MAT344 HW1 SOLUTIONS

Feb 1, 2023

Q1: There are 12 options for each of the 6 characters, so $\boxed{12^6}$ strings.

Q3: There are :

- $26 \times 2 = 52$ options for each of the first three letters
- 10 options for each of the next two digits
- 10 options for the next symbol
- 26 options for each of the next two uppercase letters
- 10 options for the next digit
- 10 options for the last symbol

so there are $\boxed{52^3 \times 10^2 \times 10 \times 26^2 \times 10 \times 10}$ passwords for Matt's website. The total number of strings of length 10 would be $(26 \times 2 + 10 + 10)^{10}$, which is around 400,000 times larger.

Q5: There are 26^3 total strings of length 3 using only letters and 25^3 of them that don't contain K. Thus there are $26^3 - 25^3$ valid strings for the $l_1 l_2 l_3$. Similarly, there are 10^3 total strings of length 3 using only digits, and 1 of them don't contain any nonzero digits (i.e. 000). Thus there are $10^3 - 1$ valid strings for the $d_1 d_2 d_3$, so in total there are $\boxed{(26^3 - 25^3)(10^3 - 1)}$ license plates.

Q7: There are 26 options for each of l_1, l_3, l_4, l_5, l_6 and 21 options for l_2. There are 10 options for d_1. After picking d_1, there are 9 options for d_2, then 8 options for d_3. There are still 10 options for d_4. Thus there are $\boxed{26^5 \times 21 \times 10 \times 9 \times 8 \times 10}$ strings.

Q9: There are $10 \times 9 \times 8$ options for the last 3 characters. There are $\binom{17}{3}$ ways to choose where the 3 letters from the set {A, E, I, O, U} appear and 5^3 ways to choose what the letters are. Each of the 14 remaining spots has 21 letter and 7 digit options, so there are 28^{14} ways to fill the rest of the string. Thus there are $\boxed{10 \times 9 \times 8 \times \binom{17}{3} \times 5^3 \times 28^{14}}$ valid strings.

Q11:

(a) Since there are 12 options for each of the 6 people, there are $\boxed{12^6}$ ways.

(b) After picking a donut for a specific person, there is one fewer option for the next person. There are 12 options for the first person, then 11 options for the second, and so on. So there are $\boxed{12 \times 11 \times 9 \times 8 \times 7}$ ways.

(c) He needs to choose 6 types of the 12 total types, so there are $\binom{12}{6}$ ways.

Q13:

(a) There are 20 options for first place. After determining the first place winner, there are 19 options for second place, then 18 options for third, and 17 for fourth. So there are $\boxed{20 \times 19 \times 18 \times 17}$ possible outcomes.

(b) There are 16 remaining individuals and the judges choose 4 of them to get honourable mentions, so there are $\boxed{\binom{16}{4}}$ ways the honourable mentions can be selected after the top four were determined, and $\boxed{20 \times 19 \times 18 \times 17 \times \boxed{\binom{16}{4}}}$ total outcomes.

Q15: We first give both Ahmed and Dieter the 1 pencil they each must get, and we give Barbara the 4 pencils she must get. There are 19 pencils remaining to distribute among the 4 people. Without the restriction on Casper, there are $\binom{22}{3}$ ways to do this. We subtract the cases where Casper gets at least 6 pencils. If Casper must get at least

6 pencils, then we also give him his 6 pencils first, and there are $\binom{16}{3}$ ways to distribute the remaining 13 pencils among the 4 people.

Thus the total number of ways to distribute pencils is $\boxed{\binom{22}{3} - \binom{16}{3}}$.

Q17: If we let $y_1 = x_1 - 1$. Then finding integer solutions to $x_1 + x_2 + x_3 + x_4 = 132$ with the conditions $x_1 \geq 1$ and $x_2, x_3, x_4 \geq 0$ is the same as finding integer solutions to $y_1 + x_2 + x_3 + x_4 = 131$ with the conditions $y_1, x_2, x_3, x_4 \geq 0$. This is the same as distributing 131 things among 4 people so there are $\boxed{\binom{134}{3}}$ solutions.

To compute the solutions satisfying $x_4 < 17$, we subtract those that satisfy $x_4 \geq 17$. If we let $y_4 = x_4 - 17$, then solving $y_1 + x_2 + x_3 + x_4 = 131$ with the conditions $x_4 \geq 17$ and $y_1, x_2, x_3 \geq 0$ is the same as solving $y_1 + x_2 + x_3 + y_4 = 114$ with the conditions $y_1, x_2, x_3, y_4 \geq 0$. This is the same as distributing 114 things among 4 people so there are $\binom{117}{3}$ solutions. Thus the number of solutions satisfying $x_1 > 0$, $x_2, x_3 \geq 0$, and $17 > x_4 \geq 0$ is $\boxed{\binom{134}{3} - \binom{117}{3}}$

Q19

(a) We first give 10 candies to the contest winner, and 1 candy each to the 34 students who insist on receiving at least 1 candy. This leaves us $450 - 10 - 34 = 406$ candies to distribute among 66 people (65 students and 1 teacher). There are $\boxed{\binom{406+65}{65}}$ ways to do this.

(b) We subtract the distributions where the diabetic student receives at least 8 candies. If the diabetic student receives at least 8 candies, then we give them 7 more (in addition to the one they already got). This leave $406 - 7 = 399$ candies to distribute among the 66 people. There are $\binom{399+65}{65}$ ways to do this. Thus there are $\boxed{\binom{406+65}{65} - \binom{399+65}{65}}$ ways to distribute the candies if we add the condition that the diabetic student can receive at most 7 candies.

Q21: Both sides count the number of ways to choose k things from $m + w$ things.

The LHS is clear. For the RHS, we can divide the $m + w$ things into a pile of m things and a pile of w things. Then we can choose k things by first choosing j things from the pile of m things, then choosing the remaining $k - j$ things from the pile of w things. For each j ranging from 0 to k, there are $\binom{m}{j}\binom{w}{k-j}$ ways to make this choice, so there are $\sum_{j=0}^{k} \binom{m}{j}\binom{w}{k-j}$ total ways to choose k things.
Note: If $k > n$ then we say $\binom{n}{k} = 0$ since there is no way to choose k things from n things.

Q23: We need to take 7 steps to the right and 7 steps up, so there are 14 total steps, and we need to choose 7 of them to be to the right. Thus there are $\boxed{\binom{14}{7}}$ ways.

Q25: The lattice path must first pass through (7,6), then (12, 9), and finally (17, 12). There are $\binom{13}{7}$ ways to go from (0, 0) to (7,6), and $\binom{8}{3}$ ways to go from (7,6) to (12, 9), and $\binom{8}{3}$ ways to go from (12, 9) to (17, 12). So in total there are $\boxed{\binom{13}{7} \cdot \binom{8}{3} \cdot \binom{8}{3}}$ paths.

Q27: Assume he is reasonably efficient, that is, he only moves in ways that bring him closer to his destination.
If we ignore the police officer, he needs to move up 6 streets and 4 avenues, so there are $\binom{10}{4}$ paths. We subtract the paths that will cross the police officer. The officer is 3 streets and 3 avenues away from the bank, so there are $\binom{6}{3}$ ways to get to the officer from the bank. The hideout is 3 streets and 1 avenue away from the officer, so there are $\binom{4}{1}$ ways to get from the officer to the hideout. Thus there are $\binom{6}{3} \cdot \binom{4}{1}$ paths that cross the police officer, so $\boxed{\binom{10}{4} - \binom{6}{3} \cdot \binom{4}{1}}$ paths that don't cross the police officer.

Q29: We look at $2x + 3y^2 + z$ as a sum of $2x$ and $3y^2 + z$ and apply the binomial theorem to get:

$$(2x + 3y^2 + z)^{100} = (2x + (3y^2 + z))^{100} = \sum_{k=0}^{100} \binom{100}{k} (2x)^k (3y^2 + z)^{100-k}$$

Since we want the coefficient of $x^{15}y^{120}z^{25}$, we only care about the terms where $k = 15$. When $k = 15$, we see that

$$(3y^2 + z)^{100-k} = (3y^2 + z)^{85} = \sum_{j=0}^{85} \binom{85}{j} (3y^2)^j z^{85-j}$$

Again since we want the coefficient of $x^{15}y^{120}z^{25}$, we want to look at $85 - j = 25$ (or, equivalently, $2j = 120$ by considering y instead of z). This happens when $j = 60$, and we get the term $\binom{85}{60}(3y^2)^{60}z^{25}$. Multiplying this with the other factors that involve $k = 15$, we get the term

$$\binom{100}{15}(2x)^{15}\binom{85}{60}(3y^2)^{60}z^{25} = \left(\binom{100}{15}2^{15}\binom{85}{60}3^{60}\right)x^{15}y^{120}z^{25}$$

so the coefficient on $x^{15}y^{120}z^{25}$ is $\boxed{\binom{100}{15}2^{15}\binom{85}{60}3^{60}}$

Q31:

(a) We can create a bijection between fully-parenthesized products on $n + 1$ factors $a_1a_2 \cdots a_n$ and sequences $\{b_i\}$ of length n as in part (c). Given a parenthesized product, the k^{th} open parenthesis will correspond to b_k in the sequence, where a_{b_k} is the closest factor to the right of the k^{th} open parenthesis. For example,

$$((a_1a_2)(a_3(a_4a_5)))$$

corresponds to the sequence

$$b_1, b_2, b_3, b_4 = 1, 1, 3, 4$$

Indeed, $b_k \leq k$ because any initial segment of a fully-parenthesized product will contain at most as many letters as open-parentheses.

Now we build a parenthesized product from a sequence. If a number i appears n_i times in the sequence, then there must be exactly n_i open parentheses directly preceding a_i in the product. Thus the placement of all the open parentheses (relative to the letters) is determined. We can now determine the full product by the following process: Look for the right-most open parenthesis and note that there must be two letters a_ia_{i+1} directly to its right. We put a closing parenthesis to the right of these two letters. Then we treat (a_ia_{i+1}) as a single "letter" and repeat the process until we are done. For example, if the sequence is $1, 1, 3, 3$, then first we add in the open-parentheses to get $((a_1a_2((a_3a_4a_5$. Then we can add the closing parenthesis to the right-most open parenthesis by

$$((a_1a_2((a_3a_4a_5 \to ((a_1a_2((a_3a_4)a_5$$

now treat (a_3a_4) as a single "letter" and repeat the process:

$$((a_1a_2((a_3a_4)a_5 \to ((a_1a_2((a_3a_4)a_5) \to ((a_1a_2)((a_3a_4)a_5) \to ((a_1a_2)((a_3a_4)a_5))$$

The condition that $b_k \leq k$ guarantees that there will always be at least as many open parentheses as letters in any initial segment of the parenthesized product, so that there will always be at least 2 "letters" to the right of the right-most open parenthesis.

(b) A step to the right on the path will correspond to a 1 in the sequence, and a step up will correspond to a -1. Not going above the diagonal line means that for any i, the first i steps cannot contain more steps upward than to the right, which is exactly the condition that in the first i elements of the sequence, there cannot be more "-1"s than "1"s. That is, the sum of the first i elements must be nonnegative.

(c) A sequence $\{a_i\}$ corresponds to a path whose horizontal segments are exactly the horizontal lines from $(i-1, a_i - 1)$ to $(i, a_i - 1)$. The conditions $a_i \leq a_{i+1}$ is exactly the condition that we never step downwards. The condition $a_i \leq i$ is the same as $a_i - 1 \leq i - 1$, which is equivalent to saying the horizontal segments never cross the diagonal, which is equivalent to saying that the entire path does not cross the diagonal.