 6 \\ &= 3n^2 + 5n + 4. \end{aligned}$$

On the other hand, $3(n+1)^2 - (n+1) + 2 = 3(n^2 + 2n + 1) - n + 1 = 3n^2 + 5n + 4$, so $f(n+1) = 3(n+1)^2 - (n+1) + 2$. Therefore, by the principle of strong induction we have $f(n) = 3n^2 - n + 2$ for all non-negative integers n.

Solution to Ex. 19: We proceed by induction. When $n = 0$, we have $(1 + x)^n = 1$ and $1 + nx = 1$, so $(1 + x)^n \geq 1 + nx$. Let n be a non-negative integer and assume that $(1 + x)^n \geq 1 + nx$. Since $1 + x > 0$ and $(1 + x)^n \geq 1 + nx$, we have

$$(1 + x)(1 + x)^n \geq (1 + x)(1 + nx).$$

Therefore

$$(1 + x)^{n+1} \geq (1 + x)(1 + nx) = 1 + nx + x + nx^2 = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x,$$

where the last inequality holds since $nx^2 \geq 0$. By the principle of induction, we have $(1 + x)^n \geq 1 + nx$ for all non-negative integers n.