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Sublinearly Morse boundary, II: Proper geodesic spaces

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We build an analogue of the Gromov boundary for any proper geodesic metric space, hence for any finitely generated group. More precisely, for any proper geodesic metric space X and any sublinear function κ, we construct a boundary for X, denoted by $\partial_\kappa X$, that is quasi-isometrically invariant and metrizable. As an application, we show that when G is the mapping class group of a finite type surface or a relatively hyperbolic group, with minimal assumptions, the Poisson boundary of G can be realized on the κ–Morse boundary of G equipped with the word metric associated to any finite generating set.

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1 Introduction

We construct an analogue of the Gromov boundary for a general proper geodesic metric space. That is, a boundary at infinity that is invariant under quasi-isometry, has good topological properties and is as large as possible. Our guiding principle is that, moving from the setting of Gromov hyperbolic spaces to general metric spaces, most key arguments still go through if we replace uniform bounds with sublinear bounds (with respect to distance to some basepoint). Examples of this philosophy have appeared in the literature before, for example in [Arzhantseva et al. 2017; Druţu 2000; Eskin et al. 2012; 2013; 2018; Kar 2011]. In a prequel to this paper [Qing and Rafi 2022], such a boundary was constructed in the setting of CAT(0) metric spaces. However, the definition of κ–Morse given in the previous paper does not work for general proper geodesic metric spaces. Hence, we use a new definition of κ–Morse that is more flexible and more fully embraces the above philosophy.

Statement of results

Let (X, d_X) be a proper geodesic metric space with a basepoint $\mathfrak{o}$. Recall that, when X is Gromov hyperbolic, the Gromov boundary of X is the set of equivalence classes of quasigeodesic rays emanating from $\mathfrak{o}$, equipped with the cone topology. Two quasigeodesic rays are considered equivalent if they stay within bounded distance from each other. In a Gromov hyperbolic space, every quasigeodesic ray β is *Morse*; that is, any other quasigeodesic segment γ with endpoints on β stays in a bounded neighborhood of β.

Similarly, we consider quasigeodesic rays in X (rays are always assumed to be emanating from $\mathfrak{o}$). Roughly speaking, we say a quasigeodesic ray β is *sublinearly Morse* if any other quasigeodesic segment γ with endpoints sublinearly close to β stays in a sublinear neighborhood of β (see Definition 3.2 for the precise definition). We group the set of sublinearly Morse quasigeodesic rays into equivalence classes by setting quasigeodesic rays α and β to be equivalent if they stay sublinearly close to each other. We call the set of equivalence classes of sublinearly Morse quasigeodesic rays, equipped with a coarse version of the cone topology, the *sublinearly Morse boundary* of X.

In fact, the above construction works for any given *sublinear function* $\kappa\colon [0,\infty) \to [1,\infty)$, where κ is a concave, increasing function with

$$\lim_{t\to\infty} \frac{\kappa(t)}{t} = 0.$$

Then we define the *κ–Morse boundary* $\partial_\kappa X$ to be the space of equivalence classes of κ–Morse quasigeodesic rays equipped with the coarse cone topology (see Definition 4.1). We obtain a possibly large family of boundaries for X, each associated to a different sublinear function κ.

We show that $\partial_\kappa X$ is metrizable and invariant under quasi-isometries; moreover, κ–boundaries associated to different sublinear functions are topological subspaces of each other.

Theorem A *Let X be a proper geodesic metric space, and let κ be a sublinear function. Then we construct a topological space $\partial_\kappa X$ with the following properties*:

(1) ***Metrizability*** *The spaces $\partial_\kappa X$ and $X \cup \partial_\kappa X$ are metrizable, and $X \cup \partial_\kappa X$ is a bordification of X.*

(2) ***QI–invariance*** *Every (k,K)–quasi-isometry $\Phi\colon X \to Y$ between proper geodesic metric spaces induces a homeomorphism $\Phi^\star\colon \partial_\kappa X \to \partial_\kappa Y$.*

(3) ***Compatibility*** *For sublinear functions κ and κ' with $\kappa \le c\cdot\kappa'$ for some $c>0$, we have $\partial_\kappa X \subset \partial_{\kappa'} X$ where the topology of $\partial_\kappa X$ is the subspace topology. Further, letting $\partial X := \bigcup_\kappa \partial_\kappa X$, we obtain a quasi-isometry invariant topological space that contains all $\partial_\kappa X$ as topological subspaces. We call ∂X the* ***sublinearly Morse boundary*** *of X.*

Note that from QI–invariance it follows that $\partial_\kappa X$ and ∂X do not depend on the basepoint $\mathfrak{o}$. Moreover, it also implies that the κ–Morse boundary of a finitely generated group G is independent of the generating set. Thus $\partial_\kappa G$ and ∂G are well defined.

We now argue that, in different settings, the κ–Morse boundary $\partial_\kappa X$ is large for an appropriate choice of κ. Recall that the *Poisson boundary* is the maximal boundary from the measurable point of view (see Section 6). In this paper, we let G be either a mapping class group or a relatively hyperbolic group and X be a Cayley graph of G and show that $\partial_\kappa X$ is a topological model for the Poisson boundary of (G,μ), where μ is any nonelementary finitely supported measure on G.

In fact, we show the following general criterion: if almost every sample path of the random walk driven by μ sublinearly tracks a κ–Morse geodesic, then the κ–Morse boundary is identified with the Poisson boundary. The following result was obtained in collaboration with Ilya Gekhtman.

Theorem B *Let G be a finitely generated group, and let (X, d_X) be a Cayley graph of G. Let μ be a probability measure on G with finite first moment with respect to d_X, such that the semigroup generated by the support of μ is a nonamenable group. Let κ be a sublinear function, and suppose that for almost every sample path $\omega = (w_n)$, there exists a κ–Morse geodesic ray γ_ω such that*

$$\lim_{n\to\infty} \frac{d_X(w_n, \gamma_\omega)}{n} = 0. \tag{1}$$

Then almost every sample path converges to a point in $\partial_\kappa X$, and moreover the space $(\partial_\kappa X, \nu)$, where ν is the hitting measure for the random walk, is a model for the Poisson boundary of (G, μ).

Theorems A and B can be applied to a large class of groups, providing a blueprint to identify their Poisson boundary. The κ–Morse boundary is always well defined, and in different situations one just needs to prove the above sublinear tracking result between geodesics and sample paths in order to obtain the Poisson boundary. In this paper, we showcase their applications to mapping class groups and relatively hyperbolic groups.

Comparison with other boundaries

Our construction provides the first example of a boundary of a general metric space which is both QI–invariant and has full measure with respect to a large class of measures. For comparison, recall that the *visual boundary* of CAT(0) spaces is not invariant by quasi-isometries [Croke and Kleiner 2000]. The *Gromov boundary* [1987], on the other hand, is QI–invariant, but only defined if the group is hyperbolic. A natural generalization is the *Morse boundary* [Cordes 2017], which is always well defined and QI–invariant, but very often it is too small; in particular, it has measure zero with respect to the hitting measure for most random walks on relatively hyperbolic groups [Cordes et al. 2022]. This is related to the fact that a typical sample path is expected to have unbounded excursions in the peripherals. Finally, the *Floyd boundary* [1980] is well behaved for relatively hyperbolic groups, but trivial for mapping class groups [Karlsson and Noskov 2004].

Recall also it is an open problem [Kaimanovich 1996, page 153] whether any two finitely supported generating measures on the same group yield isomorphic Poisson boundaries. This question is the probabilistic analogue of the quasi-isometry invariance we show in Theorem A, as two generating sets for G give rise to both two quasi-isometric metrics on G and to two finitely supported measures.

Mapping class groups

Let S be a surface of finite hyperbolic type and $\mathrm{Map}(S)$ be the mapping class group of S. Let d_w be the word metric on $\mathrm{Map}(S)$ with respect to some finite generating set. Then letting $(X, d_X) = (\mathrm{Map}(S), d_w)$ we can consider the κ–Morse boundary $\partial_\kappa \mathrm{Map}(S)$ of the mapping class group. We show the following characterization.

Theorem C *Let μ be a finitely supported, nonelementary probability measure on* $\mathrm{Map}(S)$*. Then, for* $\kappa(t) = \log(t)$,

1. *almost every sample path* (w_n) *converges to a point in* $\partial_\kappa \mathrm{Map}(S)$;
2. *the* κ*–Morse boundary* $(\partial_\kappa \mathrm{Map}(S), \nu)$ *is a model for the Poisson boundary of* $(\mathrm{Map}(S), \mu)$ *where* ν *is the hitting measure associated to the random walk driven by* μ.

The proof uses the machinery of curve complexes introduced by Masur and Minsky [2000], as well as the study of random walks in mapping class groups carried out by Maher [2010; 2012], Calegari and Maher [2015], Sisto [2017], Maher and Tiozzo [2018] and Sisto and Taylor [2019]. In particular, by [Sisto and Taylor 2019], a typical sample path makes logarithmic progress in each subsurface, which explains the function $\log(t)$.

Moreover, we also obtain the following tracking result between geodesics and sample paths in the mapping class group.

Theorem D *Let μ be a finitely supported, nonelementary probability measure on* $\mathrm{Map}(S)$*. Then, for almost every sample path, there exists a* κ*–Morse geodesic ray* γ_ω *in* $\mathrm{Map}(S)$ *such that*

$$\limsup_{n\to\infty} \frac{d_w(w_n, \gamma_\omega)}{\log n} < +\infty.$$

Sisto [2017] showed that almost every sample path in the mapping class group lies within distance $O(\sqrt{n \log n})$ of a geodesic. A more robust tracking result was proven in [Mathieu and Sisto 2020, Theorem 10.7], which gives the $\log(n)$–tracking in this setting, but not the fact that the geodesic being tracked is κ–Morse. Sublinear tracking for random walks with respect to the Teichmüller metric was obtained in [Tiozzo 2015].

Relatively hyperbolic groups

Now consider a finitely generated group G equipped with a word metric d_w associated to a finite generating set. Recall that G is relatively hyperbolic with respect to a family of subgroups $H_1, \dots, H_k$ if, after contracting the Cayley graph of G along H_i–cosets, the resulting graph equipped with the usual graph metric is Gromov hyperbolic. Further, H_i–cosets have to satisfy the technical condition of *bounded coset penetration*. Similarly to above, we show:

Theorem E *Let G be a nonelementary relatively hyperbolic group, and let μ be a probability measure whose support is finite and generates G as a semigroup. Then, for* $\kappa(t) = \log(t)$,

1. *almost every sample path* (w_n) *converges to a point in* $\partial_\kappa G$;
2. *the* κ*–Morse boundary* $(\partial_\kappa G, \nu)$ *is a model for the Poisson boundary of* (G, μ) *where* ν *is the hitting measure associated to the random walk driven by* μ.

Further applications

As mentioned above, Theorems A and B can be applied to any group with a certain degree of hyperbolicity. For example, the proof of Theorem C already works in the setting of hierarchically hyperbolic spaces [Behrstock et al. 2017]. Hierarchically hyperbolic spaces are a family of axiomatically defined spaces with properties that are modeled after the mapping class groups. Hence all the tools we use, such as subsurface projections, distance formulas and the bounded geodesic image theorem also exist in the setting of hierarchically hyperbolic spaces [Nguyen and Qing 2024].

Other groups where the κ–Morse boundary is nontrivial and these ideas can be applied to include groups that contain an element with contracting axis; for example, the groups with *statistically convex cocompact actions* as described in [Yang 2020], as well as $\mathrm{Out}(F_n)$. We shall discuss these applications in subsequent papers.

Sublinearly Morse vs sublinearly contracting

In the construction of the κ–Morse boundary $\partial_\kappa X$, several different definitions are possible for the notion of a κ–Morse quasigeodesic. The goal is always to emulate the behavior of quasigeodesics in a Gromov hyperbolic space but with sublinear errors (instead of uniform additive errors).

The definition of κ–Morse given in this paper (Definition 3.2) is equivalent to the definition of *strongly Morse* in [Qing and Rafi 2022]. Another natural condition is to require a quasigeodesic ray β to be *κ–weakly contracting* (Definition 5.3): that is, that the projection of a ball disjoint from β to β has a diameter that is bounded by a sublinear function of the distance of the center of the ball to the origin.

In the setting of CAT(0) spaces, these two notions are equivalent [Qing and Rafi 2022, Theorem 3.8], but this is no longer true for general metric spaces. In the appendix, we prove that κ–weakly contracting quasigeodesics are always κ–Morse. The converse is known not to be the case in general: for instance, when κ is the constant function, examples of Morse, but not strongly contracting geodesics arise in the "Tarski monsters" [Olshanskii et al. 2009]; moreover, a geodesic axis of a pseudo-Anosov element in the mapping class group is always Morse [Masur and Minsky 2000], but not always strongly contracting [Rafi and Verberne 2021]. However, κ–Morse quasigeodesics are always κ'–weakly contracting for a larger sublinear function κ':

Theorem F *Let X be a proper geodesic metric space, let κ be a sublinear function, and let β be a quasigeodesic ray in X. Then*

(1) *if β is κ–weakly contracting, it is κ–Morse;*
(2) *there is a sublinear function κ' such that if β is κ–Morse then it is κ'–weakly contracting.*

The definition of κ–Morse we use in this paper is the one that matches our philosophical approach the best, is the most flexible and makes the arguments simplest. Hence, we think it is the *correct* definition with minimal assumptions.

In some places, the same arguments as in [Qing and Rafi 2022] apply directly, while in others new ideas are needed; for the sake of brevity, we shall skip the proofs when the arguments are exactly the same.

History

The notion of a *Morse geodesic* is classical [Morse 1924], and much progress has been made in recent years using Morse geodesics to define boundaries of groups. Cordes [2017], inspired by the *contracting boundary* for CAT(0) spaces of Charney and Sultan [2015], constructed the *contracting boundary* for all proper geodesic spaces, where a quasigeodesic γ is *Morse* if there are uniform neighborhoods, with size depending on (q, Q), in which all (q, Q)–quasigeodesic segments with endpoints on γ lie. The Morse boundaries are equipped with a *direct limit topology* and are invariant under quasi-isometries. However, this space does not have good topological properties; for example, it is not first countable. Cashen and Mackay [2019], following the work of Arzhantseva, Cashen, Gruber and Hume [Arzhantseva et al. 2017], defined a different topology on the Morse boundary. They showed that it is Hausdorff and when there is a geometric action by a countable group, it is also metrizable; note that Theorem A does not assume any geometric action. Another notion, more closely related to our definition of κ–Morse boundary, is considered by Kar [2011]: a geodesic space is *asymptotically* CAT(0) if balls of radius r are coarsely CAT(0) with an error of $f(r)$, where the function $f(r)$ is sublinear. That is to say, a space is asymptotically CAT(0) if it has the same κ–Morse boundary as a CAT(0) space, where $\kappa = f(r)$.

The definition of a *contracting* geodesic originates from [Morse 1924] and has been brought back to attention by [Gromov 1987]. Masur and Minsky [2000] proved that axes of pseudo-Anosov elements are contracting, and since then various versions of this condition have been discussed, in particular the notions of *strongly contracting* and *weakly contracting*; see eg [Algom-Kfir 2011; Arzhantseva et al. 2015; Behrstock 2006; Bestvina and Fujiwara 2009; Rafi and Verberne 2021; Sisto 2018; Yang 2020]. Our definition of κ–weakly contracting is even weaker, and for that reason it is expected to be generic with respect to many notions of genericity. For random walks, genericity of sublinearly contracting geodesics in relatively hyperbolic groups follows from [Sisto 2017], in hierarchically hyperbolic groups it follows from [Sisto and Taylor 2019] and in CAT(0) groups it follows from [Gekhtman et al. 2022]. For the counting measure, genericity of log–weakly contracting geodesics in RAAGs has been shown in [Qing and Tiozzo 2021].

The Poisson boundary of (G, μ) is trivial for all nondegenerate measures μ on abelian groups [Choquet and Deny 1960] and nilpotent groups [Dynkin and Maljutov 1961]. Discrete subgroups of $\mathrm{SL}(d, \mathbb{R})$ are treated in [Furstenberg 1963], where the Poisson boundary is related to the space of flags. For random walks on Lie groups, the study and the description of the Poisson boundary was extensively developed in the 1970s and 1980s by many authors, most notably Furstenberg. The Poisson boundary of some Fuchsian groups has also been described by Series [1983] as being the limit set of the group. Kaimanovich [1994] identified the Poisson boundary of hyperbolic groups with their Gromov boundaries with the associated hitting measures. Karlsson and Margulis [1999] proved that visual boundaries of nonpositively curved spaces serve as models for their Poisson boundaries. Kaimanovich and Masur [1996; 1998] proved that the Poisson boundary of the mapping class group is the boundary of Thurston's compactification of Teichmüller space. Their description also applies to the Poisson boundary of the braid group; see [Farb and Masur 1998].

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2 Preliminaries

Let (X, d_X) be a proper geodesic metric space, and let $\mathfrak{o} \in X$ be a basepoint. Given a point $p \in X$, we define $\|p\| := d_X(\mathfrak{o}, p)$. Let α be a quasigeodesic ray starting at $\mathfrak{o}$. For $r > 0$, let t_r be the first time that $\|\alpha(t_r)\| = r$ and denote the point $\alpha(t_r)$ by

$$\alpha_r := \alpha(t_r),$$

while the segment from $\alpha(0)$ to $\alpha(t_r)$ will be denoted by

$$\alpha|_r := \alpha([0, t_r]).$$

We collect here the two basic geometric properties of the space that we need:

Lemma 2.1 *Let (X, d_X) be a proper geodesic metric space. Then*:

- *For any closed set $Z \subset X$ and any point $x \in X$, there is a closed set $\pi_Z(x)$ of nearest points in Z to x. We refer to any point in $\pi_Z(x)$ as a nearest point projection from x to Z.*
- *Any sequence of geodesics $\beta_n : [0, n] \to X$ with $\beta_0 = \mathfrak{o}$ has a subsequence that converges uniformly on compact sets to a geodesic ray $\beta_0 : [0, \infty) \to X$.*

The following lemma about nearest point projections will be used several times.

Lemma 2.2 *Let α and β be two quasigeodesic rays starting at the basepoint $\mathfrak{o} \in X$. Let x be a point on α, and let p be a nearest point projection of x onto β. Then*

$$\|p\| \le 2\|x\|.$$

Proof Since p is a nearest point projection of x onto β and $\mathfrak{o}$ belongs to β,

$$d_X(x,p) \le d_X(\mathfrak{o},x).$$

Hence, by the triangle inequality,

$$\|p\| = d_X(\mathfrak{o},p) \le d_X(\mathfrak{o},x) + d_X(x,p) \le 2d_X(\mathfrak{o},x) = 2\|x\|. \qquad \square$$

2.1 Sublinear functions

In this paper, a *sublinear function* will be a function $\kappa\colon[0,\infty)\to[1,\infty)$ such that

$$\lim_{t\to\infty}\frac{\kappa(t)}{t} = 0.$$

Moreover, we say that $\kappa\colon[0,\infty)\to[1,\infty)$ is a *concave sublinear function* if it is sublinear and moreover it is increasing and concave.

Remark 2.3 The assumptions that κ is increasing and concave make certain arguments cleaner, otherwise they are not really needed. One can always replace any sublinear function κ with another sublinear function $\bar\kappa$ such that $\kappa(t) \le \bar\kappa(t) \le C\kappa(t)$ for some constant C and $\bar\kappa$ is monotone increasing and concave. For example, define

$$\bar\kappa(t) := \sup\{\lambda\kappa(u) + (1-\lambda)\cdot\kappa(v) \mid 0 \le \lambda \le 1,\ u,v>0,\ \text{and } \lambda u + (1-\lambda)v = t\}.$$

Lemma 2.4 *If $\kappa\colon[0,\infty)\to[1,\infty)$ is a concave sublinear function and $\lambda>1$, then*

$$\kappa(\lambda t) \le \lambda\kappa(t)$$

for any $t \ge 0$.

Proof By concavity,

$$\kappa(t) = \kappa\Big(\frac{1}{\lambda}\lambda t + (1-\lambda^{-1})\cdot 0\Big) \ge \frac{1}{\lambda}\kappa(\lambda t) + (1-\lambda^{-1})\cdot\kappa(0) \ge \frac{1}{\lambda}\kappa(\lambda t),$$

from which the claim follows. $\square$

3 The κ–Morse boundary

We now introduce the definition of κ–Morse quasigeodesic, which will be fundamental for our construction. To set the notation, we say a quantity D *is small compared to* a radius $r>0$ if

$$D \le \frac{r}{2\kappa(r)}. \tag{2}$$

We will fix once and for all a basepoint $\mathfrak{o}\in X$, and all quasigeodesic rays we consider will be based at $\mathfrak{o}$. Given a quasigeodesic ray α and a constant m, we define

$$\mathcal{N}_\kappa(\alpha,m) := \{x\in X \mid d_X(x,\alpha) \le m\cdot\kappa(\|x\|)\}.$$

The following observation will be useful.

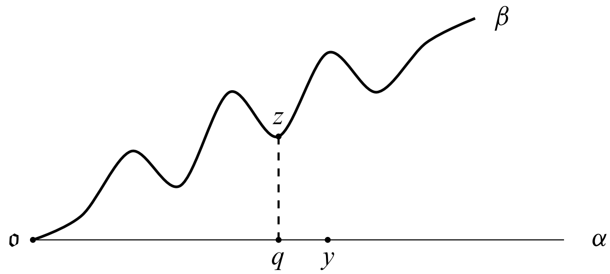

Figure 1: $\|y\| = \|z\|$ and $q \in \pi_\alpha(z)$ as in the proof of Lemma 3.1.

Lemma 3.1 *Let β be a quasigeodesic ray and α be a geodesic ray, both based at $\mathfrak{o} \in X$. Suppose that*

$$\beta \subseteq \mathcal{N}_\kappa(\alpha, m)$$

for some function κ and some constant m. Then we also have

$$\alpha \subseteq \mathcal{N}_\kappa(\beta, 2m).$$

Proof Let $y \in \alpha$ be a point and let $r := \|y\|$. Let $z \in \beta$ be a point such that $\|z\| = r$ and let q be a nearest point projection of z to α. By assumption,

$$d_X(z, q) \le m \cdot \kappa(r).$$

On the other hand,

$$\begin{aligned} d_X(y, q) &= \big| \|y\| - \|q\| \big| && \text{(since } \alpha \text{ is geodesic)} \\ &= \big| \|z\| - \|q\| \big| \\ &\le d_X(z, q) && \text{(by the triangle inequality).} \end{aligned}$$

Therefore,

$$d_X(y, \beta) \le d_X(y, z) \le d_X(y, q) + d_X(q, z) \le 2 d_X(z, q) \le 2m \cdot \kappa(r). \qquad \square$$

Definition 3.2 Let $Z \subseteq X$ be a closed set, and let κ be a concave sublinear function. We say that Z is *κ–Morse* if there exists a proper function $m_Z : \mathbb{R}^2 \to \mathbb{R}$ such that for any sublinear function κ' and for any

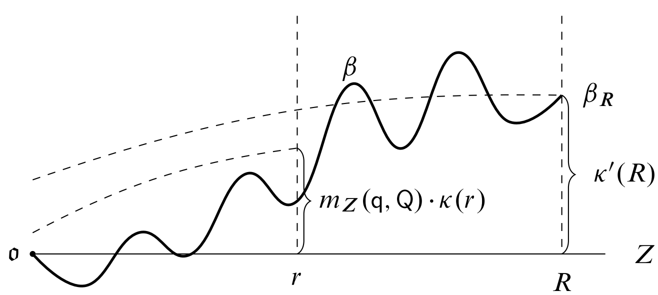

Figure 2: Definition of κ–Morse set Z: every quasigeodesic ray β has the property that there exists $R(Z, r, \mathsf{q}, \mathsf{Q}, \kappa')$ such that if β_R is distance $\kappa'(R)$ from Z, then $\beta|_r$ is in the neighborhood $\mathcal{N}_\kappa(Z, m_Z(\mathsf{q}, \mathsf{Q}))$.

$r > 0$, there exists R such that for any (q,Q)–quasigeodesic ray β with $m_Z(\mathsf{q},\mathsf{Q})$ small compared to r,

$$d_X(\beta_R, Z) \le \kappa'(R) \implies \beta|_r \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q},\mathsf{Q})).$$

The function m_Z will be called a *Morse gauge* of Z.

Note that we can always assume without loss of generality that $\max\{\mathsf{q},\mathsf{Q}\} \le m_Z(\mathsf{q},\mathsf{Q})$, and we will assume this in the following.

3.1 Equivalence classes

Definition 3.3 Given two quasigeodesic rays α and β based at $\mathfrak{o}$, we say that $\beta \sim \alpha$ if they sublinearly track each other; ie if

$$\lim_{r\to\infty} \frac{d_X(\alpha_r, \beta_r)}{r} = 0.$$

By the triangle inequality, $\sim$ is an equivalence relation on the space of quasigeodesic rays based at $\mathfrak{o}$, hence also on the space of κ–Morse quasigeodesic rays.

Lemma 3.4 *Let α be a κ–Morse quasigeodesic ray with Morse gauge m_α, and let $\beta \sim \alpha$ be a (q,Q)–quasigeodesic ray. Then*

(i) *β is κ–Morse, and moreover*

$$\beta \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathsf{q},\mathsf{Q}));$$

(ii) *if in addition α is a geodesic ray, then*

$$\alpha \subseteq \mathcal{N}_\kappa(\beta, 2m_\alpha(\mathsf{q},\mathsf{Q})).$$

Proof (i) Define $\kappa'(r) := d_X(\alpha_r, \beta_r)$. By definition of $\sim$, the function κ' is sublinear, and moreover, for any $R > 0$,

$$d_X(\beta_R, \alpha) \le \kappa'(R).$$

Hence, since α is κ–Morse, for any r there exists an $R(r, q, Q, \kappa')$ such that

$$d_X(\beta_R, \alpha) \le \kappa'(R).$$

Thus, by the definition of κ–Morse,

$$\beta|_r \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathsf{q},\mathsf{Q}))$$

and so

$$\beta \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathsf{q},\mathsf{Q})), \tag{3}$$

which proves the second part of (i). Let us now prove that β is κ–Morse. Let $r > 0$ and let β' be a $(\mathsf{q}',\mathsf{Q}')$–quasigeodesic ray that sublinearly tracks β, ie

$$d_X(\beta'_R, \beta) \le \kappa_2(R)$$

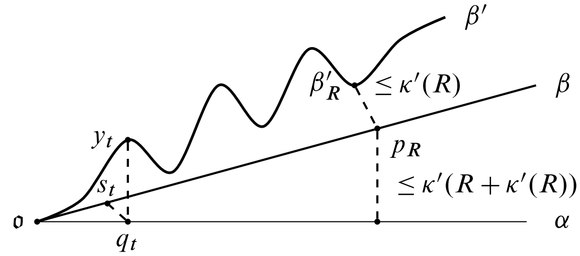

Figure 3: The setup in the proof of Lemma 3.4(i).

where κ_2 is an arbitrary sublinear function and R is sufficiently large. Let p_R be a nearest point projection of β'_R to β; by Lemma 2.2,

$$\|p_R\| \leq 2\|\beta'_R\| = 2R.$$

Then, by the triangle inequality and (3),

$$d_X(\beta'_R, \alpha) \leq d_X(\beta'_R, p_R) + d_X(p_R, \alpha) \leq \kappa_2(R) + m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|p_R\|) \leq \kappa_2(R) + m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(2R).$$

Since $\kappa''(R) := \kappa_2(R) + m(\mathsf{q}, \mathsf{Q}) \cdot \kappa(2R)$ is also a sublinear function, and since α is κ–Morse, this implies that

$$\beta|_r \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathsf{q}', \mathsf{Q}')).$$

Let y_t be any point on β' with $\|y_t\| = t \leq r$. By Lemma 2.2, if q_t is a nearest point projection of y_t to α,

$$\|q_t\| \leq 2\|y_t\| = 2t.$$

Now, if q is any point on α and s is a nearest point projection of q to β, by the triangle inequality and the Morse property,

$$\|q\| \geq \|s\| - d_X(s, q) \geq \|s\| - m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|s\|).$$

Moreover, by Lemma 2.2, $\|s\| \leq 2\|q\|$; hence by concavity

$$d_X(s, q) \leq m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|s\|) \leq 2m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|q\|).$$

Thus, if s_t is a nearest point projection of q_t to β, then the above estimate yields

$$d_X(q_t, s_t) \leq 2m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|q_t\|) \leq 4m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(t).$$

Hence, putting everything together,

$$d_X(y_t, \beta) \leq d_X(y_t, q_t) + d_X(q_t, s_t) \leq m_\alpha(\mathsf{q}', \mathsf{Q}') \cdot \kappa(t) + 4m_\alpha(\mathsf{q}, \mathsf{Q}) \cdot \kappa(t)$$

which, by setting $m_\beta(\mathsf{q}', \mathsf{Q}') := m_\alpha(\mathsf{q}', \mathsf{Q}') + 4m_\alpha(\mathsf{q}, \mathsf{Q})$, proves the claim.

(ii) This follows immediately from (i) and Lemma 3.1. □

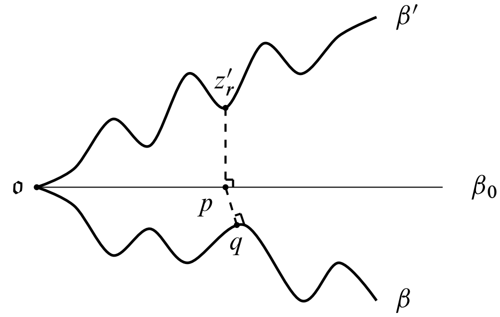

Figure 4: Corollary 3.5: $\|z'_r\| = r$, $p = \pi_{\beta_0}(z'_r)$ and $q = \pi_\beta(p)$.

Corollary 3.5 *If β is a (q, Q)–quasigeodesic ray, and β_0 is a κ–Morse geodesic ray such that $\beta \sim \beta_0$, then the function*

$$m_\beta(\cdot\,,\cdot) := m_{\beta_0}(\cdot\,,\cdot) + 4m_{\beta_0}(\mathsf{q}, \mathsf{Q})$$

is a Morse gauge for β. In particular, the Morse gauge depends only on m_{β_0}, q and Q and not on the particular quasigeodesic β.

Proof Let $\beta' \sim \beta$ be a $(\mathsf{q}', \mathsf{Q}')$–quasigeodesic ray. Let z'_r be a point along β' with norm r, let $p \in \pi_{\beta_0}(z'_r)$ and let q be a nearest point in β to p. Note that, by Lemma 2.2, $\|p\| \le 2r$. Hence, using Lemma 3.4(i) and (ii),

$$\begin{aligned} d_X(z'_r, \beta) &\le d_X(z'_r, p) + d_X(p, q) \\ &\le m_{\beta_0}(\mathsf{q}', \mathsf{Q}') \cdot \kappa(r) + 2m_{\beta_0}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(\|p\|) \\ &\le m_{\beta_0}(\mathsf{q}', \mathsf{Q}') \cdot \kappa(r) + 4m_{\beta_0}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r) \\ &\le (m_{\beta_0}(\mathsf{q}', \mathsf{Q}') + 4m_{\beta_0}(\mathsf{q}, \mathsf{Q})) \cdot \kappa(r). \end{aligned}$$ □

3.2 Surgery lemmas

We need a few technical results to splice quasigeodesic segments together. Lemma 3.6 generalizes [Qing and Rafi 2022, Lemma 2.5]. Lemma 3.7 is new and provides a way to splice a finite quasigeodesic to a quasigeodesic ray. Lemma 3.8 is already proven in [Qing and Rafi 2022, Lemma 4.3]; our statement here is slightly altered but the proof is identical, hence we skip the proof.

Lemma 3.6 *Consider a quasigeodesic segment $[x, y]_\alpha \in X$ that is (q, Q)–quasigeodesic, and another (q, Q)–quasigeodesic segment β connecting a point $z \in X$ to a point $w \in X$. Assume that for every point $u \in \alpha$, y is a point in u_β, and let γ be the concatenation of the quasigeodesic segment $[x, y]_\alpha$ and the quasigeodesic segment $[y, z]_\beta$. Then $\gamma = [x, y]_\alpha \cup [y, z]_\beta$ is a $(3\mathsf{q}, 3\mathsf{Q})$–quasigeodesic.*

Proof Consider $\gamma\colon [t_0, t_2] \to X$ and let $t_1 \in [t_0, t_2]$ be the time when $\gamma(t_1) = y$; the restriction of γ to $[t_0, t_1]$ is the parametrization of $[x, y]_\alpha$ given by arc length and the restriction of γ to $[t_1, t_2]$ is the parametrization of $[y, z]_\beta$ given by β. To show that γ is a quasigeodesic, we need to estimate the distance

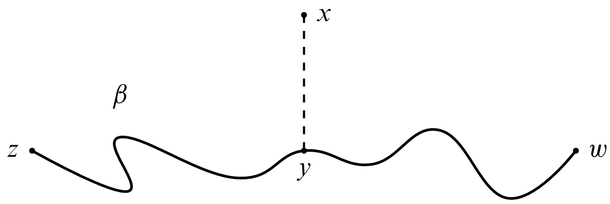

Figure 5: For $y \in x_\beta$, the concatenation of the quasigeodesic segment $[x,y]_\alpha$ and the quasigeodesic segment $[y,z]_\beta$ is a quasigeodesic.

between a point in $[x,y]_\alpha$ and a point in $[y,z]_\beta$. However, it is enough to show that $d_X(x,z)$ is comparable to $|t_2-t_0|$ because the argument for any other points along $[x,y]_\beta$ and along $[y,z]_\beta$ is the same. We argue in two cases.

Case 1 Suppose $2d_X(x,y) \geq d_X(z,y)$. Then

$$3d_X(x,y) \geq d_X(z,y) + d_X(x,y).$$

Therefore,

$$d_X(x,z) \geq d_X(x,y) \geq \tfrac{1}{3}(d_X(z,y)+d_X(x,y)) \geq \frac{1}{3}\Big(\frac{1}{\mathsf{q}}|t_2-t_1|-\mathsf{Q}+\frac{1}{\mathsf{q}}|t_1-t_0|-\mathsf{Q}\Big) \geq \frac{1}{3\mathsf{q}}|t_2-t_0|-\frac{2\mathsf{Q}}{3}.$$

Case 2 Suppose $2d_X(x,y) < d_X(z,y)$. Then

$$3d_X(x,y) \leq d_X(z,y)+d_X(x,y) \implies 2d_X(x,y) \leq \tfrac{2}{3}(d_X(z,y)+d_X(x,y)).$$

We have

$$\begin{aligned} d_X(x,z) &\geq d_X(z,y) - d_X(x,y) \\ &= d_X(z,y)+d_X(x,y)-2d_X(x,y) \\ &\geq (d_X(z,y)+d_X(x,y)) - \tfrac{2}{3}(d_X(z,y)+d_X(x,y)) \\ &\geq \tfrac{1}{3}(d_X(z,y)+d_X(x,y)) \\ &\geq \frac{1}{3}\Big(\frac{1}{\mathsf{q}}|t_2-t_1|-\mathsf{Q}+\frac{1}{\mathsf{q}}|t_1-t_0|-\mathsf{Q}\Big) \geq \frac{1}{3\mathsf{q}}|t_2-t_0|-\frac{2\mathsf{Q}}{3}. \end{aligned}$$

This established the lower bound. The upper bound follows from the triangle inequality,

$$d_X(x,z) \leq d_X(x,y)+d_X(y,z) \leq \mathsf{q}|t_1-t_0|+\mathsf{q}|t_2-t_1|+2\mathsf{Q} \leq \mathsf{q}|t_2-t_0|+2\mathsf{Q}.$$

It follows that γ is a $(3\mathsf{q}, 3\mathsf{Q})$–quasigeodesic. □

Lemma 3.7 *Consider a (q,Q)–quasigeodesic ray $\alpha\colon [0,\infty) \to X$ and a finite (q,Q)–quasigeodesic segment $\beta\colon [a,b] \to X$. Then there is an $s_0 \in [0,\infty)$ such that the following holds: for $s \in [s_0,\infty)$, let $s_\gamma \in [s,\infty)$ and $t_\gamma \in [a,b]$ be such that $[\beta(t_\gamma), \alpha(s_\gamma)]$ is a geodesic segment that realizes the set distance between $\alpha[s,\infty)$ and β; then*

$$\gamma = \beta[a,t_\gamma] \cup [\beta(t_\gamma),\alpha(s_\gamma)] \cup \alpha[s_\gamma,\infty)$$

is a $(4\mathsf{q}, 3\mathsf{Q})$–quasigeodesic.

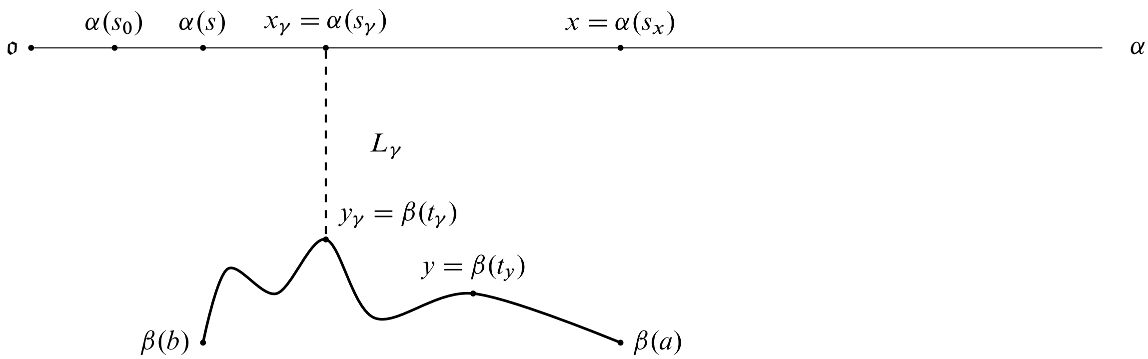

Figure 6: The proof of Lemma 3.7.

Proof Define

$$r_0 := 2\max\Big\{\sup_{y\in\beta}\|y\|, \operatorname{diam}(\beta)\Big\},$$

and let s_0 be such that, for all $s \ge s_0$ we have $\|\alpha(s)\| > r_0$ (this is possible because $\|\alpha(s)\| = \infty$ as $s \to \infty$). Now fix $s \ge s_0$ and let $s_\gamma \ge s$ and $t_\gamma \in [a,b]$ be such that $[\beta(t_\gamma), \alpha(s_\gamma)]$ is a geodesic segment that realizes the set distance between $\alpha[s,\infty)$ and β. The times s_γ and t_γ exist because the metric space X is proper. We relabel the points

$$x_\gamma := \alpha(s_\gamma) \quad \text{and} \quad y_\gamma := \beta(t_\gamma).$$

Let $L_\gamma := d(x_\gamma, y_\gamma)$. By the triangle inequality and from the definition of s_0,

$$L_\gamma \le d(\mathfrak{o}, x_\gamma) + d(\mathfrak{o}, y_\gamma) \le \|x_\gamma\| + \sup_{y\in\beta}\|y\| \le \|x_\gamma\| + \tfrac{1}{2}r_0 \le \tfrac{3}{2}\|x_\gamma\|.$$

Therefore,

$$2\|x_\gamma\| - L_\gamma \ge 2\|x_\gamma\| - \tfrac{3}{2}\|x_\gamma\| \ge \tfrac{1}{2}\|x_\gamma\| \ge \tfrac{1}{2}r_0 \ge \operatorname{diam}(\beta),$$

which we rearrange as

$$\|x_\gamma\| \ge \tfrac{1}{2}(L_\gamma + \operatorname{diam}(\beta)). \tag{4}$$

Since $\beta(t_\gamma)$ is the closest point in β to x_γ, Lemma 3.6 implies that the subsegment

$$\beta[a, t_\gamma] \cup [\beta(t_\gamma), \alpha(s_\gamma)]$$

of γ is a $(3\mathrm{q}, 3\mathrm{Q})$–quasigeodesic. Also, since $\alpha(s_\gamma)$ is the closest point in $\alpha[s,\infty)$ to y_γ, Lemma 3.6 implies that the subsegment

$$[\beta(t_\gamma), \alpha(s_\gamma)] \cup \alpha[s_\gamma, \infty)$$

of γ is a $(3\mathrm{q}, 3\mathrm{Q})$–quasigeodesic. It remains to consider a pair of points $y = \beta(t_y)$ and $x = \alpha(s_x)$ with $t_y \in [a, t_\gamma]$ and $s_x \in \alpha[s_\gamma, \infty)$. The required upper bound for $d(x,y)$ follows from the fact that γ is a concatenation of geodesics and quasigeodesics. We now find a lower bound for $d(x,y)$,

$$
\begin{aligned}
d(x,y) \ge \|x\| - \|y\| &\ge (\|x\| - \|x_\gamma\|) + \|x_\gamma\| - \tfrac{1}{2} r_0 && \text{(definition of } r_0\text{)} \\
&\ge (\|x\| - \|x_\gamma\|) + \tfrac{1}{2}\|x_\gamma\| && \text{(since } s_x \ge s_0\text{)} \\
&\ge (\|x\| - \|x_\gamma\|) + \tfrac{1}{4}(L_\gamma + \operatorname{diam}(\beta)) && \text{(by (4))} \\
&\ge (\|x_\gamma\| - \|x_\gamma\|) + \tfrac{1}{4}(L_\gamma + d(y, y_\gamma)) \\
&\ge \frac{1}{\mathsf{q}}(s_x - s_\gamma) - \mathsf{Q} + \frac{1}{4} L_\gamma + \frac{1}{4\mathsf{q}}(t_\gamma - t_y) - \frac{\mathsf{Q}}{4} && (\alpha \text{ and } \beta \text{ are } (\mathsf{q},\mathsf{Q})\text{–quasigeodesics}) \\
&\ge \frac{(s_x - s_\gamma) + L_\gamma + (t_\gamma - t_y)}{4\mathsf{q}} - \frac{5\mathsf{Q}}{4}.
\end{aligned}
$$

Thus γ is a $(4\mathsf{q}, 3\mathsf{Q})$–quasigeodesic ray. □

Lemma 3.8 (surgery lemma [Qing and Rafi 2022, Lemma 4.3]) *For every $\mathsf{q}, \mathsf{Q}, r > 0$ there exists $R > 0$ such that the following holds. Let γ be a geodesic ray of length at least R and α be a (q, Q)–quasigeodesic ray. Assume that $d_X(\gamma_r, \alpha) \le \frac{1}{2} r$. Then there exists a $(9\mathsf{q}, \mathsf{Q})$–quasigeodesic ray γ' such that*

$$\alpha|_{r/2} = \gamma'|_{r/2}$$

and the portion of γ' outside of $B(\mathfrak{o}, R)$ is the same as γ.

3.3 κ–weakly Morse rays

As in [Qing and Rafi 2022], we also define a different notion of sublinearly Morse which more closely matches the usual definition of Morse.

Definition 3.9 Let $Z \subseteq X$ be a closed set, and let κ be a concave sublinear function. We say that Z is *κ–weakly Morse* if there exists a proper function $m_Z : \mathbb{R}^2 \to \mathbb{R}$ such that for any (q, Q)–quasigeodesic $\gamma : [s,t] \to X$ with endpoints on Z,

$$\gamma([s,t]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q}, \mathsf{Q})).$$

The function m_Z will be called a *κ–weakly Morse gauge* of Z.

As in the setting of CAT(0) metric spaces, we show that the two notions are equivalent.

Proposition 3.10 *Let $\alpha : [0, \infty) \to X$ be a quasigeodesic ray. Then α is κ–Morse if and only if it is κ–weakly Morse.*

Proof First assume that α is κ–Morse. Let $\beta : [t_0, t_3] \to X$ be a (q, Q)–quasigeodesic with endpoints on α. We need to check that β stays in a κ–neighborhood of α (see Definition 3.2). Making q and Q larger if necessary, we can assume α is also a (q, Q) quasigeodesic. We will find a $(12\mathsf{q}, 9\mathsf{Q})$–quasigeodesic ray γ' that is eventually equal to α and covers a substantial portion of β; that is, the size of what remains is

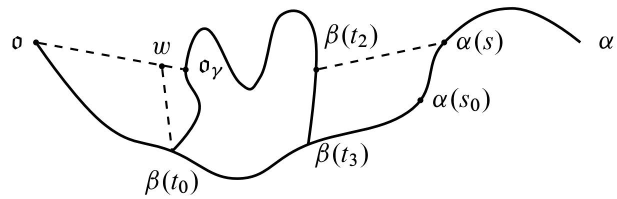

Figure 7: The path $[\beta(t_0), \mathfrak{o}_\gamma]_\beta \cup [\mathfrak{o}_\gamma, w]$ is a quasigeodesic. Hence the distance $d(\beta(t_0), w)$ is a coarse upper bound for the distance between any point in $[\beta(t_0), \mathfrak{o}_\gamma]$ and α.

κ–small. Since α is κ–Morse, γ' is contained in $\mathcal{N}_\kappa(\alpha, m_\alpha(12\mathsf{q}, 9\mathsf{Q}))$. Hence, the same is true for β for a larger κ–neighborhood.

Choose $s_0 \in [0, \infty)$ as in Lemma 3.7 for β and α. Let $\beta(t_2)$ in β and $\alpha(s)$ in $\alpha[s_0, \infty)$ be points such that $[\beta(t_2), \alpha(s)]$ is a segment that realizes the distance between β and $\alpha[s_0, \infty)$. Define γ as a concatenation

$$\gamma = \beta[t_0, t_2] \cup [\beta(t_2), \alpha(s)] \cup \alpha[s, \infty).$$

Then, by Lemma 3.7, γ is a $(4\mathsf{q}, 3\mathsf{Q})$–quasigeodesic.

Now we apply Lemma 3.6 to γ and the basepoint $\mathfrak{o}$. That is, let $\mathfrak{o}_\gamma$ be a closest point in γ to $\mathfrak{o}$. Taking s_0 large enough, we know that $\mathfrak{o}_\gamma$ is either on the segment $\beta[t_0, t_2]$ or the segment $[\beta(t_2), \alpha(s)]$. We define $t_1 \in [t_0, t_2]$ such that, in the first case $\mathfrak{o}_\gamma = \beta(t_1)$ and in the second case $t_1 = t_2$. Let

$$\gamma' = [\mathfrak{o}, \mathfrak{o}_\gamma] \cup [\mathfrak{o}_\gamma, \alpha(s)]_\gamma \cup \alpha[s, \infty).$$

By Lemma 3.6, γ' is a $(12\mathsf{q}, 9\mathsf{Q})$–quasigeodesic.

Let w be a closest point in γ' to $\beta(t_0)$. By Lemma 3.4,

$$d(\beta(t_0), w) \le 2m_\alpha(12\mathsf{q}, 9\mathsf{Q}) \cdot \kappa(\beta(t_0)).$$

Note that an initial segment of γ' is contained in the union of segments $[\mathfrak{o}, \mathfrak{o}_\gamma]$, $\beta[t_0, t_2]$ and $[\beta(t_2), \alpha(s)]$ and w can be in either of these segments. But in every case, there is a $(12\mathsf{q}, 9\mathsf{Q})$–quasigeodesic γ'' connecting $\beta(t_0)$ to w that contains $\beta[t_0, t_1]$. Namely,

$$\gamma'' := \begin{cases} \beta[t_0, t_1] \cup [\mathfrak{o}_\gamma, w] & \text{if } w \in [\mathfrak{o}, \mathfrak{o}_\gamma], \\ [\beta(t_0), w]_\beta & \text{if } w \in \beta[t_0, t_2], \\ \beta[t_0, t_2] \cup [\beta(t_2), w] & \text{if } w \in [\beta(t_2), \alpha(s)]. \end{cases}$$

The fact that γ'' is a $(12\mathsf{q}, 9\mathsf{Q})$–quasigeodesic follows again from Lemmas 3.6 and 3.7.

Then there are constants c_1, c_2 and c_3 depending only on q and Q such that, for any point $u \in \gamma''$,

$$d(\beta(t_0), u) \le c_1 d(\beta(t_0), w) + c_2 \le c_3 m_\alpha(12\mathsf{q}, 9\mathsf{Q}) \cdot \kappa(\beta(t_0)),$$

which also implies $\kappa(u) \asymp \kappa(\beta(t_0))$ (see [Qing and Rafi 2022, Lemma 3.2]). Since γ'' contains $\beta[t_0, t_1]$,

$$\beta[t_0, t_1] \subset \mathcal{N}_\kappa(\alpha, c_4 m_\alpha(12\mathsf{q}, 9\mathsf{Q}))$$

for some c_4 depending only of q and Q.

But $\beta[t_1, t_2]$ is a subset of γ' and by the definition of κ–Morse,

$$\beta[t_1, t_2] \subset \gamma' \subset \mathcal{N}_\kappa(\alpha, m_\alpha(12\mathsf{q}, 9\mathsf{Q})).$$

Hence, $\beta[t_0, t_2]$ is contained in a κ–neighborhood of α.

A similar proof shows that the segment $\beta[t_2, t_3]$ also stays in a κ–neighborhood of α. This finishes the proof of the first direction.

To see the other direction, assume α is a κ–weakly Morse quasigeodesic ray with κ–weakly Morse gauge m'_α. For a given r and a sublinear function κ', choose $R > 0$ such that

$$R > 2\kappa'(R) + r. \tag{5}$$

For a (q, Q)–quasigeodesic γ, let $s \geq 0$ be such that

$$d(\gamma_R, \alpha(s)) \leq \kappa'(R).$$

Then

$$\|\alpha(s)\| \geq \|\gamma_R\| - \kappa'(R) = R - \kappa'(R).$$

Let $\gamma(t)$ be the closest point in γ to $\alpha(s)$, By Lemma 3.6, the concatenation

$$\beta = \gamma[0, t] \cup [\gamma(t), \alpha(s)]$$

is a $(3\mathsf{q}, 3\mathsf{Q})$–quasigeodesic with end points on α. Therefore, by the definition of κ–weakly Morse, $\beta \subset \mathcal{N}_\kappa(\alpha, m'_\alpha(3\mathsf{q}, 3\mathsf{Q}))$. We now show that β contains $\gamma|_r$. That is, if $\gamma_r = \gamma(t_0)$, we show that $t \geq t_0$. Assume, for contradiction, that $t < t_0$ which implies $\|\gamma(t)\| \leq r$. Then

$$\kappa'(R) \geq d(\gamma_R, \alpha(s)) \geq d(\gamma(t), \alpha(s)) \geq \|\alpha(s)\| - \|\gamma(t)\| \geq (R - \kappa'(R)) - r.$$

Solving for R, we get $R \leq 2\kappa'(R) + r$ which contradicts (5). The contradiction implies that β contains $\gamma|_r$ and hence

$$\gamma|_r \subset \mathcal{N}_\kappa(\alpha, m'_\alpha(3\mathsf{q}, 3\mathsf{Q})).$$

That is, α is κ–Morse with the Morse gauge $m_\alpha(\mathsf{q}, \mathsf{Q}) = m'_\alpha(3\mathsf{q}, 3\mathsf{Q})$. □

4 The topology on the κ–Morse boundary

We denote by $\partial_\kappa X$ the set of all equivalence classes of κ–Morse quasigeodesic rays. In this section we will define a topology on $\partial_\kappa X$ to make it into a topological space. Even more, we will construct a bordification $X \cup \partial_\kappa X$ of X. Recall that, for topological spaces $X \subseteq Y$, we say Y is a *bordification* of X if X is a topological subspace of Y and X is open and dense in Y. Hence, we define a topology on $X \cup \partial_\kappa X$ that make it a bordification of X. We will also show that both X and $X \cup \partial_\kappa X$ have good topological properties.

4.1 Bordification

Let β be a κ–Morse quasigeodesic ray based at $\mathfrak{o}$ and let m_β be a Morse gauge for β.

Definition 4.1 We define the set $\mathcal{U}(\beta, r) \subseteq X \cup \partial_\kappa X$ as follows.

- An equivalence class $\boldsymbol{a} \in \partial_\kappa X$ belongs to $\mathcal{U}(\beta, r)$ if, for every (q, Q)–quasigeodesic ray $\alpha \in \boldsymbol{a}$, where $m_\beta(\mathsf{q}, \mathsf{Q})$ is small compared to r (in the sense of (2)), we have the inclusion
$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q})).$$
- A point $p \in X$ belongs to $\mathcal{U}(\beta, r)$ if $d_X(\mathfrak{o}, p) \geq r$ and, for every (q, Q)–quasigeodesic α between $\mathfrak{o}$ and p where $m_\beta(\mathsf{q}, \mathsf{Q})$ is small compared to r, we have
$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q})).$$

We denote $\mathcal{U}(\beta, r) \cap \partial_\kappa X$ by $\partial\mathcal{U}(\beta, r)$.

We now verify the following basic properties of $\mathcal{U}(\beta, r)$.

Lemma 4.2 *Let β be a quasigeodesic ray which belongs to the class $\boldsymbol{b}$. Then*

(1) *there exists a (not necessarily unique) geodesic ray in the class $\boldsymbol{b}$;*

(2) *the class $\boldsymbol{b}$ belongs to $\mathcal{U}(\beta, r)$ for any $r > 0$;*

(3) $\bigcap_{r>0} \mathcal{U}(\beta, r) = \{\boldsymbol{b}\}$;

(4) *if $\beta_1 \sim \beta_2$ are two κ–Morse quasigeodesic rays, then for any $r_1 > 0$ there exists $r_2 > 0$ such that*
$$\mathcal{U}(\beta_1, r_1) \supseteq \mathcal{U}(\beta_2, r_2).$$

Proof (1) Consider the geodesic segment β_n connecting $\mathfrak{o}$ and $\beta(n)$ for $n = 1, 2, 3, \ldots$; this sequence of geodesic segments has a subsequence converging to a geodesic ray β' by Arzelá–Ascoli. Since β is κ–Morse, it is κ–weakly Morse and, for $m'(\cdot\,,\cdot)$ as in Proposition 3.10,
$$\beta_n \subset \mathcal{N}_\kappa(\beta, m'_\beta(1, 0)).$$
Therefore the same is true for β'; hence β' belongs to the class $\boldsymbol{b}$.

(2) If β' is a (q, Q)–quasigeodesic ray which belongs to $\boldsymbol{b}$, then by Lemma 3.4
$$\beta' \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q}));$$
hence also
$$\beta'|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q}))$$
for any $r > 0$, as needed.

(3) Let $\boldsymbol{c} \in \partial_\kappa X$ be a class which belongs to $\mathcal{U}(\beta, r)$ for any r, let $(\mathsf{q}', \mathsf{Q}')$ be the quasigeodesic constants of β, and let $\gamma \in \boldsymbol{c}$ be a (q, Q)–quasigeodesic ray. Let β' be a geodesic ray based at $\mathfrak{o}$ with $\beta' \sim \beta$.

Take $y \in \gamma$ such that $\|y\| = r$. Then, by assumption,

$$\gamma|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q})),$$

for any r. Hence, if p is a nearest point projection of y onto β, then

$$d_X(y, \beta) = d_X(y, p) \le m_\beta(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r).$$

Moreover, by Lemma 2.2, we have $\|p\| \le 2\|y\| = 2r$ and, by Lemma 3.4(i),

$$d_X(y, \beta') \le d_X(y, p) + d_X(p, \beta') \le m_\beta(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r) + 2m_{\beta'}(\mathsf{q}', \mathsf{Q}') \cdot \kappa(r).$$

Setting

$$\widetilde{m}_{\beta'}(\mathsf{q}, \mathsf{Q}) := m_\beta(\mathsf{q}, \mathsf{Q}) + 2m_{\beta'}(\mathsf{q}', \mathsf{Q}'),$$

we get

$$\gamma|_r \subseteq \mathcal{N}_\kappa(\beta', \widetilde{m}_{\beta'}(\mathsf{q}, \mathsf{Q}))$$

for any $r > 0$.

Let now p_r be a nearest point projection of γ_r to β'. Then

$$|d_X(\mathfrak{o}, p_r) - r| = |d_X(\mathfrak{o}, p_r) - d_X(\mathfrak{o}, \gamma_r)| \le d_X(\gamma_r, p_r) \le \widetilde{m}_{\beta'}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r).$$

Since β' is geodesic,

$$d_X(p_r, \beta'_r) = |d_X(\mathfrak{o}, p_r) - r| \le m_{\beta'}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r)$$

and

$$d_X(\gamma_r, \beta'_r) \le d_X(\gamma_r, p_r) + d_X(p_r, \beta'_r) \le 2\widetilde{m}_{\beta'}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r).$$

But $\kappa(r)$ is sublinear; therefore,

$$\lim_{r \to \infty} \frac{d_X(\gamma_r, \beta'_r)}{r} = 0,$$

which implies $\gamma \in \boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{b}$. Finally, since $p \in \mathcal{U}(\beta, r)$ implies $d_X(\mathfrak{o}, p) \ge r$, the intersection $\bigcap_{r>0} \mathcal{U}(\beta, r)$ does not contain any point of X.

(4) Let β_1 be a $(\mathsf{q}_1, \mathsf{Q}_1)$–quasigeodesic ray, β_2 a $(\mathsf{q}_2, \mathsf{Q}_2)$–quasigeodesic ray with $\beta_1 \sim \beta_2$, and let $r_1 > 0$. For $r > 0$, let $\boldsymbol{a} \in \mathcal{U}(\beta_2, r)$, and pick $\alpha \in \boldsymbol{a}$ a (q, Q)–quasigeodesic ray such that $m_{\beta_2}(\mathsf{q}, \mathsf{Q})$ is small compared to r. By definition of $\mathcal{U}(\beta_2, r)$,

$$d_X(\alpha_r, \beta_2) \le m_{\beta_2}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r).$$

Let p_r be a nearest point projection of α_r to β_2. By Lemma 2.2,

$$\|p_r\| \le 2r.$$

Moreover, by Lemma 3.4 (i),

$$d_X(p_r, \beta_1) \le \kappa(\|p_r\|) m_{\beta_1}(\mathsf{q}_2, \mathsf{Q}_2) \le 2m_{\beta_1}(\mathsf{q}_2, \mathsf{Q}_2) \cdot \kappa(r);$$

hence

$$d_X(\alpha_r, \beta_1) \le m_{\beta_2}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r) + 2m_{\beta_1}(\mathsf{q}_2, \mathsf{Q}_2) \cdot \kappa(r). \tag{6}$$

Now, if we take

$$\kappa'(r) := m_{\beta_2}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r) + 2m_{\beta_1}(\mathsf{q}_2, \mathsf{Q}_2) \cdot \kappa(r),$$

by Definition 3.2, there exists r_2 such that (6) for $r = r_2$ implies

$$\alpha|_{r_1} \subseteq \mathcal{N}_\kappa(\beta_1, m_{\beta_1}(\mathsf{q}, \mathsf{Q}));$$

hence $\boldsymbol{a} \in \mathcal{U}(\beta_1, r_1)$, as required. □

We now verify that a sequence of points of X that sublinearly tracks a quasigeodesic ray γ, converges to the class of γ in $\partial_\kappa X$.

Lemma 4.3 *Let $\gamma \in \boldsymbol{c}$ be a κ–Morse quasigeodesic ray based at $\mathfrak{o} \in X$ and let $(x_n) \subseteq X$ be a sequence of points with $\|x_n\| \to \infty$. Moreover, suppose that there exists a constant $C > 0$ such that*

$$d_X(x_n, \gamma) \le C \cdot \kappa(\|x_n\|) \tag{7}$$

for all n. Then the sequence (x_n) converges to $\boldsymbol{c}$ in the topology of $X \cup \partial_\kappa X$.

Proof In order to show the claim, we need to prove that for any quasigeodesic ray $\beta \in \boldsymbol{c}$ and any $r > 0$ there exists n_0 such that for all $n \ge n_0$,

$$x_n \in \mathcal{U}(\beta, r).$$

Equivalently, we need to show that, for any r and any (q, Q)–quasigeodesic segment α joining $\mathfrak{o}$ and x_n with $m_\beta(\mathsf{q}, \mathsf{Q})$ small with respect to r,

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q})).$$

Let p_n be a nearest point projection of x_n onto γ; by Lemma 2.2, $\|p_n\| \le 2\|x_n\|$. Now, by (7) and Lemma 3.4,

$$d_X(x_n, \beta) \le d_X(x_n, p_n) + d_X(p_n, \beta) \le C \cdot \kappa(\|x_n\|) + 2m_\beta(\mathsf{q}', \mathsf{Q}') \cdot \kappa(\|x_n\|)$$

where $(\mathsf{q}', \mathsf{Q}')$ are the quasigeodesic constants of γ. Note moreover that β is κ–Morse by Lemma 3.4. Hence, consider the sublinear function

$$\kappa'(r) := (C + 2m_\beta(\mathsf{q}', \mathsf{Q}')) \cdot \kappa(r)$$

and apply the definition of κ–Morse to obtain R such that if $d_X(x_n, \beta) \le \kappa'(R)$ then $\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q}))$. Thus, if we choose n_0 such that $\|x_n\| \ge R$ for all $n \ge n_0$, the definition of κ–Morse implies

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q})).$$ □

We now show that the sets $\mathcal{U}(\beta, r)$ and $\partial\mathcal{U}(\beta, r)$ can be used as a neighborhood basis to define topologies on $X \cup \partial_\kappa X$ and $\partial_\kappa X$.

Definition 4.4 Given an equivalence class $\boldsymbol{b}$, we define the set $\mathcal{B}(\boldsymbol{b})$ as the set of subsets $\mathcal{V} \subseteq X \cup \partial_\kappa X$ such that there exists $\beta \in \boldsymbol{b}$ and $r > 0$ for which $\mathcal{U}(\beta, r) \subseteq \mathcal{V}$. Let $\partial\mathcal{B}(\boldsymbol{b})$ be the set of subsets of $\partial_\kappa X$ of the form $\mathcal{V} \cap \partial_\kappa X$ where $\mathcal{V} \in \mathcal{B}(\boldsymbol{b})$; equivalently, a set in $\partial\mathcal{B}(\boldsymbol{b})$ contains a set of the form $\partial\mathcal{U}(\beta, r)$. Also, for $x \in X$, define $\mathcal{B}(x)$ to be the set of subsets $\mathcal{V}$ of X such that $\mathcal{V}$ contains a ball $B(x, r)$ of radius r centered at x.

Lemma 4.5 *For every $\boldsymbol{b} \in \partial_\kappa X$, the set $\mathcal{B}(\boldsymbol{b})$ satisfies the following properties*:

(i) *Every subset of $X \cup \partial_\kappa X$ which contains a set belonging to $\mathcal{B}(\boldsymbol{b})$ itself belongs to $\mathcal{B}(\boldsymbol{b})$.*

(ii) *Every finite intersection of sets of $\mathcal{B}(\boldsymbol{b})$ belongs to $\mathcal{B}(\boldsymbol{b})$.*

(iii) *The element $\boldsymbol{b}$ is in every set of $\mathcal{B}(\boldsymbol{b})$.*

(iv) *If $\mathcal{V} \in \mathcal{B}(\boldsymbol{b})$ then there is a $\mathcal{W} \in \mathcal{B}(\boldsymbol{b})$ such that $\mathcal{V} \in \mathcal{B}(\boldsymbol{a})$ for every $\boldsymbol{a} \in \mathcal{W}$.*

Furthermore, *the same is true for subsets of $\partial\mathcal{B}(\boldsymbol{b})$ and $\mathcal{B}(x)$.*

Proof We prove the lemma for $\mathcal{B}(\boldsymbol{b})$. The proof for $\partial\mathcal{B}(\boldsymbol{b})$ is identical. The proof for $\mathcal{B}(x)$ is immediate from the fact that the open balls in X define a neighborhood basis for X.

(i) This is immediate from the definition of $\mathcal{B}(\boldsymbol{b})$.

(ii) It is enough to show that, for $\beta_1, \dots, \beta_k \in \boldsymbol{b}$ and $r_1, \dots, r_k > 0$, the intersection

$$\mathcal{U}(\beta_1, r_1) \cap \mathcal{U}(\beta_2, r_2) \cap \cdots \cap \mathcal{U}(\beta_k, r_k),$$

belongs to $\mathcal{B}(\boldsymbol{b})$. By Lemma 4.2(4), for any $i = 1, \dots, k$ there exists R_i such that

$$\mathcal{U}(\beta_i, r_i) \supseteq \mathcal{U}(\beta_1, R_i).$$

Thus, if we set $r := \max_{1 \le i \le k}\{R_i\}$ then

$$\bigcap_{i=1}^{k} \mathcal{U}(\beta_i, r_i) \supseteq \mathcal{U}(\beta_1, r),$$

and hence the intersection belongs to $\mathcal{B}(\boldsymbol{b})$.

(iii) Established by Lemma 4.2(2).

(iv) We need to prove the following claim.

Claim 4.6 *For any $\mathcal{U}(\beta, r)$, there exists r' (usually larger than r) such that if $\boldsymbol{a} \in \mathcal{U}(\beta, r')$ then there exists r'' (depending on α and r' but not on β) such that $\mathcal{U}(\alpha, r'') \subseteq \mathcal{U}(\beta, r)$ for some α representative of $\boldsymbol{a}$.*

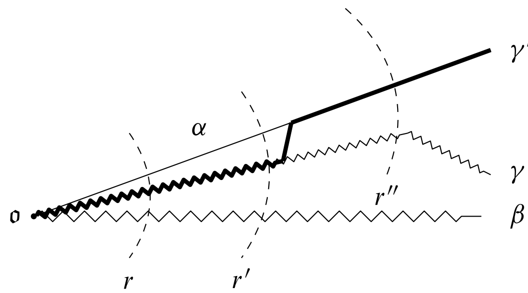

Figure 8: The proof of Claim 4.6.

In particular, we will prove the claim for a geodesic representative $\alpha \in \boldsymbol{a}$, whose existence is established by Lemma 4.2(1). We adapt here the proof from [Qing and Rafi 2022].

Let us pick a κ–Morse quasigeodesic ray β and $r > 0$. Let

$$M := \sup_{m_\beta(\mathsf{q},\mathsf{Q}) \le r} m_\beta(9\mathsf{q},\mathsf{Q})$$

and let r' be such that

(1) $r' > 2r$,

(2) $M \le r'/2\kappa(r')$,

(3) $r' > R(\beta, r, M\kappa)$.

Let α be a geodesic representative of $\boldsymbol{a} \in \mathcal{U}(\beta, r')$. Choose r'' such that

$$r'' \ge 2r' \quad \text{and} \quad \sup_{m_\beta(\mathsf{q},\mathsf{Q}) \le r} m_\alpha(\mathsf{q},\mathsf{Q}) \le \frac{r''}{4\kappa(r'')}.$$

Now consider $\boldsymbol{c} \in \mathcal{U}(\alpha, r'')$. Let $\gamma \in \boldsymbol{c}$ be a (q,Q)–quasigeodesic ray, with $m_\beta(\mathsf{q},\mathsf{Q})$ small compared to r. By the choice of r'' above and by Lemma 3.1,

$$d_X(\alpha_{r''}, \gamma) \le 2m_\alpha(\mathsf{q},\mathsf{Q}) \cdot \kappa(r'') \le \tfrac{1}{2} r''.$$

We apply Lemma 3.8, with radius being r'', to modify γ to a $(9\mathsf{q},\mathsf{Q})$–quasigeodesic ray $\gamma' \in \boldsymbol{a}$. Since $r' \le \frac{1}{2} r''$, we have $\gamma|_{r'} = \gamma'|_{r'}$.

Also, $\gamma' \in \boldsymbol{a} \in \mathcal{U}(\beta, r')$ and $m_\beta(9\mathsf{q},\mathsf{Q})$ is small compared to r' (by point (2) above); therefore

$$\gamma|_{r'} = \gamma'|_{r'} \subset \mathcal{N}_\kappa(\beta, m_\beta(9\mathsf{q},\mathsf{Q})).$$

Hence, by the choice of r' (point (3) above) and Definition 3.2, we obtain

$$\gamma|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q},\mathsf{Q})).$$

This holds for every $\gamma \in \boldsymbol{c}$ with $m_\beta(\mathsf{q},\mathsf{Q})$ small compared to r, thus $\boldsymbol{c} \in \mathcal{U}(\beta, r)$. And this argument holds for every $\boldsymbol{c} \in \mathcal{U}(\alpha, r'')$; therefore $\mathcal{U}(\alpha, r'') \subset \mathcal{U}(\beta, r)$. $\square$

These properties for $\mathcal{B}(x)$, $\mathcal{B}(\boldsymbol{b})$ and $\partial\mathcal{B}(\boldsymbol{b})$ are characteristic of the set of neighborhoods of $\boldsymbol{b}$, as stated in the following proposition.

Proposition 4.7 [Bourbaki 1989, Proposition 2] *Let Y be a set. If to each element $y \in Y$ there corresponds a set $\mathcal{B}(y)$ of subsets of Y such that properties* (i)–(iv) *from Lemma 4.5 are satisfied, then there is a unique topological structure on Y such that for each $y \in Y$, $\mathcal{B}(y)$ is the set of neighborhoods of y in this topology.*

Thus we now use the sets $\partial\mathcal{B}(b)$ to equip $\partial_\kappa X$ with a topological structure and use the sets $\mathcal{B}(b)$ and $\mathcal{B}(x)$ to equip $X \cup \partial_\kappa X$ with a topological structure. Note that, since neighborhoods in $\partial_\kappa X$ are intersections of neighborhoods of $X \cup \partial_\kappa X$ with $\partial_\kappa X$, the inclusion $\partial_\kappa X \subset X \cup \partial_\kappa X$ is a topological embedding and $X \cup \partial_\kappa X$ is a bordification of X.

Recall that a set is open if it contains a neighborhood of each of its points. Thus, a set $\mathcal{W} \subseteq \partial_\kappa X$ is open if for every $\boldsymbol{b} \in \mathcal{W}$ there is $\beta \in \boldsymbol{b}$ and $r > 0$ such that $\partial\mathcal{U}(\beta, r) \subset \mathcal{W}$. A set $\mathcal{W} \subset X \cup \partial_\kappa X$ is open if its intersection with both X and $\partial_\kappa X$ is open.

4.2 Metrizability

We now establish the metrizability of the space $\partial_\kappa X$. To begin with, we need the following property of the topology:

Lemma 4.8 *For each κ–Morse quasigeodesic ray β and $r > 0$, there exists a radius $r' > 0$ such that for any point $\boldsymbol{a} \in \partial_\kappa X$ there exists $r'' > 0$ (depending only on $\boldsymbol{a}$ and r' and not on β) such that for every geodesic representative $\alpha_0 \in \boldsymbol{a}$,*

$$\mathcal{U}(\alpha_0, r'') \cap \mathcal{U}(\beta, r') \neq \varnothing \implies \boldsymbol{a} \in \partial\mathcal{U}(\beta, r).$$

Similarly, for $x \in X$, let $B(x, 1)$ be the ball of radius 1 centered at x. Then

$$B(x, 1) \cap \mathcal{U}(\beta, r') \neq \varnothing \implies x \in \mathcal{U}(\beta, r).$$

Proof This will be done using the surgery lemma (Lemma 3.8). Pick a κ–Morse quasigeodesic ray β and $r > 0$. Let $\mathfrak{Q} := \{(\mathsf{q}, \mathsf{Q}) \mid m_\beta(\mathsf{q}, \mathsf{Q}) \le r/2\kappa(r)\}$, which is bounded by properness. Set

$$M := \sup_{(\mathsf{q},\mathsf{Q}) \in \mathfrak{Q}} m_\beta(9\mathsf{q}, \mathsf{Q} + 1),$$

and $r' := R(\beta, r, M\kappa)$. Let $\boldsymbol{a} \in \partial_\kappa X$.

By Corollary 3.5, there exists a constant $u > 0$ such that, for any geodesic ray $\star \in \boldsymbol{a}$ and any $(\mathsf{q}, \mathsf{Q}) \in \mathfrak{Q}$,

$$m_\star(1, 0) + 3m_\star(\mathsf{q}, \mathsf{Q}) \le u.$$

Let R be the radius given in Lemma 3.8 associated to q, Q and $2r'$, and let r'' be large enough that

$$r'' \ge \max\{2u \cdot \kappa(r''), 2r', R\}.$$

Let α_0 be a geodesic ray in $\boldsymbol{a}$. By assumption, there is a point $\boldsymbol{c}$ inside the intersection

$$\boldsymbol{c} \in \mathcal{U}(\alpha_0, r'') \cap \mathcal{U}(\beta, r').$$

If $\boldsymbol{c} \in \partial_\kappa X$, let $\gamma \in \boldsymbol{c}$ be a geodesic ray in this class and if $\boldsymbol{c} \in X$, let γ be a geodesic ray connecting $\mathfrak{o}$ to $\boldsymbol{c}$. In either case, $\gamma_{r''}$ is well defined since, in the second case, $d_X(\mathfrak{o}, \boldsymbol{c}) \geq r''$. Let $\alpha \in \boldsymbol{a}$ be a (q, Q)–quasigeodesic ray with $m_\beta(\mathsf{q}, \mathsf{Q})$ small compared to r. To conclude $\boldsymbol{a} \in \partial\mathcal{U}(\beta, r)$ we need to show that $\alpha|_r \subset \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q}))$.

Since $\boldsymbol{c} \in \mathcal{U}(\alpha_0, r'')$,

$$d_X(\gamma_{r''}, \alpha_0) \leq m_{\alpha_0}(1, 0) \cdot \kappa(r'').$$

Let $p \in \pi_{\alpha_0}(\gamma_{r''})$. By definition of u and r'',

$$\|p\| \leq r'' + m_{\alpha_0}(1, 0) \cdot \kappa(r'') \leq \tfrac{3}{2} r''.$$

Therefore, Lemma 3.4(ii) implies

$$d_X(p, \alpha) \leq 2m_{\alpha_0}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(p) \leq 3m_{\alpha_0}(\mathsf{q}, \mathsf{Q}) \cdot \kappa(r'').$$

Hence,

$$d_X(\gamma_{r''}, \alpha) \leq d_X(\gamma_{r''}, p) + d_X(p, \alpha) \leq u \cdot \kappa(r'') \leq \tfrac{1}{2} r''.$$

We can now apply the surgery lemma (Lemma 3.8) to α and γ with radius $2r'$ to obtain a $(9\mathsf{q}, \mathsf{Q})$–quasigeodesic ray γ' that is either in the class $\boldsymbol{c}$ if $\boldsymbol{c} \in \partial_\kappa X$ or ends in $\boldsymbol{c}$ if $\boldsymbol{c} \in X$ where $\gamma'|_{r'} = \alpha|_{r'}$. Since $\boldsymbol{c} \in \mathcal{U}(\beta, r')$,

$$\alpha|_{r'} = \gamma'|_{r'} \subset \mathcal{N}_\kappa(\beta, m_\beta(9\mathsf{q}, \mathsf{Q})).$$

Observing that $\alpha|_{r'}$ is (q, Q)–quasigeodesic and letting $\kappa' = m_\beta(9\mathsf{q}, \mathsf{Q}) \cdot \kappa$, the definition of r' and κ–Morse implies that

$$\alpha|_r \subset \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q}));$$

hence $\boldsymbol{a} \in \mathcal{U}(\beta, r)$.

To see the second assertion, assume $y \in B(x, 1) \cap \mathcal{U}(\beta, r')$. Let α be a (q, Q)–quasigeodesic ray ending at x where $m_\beta(\mathsf{q}, \mathsf{Q})$ is small compared to r. Let γ be a quasigeodesic ray that is identical to α but at the last point is sent to y instead of x. Then γ is $(\mathsf{q}, \mathsf{Q}+1)$–quasigeodesic. The definition of r' and κ–Morse implies that

$$\alpha|_r = \gamma|_r \subset \mathcal{N}_\kappa(\beta, m_\beta(\mathsf{q}, \mathsf{Q}));$$

hence $x \in \mathcal{U}(\beta, r)$. □

Our method for establishing metrizability is via the following criterion.

Theorem 4.9 [Frink 1937, Theorem 3] *Assume, for every point $\boldsymbol{b}$ of a topological space, there exists a monotonically decreasing sequence $\mathcal{V}_1(\boldsymbol{b}), \mathcal{V}_2(\boldsymbol{b}), \dots, \mathcal{V}_i(\boldsymbol{b}), \dots$ of neighborhoods whose intersection is $\boldsymbol{b}$ and such that the following holds: for every point $\boldsymbol{b}$ of the neighborhood space and every integer i, there exists an integer $j = j(\boldsymbol{b}, i) > i$ such that if $\boldsymbol{a}$ is a point for which $\mathcal{V}_j(\boldsymbol{a})$ and $\mathcal{V}_j(\boldsymbol{b})$ have a point in common then $\mathcal{V}_j(\boldsymbol{a}) \subset \mathcal{V}_i(\boldsymbol{b})$. Then the space is homeomorphic to a metric space.*

We check this condition for $\partial_\kappa X$, using as neighborhoods $\mathcal{V}_i(\boldsymbol{b})$ the sets $\partial\mathcal{U}(\beta, r)$ previously defined.

Theorem 4.10 *The space $\partial_\kappa X$ is metrizable.*

Proof Our goal is to construct, for any $i \in \mathbb{N}$ and $\boldsymbol{b} \in \partial_\kappa X$, neighborhoods $\mathcal{V}_i(\boldsymbol{b})$ which satisfy the conditions of Theorem 4.9.

Recall that, given a κ–Morse quasigeodesic ray β and $r > 0$, we can define r' as

$$r'(\beta, r) := R(\beta, r, M\kappa),$$

as in the proof of Lemma 4.8. Let $\boldsymbol{a} \in \partial_\kappa X$ and pick a geodesic representative $\alpha_0 \in \boldsymbol{a}$. Note that in both Claim 4.6 and Lemma 4.8, r'' does not depend on β or r, but it depends on α_0, r' and

$$\sup_{m_\beta(\mathsf{q},\mathsf{Q}) \le r} m_{\alpha_0}(\mathsf{q},\mathsf{Q}).$$

Since $\mathsf{q}, \mathsf{Q} \le m_\beta(\mathsf{q},\mathsf{Q}) \le r \le r'$, the maximum value of $m_{\alpha_0}(\mathsf{q},\mathsf{Q})$ can be bounded in terms of α_0 and r', without referring to β or r. Hence, we can consider the function $r''(\alpha_0, r')$ such that both Claim 4.6 and Lemma 4.8 hold.

For $i \in \mathbb{N}$ define

$$\mathcal{V}_i(\boldsymbol{a}) := \partial\mathcal{U}(\alpha_0, r_i(\boldsymbol{a})) \quad \text{where } \ r_i(\boldsymbol{a}) := \max(i, r''(\alpha_0, i)).$$

Also, given $\boldsymbol{b}$ and i, we define $\rho_i(\boldsymbol{b}) := r'(\beta_0, r_i(\boldsymbol{b}))$, and

$$j = j(\boldsymbol{b}, i) := \lceil r'(\beta_0, \rho_i(\boldsymbol{b}))\rceil,$$

where β_0 is a geodesic representative in $\boldsymbol{b}$. Assume $\mathcal{V}_j(\boldsymbol{a})$ and $\mathcal{V}_j(\boldsymbol{b})$ have a point in common; that is,

$$\partial\mathcal{U}(\alpha_0, r_j(\boldsymbol{a})) \cap \partial\mathcal{U}(\beta_0, r_j(\boldsymbol{b})) \neq \varnothing.$$

Then, since $r_j(\boldsymbol{a}) \ge r''(\alpha_0, j)$ and $r_j(\boldsymbol{b}) \ge j$, by Lemma 4.8,

$$\boldsymbol{a} \in \partial\mathcal{U}(\beta_0, \rho_i(\boldsymbol{b})).$$

Now Claim 4.6 implies

$$\partial\mathcal{U}\big(\alpha_0, r''(\alpha_0, \rho_i(\boldsymbol{b}))\big) \subset \partial\mathcal{U}(\beta_0, r_i(\boldsymbol{b})).$$

But $r_j(\boldsymbol{a}) = \max(j, r''(\alpha_0, j))$; thus

$$r_j(\boldsymbol{a}) \ge r''(\alpha_0, r'(\beta_0, \rho_i(\boldsymbol{b}))) \ge r''(\alpha_0, \rho_i(\boldsymbol{b})).$$

Therefore,

$$\partial\mathcal{U}(\alpha_0, r_j(\boldsymbol{a})) \subset \partial\mathcal{U}(\beta_0, r_i(\boldsymbol{b})),$$

which is to say $\mathcal{V}_j(\boldsymbol{a}) \subset \mathcal{V}_i(\boldsymbol{b})$. The theorem now follows from Theorem 4.9. □

Similarly, we have:

Theorem 4.11 *The space $X \cup \partial_\kappa X$ is metrizable.*

Proof For $i \in \mathbb{N}$ and $\boldsymbol{a} \in \partial_\kappa X$, let $r_i(\boldsymbol{a})$ be as in the proof of Theorem 4.10 and let

$$\mathcal{V}_i(\boldsymbol{a}) := \mathcal{U}(\alpha_0, r_i(\boldsymbol{a})).$$

For a point $x \in X$, we define $\mathcal{V}_i(x) := B(x, 1/i)$, the ball of radius $1/i$ centered around x. Since Lemma 4.8 holds for $\mathcal{U}(\alpha_0, r_i(\boldsymbol{a}))$, the same proof as above works to check the conditions of Theorem 4.9 for any point $\boldsymbol{b} \in \partial_\kappa X$.

For $x \in X$, we define $j(x, i) := 3i$. Then, if

$$\mathcal{V}_j(x) \cap \mathcal{V}_j(y) \neq \varnothing,$$

there is a point $z \in B(x, 1/3i) \cap B(y, 1/3i)$ and, by the triangle inequality,

$$\mathcal{V}_j(y) = B\Big(y, \frac{1}{3i}\Big) \subset B\Big(x, \frac{1}{i}\Big).$$

Also, if $\mathcal{V}_j(x) \cap \mathcal{V}_j(\boldsymbol{b}) \neq \varnothing$, for $x \in X$ and $\boldsymbol{b} \in \partial_\kappa X$, then $B(x,1) \cap \mathcal{U}(\beta, r_j(\boldsymbol{b})) \neq \varnothing$. By the definition of $j(\boldsymbol{b}, i)$ and the second part of Lemma 4.8, this implies that

$$\mathcal{V}_i(x) \subset B(x,1) \subset \mathcal{U}(\beta, r_i(\boldsymbol{b})).$$

Again, the theorem follows from Theorem 4.9. □

We are now ready to establish the quasi-isometric invariance of $\partial_\kappa X$.

Theorem 4.12 *Consider proper geodesic metric spaces X and Y, let $\Phi\colon X \to Y$ be a (k, K)–quasi-isometry and let κ be a concave sublinear function. Then Φ induces a homeomorphism $\Phi^\star\colon \partial_\kappa X \to \partial_\kappa Y$ where, for $\boldsymbol{b} \in \partial_s X$ and $\beta \in \boldsymbol{b}$,*

$$\Phi^\star(\boldsymbol{b}) = [\Phi \circ \beta],$$

where $[\,\cdot\,]$ denotes the equivalence class of a quasigeodesic ray.

The proof is identical to the proof of Theorem 5.1 in [Qing and Rafi 2022].

4.3 The union of $\partial_\kappa X$

We note that topologies of different sublinear boundaries are compatible.

Proposition 4.13 [Qing and Rafi 2022, Proposition 4.10] *Let κ and κ' be sublinear functions such that, for some $M > 0$,*

$$\kappa'(t) \le M \cdot \kappa(t) \quad \text{for all } t > 0.$$

Then $\partial_{\kappa'} X \subset \partial_\kappa X$ as a subspace with the subspace topology.

The proof is identical to the proof of Proposition 4.10 in [Qing and Rafi 2022] and is skipped. In view of this proposition, we can define the *sublinearly Morse boundary* of X as

$$\partial X := \bigcup_\kappa \partial_\kappa X$$

which is the space of equivalence classes (up to sublinear fellow traveling) of all sublinearly Morse quasigeodesic rays in X.

Remark 4.14 An open neighborhood $\mathcal{V}$ of a point $\boldsymbol{b} \in \partial X$ can be described as follows: assume $\boldsymbol{b} \in \partial_\kappa X$ for some κ and choose a quasigeodesic ray $\beta \in \boldsymbol{b}$ and a radius $r > 0$. Let $\mathcal{U}_\kappa(\beta, r)$ be the neighborhood of $\boldsymbol{b}$ in $(X \cup \partial_\kappa X)$ and let $\mathcal{V}$ be the closure of $\mathcal{U}_\kappa(\beta, r)$ in $(X \cup \partial X)$. That is, a point in $\mathcal{V} \cap \partial X$ is a class of κ'–sublinearly Morse quasigeodesic rays for some κ' different from κ that are eventually contained in $\mathcal{U}_\kappa(\beta, r)$. The intersection $\mathcal{V} \cap X$ equals $X \cap \mathcal{U}_\kappa(\beta, r)$.

Similar arguments as in Theorems 4.10 and 4.11 can be used to show that ∂X and $(X \cup \partial X)$ are also metrizable. We skip these, for the sake of brevity.

5 General projections and weakly sublinearly contracting sets

In order to deal with several applications to proper geodesic spaces (in particular, our applications to mapping class groups and relatively hyperbolic groups we shall see next), the usual notion of nearest point projection may be ill-suited; for instance, it is well known that nearest point projection to a closed subset of a general (eg not hyperbolic) metric space need not be, even coarsely, well defined.

Thus, we now introduce a more general notion of projection, which we call *κ–projection*, where we allow an additive error which is controlled by a sublinear function κ.

Let us denote by $\mathcal{P}(Z)$ the set of subsets of Z, and let us use the notation $\kappa(x) := \kappa(\|x\|)$.

Definition 5.1 Let (X, d_X) be a proper geodesic metric space and $Z \subseteq X$ a closed subset, and let κ be a concave sublinear function. A map $\pi_Z : X \to \mathcal{P}(Z)$ is a *κ–projection* if there exist constants D_1 and D_2, depending only on Z and κ, such that for any points $x \in X$ and $z \in Z$,

$$\operatorname{diam}_X(\{z\} \cup \pi_Z(x)) \le D_1 \cdot d_X(x, z) + D_2 \cdot \kappa(x).$$

A κ–projection differs from a nearest point projection by a uniform multiplicative error and a sublinear additive error. In particular, the nearest point projection is a κ–projection. Indeed, for $z \in Z$ and $w \in \pi_Z(x)$,

$$d(z, w) \le d(z, x) + d(x, w) \le 2d(z, x).$$

Lemma 5.2 *Given a closed set Z, for any $x \in X$,*

$$\operatorname{diam}_X(\{x\} \cup \pi_Z(x)) \le (D_1 + 1) \cdot d_X(x, Z) + D_2 \cdot \kappa(x).$$

Proof Let $z \in Z$ be a point that realizes $d_X(x, Z)$. Then, by the triangle inequality and applying Definition 5.1,

$$\operatorname{diam}_X(\{x\} \cup \pi_Z(x)) \le d_X(x, z) + \operatorname{diam}_X(\{z\} \cup \pi_Z(x)) \le (D_1 + 1) \cdot d_X(x, Z) + D_2 \cdot \kappa(x). \qquad \square$$

We now formulate a general definition of *κ–weakly contracting* with respect to a κ–projection π_Z.

Definition 5.3 (κ–weakly contracting) For a closed subspace Z of a metric space (X, d_X) and a κ–projection π_Z onto Z, we say Z is *κ–weakly contracting* with respect to π_Z if there are constants C_1 and C_2, depending only on Z and κ, such that, for every $x, y \in X$,

$$d_X(x, y) \le C_1 \cdot d_X(x, Z) \implies \operatorname{diam}_X(\pi_Z(x) \cup \pi_Z(y)) \le C_2 \cdot \kappa(x).$$

In the special case that π_Z is the nearest point projection and $C_1 = 1$, this property was called *κ–contracting* in [Qing and Rafi 2022]. It was shown in [Qing and Rafi 2022] that, in the setting of CAT(0) spaces, this is stronger than the κ–Morse condition.

With respect to any projection we prove the following analogous statement of [Qing and Rafi 2022, Theorem 3.14]:

Theorem 5.4 (κ–weakly contracting implies sublinearly Morse) *Let κ be a concave sublinear function and let Z be a closed subspace of X. Let π_Z be a κ–projection onto Z and suppose that Z is κ–weakly contracting with respect to π_Z. Then there is a function $m_Z\colon \mathbb{R}^2 \to \mathbb{R}$ such that, for every constant $r > 0$ and every sublinear function κ', there is an $R = R(Z, r, \kappa') > 0$ for which the following holds: let $\eta\colon [0, \infty) \to X$ be a (q, Q)–quasigeodesic ray such that $m_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_r be the first time $\|\eta(t_r)\| = r$ and let t_R be the first time $\|\eta(t_R)\| = R$; then*

$$d_X(\eta(t_R), Z) \le \kappa'(R) \implies \eta([0, t_r]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q}, \mathsf{Q})).$$

The proof of this result is similar to the one in [Qing and Rafi 2022], so we will postpone it to the appendix. Moreover, in the appendix we shall prove the following equivalence between κ–weakly contracting and κ–Morse (with a possibly different sublinear function) for any given closed set.

Theorem 5.5 *Let $(X, \mathfrak{o})$ be a proper geodesic metric space with a fixed basepoint and let α be a quasigeodesic ray in X. Let π be any κ–projection from X to α in the sense of Definition 5.1. Then*

1. *if α is κ–weakly contracting with respect to π, then it is κ–Morse;*
2. *the quasigeodesic α is κ–Morse if an only if it is κ–weakly Morse;*
3. *if α is κ–weakly Morse, then it is κ'–weakly contracting with respect to π for some sublinear function κ'.*

6 The Poisson boundary

We now show a general criterion (Theorem 6.2) for the κ–Morse boundary of a group to be identified with its Poisson boundary.

Random walks

Let G be a locally compact, second countable group, with left Haar measure m, and let μ be a Borel probability measure on G, which we assume to *spread-out*, ie such that there exists n for which μ^n is not singular with respect to m. Given μ, we consider the *step space* $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$, whose elements we denote by (g_n). The *random walk driven by* μ is the G–valued stochastic process (w_n), where for each n we define the product

$$w_n := g_1 g_2 \cdots g_n.$$

We denote by $(\Omega, \mathbb{P})$ the *path space*, ie the space of sequences (w_n), where $\mathbb{P}$ is the measure induced by pushing forward the measure $\mu^{\mathbb{N}}$ from the step space. Elements of Ω are called *sample paths* and will also be denoted by ω. Finally, let $T\colon \Omega \to \Omega$ be the left shift on the path space.

Background on boundaries

Let us recall some fundamental definitions from the boundary theory of random walks. We refer to [Kaimanovich 2000] for more details. Let $(B, \mathcal{A})$ be a measurable space on which G acts by measurable isomorphisms; a measure ν on B is *μ–stationary* if $\nu = \int_G g_\star \nu \, d\mu(g)$, and in that case the pair (B, ν) is called a *(G, μ)–space*. Recall that a *μ–boundary* is a measurable (G, μ)–space (B, ν) such that there exists a T–invariant, measurable map $\mathrm{bnd}\colon (\Omega, \mathbb{P}) \to (B, \nu)$, called the *boundary map*.

Moreover, a function $f\colon G \to \mathbb{R}$ is *μ–harmonic* if $f(g) = \int_G f(gh) \, d\mu(h)$ for any $g \in G$. We denote by $H^\infty(G, \mu)$ the space of bounded, μ–harmonic functions. One says a μ–boundary is the *Poisson boundary* of (G, μ) if the map

$$\Phi\colon H^\infty(G, \mu) \to L^\infty(B, \nu)$$

given by $\Phi(f)(g) := \int_B f \, dg_\star \nu$ is a bijection. The Poisson boundary (B, ν) is the maximal μ–boundary, in the sense that for any other μ–boundary (B', ν') there exists a G–equivariant, measurable map $p\colon (B, \nu) \to (B', \nu')$.

Finally, a metric d on G is *temperate* if there exists C such that

$$m(\{g \in G \mid d(1, g) \le R\}) \le C e^{CR}$$

for any $R > 0$. A measure μ has *finite first moment* with respect to d if $\int_G d(1, g) \, d\mu(g) < +\infty$.

We will use the ray approximation criterion from [Kaimanovich 2000] for the Poisson boundary (for this precise version, see [Forghani and Tiozzo 2022]).

Theorem 6.1 *Let G be a locally compact, second countable group equipped with a temperate metric d, and let μ be a spread-out probability measure on G with finite first moment with respect to d. Let (B, λ) be a μ–boundary, and suppose that there exist maps $\pi_n\colon B \to G$ for any $n \in \mathbb{N}$ such that for almost every sample path $\omega = (w_n)$ we have*

$$\lim_{n \to \infty} \frac{d(w_n, \pi_n(\mathrm{bnd}(\omega)))}{n} = 0. \tag{8}$$

Then (B, λ) is the Poisson boundary of (G, μ).

If a countable group G acts by isometries on a metric space (X, d_X) with a basepoint $\mathfrak{o}$, we define the *growth rate* of the action as

$$v := \limsup_{R\to\infty} \frac{1}{R} \log \#\{g \mid d_X(\mathfrak{o}, g\mathfrak{o}) \le R\}$$

and we say the action of G on X is *temperate* if $v < +\infty$.

The κ–Morse boundary is the Poisson boundary

We now apply this criterion to identify the Poisson boundary with the κ–Morse boundary. The following result was obtained in collaboration with Ilya Gekhtman. It implies immediately Theorem B from the introduction.

Theorem 6.2 *Let G be a countable group of isometries of a proper geodesic metric space (X, d_X), and suppose that the action of G on X is temperate. Let μ be a probability measure on G with finite first moment with respect to d_X, such that the semigroup generated by the support of μ is a nonamenable group. Let κ be a concave sublinear function, and suppose that for almost every sample path $\omega = (w_n)$, there exists a κ–Morse geodesic ray γ_ω such that*

$$\lim_{n\to\infty} \frac{d_X(w_n, \gamma_\omega)}{n} = 0. \tag{9}$$

Then almost every sample path converges to a point in $\partial_\kappa X$, and moreover the space $(\partial_\kappa X, \nu)$, where ν is the hitting measure for the random walk, is a model for the Poisson boundary of (G, μ).

Proof By the subadditive ergodic theorem and finite first moment, the limit

$$\ell := \lim_{n\to\infty} \frac{d_X(\mathfrak{o}, w_n)}{n}$$

exists almost surely and is constant, and $\ell > 0$ since the group generated by the support of μ is nonamenable.

To prove the last claim, note that, by [Blachère et al. 2008, Proposition 3.4], the asymptotic entropy h is also finite, and it satisfies $h \le \ell v$. Then $\ell = 0$ implies $h = 0$, which in turn implies that the Poisson boundary is trivial. However, the Poisson boundary is never trivial for nonamenable groups; hence $\ell > 0$.

By Lemma 4.3 and (9), almost every sequence converges to $[\gamma_\omega] \in \partial_\kappa X$. Thus, we can define $\mathrm{bnd}\colon \Omega \to \partial_\kappa X$ as

$$\mathrm{bnd}(\omega) := \lim_{n\to\infty} w_n \in \partial_\kappa X,$$

which is T–invariant by definition. Moreover, bnd is measurable, since it is a pointwise limit of the measurable functions w_n with values in the space $X \cup \partial_\kappa X$, which is metrizable by Theorem 4.11. Since G is finitely generated, any word metric d_X on it is temperate.

Finally, by (9), almost every sample path sublinearly tracks a κ–Morse quasigeodesic ray. Hence, let us define $\pi_n\colon \partial_\kappa X \to G$ as $\pi_n(\xi) := \alpha_{r_n}$, where α is a geodesic representative of the class of $\xi \in \partial_\kappa X$ and $r_n := \lfloor \ell n \rfloor$.

Now let $\omega \in \Omega$ and $\gamma = \gamma_\omega$, and let p_n be a nearest point projection of w_n onto γ. By (9), we have, for almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{d_X(w_n, p_n)}{n} = 0;$$

hence also $\|p_n\|/n \to \ell$. Since p_n and γ_{r_n} lie on the same geodesic, this implies

$$\frac{d_X(w_n,\gamma_{r_n})}{n} \le \frac{d_X(w_n,p_n)}{n} + \frac{d_X(p_n,\gamma_{r_n})}{n} \le \frac{d_X(w_n,p_n)}{n} + \frac{\big|\|p_n\| - r_n\big|}{n} \to 0 + \ell - \ell = 0 \tag{10}$$

as $n \to \infty$. Finally, we obtain

$$\frac{d_X(w_n, \pi_n(\mathrm{bnd}(\omega)))}{n} = \frac{d_X(w_n,\alpha_{r_n})}{n} \le \frac{d_X(w_n,\gamma_{r_n})}{n} + \frac{d_X(\gamma_{r_n},\alpha_{r_n})}{n},$$

and the first term tends to 0 because of (10), while the second term tends to 0 since $\alpha \sim \gamma$. Hence, by Theorem 6.1, $(\partial_\kappa X, \nu)$ is a model for the Poisson boundary of (G, μ). $\square$

7 Boundaries of mapping class groups

In this section, we show that for an appropriate choice of κ, the κ–Morse boundary of any mapping class group $G = \mathrm{Map}(S)$ can function as a topological model for the Poisson boundary of the pair (G, μ), where μ is any finitely supported nonelementary measure.

We need to show that a generic sample path of such a random walk sublinearly tracks a κ–Morse quasigeodesic ray. We will do so but showing that, in fact, the limiting quasigeodesic ray is κ–weakly contracting.

7.1 Background on mapping class groups

Let S be a surface of finite hyperbolic type, let $\mathrm{g}(S)$ be its genus and $\mathrm{b}(S)$ the number of its boundary components. Let $\mathrm{Map}(S)$ denote the mapping class group of S equipped with a word metric d_w associated to a finite generating set. That is, we are in the setting where $(X, d_X) = (\mathrm{Map}(S), d_w)$.

Ending laminations

Let $\mathscr{C}(S)$ denote the curve graph of S (see [Masur and Minsky 2000] for definition and details). The curve graph is known to be δ–hyperbolic [Masur and Minsky 2000]. By [Klarreich 2022], the Gromov boundary of $\mathscr{C}(S)$ can be identified with the space of ending laminations $\mathscr{EL}(S)$; that is, the space of minimal filling laminations after forgetting the measure.

Subsurface projections

By a subsurface Y we always mean a connected π_1–injective subsurface of S. For any subsurface Y let $\mathscr{C}(Y)$ denote the curve graph of Y. Let ∂Y denote the multicurve consisting of all boundary components of Y. There is a projection map $\pi_Y : \mathscr{C}(S) \to \mathscr{C}(Y)$ defined on a subset of $\mathscr{C}(S)$ consisting of curves that intersect Y. This is essentially a map that sends a curve $\alpha \in \mathscr{C}(S)$ to a set of curves in Y obtained from surgery between α and ∂Y. (Again, see [Masur and Minsky 2000] for details). The set $\pi_Y(\alpha)$ has a uniformly bounded diameter in $\mathscr{C}(Y)$, independent of α or Y.

We can extend this projection to a map $\pi_Y : \mathrm{Map}(S) \to \mathscr{C}(Y)$ as follows. Consider a set θ of curves on S that fill S. For example, following [Masur and Minsky 2000], we can assume θ is the union of a pants decomposition and a set of *dual curves*, one transverse to each curve in the pants decomposition. Then, for $x \in \mathrm{Map}(S)$, define

$$\pi_Y(x) := \bigcup_{\alpha \in \theta} \pi_Y(x(\alpha)).$$

Again, the set $\pi_Y(x)$ has a uniformly bounded diameter in $\mathscr{C}(Y)$. For, $x, y \in \mathrm{Map}(S)$ define

$$d_Y(x, y) := \mathrm{diam}_{\mathscr{C}(Y)}(\pi_Y(x) \cup \pi_Y(y)).$$

If $Y = S$, we define $\pi_S(x) := x$ if $x \in \mathscr{C}(S)$, and $\pi_S(x) := x(\theta)$ if $x \in \mathrm{Map}(S)$. Then we set

$$d_S(x, y) := \mathrm{diam}_{\mathscr{C}(S)}(\pi_S(x) \cup \pi_S(y)).$$

if $x, y \in \mathrm{Map}(S) \cup \mathscr{C}(S)$. Also, when Y is an annulus with core curve α, we often use $d_\alpha(x, y)$ instead of $d_Y(x, y)$.

In the discussion above, x and y can be replaced with an ending lamination $\xi \in \mathscr{EL}(S)$ since ξ has nontrivial projection to every subsurface and $\pi_Y(\xi)$ is always well defined. That is, we define

$$d_Y(x, \xi) := \mathrm{diam}_{\mathscr{C}(Y)}(\pi_Y(x) \cup \pi_Y(\xi)).$$

The distance formula

In [Masur and Minsky 2000], it was shown that the word metric on $\mathrm{Map}(S)$ can be estimated up to uniform additive and multiplicative constants by these subsurface projection distances. To simplify the exposition, we adopt the following notation. We fix S and a generating set for $\mathrm{Map}(S)$, and we say a constant M is *uniform* if it depends only on the topology of S and the generating set. For two quantities $\boldsymbol{A}$ and $\boldsymbol{B}$, we write $\boldsymbol{A} \prec \boldsymbol{B}$ if there is a uniform constant M such that

$$\boldsymbol{A} \le M \cdot \boldsymbol{B} + M.$$

We write $\boldsymbol{A} \asymp \boldsymbol{B}$ if $\boldsymbol{A} \prec \boldsymbol{B}$ and $\boldsymbol{B} \prec \boldsymbol{A}$ and we use the notation $O(\boldsymbol{A})$ for a quantity that has an upper bound of $M \cdot \boldsymbol{A}$. Also, recall that, for $K > 0$,

$$\lfloor x \rfloor_K := \begin{cases} x & \text{if } x \ge K, \\ 0 & \text{if } x < K. \end{cases}$$

The following inequality is useful: if $K \ge 2C \ge 0$, then

$$\lfloor a + C \rfloor_K \le 2 \lfloor a \rfloor_{K/2} \quad \text{for all } a \ge 0.$$

Now the Masur–Minsky distance formula can be stated as follows: for any K sufficiently large, for any $x, y \in \mathrm{Map}(S)$,

$$d_w(x, y) \asymp \sum_{Y \subseteq S} \lfloor d_Y(x, y) \rfloor_K, \tag{11}$$

where the implicit constant depends on K. Given $K \geq 0$, for any $x, y \in \mathrm{Map}(S) \cup \mathcal{EL}(S)$ and α a simple closed curve, let us define

$$D_{\alpha,K}(x,y) := \sum_{\alpha \cap Y = \varnothing} \lfloor d_Y(x,y) \rfloor_K.$$

The hierarchy of geodesics

To every pair of points $x, y \in \mathrm{Map}(S)$ one can associate a *hierarchy of geodesics* connecting $x(\theta)$ to $y(\theta)$ [Masur and Minsky 2000, Theorem 4.6]. The hierarchy $H = H(x,y)$ consists of a geodesic $[x,y]_S$ in $\mathcal{C}(S)$ connecting $x(\theta)$ to $y(\theta)$ and other geodesics $[x,y]_Y$ in various curve graphs $\mathcal{C}(Y)$, where $[x,y]_Y$ is essentially a geodesic connecting $\pi_Y(x)$ to $\pi_Y(y)$. Hence we write $H = \{[x,y]_Y\}$. Besides S, other subsurfaces that appear in H are described as follows: for every curve α in $[x,y]_S$, we include every component of $S - \alpha$ that is not a pair of pants and the annulus A_α (the annulus whose core curve is α); also, if a subsurface Y appears in H, then for every β that appears in $[x,y]_Y$, we also include every component of $Y - \beta$ that is not a pair of pants and the annulus A_β. The length $|H|$ is defined to be the sum of the lengths $|[x,y]_Y|$ of geodesics $[x,y]_Y$. By [Masur and Minsky 2000, Theorem 3.1], there exists K, depending only on the topology of S, such that for every subsurface Y, if $d_Y(x,y) \geq K$ then Y is included in H. Furthermore, by [Masur and Minsky 2000, Theorems 6.10, 6.12 and 7.1],

$$d_w(x,y) \asymp |H(x,y)| := \sum_{Y \text{ in } H} |[x,y]_Y|. \tag{12}$$

A *resolution* $\mathcal{G}(x,y)$ of a hierarchy $H(x,y)$ is a uniform quasigeodesic in $\mathrm{Map}(S)$ connecting x to y where, for any subsurface Y, the projection of $\mathcal{G}(x,y)$ to $\mathcal{C}(Y)$ is contained in a uniformly bounded neighborhood of the geodesic segment $[x,y]_Y$.

We can also replace x or y with a point $\xi \in \mathcal{EL}$. We start with a (tight) geodesic $\gamma = [x,\xi)_S$ in $\mathcal{C}(S)$ and build $H(x,\xi)$ the same as before replacing, for every subsurface Y, $\pi_Y(y(\theta))$ with $\pi_Y(\xi)$. The resolution $\mathcal{G}(x,\xi)$ of $H(x,\xi)$ is then a uniform quasigeodesic in $\mathrm{Map}(S)$ starting from x; we say that γ, which is a geodesic in $\mathcal{C}(S)$ that converges to ξ, is the *shadow* of $\mathcal{G}(x,\xi)$ in $\mathcal{C}(S)$.

We use the hierarchy paths to show:

Proposition 7.1 *Let $p := 3\,\mathrm{g}(S) - 3 + \mathrm{b}(S)$ be the* ***complexity*** *of S. For any $x, y \in \mathrm{Map}(S)$, assume that $d_Y(x,y) \leq E$ for all $Y \subsetneq S$ and some $E > 1$. Then*

$$d_w(x,y) \prec d_S(x,y) \cdot E^p.$$

Proof This is essentially contained in [Masur and Minsky 2000]. We sketch the proof here and refer the reader to [Masur and Minsky 2000] for definitions and details. In view of (12), we need to show

$$|H(x,y)| \prec d_S(x,y) \cdot E^p.$$

The restriction of $H(x,y)$ to a subsurface Y is again a hierarchy which we denote by $H_Y(x,y)$. We check the proposition inductively. When S is $S_{1,1}$ or $S_{0,4}$, we have $p = 1$ and for every curve α in S, $S - \alpha$

does not have any complementary component that is not a pair of pants. Also, by assumption, for every $\alpha \in [x,y]_S$, we have $|[x,y]_\alpha| \prec D_{\alpha,K}(x,y) \le E$. Therefore, $|H(x,y)| \prec E \cdot |[x,y]_S| \prec E \cdot d_S(x,y) \le E$, as required.

Now let S be a larger surface and assume, by induction, that for every subsurface Y, the hierarchy $H_Y(x,y)$ satisfies $|H_Y(x,y)| \prec d_Y(x,y) \cdot E^{p-1}$. Then

$$|H(x,y)| \prec \sum_{\alpha \in [x,y]_S} \biggl(|[x,y]_\alpha| + \sum_{Y \subset S - \alpha} |H_Y(x,y)| \biggr) \prec |[x,y]_S| \cdot (E + 2 d_Y(x,y) \cdot E^{p-1}).$$

But $d_Y(x,y) \le E$ and $|[x,y]_S| \prec d_S(x,y)$; thus $|H(x,y)| \prec d_S(x,y) \cdot E^p$. □

Projections in mapping class groups

Here, we recall the construction of the center of a triangle in $\mathrm{Map}(S)$ according to Eskin, Masur and Rafi [Eskin et al. 2017]. For $x \in \mathrm{Map}(S)$ and a subsurface Y, we denote $\pi_Y(x(\theta))$ simply by x_Y. Also, as before, for $x, y \in \mathrm{Map}(S)$, the geodesic segment in $\mathscr{C}(Y)$ connecting x_Y and y_Y is denoted by $[x,y]_Y$. For any subsurface Y, the curve graph $\mathscr{C}(Y)$ is δ–hyperbolic for some uniform constant δ. Thus, for any three points $x, y, z \in \mathrm{Map}(S)$ and every subsurface Y, there exists a point $\mathrm{ctr}_Y(x,y,z)$ in $\mathscr{C}(Y)$ that is δ–close to all three geodesic segments $[x,y]_Y$, $[x,z]_Y$ and $[y,z]_Y$. We refer to $\mathrm{ctr}_Y(x,y,z)$ as the *center* of the triple $x_Y, y_Y, z_Y \in \mathscr{C}(Y)$.

It was shown in [Eskin et al. 2017] that there is an element $\eta \in \mathrm{Map}(S)$ that projects near the center of x_Y, y_Y and z_Y for every subsurface Y. More precisely:

Lemma 7.2 [Eskin et al. 2017, Lemma 4.11] *There exists a constant D such that the following holds. For any $x, y, z \in \mathrm{Map}(S)$, there exists a point $\eta \in \mathrm{Map}(S)$ such that, for any subsurface $Y \subseteq S$,*

$$d_Y(\eta_Y, \mathrm{ctr}_Y(x,y,z)) \le D.$$

We call η the ***center*** *of x, y and z and we denote it by $\mathrm{ctr}(x,y,z)$.*

Note that, as always, we can replace each of x, y and z with an ending lamination $\xi \in \mathscr{PML}$. That is, $\mathrm{ctr}(x,y,\xi)$ is a well-defined element of $\mathrm{Map}(S)$. From now on, we will denote by $\mathfrak{o}$ the identity element in $\mathrm{Map}(S)$, which will function as basepoint.

Definition 7.3 Let D be given from Lemma 7.2, and let $\xi \in \mathscr{EL}$ be an ending lamination on S. We define a *D–cloud of a ray in the direction of ξ* to be

$$\mathscr{X}(\mathfrak{o}, \xi) := \{z \in \mathrm{Map}(S) \mid d_{\mathscr{C}(Y)}(z_Y, [\mathfrak{o}, \xi)_Y) \le D \text{ for all } Y\}.$$

By construction, the resolution $\mathscr{G}(\mathfrak{o}, \xi)$ of the hierarchy $H(\mathfrak{o}, \xi)$ is contained in $\mathscr{X}(\mathfrak{o}, \xi)$. Fixing $\xi \in \mathscr{EL}$, we define a projection map

$$\Pi_\xi \colon \mathrm{Map}(S) \to \mathscr{X}(\mathfrak{o}, \xi) \quad \text{where } \Pi_\xi(x) := \mathrm{ctr}(\mathfrak{o}, x, \xi),\ x \in \mathrm{Map}(S).$$

We now check that Π_ξ satisfies the usual properties of a projection; in particular, it is a κ–projection according to Definition 5.1.

Lemma 7.4 *For any $\xi \in \mathscr{EL}$, the map Π_ξ is coarsely Lipschitz with respect to d_w. Furthermore, if $x \in \mathfrak{X}(\mathfrak{o}, \xi)$, then $d_w(x, \Pi_\xi(x))$ is uniformly bounded. As a consequence, Π_ξ is a κ–projection.*

Proof Consider points $x, x' \in \mathrm{Map}(S)$ where $d_w(x, x') \le 1$. Then $x(\theta)$ and $x'(\theta)$ have a uniformly bounded intersection number, which implies that there exists a uniform constant $C_1 > 0$ such that

$$d_Y(x_Y, x'_Y) \le C_1 \quad \text{for all } Y.$$

Let $\eta := \mathrm{ctr}(\mathfrak{o}, x, \xi)$ and $\eta' := \mathrm{ctr}(\mathfrak{o}, x', \xi)$. Since $\mathscr{C}(Y)$ is hyperbolic, the dependence of η_Y on x_Y is Lipschitz; that is, there exists a uniform constant $C_2 > 0$ such that

$$d_Y(\eta_Y, \eta'_Y) \le C_2 \quad \text{for all } Y \subseteq S.$$

Now Proposition 7.1 implies that

$$d_w(\Pi_\xi(x), \Pi_\xi(x')) \prec (C_2)^{p+1},$$

which means Π_ξ is coarsely Lipschitz. Similarly, if $x \in \mathfrak{X}(\mathfrak{o}, \xi)$ then for $\eta = \mathrm{ctr}(\mathfrak{o}, x, \xi)$ we have $d_Y(x_Y, \eta_Y) \le C_2$ for all subsurfaces Y, and hence $d_Y(x, \Pi_\xi(x)) \prec (C_2)^{p+1}$. □

This projection has the following desirable property as shown by Duchin and Rafi [2009].

Theorem 7.5 [Duchin and Rafi 2009, Theorem 4.2] *There exist constants B_1 and B_2 depending on the topology of the surface S and D such that, for $x, y \in \mathrm{Map}(S)$,*

$$d_w(x, \mathfrak{X}(\mathfrak{o}, \xi)) \ge B_1 \cdot d_w(x, y) \implies d_S(\Pi_\xi(x), \Pi_\xi(y)) \le B_2.$$

In [Duchin and Rafi 2009], the theorem is proven under the assumption that the geodesic (or cloud) is cobounded. However, the result holds in general: we will see in Proposition 8.5 a detailed proof for relatively hyperbolic groups, which can be easily adapted to mapping class groups.

Logarithmic projections

We now consider the set of points in $\mathscr{EL}$ that have *logarithmically bounded projection* to all subsurfaces. Given a proper subsurface $Y \subsetneq S$, let ∂Y denote the multicurve of boundary components of Y and define

$$\|Y\|_S := d_S(\theta, \partial Y).$$

Similarly, for $x \in \mathrm{Map}(S)$, define

$$\|x\|_S := d_S(\theta, x(\theta)).$$

We now choose a constant $K \ge 4D$ and large enough that (11) and (12) hold.

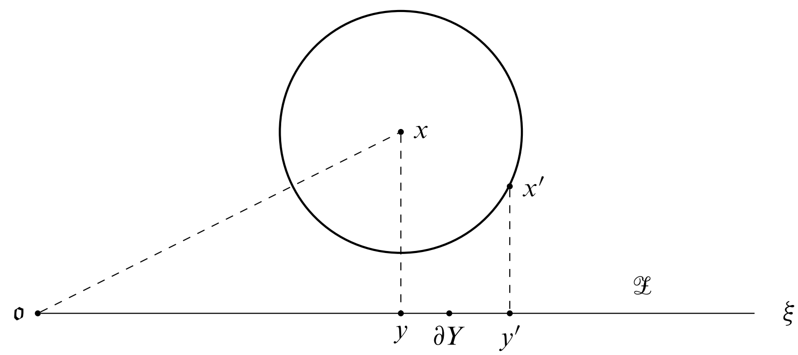

Figure 9: The projections of x and x' to $\mathfrak{X}$ are κ–close.

Definition 7.6 For a constant $c > 0$, let $\mathscr{L}_c$ be the set of points $\xi \in \mathscr{E}\mathscr{L}$ such that

$$D_{\alpha,K}(\mathfrak{o},\xi) \le c \cdot \log d_S(\mathfrak{o},\alpha) \tag{13}$$

for every simple closed curve α in S.

Proposition 7.7 *For any $\xi \in \mathscr{L}_c$, the set $\mathfrak{X}(\mathfrak{o},\xi)$ is κ–weakly contracting, where $\kappa(r) = \log(r)$. Furthermore, any resolution $\mathscr{G}(\mathfrak{o},\xi)$ of the hierarchy $H(\mathfrak{o},\xi)$ is also κ–Morse.*

Proof In this proof, we use the notations $\prec_c$ and O_c to mean that the implicit constants additionally depend on c. Let $\mathfrak{X} = \mathfrak{X}(\mathfrak{o},\xi)$. Given $x, x' \in X$ where

$$B_1 \cdot d_w(x,x') < d_w(x,\mathfrak{X}),$$

let $y = \Pi_\xi(x)$, $y' = \Pi_\xi(x')$. We claim that, for every simple closed curve α,

$$D_{\alpha,2K}(y,y') \prec_c \log \|x\|_S.$$

Since $\mathscr{C}(S)$ is hyperbolic, nearest point projection in $\mathscr{C}(S)$ is coarsely distance decreasing; hence $\|y\|_S \prec \|x\|_S$. Also, by Theorem 7.5,

$$d_S(y,y') \le B_2; \tag{14}$$

therefore, $\|y'\|_S \prec \|x\|_S$. Which means, for every curve α in the geodesic segment $[y,y']_S$ in $\mathscr{C}(S)$, we have $d_S(\mathfrak{o},\alpha) \prec \|x\|_S$. The bounded geodesic image theorem [Masur and Minsky 2000, Theorem 3.1] implies that if $d_Y(y,y')$ is large then $d_{\mathscr{C}(S)}([y,y']_S,\partial Y) \prec 1$; hence

$$\|Y\|_S \prec \|x\|_S.$$

By the definition of Π_ξ, y_Y and y'_Y are D–close to the geodesic segment $[\mathfrak{o},\xi]_Y$ in $\mathscr{C}(Y)$. Now, by summing over all Y disjoint from α, and using the definition of $\mathscr{L}_c$,

$$D_{\alpha,2K}(y,y') = \sum_{Y\cap\alpha=\varnothing} \lfloor d_Y(y,y')\rfloor_{2K} \prec \sum_{Y\cap\alpha=\varnothing} \lfloor d_Y(\mathfrak{o},\xi)\rfloor_K \prec_c \log d_S(\mathfrak{o},\alpha) \prec \log\|x\|_S.$$

Now we apply (12), recalling that every subsurface in the hierarchy is disjoint from some curve α in $[y,y']_S$; hence only the curves in $[y,y']_S$ contribute to the sum. Hence, in view of (14), there are at most B_2 of them, so we get

$$|H(y,y')| \prec \sum_{\alpha\in[y,y']_S} \sum_{Y\cap\alpha=\varnothing} \lfloor d_Y(y,y')\rfloor_{2K} = \sum_{\alpha\in[y,y']_S} D_{\alpha,2K}(y,y') \prec_c \log\|x\|_S. \tag{15}$$

Thus, by (11),

$$d_w(y,y') \asymp |H(y,y')| \prec_c \log\|x\|_S.$$

Now, by Theorem 5.5, $\mathfrak{X}(\mathfrak{o},\xi)$ is κ–Morse. Let $m_{\mathfrak{X}}$ be the associated Morse gauge for $\mathfrak{X}(\mathfrak{o},\xi)$.

Now we show $\mathcal{G}(\mathfrak{o},\xi)$ is also κ–Morse. Assume κ' and $r>0$ be given (see Definition 3.2) and, using the fact that $\mathfrak{X}(\mathfrak{o},\xi)$ is κ–Morse, let R be a radius such that, for any (q,Q)–quasigeodesic ray β in $\mathrm{Map}(S)$ with $m_{\mathfrak{X}}(q,Q)$ small compared to r,

$$d_w(\beta_R,\mathfrak{X}(\mathfrak{o},\xi)) \le \kappa'(R) \implies \beta|_r \subset \mathcal{N}(\mathfrak{X}(\mathfrak{o},\xi), m_{\mathfrak{X}}(q,Q)).$$

Also, assume

$$d_w(\beta_R,\mathcal{G}(\mathfrak{o},\xi)) \le \kappa'(R).$$

We need to show that every $x\in\beta|_r$ is close to $\mathcal{G}(\mathfrak{o},\xi)$.

Since $\mathcal{G}(\mathfrak{o},\xi)\subset\mathfrak{X}(\mathfrak{o},\xi)$, we can still conclude that there is a point $y\in\mathfrak{X}(\mathfrak{o},\xi)$ with

$$d_w(x,y) \le m_{\mathfrak{X}}(\mathsf{q},\mathsf{Q})\cdot\kappa(x).$$

In fact, y can be taken to be $\Pi_\xi(x)$, and hence $\|y\|_S \prec \|x\|_S$. Let z be a point in $\mathcal{G}(\mathfrak{o},\xi)$ where $d_S(z_S,y_S)\le D$ (such a point exists since the shadow of $\mathcal{G}(\mathfrak{o},\xi)$ to $\mathcal{C}(S)$ is the geodesic ray $[\mathfrak{o},\xi)_S$). Since $y,z\in\mathfrak{X}(\mathfrak{o},\xi)$, we have for every curve α on $[y,z]_S$ that

$$\sum_{Y\cap\alpha=\varnothing} \lfloor d_Y(y,z)\rfloor_{2K} \le \sum_{Y\cap\alpha=\varnothing} \lfloor d_Y(\mathfrak{o},\xi)+2D\rfloor_{2K} \prec \sum_{Y\cap\alpha=\varnothing} \lfloor d_Y(\mathfrak{o},\xi)\rfloor_K \prec_c \log d_S(\mathfrak{o},\alpha) \prec \log\|x\|_S.$$

Therefore, by summing over all α as in (15),

$$d_w(y,z) \prec_c \log\|x\|_S \prec \kappa(x).$$

And, hence,

$$d_w(x,z) \le d_w(x,y)+d_w(y,z) \prec_c m_{\mathfrak{X}}(q,Q)\cdot\kappa(x).$$

We have shown

$$\beta|_r \subset \mathcal{N}\big(\mathcal{G}(\mathfrak{o},\xi), O_c(m_{\mathfrak{X}}(q,Q))\big).$$

That is, $\mathcal{G}(\mathfrak{o},\xi)$ is κ–Morse with a Morse gauge $m_{\mathcal{G}}=O_c(m_{\mathfrak{X}})$. □

7.2 Convergence to the κ–Morse boundary

Let G be a countable group of isometries of a δ–hyperbolic space X, and let μ be a probability measure on G. We say that μ is *nonelementary* if the semigroup generated by its support contains two loxodromic elements with disjoint fixed sets in ∂X, the Gromov boundary of X. Let us recall some useful facts on random walks on such groups.

Theorem 7.8 *Let G be a countable group of isometries of a geodesic, δ–hyperbolic space X, and let μ be a finitely supported, nonelementary probability measure on G. Then:*

1. *For almost every sample path $\omega = (w_n)$, the sequence (w_n) converges to a point ξ_ω in the Gromov boundary of X.*
2. *Moreover, there exists $\ell > 0$ and $c < 1$ such that*
$$\mathbb{P}(d_X(\mathfrak{o}, w_n) \geq \ell n) \geq 1 - c^n$$
for any n.
3. *Further, for any $k > 0$ there exists $C > 0$ such that*
$$\mathbb{P}(d_X(w_n, \gamma_\omega) \geq C \log n) \leq C n^{-k}$$
for any n, where $\gamma_\omega = [\mathfrak{o}, \xi_\omega)_X$ is any geodesic in X connecting $\mathfrak{o}$ and ξ_ω.

Note that this theorem applies to the mapping class group $G = \mathrm{Map}(S)$, if we take $X = \mathscr{C}(S)$, whose Gromov boundary is $\mathscr{E}\mathscr{L}(S)$. It also applies if G is a relatively hyperbolic group, by taking as X the coned-off Cayley graph $\widehat{G}$ (see Section 8.1).

Claim (2) is proven by Maher [2010; 2012], while (1) and (3) are proven in [Maher and Tiozzo 2018]. By adapting the proof of [Qing and Rafi 2022, Theorem A.17] (inspired by [Sisto and Taylor 2019] and [Sisto 2017, Lemma 4.4]), we obtain for the mapping class group:

Theorem 7.9 *Let μ be a finitely supported, nonelementary probability measure on $\mathrm{Map}(S)$. Then, for any $k > 0$ and for any K sufficiently large, there exists $C > 0$ such that for all n,*
$$\mathbb{P}\Big(\sup_\alpha D_{\alpha,K}(\mathfrak{o}, w_n) \geq C \log n\Big) \leq C n^{-k},$$
where the supremum is taken over all simple closed curves α in S. As a consequence, for almost every sample path there exists $C > 0$ such that for all n,
$$\sup_\alpha D_{\alpha,K}(\mathfrak{o}, w_n) \leq C \log n.$$

Proof The idea of the proof is that in order to make progress with respect to $D_{\alpha,K}$, the sample path must project close to α in the curve graph $\mathscr{C}(S)$. However, linear progress with exponential decay implies that the sample path can stay close to the projection of α only for a time of order $\log n$, which completes the proof.

Let us see the details. By the bounded geodesic image theorem, there exists B (independent of Y) such that if $[x, y]$ is a geodesic segment in $\mathscr{C}(S)$ and ∂Y is far from it, namely
$$d_S(\partial Y, [x, y]) \geq 2,$$
then $d_Y(x, y) \leq B$ is uniformly bounded.

By linear progress with exponential decay (Theorem 7.8(2)), replacing n by $A\log n$ we obtain for any $A > 0$ and $n \ge e^{2/\ell A}$,

$$\mathbb{P}\big(d_S(\mathfrak{o}, w_{A\log n}) \le 4\big) \le n^{A\log c}. \tag{16}$$

Consider now the path of vertices $(w_i)_{i\le n}$ in $\mathrm{Map}(S)$, and suppose that $D_{\alpha,K}(\mathfrak{o}, w_n) \ge C\log n$ for a curve α. Let

$$\begin{aligned} i_1 &:= \min\{0 < i \le n \mid d_S(w_i, \alpha) \le 2\}, \\ i_2 &:= \max\{0 \le i < n \mid d_S(w_i, \alpha) \le 2\}, \\ D &:= \max\{d_w(\mathfrak{o}, g) \mid g \in \operatorname{supp}\mu\}. \end{aligned}$$

Then, for any subsurface Y disjoint from α, we obtain $d_S(w_{i_1-1}, \partial Y) \ge d_S(w_{i_1-1}, \alpha) - 1 \ge 2$; hence

$$d_Y(\mathfrak{o}, w_n) \le d_Y(\mathfrak{o}, w_{i_1-1}) + d_Y(w_{i_1-1}, w_{i_2+1}) + d_Y(w_{i_2+1}, w_n) \le d_Y(w_{i_1-1}, w_{i_2+1}) + 2B.$$

Therefore, for $K \ge 4B$,

$$\lfloor d_Y(\mathfrak{o}, w_n)\rfloor_K \le \lfloor d_Y(w_{i_1-1}, w_{i_2+1}) + 2B\rfloor_K \le 2\lfloor d_Y(w_{i_1-1}, w_{i_2+1})\rfloor_{K/2}$$

and, summing over Y and using the distance formula, there exists C_1 such that

$$\begin{aligned} D_{\alpha,K}(\mathfrak{o}, w_n) &= \sum_{\alpha\cap Y=\varnothing} \lfloor d_Y(\mathfrak{o}, w_n)\rfloor_K \\ &\le 2\sum_{\alpha\cap Y=\varnothing} \lfloor d_Y(w_{i_1-1}, w_{i_2+1})\rfloor_{K/2} \le 2(C_1 d_w(w_{i_1-1}, w_{i_2+1}) + C_1). \end{aligned}$$

Thus,

$$C\log n \le D_{\alpha,K}(\mathfrak{o}, w_n) \le 2(C_1 d_w(w_{i_1-1}, w_{i_2+1}) + C_1) \le 2(C_1 D(i_2 - i_1 + 2) + C_1),$$

so, for n large enough,

$$|i_1 - i_2| \ge \frac{C\log n}{4C_1 D}.$$

Hence

$$\mathbb{P}\big(\exists\,\alpha \mid D_{\alpha,K}(\mathfrak{o}, w_n) \ge C\log n\big) \le \mathbb{P}\Big(\exists\, i_1 \le i_2 \le n, i_2 - i_1 \ge \frac{C}{4C_1 D}\log n \;\Big|\; d_S(w_{i_1}, w_{i_2}) \le 4\Big)$$

and by (16) this is bounded above by

$$n^2 \cdot n^{-C\log(1/c)/(4C_1 D)},$$

which tends to 0 as $n \to \infty$, as long as $C > 8C_1 D/\log(1/c)$.

The second claim follows immediately from the first one for $k = 2$ by Borel–Cantelli. □

We now prove that almost every sample path converges to a point in the κ–Morse boundary of the mapping class group, where $\kappa(r) = \log(r)$.

Theorem 7.10 *Let μ be a finitely supported, nonelementary probability measure on* $\mathrm{Map}(S)$, *and let* $\kappa(r) := \log(r)$. *Then*

(1) *almost every sample path $\omega = (w_n)$ converges to a point in the κ–Morse boundary with respect to the topology of $X \cup \partial_\kappa X$;*

(2) *moreover, for almost every sample path there exists a κ–Morse geodesic ray $\mathcal{G}_\omega$ in* $\mathrm{Map}(S)$ *such that*

$$\limsup_{n\to\infty} \frac{d(w_n, \mathcal{G}_\omega)}{\log n} < +\infty.$$

Proof By Theorem 7.8, for almost every sample path, $w_n(\theta)$ converges to a point ξ_ω on the boundary of $\mathscr{C}(S)$ (with respect to the topology on $\mathscr{C}(S) \cup \mathscr{EL}(S)$).

Let $\mathcal{G} = \mathcal{G}(\mathfrak{o}, \xi_\omega)$ be a resolution of a hierarchy towards ξ_ω and let $\gamma_\omega := [\mathfrak{o}, \xi_\omega)_S$ be the shadow of $\mathcal{G}$, which is a geodesic ray in $\mathscr{C}(S)$ starting from θ and limiting to ξ_ω. Let p_n be a nearest point projection (in $\mathscr{C}(S)$) of $w_n(\theta)$ to γ_ω and

$$c_n := \mathrm{ctr}(\mathfrak{o}, w_n, \xi_\omega).$$

By the definition of center and the hyperbolicity of $\mathscr{C}(S)$, there is a $D' > 0$ depending on D and δ such that $d_S(c_n, p_n) \le D'$ for any n.

By Proposition 7.7, in order to prove that $\mathcal{G}$ is κ–Morse, it is sufficient to prove that there is a constant c such that

$$D_{\alpha,K}(\mathfrak{o}, \xi_\omega) \le c \log d_S(\mathfrak{o}, \alpha) \tag{17}$$

for any simple closed curve α.

Since the drift of the random walk is positive with exponential decay (Theorem 7.8(2)), we have by the Markov property that there exists $0 < C_0 < 1$ such that

$$\mathbb{P}(d_S(w_n, w_{n+m}) \le \ell m) = \mathbb{P}(d_S(\mathfrak{o}, w_m) \le \ell m) \le (C_0)^m \quad \text{for all } n, m \ge 1 \tag{18}$$

with $\ell > 0$ a constant larger than the drift of the random walk. By setting $m = A \log n$, we get

$$\mathbb{P}(d_S(w_n, w_{n+A\log n}) \le \ell A \log n) \le n^{A \log C_0} \quad \text{for all } n;$$

hence, if is A sufficiently large,

$$\mathbb{P}(d_S(w_n, w_{n+A\log n}) \le \ell A \log n) \le n^{-2} \quad \text{for all } n. \tag{19}$$

By Theorem 7.8(3), there exists $C_1 > 0$ such that

$$\mathbb{P}(d_S(w_n, p_n) \ge C_1 \log n) \le C_1 n^{-2} \quad \text{for all } n. \tag{20}$$

Finally, by Theorem 7.9, there exists $C_2 > 0$ such that

$$\mathbb{P}\bigl(\sup_\alpha D_{\alpha,K/4}(\mathfrak{o}, w_n) \ge C_2 \log n\bigr) \le C_2 n^{-2} \quad \text{for all } n. \tag{21}$$

Now the complement of the union of all events expressed by (18), (19), (20) and (21) has measure at least $1 - C_3 n^{-2}$ for some new C_3. We will consider from now on a sample path in such a set.

Step 1 We claim that there exists $C_4 > 0$ such that

$$\mathbb{P}\big(d_w(w_n, c_n) \geq C_4 \log n\big) \leq C_4 n^{-2} \tag{22}$$

for all n.

Proof Let us pick a subsurface Y, and for any n, let us pick a geodesic segment $\gamma_n = [w_n(\theta), p_n]$ in $\mathscr{C}(S)$. By the bounded geodesic image theorem, there exists B (independent of Y) such that if ∂Y is far from γ_n in $\mathscr{C}(S)$, namely

$$d_S(\partial Y, \gamma_n) \geq 2,$$

then $d_Y(w_n, c_n) \leq B$ is uniformly bounded.

On the other hand, if $d_S(\partial Y, \gamma_n) \leq 1$, let us denote by q_1 a nearest point projection of ∂Y onto $\gamma_n = [w_n(\theta), p_n]$ and by q_2 a nearest point projection of ∂Y onto $\gamma_{n+m} = [w_{n+m}(\theta), p_{n+m}]$. Then

$$\begin{aligned} d_S(\partial Y, \gamma_{n+m}) = d_S(\partial Y, q_2) &\geq d_S(w_n, w_{n+m}) - d_S(\partial Y, q_1) - d_S(q_1, w_n) - d_S(w_{n+m}, q_2) \\ &\geq d_S(w_n, w_{n+m}) - d_S(\partial Y, q_1) - d_S(w_n, p_n) - d_S(w_{n+m}, p_{n+m}) \end{aligned}$$

and, if we set $m = A \log n$ with $A > 2C_1/\ell$, we obtain

$$\geq \ell A \log n - 1 - C_1 \log n - C_1 \log(n + A \log n) \geq 2$$

for n sufficiently large (independently of Y and ω). Hence, by the bounded geodesic image theorem,

$$d_Y(w_{n+m}, c_{n+m}) \leq B.$$

Now, since projection in a δ–hyperbolic space is coarsely distance-decreasing,

$$d_Y(c_n, c_{n+m}) \leq d_Y(w_n, w_{n+m}) + B_1,$$

where the constant B_1 is independent of Y. By putting these estimates together and by the triangle inequality,

$$d_Y(w_n, c_n) \leq d_Y(w_n, w_{n+m}) + d_Y(w_{n+m}, c_{n+m}) + d_Y(c_{n+m}, c_n) \leq 2d_Y(w_n, w_{n+m}) + B + B_1;$$

hence,

$$\lfloor d_Y(w_n, c_n) \rfloor_K \leq \lfloor 2d_Y(w_n, w_{n+m}) + B + B_1 \rfloor_K \leq 4 \lfloor d_Y(w_n, w_{n+m}) \rfloor_{K/4}.$$

Thus, by the distance formula, there exists a constant C_5 such that

$$\begin{aligned} d_w(w_n, c_n) &\leq C_5 \sum_{Y \subseteq S} \lfloor d_Y(w_n, c_n) \rfloor_K + C_5 \\ &\leq 4C_5 \sum_{Y \subseteq S} \lfloor d_Y(w_n, w_{n+m}) \rfloor_{K/4} + C_5 \\ &\leq 4C_5 (C_5 d_w(w_n, w_{n+m}) + C_5) + C_5 \\ &\leq 4C_5^2 AD \cdot \log n + (4C_5^2 + C_5). \end{aligned}$$

□

Step 2 We show that there exists a constant C_6 such that for any n,

$$\mathbb{P}(\sup_{\alpha} D_{\alpha,K/2}(\mathfrak{o}, c_n) \geq C_6 \log d_S(\mathfrak{o}, p_n) + C_6) \leq C_6 n^{-2}. \tag{23}$$

Proof First note that by the triangle inequality if $d_S(\mathfrak{o}, w_n) \geq \ell n$ and $d_S(w_n, p_n) \leq C_1 \log n$ then

$$d_S(\mathfrak{o}, p_n) \geq d_S(\mathfrak{o}, w_n) - d_S(w_n, p_n) \geq \ell n - C_1 \log n \geq \ell \tfrac{1}{2} n$$

for n sufficiently large; hence by (18) and (20) there exists C_7 for which

$$\mathbb{P}\big(d_S(\mathfrak{o}, p_n) \geq \tfrac{1}{2}\ell n\big) \geq 1 - C_7 n^{-2} \quad \text{for all } n.$$

Then, since p_n is the closest point projection of w_n onto $[\mathfrak{o}, \xi_\omega)$ in the δ–hyperbolic space $\mathscr{C}(Y)$,

$$d_Y(\mathfrak{o}, c_n) \leq d_Y(\mathfrak{o}, w_n) + B_1. \tag{24}$$

Then, by summing over Y, for $K \geq 4B_1$,

$$\begin{aligned}
D_{\alpha,K/2}(\mathfrak{o}, c_n) = \sum_{Y \cap \alpha = \varnothing} \lfloor d_Y(\mathfrak{o}, c_n) \rfloor_{K/2} &\leq \sum_{Y \cap \alpha = \varnothing} \lfloor d_Y(\mathfrak{o}, w_n) + B_1 \rfloor_{K/2} \\
&\leq \sum_{Y \cap \alpha = \varnothing} 2\lfloor d_Y(\mathfrak{o}, w_n) \rfloor_{K/4} \\
&\leq 2D_{\alpha,K/4}(\mathfrak{o}, w_n) \\
&\leq 2C_2 \log n && \text{(by (21))} \\
&\leq 2C_2 \log d_S(\mathfrak{o}, p_n) + 2C_2 \log(2/\ell) \\
&\leq C_6 \log d_S(\mathfrak{o}, p_n) + C_6
\end{aligned}$$

for the appropriate choice of C_6. □

Step 3 For almost every ω, there exists $C_8(\omega)$ such that

$$D_{\alpha,K}(\mathfrak{o}, \xi_\omega) \leq C_8(\omega) \log d_S(\mathfrak{o}, \alpha) \tag{25}$$

for every simple closed curve α on S.

Proof By Step 2 and Borel–Cantelli, for almost every ω there exists $n_0 = n_0(\omega)$ such that

$$\sup_{\alpha} D_{\alpha,K/2}(\mathfrak{o}, c_n) \leq C_6 \log d_S(\mathfrak{o}, p_n) + C_6 \tag{26}$$

for any $n \geq n_0$.

Now observe that the random walk is finitely supported, so $d_S(w_n, w_{n+1})$ is uniformly bounded; hence, since nearest point projection is coarsely distance decreasing, there exists $D_1 > 0$ such that

$$d_S(p_n, p_{n+1}) \leq D_1$$

is also uniformly bounded.

Now let α be a simple closed curve, and Y a subsurface disjoint from α. By the bounded geodesic image theorem, $d_Y(\mathfrak{o}, \xi_\omega) \le B$ unless ∂Y lies in a 1–neighborhood of $[\mathfrak{o}, \xi_\omega)_S$; hence α lies in a 2–neighborhood of $[\mathfrak{o}, \xi_\omega)_S$.

Let us suppose that α lies in a 2–neighborhood of $[\mathfrak{o}, \xi_\omega)_S$ and let $n = n(\alpha, \omega)$ be the smallest integer such that $d_S(\mathfrak{o}, p_k) \ge d_S(\mathfrak{o}, \alpha) + 3$ for every $k \ge n$. Note that by minimality,

$$d_S(\mathfrak{o}, p_n) - D_1 \le d_S(\mathfrak{o}, p_{n-1}) \le d_S(\mathfrak{o}, \alpha) + 3 \le d_S(\mathfrak{o}, p_n);$$

hence $d_S(\mathfrak{o}, p_n) \le d_S(\mathfrak{o}, \alpha) + D_1 + 3$. Moreover, by the bounded geodesic image theorem,

$$|d_Y(\mathfrak{o}, \xi_\omega) - d_Y(\mathfrak{o}, c_n)| \le B$$

since the distance in $\mathscr{C}(S)$ between ∂Y and $[p_n, \xi_\omega)_S$ is at least 2; hence the projection of $[p_n, \xi_\omega)_S$ to $\mathscr{C}(Y)$ is uniformly bounded. Therefore, by (26), if $n \ge n_0(\omega)$ then

$$d_Y(\mathfrak{o}, \xi_\omega) \le d_Y(\mathfrak{o}, c_n) + B,$$

so for $K \ge 2B$ we obtain

$$\begin{aligned} D_{\alpha,K}(\mathfrak{o}, \xi_\omega) = \sum_{Y \cap \alpha = \varnothing} \lfloor d_Y(\mathfrak{o}, \xi_\omega) \rfloor_K &\le 2 \sum_{Y \cap \alpha = \varnothing} \lfloor d_Y(\mathfrak{o}, c_n) \rfloor_{K/2} \\ &= 2 D_{\alpha, K/2}(\mathfrak{o}, c_n) \\ &\le 2C_6 \log d_S(\mathfrak{o}, p_n) + 2C_6 \\ &\le 2C_6 \log(d_S(\mathfrak{o}, \alpha) + D_1 + 3) + 2C_6. \end{aligned}$$

On the other hand, if $n \le n_0(\omega)$ then $d_S(\mathfrak{o}, \alpha) \le d_S(\mathfrak{o}, p_n) \le D_1 n \le D_1 n_0(\omega)$, and there are at most finitely many such curves α and subsurfaces Y disjoint from them for which the projection of $[\mathfrak{o}, \xi_\omega)_S$ onto $\mathscr{C}(Y)$ is large. Hence

$$\sup_{\alpha : n(\alpha,\omega) \le n_0} D_{\alpha,K}(\mathfrak{o}, \xi_\omega) < +\infty;$$

thus the claim follows by adjusting the constant to a constant $C_8(\omega)$, which depends on ω, to take into account the initial part. $\square$

Let us now prove the second claim, namely the tracking estimate between the sample path and the geodesic ray. From that and Lemma 4.3, it follows that almost every sample path converges in the topology of $X \cup \partial_\kappa X$.

For a given sample path $\omega = (w_n)$, let $\xi \in \mathscr{EL}(S)$ be the ending lamination the path (w_n) is converging to, and let $c_n := \operatorname{ctr}(\mathfrak{o}, w_n, \xi)$. Note that by construction the projection of c_n onto $\mathscr{C}(S)$ is within uniformly bounded distance of p_n. By Step 1, there exists $C_4 > 0$ such that

$$\mathbb{P}(d_w(w_n, c_n) \ge C_4 \log n) \le C_4 n^{-2} \tag{27}$$

for any n. As a consequence, the Borel–Cantelli lemma implies that for almost every $\omega \in \Omega$ there exists n_0 such that

$$d_w(w_n, c_n) \leq C_4 \log n \tag{28}$$

for all $n \geq n_0$. The claim follows by noting that $d_w(w_n, \mathcal{G}) \leq d_w(w_n, c_n) + O(1)$. □

We now complete the proof of Theorem C by identifying the κ–Morse boundary with the Poisson boundary.

Theorem 7.11 *Let μ be a nonelementary, finitely supported measure on $G = \mathrm{Map}(S)$. Then, for $\kappa(r) := \log(r)$, the κ–Morse boundary is a model for the Poisson boundary of (G, μ).*

Proof By Theorem 7.10, almost every sample path sublinearly tracks a κ–Morse geodesic ray, with $\kappa(r) = \log(r)$, so we can apply Theorem 6.2. □

8 Relatively hyperbolic groups

8.1 Background

Let G be a finitely generated group, and let $P_1, P_2, \ldots, P_k$ be a set of subgroups of G which we call *peripheral subgroups*. Fix a finite generating set S. Let (G, d_G) denote the Cayley graph of G with respect to S, equipped with the word metric. Following [Sisto 2013], we denote by $(\widehat{G}, d_{\widehat{G}})$ the metric graph obtained from the Cayley graph of G by adding an edge between every pair of distinct vertices contained in a left coset P of a peripheral subgroup. Setting up this way, G and $\widehat{G}$ have the same vertex set, so any set in G can also be considered as a set in $\widehat{G}$. However, for $x, y \in G$, $d_{\widehat{G}}(x, y) \leq d_G(x, y)$. In this section, $[x, y]$ will denote a geodesic segment between x and y in the metric $d_{\widehat{G}}$, and $[\![x, y]\!]$ a geodesic segment in d_G.

Definition 8.1 A group G is *relatively hyperbolic*, relative to peripheral subgroups $P_1, P_2, \ldots, P_k$, if the graph $\widehat{G}$ is

- δ–hyperbolic, and
- *fine*; ie for each integer n, every edge belongs to only finitely many simple cycles of length n.

The collection of all cosets of peripheral subgroups is denoted by $\mathcal{P}$. We shall denote by $\mathcal{N}_D(P)$ the D–neighborhood of P in the metric d_G.

Moreover, we say a relatively hyperbolic group is *nonelementary* if it is infinite, not virtually cyclic, and each P_i is infinite and with infinite index in G. We now fix a nonelementary relatively hyperbolic group G and a generating set. We shall use the symbols $\prec$ and $\asymp$ as before, where the implicit constants depend only on G and the generating set; we shall write $\asymp_K$ if the implicit constants additionally depend on K.

By definition of relative hyperbolicity, the graph $\widehat{G}$ is δ–hyperbolic for some $\delta > 0$. Moreover, for $P \in \mathcal{P}$ we let $\pi_P\colon G \to P$ be a nearest point projection to P, and define

$$d_P(x, y) := d_G(\pi_P(x), \pi_P(y)).$$

We need some properties of relatively hyperbolic groups from [Sisto 2013; 2017]. A *lift* of a geodesic ray in $\widehat{G}$ is a path in G obtained by substituting edges labeled by an element of some P_i, and possibly their endpoints, with a geodesic in the corresponding left coset. Given D, R and a geodesic segment γ in G, we define the set $\mathrm{deep}_{D,R}(\gamma)$ as the set of points p of γ that belong to some subgeodesic $[\![x_1, y_1]\!]$ of γ with endpoints in $\mathcal{N}_D(P)$ for some $P \in \mathcal{P}$ and with $d_G(x_1, p) > R, d_G(y_1, p) > R$. A point which does not belong to $\mathrm{deep}_{D,R}(\gamma)$ is called a *transition point*.

Proposition 8.2 *Let G be a relatively hyperbolic group, and fix a generating set. Then*:

(1) ***Coarse lifting property*** [Sisto 2013, Proposition 1.14] *There exist uniform constants $q_0, Q_0 > 0$ such that if α is a geodesic in $\widehat{G}$, then its lifts are (q_0, Q_0)–quasigeodesics in G.*

(2) ***Distance formula*** [Sisto 2013, Theorem 0.1] *There exists K_0 such that, for any $K \ge K_0$ and every pair of points $x, y \in G$,*
$$d_G(x, y) \asymp_K \sum_{P \in \mathcal{P}} \lfloor d_P(x, y) \rfloor_K + d_{\widehat{G}}(x, y).$$

(3) ***Bounded geodesic image theorem*** [Sisto 2013, Lemma 1.15] *There exists $L_0 \ge 0$ such that, if $d_P(x, y) \ge L_0$ for some $P \in \mathcal{P}$, all geodesics in $\widehat{G}$ connecting x to y contain an edge in P. Moreover, for every $\mathsf{q}, \mathsf{Q} > 0$ there is a $D = D(\mathsf{q}, \mathsf{Q})$ such that every (q, Q)–quasigeodesic in G connecting x and y intersect the balls of radius D, $B_D(\pi_P(x))$ and $B_D(\pi_P(y))$. Let $D_0 = D(1, 0)$ be the constant associated to geodesics.*

(4) ***Deep components*** [Sisto 2017, Lemma 3.3] *There exist D, t and R such that for any $x, y \in G$, the set $\mathrm{deep}_{D,R}([\![x, y]\!])$ is contained in a disjoint union of subgeodesics of $[\![x, y]\!]$, each contained in $\mathcal{N}_{tD}(P)$ for some $P \in \mathcal{P}$. We call each subgeodesic a **deep component of $[\![x, y]\!]$ along P**. Moreover, if $d_P(x, y) \ge L_0$, then $[\![x, y]\!]$ contains a deep component along P.*

Let $\mathfrak{o} \in G$ be the vertex representing the identity element and consider an infinite geodesic ray γ in (G, d_G) starting at $\mathfrak{o}$. For $x \in G$, define $\|x\|_{\widehat{G}} := d_{\widehat{G}}(\mathfrak{o}, x)$. Also, for $P \in \mathcal{P}$, define $\|P\|_{\widehat{G}} := d_{\widehat{G}}(\mathfrak{o}, P)$. We will include in our κ–Morse boundary the geodesic rays which have excursion in each peripheral set bounded by a multiple of κ of its $\widehat{G}$–norm. To be precise, we have the following.

Definition 8.3 Let D_0 be given by Proposition 8.2(3). We say that γ has *κ–excursion* with respect to $\mathcal{P}$ if there exists a constant E_γ such that, for each $P \in \mathcal{P}$,

$$\mathrm{diam}_G(\gamma \cap \mathcal{N}_{D_0}(P)) \le E_\gamma \cdot \kappa(\|P\|_{\widehat{G}}),$$

where $\mathcal{N}_{D_0}(P)$ is the D_0–neighborhood of P in (G, d_G). That is, the amount of time γ stays near P grows sublinearly with $\|P\|_{\widehat{G}}$.

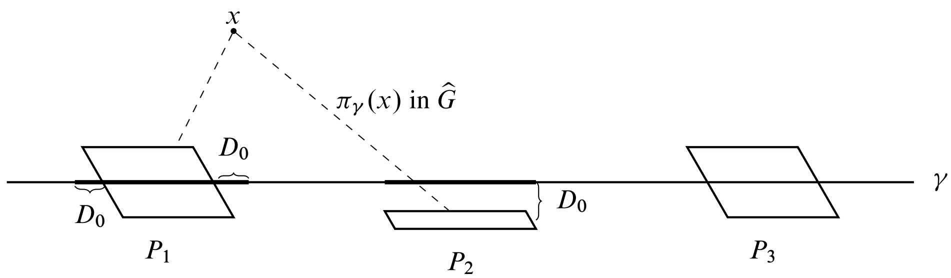

Figure 10: Definition of $\Pi_\gamma(x)$. Suppose P_1, P_2 and P_3 are peripheral sets that are within distance D_0 of γ in G, and that, out of the three, $P_1, P_2 \in \mathcal{P}_{\gamma,x}$; then we consider the union of intersections of γ with the D_0–neighborhoods of P_1 and P_2, which are the bold subsegments.

Our goal is to prove that if γ has κ–excursion, then it is κ–Morse. In fact, we will first show that γ is κ–weakly contracting (see Definition 5.3) and then use Theorem 5.4 to conclude that γ is κ–Morse.

Note that in general the converse inclusion is not true, as there are κ–Morse rays that do not have κ–excursion. The easiest case would be the ray $aca^2ca^3ca^4c\cdots$ in the group $\mathbb{Z}^2 * \mathbb{Z} = \langle a, b, c \mid [a, b] = 1\rangle$, which is $\sqrt{t}$–Morse but does not have κ–excursion for any sublinear function κ. The main reason is that in Definition 8.3 we bound the diameter by $\kappa(\|P\|_{\widehat{G}})$; an alternative definition would be to replace $\kappa(\|P\|_{\widehat{G}})$ by $\kappa(d_G(\mathfrak{o}, P))$, and in that case the above ray would still have $\sqrt{t}$–excursion. Since we shall prove that the set of rays with κ–excursion according to the stronger definition has full measure, the same holds if we adopt the second, weaker, definition.

Another, different reason why κ–Morse does not in general imply κ–excursion is because excursion is defined with respect to the peripheral subgroups. For instance, if the peripheral groups are Gromov hyperbolic, then all geodesics are 1–Morse, while not all geodesic rays have the 1–excursion property.

Let γ be a geodesic ray in G, and let us define $\pi_\gamma \colon G \to \gamma$ to be a nearest point projection onto γ in the metric $d_{\widehat{G}}$. Note that this is well defined up to bounded distance in $\widehat{G}$, but we will fix one such choice for the remainder of this section.

By [Sisto 2012, Proposition 4.11], there exists a constant R_1 such that the set of vertices of γ lies within distance R_1 from a $d_{\widehat{G}}$–geodesic. Thus, by hyperbolicity of $\widehat{G}$, there exist L_1 and R_0 (depending on δ) such that for any $x, y \in G$, if $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L_1$, the geodesic $[x, y]$ in $\widehat{G}$ and the broken geodesic $\gamma' = [x, \pi_\gamma(x)] \cup [\pi_\gamma(x), \pi_\gamma(y)] \cup [\pi_\gamma(y), y]$ lie within a R_0–neighborhood of each other in the metric $d_{\widehat{G}}$ (see eg [Maher 2010, Proposition 3.4]). Moreover, any geodesic segments $[x, \bar{x}]$ and $[y, \bar{y}]$, with $[\bar{x}, \bar{y}]$ a subsegment of $[\pi_\gamma(x), \pi_\gamma(y)]$, belongs to an R_0–neighborhood of γ'.

We denote by $\mathcal{P}_{\gamma,x}$ the set of $P \in \mathcal{P}$ such that P (which has diameter 1 in $\widehat{G}$) lies within $d_{\widehat{G}}$–distance $R_1 := 1 + 4R_0$ of $\pi_\gamma(x)$. Now define a projection $\Pi_\gamma \colon G \to \gamma$ by

$$\Pi_\gamma(x) := \bigcup_{P \in \mathcal{P}_{\gamma,x}} (\gamma \cap \mathcal{N}_{D_0}(P)).$$

As usual, the image of Π_γ is a subset of γ that could have a large diameter.

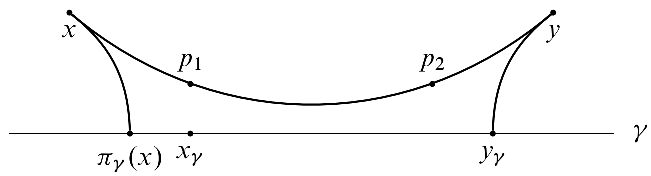

Figure 11: A thin quadrilateral in $\widehat{G}$.

Finally, given $x, y \in G$, consider the subsegment $\eta = [\![\pi_\gamma(x), \pi_\gamma(y)]\!]$ of γ; we define $c_{\gamma,y}(x)$ to be the point of η closest to $\pi_\gamma(x)$ such that the subsegment $[\![c_{\gamma,y}(x), \pi_\gamma(y)]\!] \subseteq \eta$ does not intersect any deep component of any P in $\mathcal{P}_{\gamma,x}$.

Lemma 8.4 *There exists an $L > 0$ such that, for any $x, y \in G$ and any geodesic ray γ based at $\mathfrak{o}$, if $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L$, then*

(1) *$d_{\widehat{G}}(x, x_\gamma) \leq d_{\widehat{G}}(x, y) + L$, where $x_\gamma := c_{\gamma,y}(x)$;*

(2) *for any $P \in \mathcal{P}$, if $d_P(x, x_\gamma) \geq L$, then $d_P(x, x_\gamma) \leq d_P(x, y) + L$.*

Proof Let $y_\gamma := \pi_\gamma(y)$.

(1) By the choice of L_1 and R_0, if $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L_1$, the geodesic $\gamma_1 := [x, y]$ in $\widehat{G}$ and the broken geodesic

$$\gamma_2 := [x, \pi_\gamma(x)] \cup [\pi_\gamma(x), \pi_\gamma(y)] \cup [\pi_\gamma(y), y]$$

lie in a R_0–neighborhood of each other for the metric $d_{\widehat{G}}$. Let η be the segment of γ with endpoints $\pi_\gamma(x)$ and $\pi_\gamma(y)$; by [Sisto 2012, Proposition 4.11], every point of η lies within distance R_1 of the segment $[\pi_\gamma(x), \pi_\gamma(y)]$. Let p_1 and p_2 be nearest point projections (in $\widehat{G}$), respectively, of x_γ and y_γ onto $[x, y]$. This implies

$$d_{\widehat{G}}(x, y) \geq d_{\widehat{G}}(x, p_1) \geq d_{\widehat{G}}(x, x_\gamma) - R_0 - R_1, \tag{29}$$

which proves (1), as long as $L \geq R_0 + R_1$.

(2) To prove (2), suppose that $d_P(x, x_\gamma) \geq 2L_0$ is large, where L_0 is given by Proposition 8.2(3). First, we claim that $d_P(x_\gamma, y_\gamma) \leq L_0$. By the triangle inequality,

$$d_P(x, x_\gamma) \leq d_P(x, \pi_\gamma(x)) + d_P(\pi_\gamma(x), x_\gamma);$$

hence there are two cases: either $d_P(x, \pi_\gamma(x)) \geq L_0$ or $d_P(\pi_\gamma(x), x_\gamma) \geq L_0$.

If $d_P(\pi_\gamma(x), x_\gamma) \geq L_0$, then by Proposition 8.2(4), the geodesic segment $[\![\pi_\gamma(x), x_\gamma]\!]$ in G contains a deep component along P. Then, since x_γ is a transition point and deep components are disjoint, the segment $[\![x_\gamma, y_\gamma]\!]$ in G has no deep component along P. This implies by Proposition 8.2(4) that $d_P(x_\gamma, y_\gamma) \leq L_0$, as claimed.

Otherwise, we can assume that $d_P(\pi_\gamma(x), x_\gamma) \leq L_0$ and $d_P(x, \pi_\gamma(x)) \geq L_0$. First, by the bounded geodesic image theorem, the geodesic segment $[x, \pi_\gamma(x)]$ in $\widehat{G}$ contains an edge in P; let $p \in P$ be a vertex of this edge. Now, by contradiction, suppose that $d_P(x_\gamma, y_\gamma) \ge L_0$; then, by the bounded geodesic image theorem, $[x_\gamma, y_\gamma]$ also contains an edge in P. Thus, there exists $p' \in [x_\gamma, y_\gamma] \cap P$ with $d_{\widehat{G}}(p, p') \le 1$. Thus, using that γ_1 and γ_2 lie in a R_0–neighborhood of each other,

$$d_{\widehat{G}}(p, \pi_\gamma(x)) \le d_{\widehat{G}}(p, p') + 4R_0 \le 1 + 4R_0 = R_1.$$

Hence, P belongs to $\mathcal{P}_{\gamma,x}$, which, as above, implies $d_P(x_\gamma, y_\gamma) \le L_0$.

We now claim that

$$d_{\widehat{G}}([x, x_\gamma], [y, y_\gamma]) \ge 2,$$

which again by Proposition 8.2(3) implies $d_P(y, y_\gamma) \le L_0$. Indeed, let q_1 be a point in $[x, x_\gamma]$ and q_2 be a point in $[y, y_\gamma]$; by hyperbolicity, $d_{\widehat{G}}(q_1, [x, p_1]) \le R_0$ and also $d_{\widehat{G}}(q_2, [y, p_2]) \le R_0$; hence,

$$d_{\widehat{G}}(q_1, q_2) \ge d_{\widehat{G}}(p_1, p_2) - 2R_0 \ge d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) - 2R_2 \ge L - 2R_2 \ge 2$$

where $R_2 = 2R_0 + D_0 + R_1$, provided that we choose $L \ge 2R_2 + 2$.

Finally, by the triangle inequality,

$$d_P(x, y) \ge d_P(x, x_\gamma) - d_P(x_\gamma, y_\gamma) - d_P(y_\gamma, y) \ge d_P(x, x_\gamma) - 2L_0.$$

Thus, if we choose $L := \max\{L_1, R_0, 2R_2 + 2, 2L_0\}$, both (1) and (2) hold. □

Proposition 8.5 *Given any geodesic ray γ, there exist $D_1 < 1$ and $D_2 > 1$ such that for any two points $x, y \in G$,*

$$d_G(x, y) \le D_1 \cdot d_G(x, \gamma) \implies d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \le D_2.$$

Proof We now fix L as given by Lemma 8.4, and we start by contradiction, by assuming that $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \ge L$. By Lemma 8.4, $d_{\widehat{G}}(x, x_\gamma) \prec d_{\widehat{G}}(x, y)$, and $d_P(x, x_\gamma) \prec d_P(x, y)$ whenever $d_P(x, x_\gamma)$ is large enough. Now, applying the distance formula (Proposition 8.2(2)) to the pair of points (x, y), we have

$$\begin{aligned} d_G(x, y) &\asymp \sum_{P \in \mathcal{P}} \lfloor d_P(x, y) \rfloor_L + d_{\widehat{G}}(x, y) \\ &\succ \sum_{P \in \mathcal{P}} \lfloor d_P(x, x_\gamma) \rfloor_L + d_{\widehat{G}}(x, x_\gamma) - O(\delta) \quad \text{(by (29))} \\ &\asymp d_G(x, x_\gamma). \end{aligned}$$

That is to say, there exists $D_1 = D_1(L, \delta)$ such that

$$d_G(x, y) \ge D_1 \cdot d_G(x, x_\gamma),$$

which is a contradiction since $x_\gamma \in \gamma$. Therefore, setting $D_2 = L$ yields

$$d_G(x, y) \le D_1 \cdot d_G(x, \gamma) \implies d_{\widehat{G}}((\pi_\gamma(x), \pi_\gamma(y)) \le D_2. \qquad \square$$

We now show that every κ–excursion geodesic ray is κ–weakly contracting.

Proposition 8.6 *Every κ–excursion geodesic ray $\gamma \in (G, d_G)$ is κ–weakly contracting. That is to say, there exists $D_1 < 1$ and $D_2 > 1$ such that for any two points $x, y \in G$,*

$$d_G(x, y) \le D_1 \cdot d_G(x, \gamma) \implies \operatorname{diam}_G(\Pi_\gamma(x) \cup \Pi_\gamma(y)) \le D_2 \cdot \kappa(x).$$

As a consequence, every κ–excursion geodesic ray is κ–Morse.

Proof By Proposition 8.5, $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \le D_2$, so there are only boundedly many P intersecting $[\pi_\gamma(x), \pi_\gamma(y)]$ in $\widehat{G}$; let $\mathcal{P}_0$ denote the set of such P. By definition of κ–excursion, for each $P \in \mathcal{P}_0$, $d_P(\pi_\gamma(x), \pi_\gamma(y))$ is bounded above by $\kappa(\|P\|_{\widehat{G}})$.

We claim that for all $P \in \mathcal{P}_0$ we have $\|P\|_{\widehat{G}} \prec \|x\|$; hence also $\kappa(\|P\|_{\widehat{G}}) \prec \kappa(x)$. Indeed, since P intersects $[\pi_\gamma(x), \pi_\gamma(y)]$ and $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y))$ is bounded,

$$\|P\|_{\widehat{G}} = d_{\widehat{G}}(\mathfrak{o}, P) \prec d_{\widehat{G}}(\mathfrak{o}, \pi_\gamma(x)).$$

Then, since nearest point projection in the δ–hyperbolic space $\widehat{G}$ is coarsely distance decreasing, $d_{\widehat{G}}(\mathfrak{o}, \pi_\gamma(x)) \prec d_{\widehat{G}}(\mathfrak{o}, x)$, and finally

$$d_{\widehat{G}}(\mathfrak{o}, x) \prec d_G(\mathfrak{o}, x)$$

since the inclusion $G \to \widehat{G}$ is Lipschitz.

Thus, the claim together with the previous estimates and the distance formula yields

$$d_G(\pi_\gamma(x), \pi_\gamma(y)) \asymp \sum_{P \in \mathcal{P}} \lfloor d_P(\pi_\gamma(x), \pi_\gamma(y)) \rfloor_L \prec \sum_{P \in \mathcal{P}_0} \kappa(\|P\|_{\widehat{G}}) \prec D_2 \cdot \kappa(x).$$

Finally, by Theorem 5.4, a κ–weakly contracting geodesic ray is κ–Morse. □

We now show that the map Π_γ is a κ–projection.

Proposition 8.7 *Let γ be a geodesic ray with κ–excursion. Then the map Π_γ defined above is a κ–projection map.*

Proof Let $x \in X$ and $z \in \gamma$, and let $x_\gamma := c_{\gamma,z}(x)$. Then, if $d_{\widehat{G}}(x_\gamma, z) \ge L$, we have, as in the proof of Lemma 8.4, that $d_{\widehat{G}}(x_\gamma, z) \prec d_{\widehat{G}}(x, z)$ and $d_P(x_\gamma, z) \prec d_P(x, z)$ for any $P \in \mathcal{P}$; hence by the distance formula (Proposition 8.2(2)),

$$d_G(x_\gamma, z) \prec d_G(x, z).$$

On the other hand, if $d_{\widehat{G}}(x_\gamma, z) \le L$, then, as in the proof of Proposition 8.6,

$$d_G(x_\gamma, z) \prec \kappa(\|x_\gamma\|_{\widehat{G}}) \prec \kappa(\|x\|_{\widehat{G}}).$$

Moreover, since γ has κ–excursion, $\operatorname{diam}_G(\Pi_\gamma(x)) \prec E_\gamma \cdot \kappa(\|x_\gamma\|_{\widehat{G}}) \prec E_\gamma \cdot \kappa(\|x\|_{\widehat{G}})$. Thus,

$$\operatorname{diam}_G(\Pi_\gamma(x) \cup \{z\}) \le C_1 \cdot d_G(x, z) + C_2 \cdot \kappa(x),$$

where C_1 and C_2 depend only on γ. □

The following is our main result on relatively hyperbolic groups.

Theorem 8.8 *Let μ be a finitely supported probability measure on a relatively hyperbolic group G, and let S be a finite generating set. Let $\kappa(r) := \log r$, and let $\partial_\kappa X$ be the κ–Morse boundary of the Cayley graph X of G with respect to S. Then*

1. *almost every sample path (w_n) converges to a point in $\partial_\kappa X$;*
2. *the pair $(\partial_\kappa X, \nu)$, where ν is the hitting measure of the random walk on $\partial_\kappa X$, is a model for the Poisson boundary of (G, μ).*

Proof The proof is fairly similar to that of Theorem 7.10 for the mapping class group, after replacing subsurfaces Y by peripherals $P \in \mathcal{P}$, d_Y by d_P, and the curve complex $\mathcal{C}(S)$ by $\widehat{G}$. Note, however, that one difference is that there need not be a hyperbolic space analogous to $\mathcal{C}(Y)$ for each $P \in \mathcal{P}$; moreover, the concept of *center* is not well defined. Essentially, the only step where the proof as written does not immediately generalize is the proof of (24), where we do *not* know that nearest point projections are distance decreasing (not even coarsely). We shall give an alternative proof of this point, not using the hyperbolicity of $\mathcal{C}(Y)$.

By Theorem 7.8, almost every sample path $\omega = (w_n)$ converges to a point ξ_ω in the Gromov boundary of $\widehat{G}$. Consider a geodesic ray in $\widehat{G}$ joining the basepoint $\mathfrak{o}$ and ξ_ω, and let $\gamma = \gamma_\omega$ be a lift to G.

Let $c_n := c_{\gamma, w_{2n}}(w_n)$, following the notation before Lemma 8.4. We replace Step 1 in the proof of Theorem 7.10 with the following.

Step 1 We claim that there exists $C > 0$ such that

$$\mathbb{P}\Big(\sup_P d_P(w_n, c_n) \geq C \log n\Big) \leq C n^{-2} \tag{30}$$

for all n.

Proof Since the drift of the random walk is positive with exponential decay (Theorem 7.8(2)), by the Markov property there exists $0 < C_0 < 1$ such that

$$\mathbb{P}\big(d_{\widehat{G}}(w_n, w_{2n}) \leq \ell n\big) = \mathbb{P}\big(d_{\widehat{G}}(\mathfrak{o}, w_n) \leq \ell n\big) \leq (C_0)^n \quad \text{for all } n, \tag{31}$$

where $\ell > 0$ is the drift of the random walk. By Theorem 7.8(3), and recalling that c_n projects close to $\pi_\gamma(w_n)$, there exists $C_1 > 0$ such that

$$\mathbb{P}\big(d_{\widehat{G}}(w_n, c_n) \geq C_1 \log n\big) \leq C_1 n^{-2} \quad \text{for all } n. \tag{32}$$

If a sample path lies in the complement of the union of the events expressed by (31) and (32), then

$$d_{\widehat{G}}(c_n, c_{2n}) \geq d_{\widehat{G}}(w_n, w_{2n}) - d_{\widehat{G}}(w_n, c_n) - d_{\widehat{G}}(w_{2n}, c_{2n}) \geq \ell n - C_1 \log n - C_2 \log(2n) \geq L,$$

where L is given by Lemma 8.4; hence, by Lemma 8.4,

$$d_P(w_n, c_n) \leq d_P(w_n, w_{2n}) + L \tag{33}$$

for any $P \in \mathcal{P}$. Moreover, by [Sisto 2017, Lemma 4.4], there exists $C_2 > 0$ for which

$$\mathbb{P}\big(\sup_P d_P(w_n, w_{2n}) \geq C_2 \log n\big) = \mathbb{P}\big(\sup_P d_Y(\mathfrak{o}, w_n) \geq C_2 \log n\big) \leq C_2 n^{-2} \quad \text{for all } n. \tag{34}$$

Hence, by combining (33) and (34) we obtain (30). □

Then we proceed exactly as in Theorem 7.10 (Steps 2 and 3), proving that for almost every $\omega \in \Omega$, there is a constant c such that

$$\sup_{P \in \mathcal{P}} \operatorname{diam}_G(\gamma \cap \mathcal{N}_{D_0}(P)) \leq c \log d_{\widehat{G}}(\mathfrak{o}, P). \tag{35}$$

Hence, by Proposition 8.6, the geodesic ray γ_ω has κ–excursion; hence it is κ–Morse. This shows (1).

Finally, (2) follows by Theorem 6.1 using Theorem 6.2. □

Appendix General projection and the weakly κ–contracting property

We begin this appendix by proving the following, announced in Section 5.

Theorem 5.4 (κ–weakly contracting implies sublinearly Morse) *Let κ be a concave sublinear function and let Z be a closed subspace of X. Let π_Z be a κ–projection onto Z and suppose that Z is κ–weakly contracting with respect to π_Z. Then there is a function $m_Z\colon \mathbb{R}^2 \to \mathbb{R}$ such that, for every constant $r > 0$ and every sublinear function κ', there is an $R = R(Z, r, \kappa') > 0$ for which the following holds: let $\eta\colon [0, \infty) \to X$ be a (q, Q)–quasigeodesic ray such that $m_Z(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_r be the first time $\|\eta(t_r)\| = r$ and let t_R be the first time $\|\eta(t_R)\| = R$; then*

$$d_X(\eta(t_R), Z) \leq \kappa'(R) \implies \eta([0, t_r]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q}, \mathsf{Q})).$$

Proof Let C_1, C_2, D_1 and D_2 be the constants which appear in the definitions of κ–projection and κ–weakly contracting (Definitions 5.1 and 5.3). Note that the condition of being κ–weakly contracting becomes weaker as C_1 gets smaller; hence we can assume that $C_1 \leq \frac{1}{2}$. We first set

$$m_0 := \max\left\{\frac{\mathsf{q}(\mathsf{q}C_2 + \mathsf{q} + 1) + \mathsf{Q}}{C_1}, \frac{2C_2(D_1 + 1)}{\mathsf{q} - 1}, \mathsf{Q}\right\}, \quad m_1 := \mathsf{q}(C_2 + 1)(D_1 + 1). \tag{36}$$

Claim A.1 *Consider a time interval $[s, s']$ during which η is outside of $\mathcal{N}_\kappa(Z, m_0)$. Then there exists a constant $\mathfrak{A}$ depending only on $\{C_1, C_2, D_1, D_2, \mathsf{q}, \mathsf{Q}\}$ such that*

$$|s' - s| \leq m_1\big(d_X(\eta(s), Z) + d_X(\eta(s'), Z)\big) + \mathfrak{A} \cdot \kappa(\eta(s')). \tag{37}$$

Proof Let

$$s = t_0 < t_1 < t_2 < \cdots < t_\ell = s'$$

be a sequence of times such that, for $i = 0, \ldots, \ell - 2$, we have t_{i+1} is a first time after t_i where

$$d_X(\eta(t_i), \eta(t_{i+1})) = C_1 d_X(\eta(t_i), Z) \quad \text{and} \quad d_X(\eta(t_{\ell-1}), \eta(t_\ell)) \leq C_1 d_X(\eta(t_{\ell-1}), Z). \tag{38}$$

To simplify the notation, we define

$$\eta_i := \eta(t_i), \quad r_i := \|\eta(t_i)\|,$$

and moreover, we pick some $\pi_i \in \pi_Z(\eta_i)$ and let

$$d_i^\pi := d_X(\eta_i, \pi_i), \quad d_i := d_X(\eta_i, Z).$$

Note that, by assumption

$$d_i^\pi \ge d_i = d_X(\eta_i, Z) \ge m_0 \cdot \kappa(r_i). \tag{39}$$

Claim A.2 $$d_{\ell-1}^\pi \le 2(D_1+1)d_\ell^\pi + D_2 \cdot \kappa(\eta_{\ell-1}).$$

Proof By the triangle inequality and (38),

$$d_X(\eta_{\ell-1}, Z) \le d_X(\eta_{\ell-1}, \eta_\ell) + d_X(\eta_\ell, Z) \le C_1 d_X(\eta_{\ell-1}, Z) + d_X(\eta_\ell, Z);$$

hence, using $C_1 \le \frac{1}{2}$,

$$d_X(\eta_{\ell-1}, Z) \le \frac{1}{1-C_1} d_X(\eta_\ell, Z) \le 2 d_X(\eta_\ell, Z).$$

Thus, by Lemma 5.2 and the above equation,

$$\begin{aligned} d_{\ell-1}^\pi &\le (D_1+1) d_X(\eta_{\ell-1}, Z) + D_2 \cdot \kappa(\eta_{\ell-1}) \\ &\le 2(D_1+1) d_X(\eta_\ell, Z) + D_2 \cdot \kappa(\eta_{\ell-1}) \\ &\le 2(D_1+1) d_\ell^\pi + D_2 \cdot \kappa(\eta_{\ell-1}). \end{aligned}$$ □

Now, since Z is κ–weakly contracting, by Definition 5.3 we get

$$d_X(\pi_0, \pi_\ell) \le \sum_{i=0}^{\ell-1} d_X(\pi_i, \pi_{i+1}) \le \sum_{i=0}^{\ell-1} C_2 \cdot \kappa(r_i).$$

But η is (q, Q)–quasigeodesic; hence,

$$|s'-s| \le \mathsf{q} d_X(\eta_0, \eta_\ell) + \mathsf{Q} \le \mathsf{q}\big(d_0^\pi + d_X(\pi_0, \pi_\ell) + d_\ell^\pi\big) + \mathsf{Q} \le \mathsf{q} C_2 \left(\sum_{i=0}^{\ell-1} \kappa(r_i)\right) + \mathsf{q}(d_0^\pi + d_\ell^\pi) + \mathsf{Q}. \tag{40}$$

On the other hand,

$$|s'-s| = \sum_{i=0}^{\ell-1} |t_{i+1} - t_i| \ge \frac{1}{\mathsf{q}} \sum_{i=0}^{\ell-1} (d_X(\eta_i, \eta_{i+1}) - \mathsf{Q}).$$

Meanwhile, for $i = 0, \dots, \ell-2$, we have $d_X(\eta_i, \eta_{i+1}) = C_1 d_X(\eta_i, Z)$. Furthermore, by the triangle inequality,

$$d_X(\eta_{\ell-1}, \eta_\ell) + d_\ell^\pi + d_X(\pi_{\ell-1}, \pi_\ell) \ge d_{\ell-1}^\pi \ge d_X(\eta_{\ell-1}, Z),$$

which gives

$$d_X(\eta_{\ell-1}, \eta_\ell) \ge d_X(\eta_{\ell-1}, Z) - d_\ell^\pi - C_2 \cdot \kappa(r_{\ell-1}).$$

Hence, together with (39) and using $C_1 \le 1$,

$$\begin{aligned} |s'-s| &\ge \frac{1}{\mathsf{q}} \sum_{i=0}^{\ell-1} (C_1 d_X(\eta_i, Z) - \mathsf{Q}) - \frac{d_\ell^\pi + C_2 \cdot \kappa(r_{\ell-1})}{\mathsf{q}} \\ &\ge \frac{1}{\mathsf{q}} \sum_{i=0}^{\ell-1} (C_1 m_0 \cdot \kappa(r_i) - \mathsf{Q}) - \frac{d_\ell^\pi}{\mathsf{q}} - C_2 \frac{\kappa(r_{\ell-1})}{\mathsf{q}} && \text{(by (39))} \\ &\ge \frac{1}{\mathsf{q}} \sum_{i=0}^{\ell-1} (C_1 m_0 \cdot \kappa(r_i) - \mathsf{Q} \cdot \kappa(r_i)) - \frac{d_\ell^\pi}{\mathsf{q}} - C_2 \frac{\kappa(r_{\ell-1})}{\mathsf{q}} && (\kappa(t) \ge 1 \text{ for all } t) \\ &\ge \left(\frac{C_1 m_0 - \mathsf{Q}}{\mathsf{q}} \right) \sum_{i=0}^{\ell-1} \kappa(r_i) - \frac{d_\ell^\pi}{\mathsf{q}} - C_2 \frac{\kappa(r_{\ell-1})}{\mathsf{q}}. \end{aligned}$$

Combining the above inequality with (40), we get

$$q(d_0^\pi + d_\ell^\pi) + \mathsf{Q} + \frac{d_\ell^\pi}{\mathsf{q}} + C_2 \frac{\kappa(r_{\ell-1})}{\mathsf{q}} \ge \left(\frac{C_1 m_0 - \mathsf{Q}}{\mathsf{q}} - \mathsf{q} C_2 \right) \sum_{i=0}^{\ell-1} \kappa(r_i) \ge (\mathsf{q}+1) \sum_{i=0}^{\ell-1} \kappa(r_i), \tag{41}$$

where in the last step we plugged in the definition of m_0 from (36).

By (39), we also have

$$q(d_0^\pi + d_\ell^\pi) + \mathsf{Q} + \frac{d_\ell^\pi}{\mathsf{q}} + C_2 \frac{\kappa(r_{\ell-1})}{\mathsf{q}} \le \mathsf{q}(d_0^\pi + d_\ell^\pi) + \mathsf{Q} + \frac{d_\ell^\pi}{\mathsf{q}} + C_2 \frac{d_{\ell-1}^\pi}{m_0 \mathsf{q}}.$$

By the expression of m_0 in (36), $Q \le m_0$ and by (39), $m_0 \le d_0^\pi$. Thus we have $Q \le d_0^\pi$ and again by plugging in m_0 and using Claim A.2, we obtain

$$\mathsf{q}(d_0^\pi + d_\ell^\pi) + \mathsf{Q} + \frac{d_\ell^\pi}{\mathsf{q}} + C_2 \frac{d_{\ell-1}^\pi}{m_0 \mathsf{q}} \le (\mathsf{q}+1)(d_0^\pi + d_\ell^\pi) + \frac{C_2 D_2}{m_0 \mathsf{q}} \cdot \kappa(\eta_{\ell-1}).$$

Plugging this inequality into (41), we get

$$\begin{aligned} \sum_{i=0}^{\ell-1} \kappa(r_i) &\le d_0^\pi + d_\ell^\pi + \frac{C_2 D_2}{m_0 \mathsf{q}(\mathsf{q}+1)} \cdot \kappa(\eta_{\ell-1}) \\ &\le (D_1+1)(d_0 + d_\ell) + D_2(\kappa(\eta_0) + \kappa(\eta_\ell)) + \frac{C_2 D_2}{m_0 \mathsf{q}(\mathsf{q}+1)} \cdot \kappa(\eta_{\ell-1}), \end{aligned}$$

where we recall $d_i = d_X(\eta_i, Z)$, and the last inequality comes from Lemma 5.2 (note the difference between d_i and $d_i^\pi = d_X(\eta_i, \pi_i)$).

Finally, we claim that since η is a quasigeodesic ray, there is a constant C_3, related to q and Q, such that $\|\eta_i\| \le C_3 \cdot \|\eta(s')\| + C_3$ for $i = 0, \ldots, \ell$; hence also $\kappa(\eta_i) \le 2C_3 \cdot \kappa(\eta(s')) + \kappa(2C_3)$. Thus, to shorten the preceding expression, let $\mathfrak{A}$ be a constant, depending on $\{C_1, C_2, D_1, D_2, \mathsf{q}, \mathsf{Q}, \kappa\}$, such that

$$q(C_2+1) D_2(\kappa(\eta_0) + \kappa(\eta_\ell)) + \mathsf{Q} + \frac{C_2^2 D_2}{m_0(\mathsf{q}+1)} \cdot \kappa(\eta_{\ell-1}) \le \mathfrak{A} \cdot \kappa(\eta(s')).$$

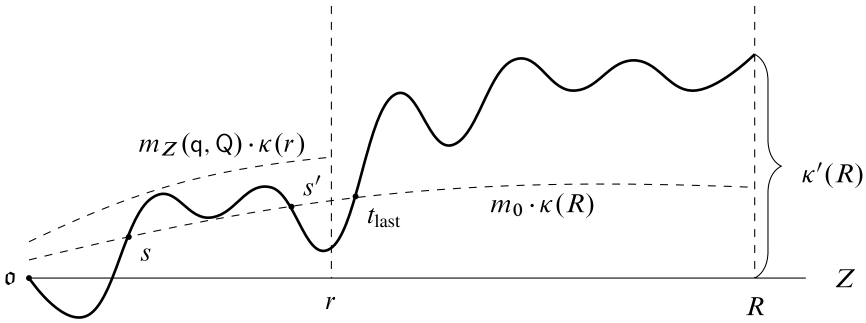

Figure 12: The proof of Theorem 5.4.

By (40) and the definition $m_1=\mathsf{q}(C_2+1)(D_1+1)$ from (36),

$$|s'-s|\le m_1(d_0+d_\ell)+\mathfrak{A}\cdot\kappa(\eta(s')).$$

This proves Claim A.1. □

Now let t_{last} be the last time η is in $\mathcal{N}_\kappa(Z,m_0)$ and consider the quasigeodesic path $\eta([t_{\text{last}},t_R])$. Since this path is outside of $\mathcal{N}_\kappa(Z,m_0)$, we can use (37) to get

$$|t_R-t_{\text{last}}|\le m_1\big(d_X(\eta(t_{\text{last}}),Z)+d_X(\eta(t_R),Z)\big)+\mathfrak{A}\cdot\kappa(R).$$

But

$$\begin{aligned} d_X(\eta(t_{\text{last}}),Z) &\le m_0\cdot\kappa(\eta(t_{\text{last}})) && \text{(by the choice of } t_{\text{last}})\\ &\le m_0\cdot\kappa(R) && \text{(since } \kappa \text{ is monotone)} \end{aligned}$$

and we have by assumption $d_X(\eta(t_R),Z)\le\kappa'(R)$. Therefore,

$$|t_R-t_{\text{last}}|\le m_0m_1\cdot\kappa(R)+m_1\cdot\kappa'(R)+\mathfrak{A}\cdot\kappa(R).$$

Since η is (q,Q)–quasigeodesic, we obtain $R=d_X(\eta(0),\eta(t_R))\le\mathsf{q}t_R+Q$; hence

$$t_R\ge\frac{R-\mathsf{Q}}{\mathsf{q}}.$$

Since m_0 and m_1 are given and κ and κ' are sublinear, there is a value of R depending on m_0, m_1, r, $\mathfrak{A}$, κ and κ' such that

$$m_0\cdot m_1\cdot\kappa(R)+m_1\cdot\kappa'(R)+\mathfrak{A}\cdot\kappa(R)\le\frac{R-\mathsf{Q}}{\mathsf{q}}-r.$$

For any such R, we then have

$$t_{\text{last}}\ge t_R-\frac{R-\mathsf{Q}}{\mathsf{q}}+r\ge r.$$

We show that $\eta([0,t_{\text{last}}])$ stays in a larger κ–neighborhood of Z. Consider any other subinterval $[s,s']\subset[0,t_{\text{last}}]$ where η exits $\mathcal{N}_\kappa(Z,m_0)$. By taking $[s,s']$ as large as possible, we can assume $\eta(s),\eta(s')\in\mathcal{N}_\kappa(Z,m_0)$. In this case,

$$d_X(\eta(s),Z)\le m_0\cdot\kappa(\eta(s))\quad\text{and}\quad d_X(\eta(s'),Z)\le m_0\cdot\kappa(\eta(s')).$$

Again applying (37), we get

$$|s'-s| \le m_0 m_1 \cdot \bigl(\kappa(\eta(s)) + \kappa(\eta(s'))\bigr) + \mathfrak{A}\cdot\kappa(\eta(s')), \tag{42}$$

and thus

$$\begin{aligned} d_X(\eta(s'),\eta(s)) &\le \mathsf{q} m_0 m_1 \cdot \bigl(\kappa(\eta(s)) + \kappa(\eta(s'))\bigr) + \mathsf{q}\mathfrak{A}\cdot\kappa(\eta(s')) + \mathsf{Q} \\ &\le (2\mathsf{q} m_0 m_1 + \mathsf{q}\mathfrak{A} + \mathsf{Q}) \cdot \max\bigl(\kappa(\eta(s)), \kappa(\eta(s'))\bigr). \end{aligned}$$

Applying the sublinear estimation lemma [Qing and Rafi 2022, Lemma 3.2], we obtain

$$\kappa(\eta(s')) \le m_2 \cdot \kappa(\eta(s))$$

for some m_2 depending on q, Q and κ. Therefore, by plugging this inequality back into (42), we have for any $t \in [s,s']$,

$$|t-s| \le (m_0 m_1(1+m_2) + \mathfrak{A} m_2)\cdot\kappa(\eta(s)) = m_3\cdot\kappa(\eta(s)) \tag{43}$$

with $m_3 = (m_0 m_1(1+m_2) + \mathfrak{A} m_2)$. As before, this implies

$$d_X(\eta(t),\eta(s)) \le \mathsf{q} m_3\cdot\kappa(\eta(s)) + \mathsf{Q} \le (\mathsf{q} m_3 + Q)\cdot\kappa(\eta(s)).$$

Applying [Qing and Rafi 2022, Lemma 3.2] again, we have

$$\kappa(\eta(s)) \le m_4\cdot\kappa(\eta(t)) \tag{44}$$

for some m_4 depending on q, Q and κ.

Now, for any $t \in [s,s']$,

$$\begin{aligned} d_X(\eta(t), Z) &\le d_X(\eta(t),\eta(s)) + r_0 \\ &\le \mathsf{q}|t-s| + \mathsf{Q} + m_0\cdot\kappa(\eta(s)) \\ &\le (\mathsf{q} m_3 + \mathsf{Q} + m_0)\cdot\kappa(\eta(s)) && \text{(by (43))} \\ &\le (\mathsf{q} m_3 + \mathsf{Q} + m_0) m_4\cdot\kappa(\eta(t)) && \text{(by (44))}. \end{aligned}$$

Setting

$$m_Z(\mathsf{q},\mathsf{Q}) = (\mathsf{q} m_3 + \mathsf{Q} + m_0) m_4, \tag{45}$$

we have the inclusion

$$\eta([s,s']) \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q},\mathsf{Q})) \quad \text{and hence} \quad \eta([0,t_{\mathrm{last}}]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathsf{q},\mathsf{Q})).$$

The R we have chosen depends on the value of q and Q. However, the assumption that $m_Z(\mathsf{q},\mathsf{Q})$ is small compared to r (see (2)) gives an upper bound for the values of q and Q. Hence, we can choose R to be the radius associated to the largest possible value for q and the largest possible value for Q. This finishes the proof.

Note that, the assumption that $m_Z(\mathsf{q},\mathsf{Q})$ is small compared to r is not really needed here and any upper bound on the values of q and Q would suffice. But this is the assumption we will have later on and hence it is natural to state the theorem this way. □

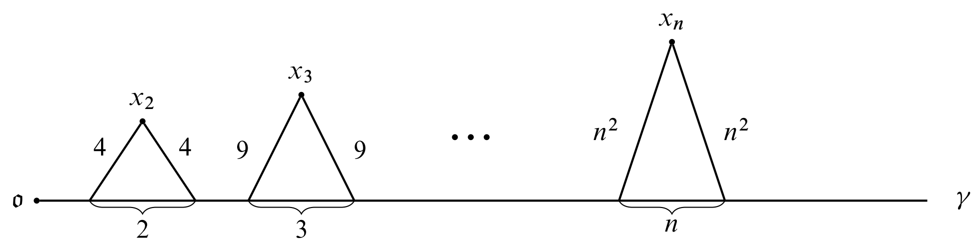

Figure 13: A 1–Morse geodesic ray which is $\sqrt{t}$–contracting.

Next, we show that being κ–Morse implies being κ'–weakly contracting, with respect to κ–projection maps, for a sublinear function κ'. Note that κ' is not assumed to be the same function as κ. This is parallel to [Qing and Rafi 2022, Theorem 3.8], where we show that in a CAT(0) space being κ–contracting is equivalent to being κ–Morse (with the same κ); an identical statement cannot hold in general for proper geodesic spaces, as evidenced by the following example.

Example A.3 We give here a folklore example of a geodesic ray in proper metric space which is 1–Morse but not 1–contracting (see also the related [Arzhantseva et al. 2017, Example 3.4]). The points x_i form loops with the geodesic ray γ such that each path going through x_i represents a detour that is locally an $(i, 0)$ quasigeodesic segment. This geodesic is

- 1–Morse, as any (q, Q)–quasigeodesic ray must lie in a q–neighborhood of γ (in particular, it only goes through finitely many loops);
- not 1–contracting, but is $\sqrt{t}$–contracting, as $\|x_i\| \succcurlyeq i^2$ and, if π_γ is the nearest point projection to γ, we have $\operatorname{diam}(\pi_\gamma(x_i)) = i$.

Thus it makes sense for us to prove in general that a κ–Morse set is κ'–weakly contracting for some sublinear function κ'.

To begin with, a set Z is Morse if it is κ–Morse for $\kappa = 1$, and its Morse gauge is denoted by $m_Z(\mathsf{q}, \mathsf{Q})$. We say that a closed set Z is *ρ–radius-contracting* if there exists a function ρ such that, for each ball B that is disjoint from Z, the nearest point projection of B to Z is a set whose diameter is bounded above by $\rho(d_X(x, Z))$. We present [Arzhantseva et al. 2017, Proposition 4.2] here with notation adapted to that of this paper. Furthermore, we take into account that in the setting of this paper, nearest point projection is nonempty:

Proposition A.4 [Arzhantseva et al. 2017, Proposition 4.2] *Let Z be a closed subspace of a geodesic metric space X, with $\mathfrak{o} \in Z$. Suppose Z is Morse with Morse gauge $m_Z(\mathsf{q}, \mathsf{Q})$. Let $B(x, r)$ denote a ball centered at x with radius r and disjoint from Z. Moreover, let $r_d = d_X(x, Z)$. Then there is a sublinear function ρ, depending on $m_Z(\mathsf{q}, \mathsf{Q})$, such that the nearest-point projection of the ball $B(x, r)$ onto Z is bounded above by $\rho(r_d)$. Specifically, ρ is obtained as*

$$\rho(r_d) := \sup_s \left\{ s \le 4r_d \text{ and } s \le 18 m_Z\left(\frac{12 r_d}{s}, 0\right) \right\}.$$

Given a point $x \in X$ and a sublinearly Morse set $Z \subseteq X$, first we note that

$$r_d = d_X(x, Z) \le d_X(x, \mathfrak{o}) = \|x\|.$$

We call a set Z *sublinearly weakly contracting* if it is κ–weakly contracting for some sublinear function κ.

Proposition A.5 *Let α be a quasigeodesic ray in a proper geodesic space X that is κ–weakly Morse. Then there exists a sublinear function κ' such that α is κ'–weakly contracting.*

Proof It suffices to prove the statement for nearest point projections, since by Lemma 5.2, the uniform multiplicative error and the sublinear additive error will not contradict the conclusion. Consider a ball $B(x,r)$, disjoint from α, and centered at x with radius r. Observe first of all that $B(x,r)$ is inside the ball of radius $\|x\| + r_d \le 2\|x\|$. Thus the distance between any point $y \in B(x,r)$ and a point in its nearest point projection $y_1 \in \pi_\alpha^{(\text{near})}(y)$ is also bounded above by $2\|x\|$. We have by triangle inequality

$$\|y_1\| \le 4\|x\|.$$

That is to say, given a ball $B(x,r)$, we only have to consider its projection to $\alpha \cap B(\mathfrak{o}, 4\|x\|)$. By Proposition 3.10, since α is κ–Morse, it is also κ–weakly Morse in the sense of Definition 3.9. Let m_α be the κ–weakly Morse gauge of α. Thus the set $\alpha \cap B(\mathfrak{o}, 4\|x\|)$ is Morse with its Morse gauge $m_\alpha(\mathsf{q},\mathsf{Q}) \cdot \kappa(4x)$. Now let s denote the function that measures the diameter of the projection of disjoint ball $B(x,r)$ to α. By Proposition A.4,

$$s \le 18 m_\alpha\left(\frac{12 r_d}{s}, 0\right) \cdot \kappa(4x) \quad \text{by definition of } \kappa\text{–Morse.}$$

Since $r_d \le \|x\|$,

$$s \le 18 m_\alpha\left(\frac{12\|x\|}{s}, 0\right) \cdot \kappa(4x).$$

Suppose s is not a sublinear function of $\|x\|$; that is to say, as $\|x\| \to \infty$, there exists a sequence of disjoint balls $\{B_i\}$ with centers $\{x_i\}$ and projections $\{s_i\}$ such that there exists a positive number c such that

$$\lim_{i\to\infty} \frac{s_i}{\|x_i\|} \ge c.$$

Then, for every ϵ, there exists N such that for all $j > N$,

$$s_j \le 18 m_\alpha\left(\frac{12\|x_j\|}{s_j}, 0\right) \cdot \kappa(4x_j) \le 18 m_\alpha\left(12\left(\frac{1}{c} - \epsilon\right), 0\right) \cdot \kappa(4x_j).$$

That is to say, s is bounded above by a sublinear function of $\|x\|$, which means s itself is a sublinear function of $\|x\|$, which is contrary to our assumption. Therefore, there does not exist such a sequence of balls, and thus s is a sublinear function of $\|x\|$, which we denote by $\kappa'(x)$.

Finally, by Lemma 5.2, the same claim holds for all κ–projection maps. $\square$

To summarize, we have proven the equivalence between κ–weakly contracting, κ–weakly Morse and κ–Morse (for a possibly different κ) for any quasigeodesic:

Theorem 5.5 *Let $(X, \mathfrak{o})$ be a proper geodesic metric space with a fixed basepoint and let α be a quasigeodesic ray in X. Let π be any κ–projection from X to α in the sense of Definition 5.1. Then*

1. *if α is κ–weakly contracting with respect to π, then it is κ–Morse;*
2. *the quasigeodesic α is κ–Morse if an only if it is κ–weakly Morse;*
3. *if α is κ–weakly Morse, then it is κ'–weakly contracting with respect to π for some sublinear function κ'.*

Proof If α is κ–weakly contracting, then by Theorem 5.4 it is κ–Morse. The second assertion is the statement of Proposition 3.10. Finally, if α is κ–Morse, by Proposition A.5 there exists a sublinear function κ' for which α is κ'–weakly contracting. □

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