University of Toronto

Department of Mathematics
Faculty of arts and science

MAT 1300, Topology I

Midterm, October 15, 2019

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Duration: 2 hours

First ____________ Last ____________ Student Number ____________

Instructions: No aids allowed. This exam has 6 questions and 40 points. All maps are always assumed to be smooth.

Problem #	Grade
Total	

1. (15 points) Mark True of False. Justify your answer with a brief argument or a counter example.

   (a) ☐ If $f\colon X \to Y$ is one-to-one and onto then it is a diffeomorphism.

   (b) ☐ For any $f\colon \mathbb{R}^3 \to \mathbb{R}$, the set $\{x \in \mathbb{R}^3 \mid f(x) \geq 0\}$ is a manifold with boundary.

   (c) ☐ For $f\colon X \to Y$, if $\deg_2(f) = 1$, then f is onto.

   (d) ☐ The set of critical points of a map $f\colon X \to Y$ has measure zero.

2. (7 points) (a) Find all the critical values of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = x^2 + y^2 + xz - 4z + 3.$$

   (b) Let

$$X = \big\{(x, y, z) \mid f(x, y, z) \geq 0\big\} \subset \mathbb{R}^3.$$

   Use part (a) to show that X is a manifold with boundary. (State any theorems you use.)

   (c) What is the dimension of $T_x(\partial X)$ for $x \in \partial X$?

   (d) Verify that $(0, 1, 1) \in \partial X$ and find a basis for $T_x(\partial X)$ at $x = (0, 1, 1)$.

3. (6 points) Let W be a sub-manifold of Y with boundary and let $\partial W = X \sqcup Z$. Let C be any closed sub-manifold of Y with dimension complementary to X and Z. Show that

$$I_2(X, C) = I_2(Z, C).$$

4. (6 points) Let $M_2(\mathbb{R})$ be the space of 2×2 matrices identified with $\mathbb{R}^4$. Let $\mathrm{SL}(2, \mathbb{R})$ be the subspace of $M_2(\mathbb{R})$ consisting of matrices of determinant 1. Show that $\mathrm{SL}(2, \mathbb{R})$ is a sub-manifold of $M_2(\mathbb{R})$. What is the dimension of $\mathrm{SL}(2, \mathbb{R})$?

5. (6 points) Let S^1 be the unit circle in $\mathbb{R}^2$, S^2 be the unit sphere in $\mathbb{R}^3$ and B^2 be the unit ball in $\mathbb{R}^3$. For a point $a \in S^1$, consider the map

$$f\colon S^2 \to S^2 \times S^1, \qquad f(x) = (x, a).$$

   Show that f does not extend to a map $F\colon B^2 \to S^2 \times S^1$.

6. (Extra Credit) Consider

$$S^3 = \Big\{(z_1, z_2) \in \mathbb{C}^2 \;\Big|\; |z_1|^2 + |z_2|^2 = 1\Big\}$$

   and

$$S^2 = \Big\{(z, x) \in \mathbb{C} \times \mathbb{R} \;\Big|\; |z|^2 + x^2 = 1\Big\}.$$

   Define a map

$$f\colon S^3 \to S^2 \qquad f(z_1, z_2) = \big(2z_1\overline{z_2}, |z_1|^2 - |z_2|^2\big).$$

   (a) Verify that the image of f is in fact S^2 and show that f is a submersion.

   (b) Show that the pre-image of every point in S^2 is a (great) circle in S^3.