University of Toronto

Department of Mathematics
Faculty of arts and science

MAT332H1F, Graph Theory

Final Examination, 18 December 2017

Instructor: Kasra Rafi
Duration: 3 hours

First ______ Last ______ Student Number ______

Instructions: No aids allowed. Write solutions on the space provided. To receive full credit you must show all your work. If you run out of room for an answer, continue on the back of the page. This exam has 8 questions, for a total of 100 points.

Problem #	Grade
1	
2	
3	
4	
5	
6	
7	
Bonus	
Total	

1. (24 points) Define the following terms and expressions:

   (a) Vertex cover

   (b) Feasible flow

   (c) Planar graph

   (d) Connected component

(e) Dual of a plane graph

(f) Edge cut

(g) M–augmenting path

(h) Tree

2. (20 points) Answer true of false. Justify your answer with an argument or a counter example.

(a) $\boxed{}$ If $n(G)$ is odd, then some vertex v in G has an even degree.

(b) $\boxed{}$ If $n(g) = 7$ and $e(g) = 13$ then G has an odd cycle.

(c) $\boxed{}$ Every graph is 4–vertex colourable.

(d) $\boxed{}$ If G is k–connected then it is k–edge-connected.

3. (12 points) State and prove the Euler's formula.

4. (12 points) Recall that a k–dimensional cube Q_k is a simple graph whose vertices are k–tuples with entries in $\{0, 1\}$ and whose edges are pairs of k–tuples that differ in exactly one position.

   (a) For which values of k is Q_k bi-partite?

   (b) For which values of k is Q_k Eulerian?

5. (12 points) For each item, give an example of a graph described or prove no such graph exists.

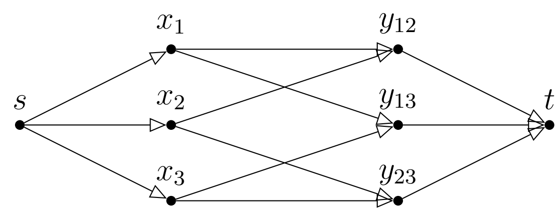
- A simple bi-partite planar 4–regular graph.

- A 3–regular planar graph of diameter 3 with 12 vertices.

6. (10 points) Let G be a graph with 11 vertices. Show that either G or the complement of G is not planar. (Hint: Use Euler's formula.)

7. (10 points) Consider a local baseball league with four teams x_0, x_1, x_2, x_3. You are rooting for the team x_0 and, at some time during the season, you are wondering if x_0 can still win the championship. Each pair of teams play 10 games during the season. The table below shows the results so far; the entry in row x_i and column x_j is the number of times the team x_i has defeated the team x_j so far (there are no draws).

	x_0	x_1	x_2	x_3
x_0	-	6	4	3
x_1	3	-	3	8
x_2	5	2	-	7
x_3	5	1	2	-



- If x_0 wins all its remaining games, how many wins will it have altogether?

- Assuming x_0 does win all its remaining games, how many more games can each of x_1, x_2 and x_3 win so that x_0 is the champion (has more wins than every other team)?

- How many more times will x_i play x_j, for $1 \leq i < j \leq 3$?

- Create a model for this problem by assigning capacities to the above network. What should the value of max flow be so that x_0 has a chance to be the champion?

- Find the minimum cut for the model network you have created. What conclusion can you make from value of the min cut?

8. (Bonus Problem) Let G be an even plane graph. Show that $G^\star$ is 2–colorable. (Recall that a graph is even if the degree of every vertex is even.)