# University of Toronto

Department of Mathematics

**MAT344, Introduction to Combinatorics**

**Practice Midterm 1**

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1. Each of 15 red balls and 15 green balls is marked with an integer between 1 and 100 inclusive; no integer appears on more than one ball. Show there are at least two pairs (pair = one red and one green ball) where the sum of numbers on the balls is the same. Show that this is not necessarily true if there are 13 balls of each color.

2. Prove that, for every $n \geq 1$, we have

$$\sum_{k \text{ is odd}} \binom{n}{k} = \sum_{k \text{ is even}} \binom{n}{k}$$

3. Which complete graphs $K_n$, $n \geq 2$ are Eulerian? A walk that uses all the edges is called an *Eulerian walk*. Which ones have Eulerian walks? Justify your answers.

4. Suppose $(a_1, a_2, \ldots, a_{52})$ are integers, not necessarily distinct. Show that there are two, $a_i$ and $a_j$ with $i \neq j$ , such that either $a_i + a_j$ or $a_i - a_j$ is divisible by 100.

5. A poker hand consists of five cards from a standard 52 card deck with four suits and thirteen values in each suit; the order of the cards in a hand is irrelevant. How many hands consist of 2 cards with one value and 3 cards of another value (a full house)? How many consist of 5 cards from the same suit (a flush)?

6. How many ways can the score go in an ice hockey game if the Leafs are always ahead and it ends up a 6-6 tie?

7. How many binary strings are there of length 8 that contain at least two 1's? How many that contain two 1's in a row?

8. Show that if the edges of the complete graph $K_6$ are colored with two colors, there are at least two monochromatic triangles. (Two triangles are different if each contains at least one vertex not in the other. For example, two red triangles that share an edge count as two triangles.)

9. Prove that a graph $G$ is bipartite if and only if every cycle is of even length.

10. A regular graph is one in which the degree of every vertex is the same. Show that if $G$ is a regular bipartite graph, and the common degree of the vertices is at least 1, then the two parts are the same size.

11. Prove that $G$ is a tree if and only if there is a unique path between any two vertices.

12. The independence number $\alpha(G)$ of a graph $G$ is the maximum size of an independent set. Prove that for a graph $G$ with $n$ vertices
$$n \leq \alpha(G)\chi(G)$$
where $\chi(G)$ is the chromatic number.

13. What is the number of faces of a planar graph that has 6 edges and 4 vertices?

14. How many ways can you distribute 20 folders amoung 5 employees? 1. If the folders are distinguishable, 2. If the folders are indistinguishable. (We assume the employees are distinguishable.)

15. A Hamiltonian path on a graph is a path which uses each vertex exactly once. How many Hamiltonian paths are there on the complete graph $K_n$?

16. Suppose you are given a list of $n$ integers, each of size at most $100n$. Describe an algorithm to sort the list in a decreasing order (use the fastest algorithm you know). How many operations would it take? You can use the Big "Oh" notation and ignore multiplicative constants.