Classifying Subgroups of $\text{PSL}_2(\mathbb{R})$ by Hyperbolic Isometries

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1 Classification of Hyperbolic Isometries

We can realize elements of $\text{PSL}_2(\mathbb{R})$ as orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$. Specifically, a matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}) \) acts on $\mathbb{H}^2$ by the Möbius transformation \( \frac{az+b}{cz+d} \), which sends the upper-half plane to itself. Notice that $A$ and $-A$ would give the same Möbius transformation, so this action is well defined.

This allows us to study the behaviour elements of $\text{PSL}_2(\mathbb{R})$ by examining hyperbolic isometries. The following figure summarizes the three types of hyperbolic isometries

(a) A hyperbolic isometry has two fixed points on the boundary. Points move from one fixed points to the other. Hyperbolic isometries correspond to $A \in \text{PSL}_2(\mathbb{R})$ where $|\text{tr}(A)| > 2$

(b) An elliptic isometry has one fixed point in the interior. Points move around the fixed point. Elliptic isometries correspond to $A \in \text{PSL}_2(\mathbb{R})$ where $|\text{tr}(A)| = 2$

(c) A parabolic isometry has one fixed point on the boundary. Points move around the fixed point. Parabolic isometries correspond to $A \in \text{PSL}_2(\mathbb{R})$ where $|\text{tr}(A)| < 2$

Figure 1: Classification of hyperbolic isometries
We will now use elements in $\text{PSL}_2(\mathbb{R})$ and their corresponding isometries interchangeably. We will now examine the subgroups generated by two hyperbolic, two parabolic, and two elliptic elements.
2 Hyperbolic and Parabolic Case

If we have two hyperbolic or parabolic elements $a$ and $b$, we can use the Ping-pong lemma to show that some power of them generate a free group.

**Theorem 2.1** (Ping-pong Lemma). Suppose $a$ and $b$ generate a group $G$ that acts on a set $X$. If $X$ has disjoint nonempty subsets $X_a$ and $X_b$ such that $a^k(X_b) \subset X_a$ and $b^k(X_a) \subset X_b$ for all nonzero $k$, then $G \cong \mathbb{F}_2$.

When we have two hyperbolic elements $a$ and $b$, each of them has two fixed points. Let $X_a$ be the union of some neighborhoods of fixed points of $a$, and let $X_b$ be the union of some neighborhoods of fixed points of $b$. Choose the neighborhoods small enough so that they are disjoint.

![Diagram](image)

**Figure 2:** Applying ping-pong lemma to two hyperbolic elements

Since hyperbolic isometries pushes all points towards one endpoint on the boundary, $a^k(X_b) \subset X_a$ for $k$ large enough and $b^k(X_a) \subset X_b$ for $k$ large enough. And $a$ and $b$ push all points towards the other two endpoints, so $a^{-k}(X_b) \subset X_a$ and $b^{-k}(X_a) \subset X_b$ for $k$ large enough. Hence we can apply the ping-pong lemma and conclude that $\langle a^k, b^k \rangle = \mathbb{F}_2$ for large enough $k$.

The argument for parabolic elements is similar. If we have two parabolic elements $a$ and $b$, each of them has one fixed point on the boundary. Then both $a$ and $\tilde{a}$ push all points on the disk towards the fixed point for $a$ and both $b$ and $\tilde{b}$ push all points on the disk towards the fixed point for $b$. Hence we can take a neighborhood of the fixed point of $a$ to be $X_a$ and the neighborhood of the fixed point of $b$ to be $X_b$. Then $a^k, b^k$ satisfy the criterion for the ping-pong lemma for large enough $k$. 

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3 The Elliptic Case

The ping-pong lemma argument will not work for elliptic isometries because elliptic isometries do not push all points into a neighborhood. We will instead use the following theorem:

**Theorem 3.1.** Suppose that a group $G$ acts without inversions on a tree $T$ in such a way that $G$ acts freely and transitively on edges. Choose one edge $e$ of $T$ and say that the stabilizers of its vertices are $H_1$ and $H_2$. Then

$$G \cong H_1 * H_2.$$ 

We will focus on the case where $a$ corresponds to a rotation by $2\pi/m$ and $b$ corresponds to a rotation by $2\pi/n$, and the two fixed points are sufficiently far apart. Let $v_1$ and $v_2$ be the fixed points of $a$ and $b$ respectively, and let $e$ be the geodesic connecting them.

Clearly in this case $\langle a, b \rangle$ is not free, as both $a$ and $b$ have finite order. We can construct a tree: let $T = \{g \cdot e \mid g \in \langle a, b \rangle\}$, i.e. the orbit of $e$. Note that $a$ has order $m$ and $b$ has order $n$. So an edge in $T$ is of the form $a^{e_1} b^{f_1} a^{e_2} b^{f_2} \ldots a^{e_k} b^{f_k} \cdot e$, where $1 \leq e_i \leq m - 1$ and $1 \leq f_i \leq n - 1$, and $e_1$ and $f_k$ can be 0. We can see that $T$ is a tree provided that all images of $e$ are disjoint.

A sufficient condition is that $v_1$ and $v_2$ are far apart, so that the angle bisector of $e$ and $a \cdot e$ and the angle bisector of $e$ and $b \cdot e$ do not intersect. We can compute the critical distance between $v_1$ and $v_2$.

We may do the calculation in the upper half plane. Moreover, up to isometry, we may assume that $v_2$ is $i$ in the upper half plane and $v_1$ lies on the imaginary axis. (See figure 5b). Then a Euclidean geometry calculation gives the Euclidean distance between $v_1$ and $v_2$ (in the upper half plane) is

$$\frac{\sqrt{2}}{2} \frac{\sin(\pi/(m\cdot n))}{\sin(\pi/m) - \cos(\pi/n)\sin(\pi/n)}.$$
Since $v_2 = i$, $v_1$ is on the imaginary axis, the hyperbolic distance is simply
\[
\log \left( \frac{1 + \cos(\pi/n)}{(1 - \cos(\pi/n))\sin(\pi/m)} \right)
\]

Assuming the minimal distance condition is satisfied, then $T$ is a tree. We see that $G$ acts without inversion on $T$ because for any $g \in G$, $gv_1$ is odd distance away from $v_2$, so it is impossible for $gv_1 = v_2$. The action is edge transitive on $T$ since $T$ is the orbit of an edge.

The action is free because if $ge = e$, then $g$ fixes $v_1$ and $v_2$ since it is acting without inversion. An orientation preserving isometry that fixes two points must be trivial.