

SUBLINEARLY MORSE BOUNDARY II: PROPER GEODESIC SPACES

YULAN QING, KASRA RAFI, AND GIULIO TIOZZO

ABSTRACT. We build an analogue of the Gromov boundary for any proper geodesic metric space, hence for any finitely generated group. More precisely, for any proper geodesic metric space X and any sublinear function κ , we construct a boundary for X , denoted $\partial_\kappa X$, that is quasi-isometrically invariant and metrizable. As an application, we show that when G is the mapping class group of a finite type surface, or a relatively hyperbolic group, then with minimal assumptions the Poisson boundary of G can be realized on the κ -Morse boundary of G equipped with the word metric associated to any finite generating set.

1. INTRODUCTION

In this paper, we construct an analogue of the Gromov boundary for a general proper geodesic metric space. That is, a boundary at infinity that is invariant under quasi-isometry, has good topological properties and is as large as possible. Our guiding principle is that, moving from the setting of Gromov hyperbolic spaces to general metric spaces, most key arguments still go through if we replace uniform bounds with sublinear bounds (with respect to distance to some base point). Examples of this philosophy have appeared in the literature before, for example in [Dru00, Kar11, EFW12, EFW13, ACGH17, EMR18]. In a prequel to this paper [QRT19], such a boundary was constructed in the setting of CAT(0) metric spaces. However, the definition of κ -Morse given in the previous paper does not work for general proper geodesic metric spaces. Hence, we use a new definition of κ -Morse that is more flexible and more fully embraces the above philosophy.

Statement of Results. Let (X, d_X) be a proper, geodesic metric space with a base point \mathfrak{o} . Recall that, when X is Gromov hyperbolic, the Gromov boundary of X is the set of equivalence classes of quasi-geodesic rays emanating from \mathfrak{o} , equipped with the cone topology. Two quasi-geodesic rays are considered equivalent if they stay within bounded distance from each other. In a Gromov hyperbolic space, every quasi-geodesic ray β is *Morse*: that is, any other quasi-geodesic segment γ with endpoints on β stays in a bounded neighborhood of β .

Similarly, we consider quasi-geodesic rays in X (rays are always assumed to be emanating from \mathfrak{o}). Roughly speaking, we say a quasi-geodesic ray β is *sublinearly Morse* if any other quasi-geodesic segment γ with endpoints sublinearly close to β stays in a sublinear neighborhood of β (see Definition 3.2 for the precise definition). We group the set of sublinearly Morse quasi-geodesic rays into equivalence classes by setting quasi-geodesic rays α and β to be equivalent if they stay sublinearly close to each other. We call the set of equivalence classes of sublinearly Morse quasi-geodesic rays, equipped with a coarse version of the cone topology, the *sublinearly Morse boundary* of X .

In fact, the above construction works for any given *sublinear function* $\kappa: [0, \infty) \rightarrow [1, \infty)$, where κ is a concave, increasing function with

$$\lim_{t \rightarrow \infty} \frac{\kappa(t)}{t} = 0.$$

Then, we define the κ -Morse boundary $\partial_\kappa X$ to be the space of equivalence classes of κ -Morse quasi-geodesic rays equipped with the coarse cone topology (see Definition 4.1). We obtain a possibly large family of boundaries for X , each associated to a different sublinear function κ .

We show that $\partial_\kappa X$ is metrizable and invariant under quasi-isometries; moreover, κ -boundaries associated to different sublinear functions are topological subspaces of each other.

Theorem A. *Let X be a proper, geodesic metric space, and let κ be a sublinear function. Then we construct a topological space $\partial_\kappa X$ with the following properties:*

- (1) (Metrizability) *The spaces $\partial_\kappa X$ and $X \cup \partial_\kappa X$ are metrizable, and $X \cup \partial_\kappa X$ is a bordification of X ;*
- (2) (QI-invariance) *Every (k, K) -quasi-isometry $\Phi: X \rightarrow Y$ between proper geodesic metric spaces induces a homeomorphism $\Phi^*: \partial_\kappa X \rightarrow \partial_\kappa Y$;*
- (3) (Compatibility) *For sublinear functions κ and κ' where $\kappa \leq c \cdot \kappa'$ for some $c > 0$, we have $\partial_\kappa X \subset \partial_{\kappa'} X$ where the topology of $\partial_\kappa X$ is the subspace topology. Further, letting $\partial X := \bigcup_\kappa \partial_\kappa X$, we obtain a quasi-isometry invariant topological space that contains all $\partial_\kappa X$ as topological subspaces. We call ∂X the sublinearly Morse boundary of X .*

Note that from QI-invariance it follows that $\partial_\kappa X$ and ∂X do not depend on the base point \mathfrak{o} . Moreover, it also implies that the κ -Morse boundary of a finitely generated group G is independent of the generating set. Thus $\partial_\kappa G$ and ∂G are well defined.

We now argue that, in different settings, the κ -Morse boundary $\partial_\kappa X$ is large for an appropriate choice of κ . Recall that the *Poisson boundary* is the maximal boundary from the measurable point of view (see Section 6). In this paper, we let G be either a mapping class group or a relatively hyperbolic group and X be a Cayley graph of G and show that $\partial_\kappa X$ is a topological model for the Poisson boundary of (G, μ) , where μ is any non-elementary finitely supported measure on G .

In fact, we show the following general criterion: if almost every sample path of the random walk driven by μ sublinearly tracks a κ -Morse geodesic, then the κ -Morse boundary is identified with the Poisson boundary. The following result was obtained in collaboration with Ilya Gekhtman.

Theorem B. *Let G be a finitely generated group, and let (X, d_X) be a Cayley graph of G . Let μ be a probability measure on G with finite first moment with respect to d_X , such that the semigroup generated by the support of μ is a non-amenable group. Let κ be a sublinear function, and suppose that for almost every sample path $\omega = (w_n)$, there exists a κ -Morse geodesic ray γ_ω such that*

$$(1) \quad \lim_{n \rightarrow \infty} \frac{d_X(w_n, \gamma_\omega)}{n} = 0.$$

Then almost every sample path converges to a point in $\partial_\kappa X$, and moreover the space $(\partial_\kappa X, \nu)$, where ν is the hitting measure for the random walk, is a model for the Poisson boundary of (G, μ) .

Theorem A and Theorem B can be applied to a large class of groups, providing a blueprint to identify their Poisson boundary. The κ -Morse boundary is always well-defined, and in different situations one just needs to prove the above sublinear tracking result between geodesics and sample paths in order to obtain the Poisson boundary. In this paper, we showcase their applications to mapping class groups and relatively hyperbolic groups.

Comparison with other boundaries. Our construction provides the first example of a boundary of a general metric space which is both QI-invariant and has full measure with respect to a large class of measures. For comparison, recall that the *visual boundary* of CAT(0) spaces is not invariant by quasi-isometries [CK00]. The *Gromov boundary* [Gro87], on the other hand, is QI-invariant, but only defined if the group is hyperbolic; a natural generalization is the *Morse boundary* [Cor18], which is always well-defined and QI-invariant, but very often it is too small: in particular, it has measure zero with respect to the hitting measure for most random walks on relatively hyperbolic groups

[CDG20]. This is related to the fact that a typical sample path is expected to have unbounded excursions in the peripherals. Finally, the *Floyd boundary* [Flo80] is well-behaved for relatively hyperbolic groups, but trivial for mapping class groups [KN04].

Recall also it is an open problem [Kai96, page 153] whether any two finitely supported generating measures on the same group yield isomorphic Poisson boundaries. This question is the probabilistic analog of the quasi-isometry invariance we show in Theorem A, as two generating sets for G give rise to both two quasi-isometric metrics on G and to two finitely supported measures.

Mapping class groups. Let S be a surface of finite hyperbolic type and $\text{Map}(S)$ be the mapping class group of S . Let d_w be the word metric on $\text{Map}(S)$ with respect to some finite generating set. Then letting $(X, d_X) = (\text{Map}(S), d_w)$ we can consider the κ -Morse boundary $\partial_\kappa \text{Map}(S)$ of the mapping class group. We show the following characterization.

Theorem C. *Let μ be a finitely supported, non-elementary probability measure on $\text{Map}(S)$. Then for $\kappa(t) = \log(t)$, we have:*

- (1) *Almost every sample path (w_n) converges to a point in $\partial_\kappa \text{Map}(S)$;*
- (2) *The κ -Morse boundary $(\partial_\kappa \text{Map}(S), \nu)$ is a model for the Poisson boundary of $(\text{Map}(S), \mu)$ where ν is the hitting measure associated to the random walk driven by μ .*

The proof uses the machinery of curve complexes introduced by Masur-Minsky [MM00], as well the study of random walks in mapping class groups carried by Maher [Mah10], [Mah12], Calegari-Maher [CM15], Sisto [Sis17], Maher-Tiozzo [MT18] and Sisto-Taylor [ST19]. In particular, by [ST19], a typical sample path makes logarithmic progress in each subsurface, which explains the function $\log(t)$.

Moreover, we also obtain the following tracking result between geodesics and sample paths in the mapping class group.

Theorem D. *Let μ be a finitely supported, non-elementary probability measure on $\text{Map}(S)$. Then, for almost every sample path there exists a κ -Morse geodesic ray γ_w in $\text{Map}(S)$ such that*

$$\limsup_{n \rightarrow \infty} \frac{d_w(w_n, \gamma_w)}{\log n} < +\infty.$$

Sisto [Sis17] showed that almost every sample path in the mapping class group lies within distance $O(\sqrt{n \log n})$ of a geodesic. A more robust tracking result was proven in [MS20, Theorem 10.7], which gives the $\log(n)$ -tracking in this setting, but not the fact that the geodesic being tracked is κ -Morse. Sublinear tracking for random walks with respect to the Teichmüller metric was obtained in [Tio15].

Relatively hyperbolic groups. Now consider a finitely generated group G equipped with a word metric d_w associated to a finite generating set. Recall that G is relatively hyperbolic with respect to a family of subgroups H_1, \dots, H_k if, after contracting the Cayley graph of G along H_i -cosets, the resulting graph equipped with the usual graph metric is Gromov hyperbolic. Further, H_i -cosets have to satisfy the technical condition of *bounded coset penetration*. Similarly to above, we show:

Theorem E. *Let G be a non-elementary relatively hyperbolic group, and let μ be a probability measure whose support is finite and generates G as a semigroup. Then for $\kappa(t) = \log(t)$, we have:*

- (1) *Almost every sample path (w_n) converges to a point in $\partial_\kappa G$;*
- (2) *The κ -Morse boundary $(\partial_\kappa G, \nu)$ is a model for the Poisson boundary of (G, μ) where ν is the hitting measure associated to the random walk driven by μ .*

Further applications. As mentioned above, Theorem A and Theorem B can be applied to any group with a certain degree of hyperbolicity. For example, the proof of Theorem C already works in the setting of hierarchically hyperbolic spaces [BHS17]. Hierarchically hyperbolic spaces are a family of axiomatically defined spaces with properties that are modelled after the mapping class groups. Hence, all the tools we use, such as subsurface projections, distance formulas and the Bounded Geodesic Image Theorem also exist in the setting of hierarchically hyperbolic spaces.

Other groups where the κ -Morse boundary is non-trivial and these ideas can be applied to include groups that contain an element with contracting axis, for example, the groups with *statistically convex co-compact actions* as described in [Yan20], as well as $\text{Out}(F_n)$. We shall discuss these applications in subsequent papers.

Sublinearly Morse vs. Sublinearly Contracting. In the construction of the κ -Morse boundary $\partial_\kappa X$, several different definitions are possible for the notion of a κ -Morse quasi-geodesic. The goal is always to emulate the behavior of quasi-geodesics in a Gromov hyperbolic space but with sublinear errors (instead of uniform additive errors).

The definition of κ -Morse given in this paper (Definition 3.2) is equivalent to the definition of *strongly Morse* in [QRT19]. Another natural condition is to require a quasi-geodesic ray β to be κ -*weakly contracting* (Definition 5.3): that is, that the projection of a ball disjoint from β to β has a diameter that is bounded by a sublinear function of the distance of the center of the ball to the origin.

In the setting of CAT(0) spaces, these two notions are equivalent ([QRT19, Theorem 3.8]), but this is no longer true for general metric spaces. In the Appendix, we prove that κ -weakly contracting quasi-geodesics are always κ -Morse. The converse is known not to be the case in general: for example, when κ is the constant function, a geodesic axis of a pseudo-Anosov element in the mapping class group is always Morse [MM00], but not always strongly contracting [RV18]. However, κ -Morse quasi-geodesics are always κ' -weakly contracting for a larger sublinear function κ' :

Theorem F. *Let X be a proper geodesic metric space, let κ be a sublinear function, and let β be a quasi-geodesic ray in X . Then:*

- (1) *If β is κ -weakly contracting, it is κ -Morse;*
- (2) *There is a sublinear function κ' such that if β is κ -Morse then it is κ' -weakly contracting.*

The definition of κ -Morse we use in this paper is the one that matches our philosophical approach the best, is the most flexible and makes the arguments simplest. Hence, we think it is the *correct* definition with minimal assumptions.

In some places, the same arguments as in [QRT19] apply directly, while in others new ideas are needed; for the sake of brevity, we shall skip the proofs when the arguments are exactly the same.

History. The notion of a *Morse geodesic* is classical [Mor24], and much progress has been made in recent years using Morse geodesics to define boundaries of groups. In [Cor18], Cordes, inspired by the *contracting boundary* for CAT(0) spaces of Charney and Sultan [CS15], constructed the *contracting boundary* for all proper geodesic spaces, where a quasi-geodesic γ is *Morse* if there are uniform neighborhoods, with size depending on (q, Q) , in which all (q, Q) -quasi-geodesic segments with endpoints on γ lie. The Morse boundaries are equipped with a *direct limit topology* and are invariant under quasi-isometries. However, this space does not have good topological properties; for example, it is not first countable. Cashen-Mackay [CM19], following the work of Arzhantseva-Cashen-Gruber-Hume [ACGH17], defined a different topology on the Morse boundary. They showed that it is Hausdorff and when there is a geometric action by a countable group, it is also metrizable; note that Theorem A does not assume any geometric action. Another notion, more closely related to our definition of κ -Morse boundary, is considered by Kar [Kar11]: a geodesic space is *asymptotically CAT(0)* if balls of radius r are coarsely CAT(0) with an error of $f(r)$, where the function $f(r)$ is

sublinear. That is to say, a space is asymptotically CAT(0) if it has the same κ -Morse boundary as a CAT(0) space, where $\kappa = f(r)$.

The definition of a *contracting* geodesic originates from [Mor24] and has been brought back to attention by [Gro87]. In [MM00], Masur-Minsky proved that axes of pseudo-Anosov elements are contracting, and since then various versions of this condition have been discussed, in particular the notions of *strongly contracting* and *weakly contracting*, see e.g. [Beh06], [BF09], [AK11], [ACT15], [Sis18], [Yan20], and [RV18]. Our definition of κ -weakly contracting is even weaker, and for that reason it is expected to be generic with respect to many notions of genericity. For random walks, genericity of sublinearly contracting geodesics in relatively hyperbolic groups follows from [Sis17], in hierarchically hyperbolic groups it follows from [ST19] and in CAT(0) groups it follows from [GQR20]. For the counting measure, genericity of log-weakly contracting geodesics in RAAGs has been shown in [QT19].

The Poisson boundary of (G, μ) is trivial for all non-degenerate measures μ on abelian groups [CD60] and nilpotent groups [DM60]. Discrete subgroups of $SL(d, \mathbb{R})$ are treated in [Fur63], where the Poisson boundary is related to the space of flags. For random walks on Lie groups, the study and the description of the Poisson boundary was extensively developed in the 70's and the 80's by many authors, most notably Furstenberg. The Poisson boundary of some Fuchsian groups has also been described by Series [Ser83] as being the limit set of the group. Kaimanovich [Kai94] identified the Poisson boundary of hyperbolic groups with their Gromov boundaries with the associated hitting measures. Karlsson and Margulis proved that visual boundaries of nonpositively curved spaces serve as models for their Poisson boundaries [KM99]. Kaimanovich and Masur [KM96, KM98] proved that the Poisson boundary of the mapping class group is the boundary of Thurston's compactification of Teichmüller space. Their description also runs for the Poisson boundary of the braid group; see [FM98].

Acknowledgement. Y.Q. thanks the University of Toronto for its hospitality in Fall 2020. K.R. is partially supported by NSERC. G.T. is partially supported by NSERC and the Alfred P. Sloan foundation. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while G.T. participated in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2020 semester. We thank Ilya Gekhtman for several useful conversations. We also thank Alessandro Sisto and Abdul Zalloum for helpful comments on an earlier version of this paper.

2. PRELIMINARIES

Let (X, d_X) be a proper, geodesic metric space, and let $\mathfrak{o} \in X$ be a base point. Given a point $p \in X$, we denote $\|p\| := d_X(\mathfrak{o}, p)$. Let α be a quasi-geodesic ray starting at \mathfrak{o} . For $r > 0$, let t_r be the first time where $\|\alpha(t_r)\| = r$ and denote the point $\alpha(t_r)$ with

$$\alpha_r := \alpha(t_r),$$

while the segment from $\alpha(0)$ to $\alpha(t_r)$ will be denoted as

$$\alpha|_r := \alpha([0, t_r]).$$

We collect here the two basic geometric properties of the space that we need:

Lemma 2.1. *Let (X, d_X) be a proper, geodesic metric space. Then:*

- *For any closed set $Z \subset X$ and any point $x \in X$, there is a closed set $\pi_Z(x)$ of nearest points in Z to x . We refer to any point in $\pi_Z(x)$ as a nearest point projection from x to Z .*
- *Any sequence of geodesics $\beta_n: [0, n] \rightarrow X$ with $\beta_0 = \mathfrak{o}$ has a subsequence that converges uniformly on compact sets to a geodesic ray $\beta_0: [0, \infty) \rightarrow X$.*

The following lemma about nearest point projections will be used several times.

Lemma 2.2. *Let α, β be two quasi-geodesic rays starting at the base point $\mathfrak{o} \in X$. Let x be a point on α , and let p be a nearest point projection of x onto β . Then we have*

$$\|p\| \leq 2\|x\|.$$

Proof. Note that, since p is a nearest point projection of x onto β and \mathfrak{o} belongs to β , we have

$$d_X(x, p) \leq d_X(\mathfrak{o}, x).$$

Hence, by the triangle inequality,

$$\|p\| = d_X(\mathfrak{o}, p) \leq d_X(\mathfrak{o}, x) + d_X(x, p) \leq 2d_X(\mathfrak{o}, x) = 2\|x\|. \quad \square$$

2.1. Sublinear functions. In this paper, a *sublinear function* will be a function $\kappa: [0, \infty) \rightarrow [1, \infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{\kappa(t)}{t} = 0.$$

Moreover, we say that $\kappa: [0, \infty) \rightarrow [1, \infty)$ is a *concave sublinear function* if it is sublinear and moreover it is increasing and concave.

Remark 2.3. The assumptions that κ is increasing and concave make certain arguments cleaner, otherwise they are not really needed. One can always replace any sublinear function κ with another sublinear function $\bar{\kappa}$ so that $\kappa(t) \leq \bar{\kappa}(t) \leq C\kappa(t)$ for some constant C and $\bar{\kappa}$ is monotone increasing and concave. For example, define

$$\bar{\kappa}(t) := \sup \left\{ \lambda\kappa(u) + (1 - \lambda) \cdot \kappa(v) \mid 0 \leq \lambda \leq 1, u, v > 0, \text{ and } \lambda u + (1 - \lambda)v = t \right\}.$$

Lemma 2.4. *If $\kappa: [0, \infty) \rightarrow [1, \infty)$ is a concave sublinear function and $\lambda > 1$, then*

$$\kappa(\lambda t) \leq \lambda\kappa(t)$$

for any $t \geq 0$.

Proof. By concavity,

$$\kappa(t) = \kappa\left(\frac{1}{\lambda}\lambda t + (1 - \lambda^{-1}) \cdot 0\right) \geq \frac{1}{\lambda}\kappa(\lambda t) + (1 - \lambda^{-1}) \cdot \kappa(0) \geq \frac{1}{\lambda}\kappa(\lambda t)$$

from which the claim follows. □

3. THE κ -MORSE BOUNDARY

We now introduce the definition of κ -Morse quasi-geodesic, which will be fundamental for our construction. To set the notation, we say a quantity D is *small compared to* a radius $r > 0$ if

$$(2) \quad D \leq \frac{r}{2\kappa(r)}.$$

We will fix once and for all a base point $\mathfrak{o} \in X$, and all quasi-geodesic rays we consider will be based at \mathfrak{o} . Given a quasi-geodesic ray α and a constant m , we define

$$\mathcal{N}_\kappa(\alpha, m) := \left\{ x \in X : d_X(x, \alpha) \leq m \cdot \kappa(\|x\|) \right\}.$$

The following observation will be useful.

Lemma 3.1. *Let β be a closed set and α be a geodesic ray, both based at $\mathfrak{o} \in X$. Suppose that*

$$\beta \subseteq \mathcal{N}_\kappa(\alpha, m)$$

for some function κ and some constant m . Then we also have

$$\alpha \subseteq \mathcal{N}_\kappa(\beta, 2m).$$

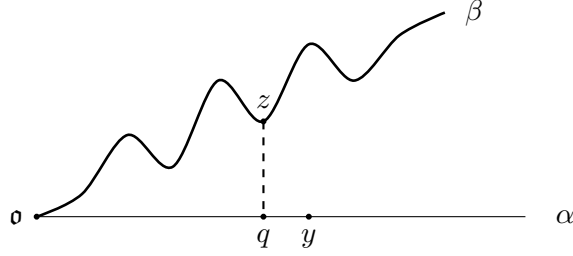


FIGURE 1. $\|y\| = \|z\|$ and $q \in \pi_\alpha(z)$ as in the proof of Lemma 3.1.

Proof. Let $y \in \alpha$ be a point and let $r := \|y\|$. Let $z \in \beta$ be a point such that $\|z\| = r$ and let q be a nearest point projection of z to α . By assumption,

$$d_X(z, q) \leq m \cdot \kappa(r).$$

On the other hand,

$$\begin{aligned} d_X(y, q) &= \left| \|y\| - \|q\| \right| && \text{since } \alpha \text{ is geodesic} \\ &= \left| \|z\| - \|q\| \right| \\ &\leq d_X(z, q) && \text{by triangle inequality.} \end{aligned}$$

Therefore we have

$$\begin{aligned} d_X(y, \beta) &\leq d_X(y, z) \leq d_X(y, q) + d_X(q, z) \\ &\leq 2d_X(z, q) \\ &\leq 2m \cdot \kappa(r) \end{aligned}$$

which completes the proof. \square

Definition 3.2. Let $Z \subseteq X$ be a closed set, and let κ be a concave sublinear function. We say that Z is κ -Morse if there exists a proper function $m_Z : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for any sublinear function κ' and for any $r > 0$, there exists R such that for any (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray β with $m_Z(\mathbf{q}, \mathbf{Q})$ small compared to r , if

$$d_X(\beta_R, Z) \leq \kappa'(R) \quad \text{then} \quad \beta|_r \subset \mathcal{N}_\kappa(Z, m_Z(\mathbf{q}, \mathbf{Q}))$$

The function m_Z will be called a *Morse gauge* of Z .

Note that we can always assume without loss of generality that $\max\{\mathbf{q}, \mathbf{Q}\} \leq m_Z(\mathbf{q}, \mathbf{Q})$, and we will assume this in the following.

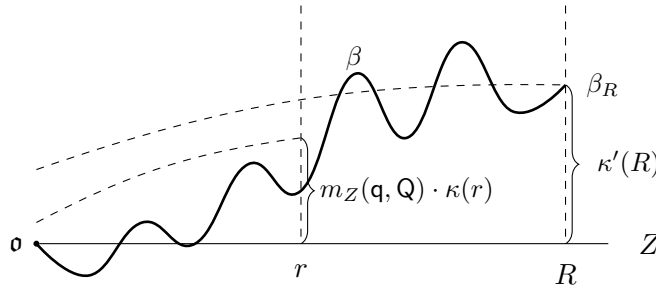


FIGURE 2. Definition of κ -Morse set Z : Every quasi-geodesic ray β has the property that there exists $R(Z, r, \mathbf{q}, \mathbf{Q}, \kappa')$, such that if β_R is distance $\kappa'(R)$ from Z , then $\beta|_r$ is in the neighborhood $\mathcal{N}_\kappa(Z, m_Z(\mathbf{q}, \mathbf{Q}))$.

3.1. Equivalence classes.

Definition 3.3. Given two quasi-geodesic rays α, β based at \mathfrak{o} , we say that $\beta \sim \alpha$ if they sublinearly track each other: i.e. if

$$\lim_{r \rightarrow \infty} \frac{d_X(\alpha_r, \beta_r)}{r} = 0.$$

By the triangle inequality, \sim is an equivalence relation on the space of quasi-geodesic rays based at \mathfrak{o} , hence also on the space of κ -Morse quasi-geodesic rays.

Lemma 3.4. *Let α be a κ -Morse quasi-geodesic rays with Morse gauge m_α , and let $\beta \sim \alpha$ be a $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray. Then:*

(i) β is κ -Morse, and moreover

$$\beta \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathfrak{q}, \mathfrak{Q}));$$

(ii) if in addition α is a geodesic ray, then

$$\alpha \subseteq \mathcal{N}_\kappa(\beta, 2m_\alpha(\mathfrak{q}, \mathfrak{Q})).$$

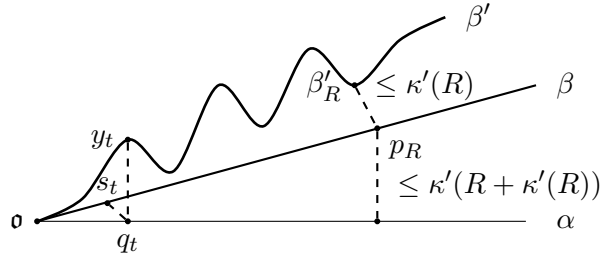


FIGURE 3. The setup in the proof of Lemma 3.4 (i).

Proof. (i) Define $\kappa'(r) := d_X(\alpha_r, \beta_r)$. By definition of \sim , the function κ' is sublinear, and moreover, for any $R > 0$,

$$d_X(\beta_R, \alpha) \leq \kappa'(R).$$

Hence, since α is κ -Morse, for any r we have

$$\beta|_r \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathfrak{q}, \mathfrak{Q}))$$

thus

$$(3) \quad \beta \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathfrak{q}, \mathfrak{Q})),$$

which proves the second part of (i). Let us now prove that β is κ -Morse. Let $r > 0$ and let β' be a $(\mathfrak{q}', \mathfrak{Q}')$ -quasi-geodesic ray such that

$$d_X(\beta'_R, \beta) \leq \kappa'(R)$$

for some sufficiently large R . Let p_R be a nearest point projection of β'_R to β ; by Lemma 2.2 we have

$$\|p_R\| \leq 2\|\beta'_R\| = 2R.$$

Then, by the triangle inequality and equation (3),

$$\begin{aligned} d_X(\beta'_R, \alpha) &\leq d_X(\beta'_R, p_R) + d_X(p_R, \alpha) \\ &\leq \kappa'(R) + m_\alpha(\mathfrak{q}, \mathfrak{Q}) \cdot \kappa(\|p_R\|) \\ &\leq \kappa'(R) + m_\alpha(\mathfrak{q}, \mathfrak{Q}) \cdot \kappa(2R). \end{aligned}$$

Since $\kappa''(R) := \kappa'(R) + m(\mathbf{q}, \mathbf{Q}) \cdot \kappa(2R)$ is also a sublinear function, and since α is κ -Morse, this implies that

$$\beta|_r \subseteq \mathcal{N}_\kappa(\alpha, m_\alpha(\mathbf{q}', \mathbf{Q}')).$$

Let y_t be any point on β' with $\|y_t\| = t \leq r$. By Lemma 2.2, if q_t is a nearest point projection of y_t to α , we have

$$\|q_t\| \leq 2\|y_t\| = 2t.$$

Now, if q is any point on α and s is a nearest point projection of q to β , by the triangle inequality and the Morse property,

$$\|q\| \geq \|s\| - d_X(s, q) \geq \|s\| - m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(\|s\|).$$

Moreover, by Lemma 2.2, $\|s\| \leq 2\|q\|$, hence by concavity

$$d_X(s, q) \leq m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(\|s\|) \leq 2m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(\|q\|).$$

Thus, let s_t be a nearest point projection of q_t to β , the above estimate yields

$$\begin{aligned} d_X(q_t, s_t) &\leq 2m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(\|q_t\|) \\ &\leq 4m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(t) \end{aligned}$$

hence, putting everything together,

$$\begin{aligned} d_X(y_t, \beta) &\leq d_X(y_t, q_t) + d_X(q_t, s_t) \\ &\leq m_\alpha(\mathbf{q}', \mathbf{Q}') \cdot \kappa(t) + 4m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(t) \end{aligned}$$

which, by setting $m_\beta(\mathbf{q}', \mathbf{Q}') := m_\alpha(\mathbf{q}', \mathbf{Q}') + 4m_\alpha(\mathbf{q}, \mathbf{Q})$, proves the claim.

(ii) It follows immediately from (i) and Lemma 3.1. \square

Corollary 3.5. *If β is a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray, and β_0 is a κ -Morse geodesic ray such that $\beta \sim \beta_0$, then the function*

$$m_\beta(\cdot, \cdot) := m_{\beta_0}(\cdot, \cdot) + 4m_{\beta_0}(\mathbf{q}, \mathbf{Q})$$

is a Morse gauge for β . In particular, the Morse gauge depends only on m_{β_0} , \mathbf{q} and \mathbf{Q} and not on the particular quasi-geodesic β .

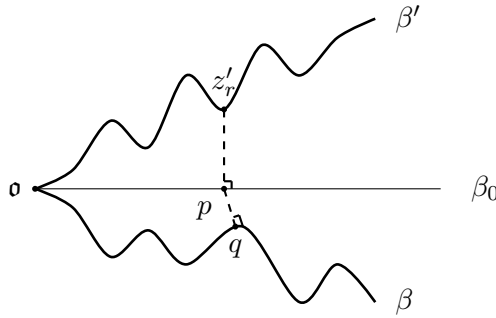


FIGURE 4. Corollary 3.5: $\|z'_r\| = r$ and $p = \pi_{\beta_0}(z'_r)$ and $q = \pi_\beta(p)$.

Proof. Let $\beta' \sim \beta$ be a $(\mathbf{q}', \mathbf{Q}')$ -quasi-geodesic ray. Let z'_r be a point along β' with norm r , let $p \in \pi_{\beta_0}(z'_r)$ and let q be a nearest point in β to p . Note that, by Lemma 2.2, $\|p\| \leq 2r$. Hence,

$$d_X(z'_r, \beta) \leq d_X(z'_r, p) + d_X(p, q)$$

and, by Lemma 3.4 (i) and (ii),

$$\begin{aligned}
&\leq m_{\beta_0}(\mathbf{q}', \mathbf{Q}') \cdot \kappa(r) + 2m_{\beta_0}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(\|p\|) \\
&\leq m_{\beta_0}(\mathbf{q}', \mathbf{Q}') \cdot \kappa(r) + 4m_{\beta_0}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r) \\
&\leq (m_{\beta_0}(\mathbf{q}', \mathbf{Q}') + 4m_{\beta_0}(\mathbf{q}, \mathbf{Q})) \cdot \kappa(r).
\end{aligned}$$

This finishes the proof. \square

3.2. Surgery Lemmas. We need a few technical results to splice quasi-geodesic segments together. Lemma 3.6 generalizes [QRT19, Lemma 2.5]. Lemma 3.7 is new and provides a way to splice a finite quasi-geodesic to a quasi-geodesic ray. Lemma 3.8 is already proven in [QRT19, Lemma 4.3]; our statement here is slightly altered but the proof is identical hence we skip the proof.

Lemma 3.6. *Consider a quasi-geodesic segment $[x, y]_\alpha \in X$ that is (\mathbf{q}, \mathbf{Q}) -quasi-geodesic and another (\mathbf{q}, \mathbf{Q}) -quasi-geodesic segment β connecting a point $z \in X$ to a point $w \in X$. Assume that for every point $u \in \alpha$, y is a point in u_β , and let γ be the concatenation of the quasi-geodesic segment $[x, y]_\alpha$ and the quasi-geodesic segment $[y, z]_\beta$. Then $\gamma = [x, y]_\alpha \cup [y, z]_\beta$ is a $(3\mathbf{q}, 3\mathbf{Q})$ -quasi-geodesic.*

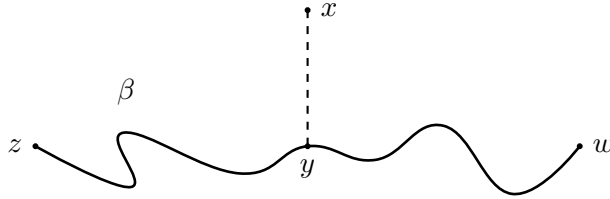


FIGURE 5. For $y \in x_\beta$, the concatenation of the quasi-geodesic segment $[x, y]_\alpha$ and the quasi-geodesic segment $[y, z]_\beta$ is a quasi-geodesic.

Proof. Consider $\gamma: [t_0, t_2] \rightarrow X$ and let $t_1 \in [t_0, t_2]$ be the time when $\gamma(t_1) = y$, the restriction of γ to $[t_0, t_1]$ is the parametrization of $[x, y]_\alpha$ given by arc length and the restriction of γ to $[t_1, t_2]$ is the parametrization of $[y, z]_\beta$ given by β . To show that γ is a quasi-geodesic, we need to estimate the distance between a point in $[x, y]_\alpha$ and a point in $[y, z]_\beta$. However, it is enough to show that $d_X(x, z)$ is comparable to $|t_2 - t_0|$ because the argument for any other points along $[x, y]_\beta$ and along $[y, z]_\beta$ is the same. We argue in two cases.

Case 1. Suppose $2d_X(x, y) \geq d_X(z, y)$. Then,

$$3d_X(x, y) \geq d_X(z, y) + d_X(x, y)$$

Therefore,

$$\begin{aligned}
d_X(x, z) &\geq d_X(x, y) \geq \frac{1}{3}(d_X(z, y) + d_X(x, y)) \\
&\geq \frac{1}{3} \left(\frac{1}{\mathbf{q}}|t_2 - t_1| - \mathbf{Q} + \frac{1}{\mathbf{q}}|t_1 - t_0| - \mathbf{Q} \right) \\
&\geq \frac{1}{3\mathbf{q}}|t_2 - t_0| - \frac{2\mathbf{Q}}{3}.
\end{aligned}$$

Case 2. Suppose $2d_X(x, y) < d_X(z, y)$, then

$$3d_X(x, y) \leq d_X(z, y) + d_X(x, y) \quad \implies \quad 2d_X(x, y) \leq \frac{2}{3}(d_X(z, y) + d_X(x, y)).$$

We have

$$\begin{aligned} d_X(x, z) &\geq d_X(z, y) - d_X(x, y) = d_X(z, y) + d_X(x, y) - 2d_X(x, y) \\ &\geq (d_X(z, y) + d_X(x, y)) - \frac{2}{3}(d_X(z, y) + d_X(x, y)) \\ &\geq \frac{1}{3}(d_X(z, y) + d_X(x, y)) \\ &\geq \frac{1}{3} \left(\frac{1}{\mathfrak{q}}|t_2 - t_1| - \mathfrak{Q} + \frac{1}{\mathfrak{q}}|t_1 - t_0| - \mathfrak{Q} \right) \geq \frac{1}{3\mathfrak{q}}|t_2 - t_0| - \frac{2\mathfrak{Q}}{3}. \end{aligned}$$

This established the lower-bound. The upper-bound follows from the triangle inequality:

$$d_X(x, z) \leq d_X(x, y) + d_X(y, z) \leq \mathfrak{q}|t_1 - t_0| + \mathfrak{q}|t_2 - t_1| + 2\mathfrak{Q} \leq \mathfrak{q}|t_2 - t_0| + 2\mathfrak{Q}.$$

It follows that γ is a $(3\mathfrak{q}, 3\mathfrak{Q})$ -quasi-geodesic. \square

Lemma 3.7. *Consider a $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray $\alpha: [0, \infty) \rightarrow X$ and a finite $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic segment $\beta: [a, b] \rightarrow X$. Then there is $s_0 \in [0, \infty)$ such that the following holds: for $s \in [s_0, \infty)$ let $s_\gamma \in [s, \infty)$ and $t_\gamma \in [a, b]$ be such that $[\beta(t_\gamma), \alpha(s_\gamma)]$ is a geodesic segment that realizes the set distance between $\alpha[s, \infty)$ and β . Then*

$$\gamma = \beta[a, t_\gamma] \cup [\beta(t_\gamma), \alpha(s_\gamma)] \cup \alpha[s_\gamma, \infty)$$

is a $(4\mathfrak{q}, 3\mathfrak{Q})$ -quasi-geodesic.

Proof. Define

$$r_0 := 2 \max \left\{ \sup_{y \in \beta} \|y\|, \text{diam}(\beta) \right\},$$

and let s_0 to be such that, for all $s \geq s_0$ we have $\|\alpha(s)\| > r_0$ (this is possible because $\|\alpha(s)\| = \infty$ as $s \rightarrow \infty$). Now fix $s \geq s_0$ and let $s_\gamma \geq s$ and $t_\gamma \in [a, b]$ be such that $[\beta(t_\gamma), \alpha(s_\gamma)]$ is a geodesic segment that realizes the set distance between $\alpha[s, \infty)$ and β . The times s_γ and t_γ exist because the metric space X is proper. We relabel the points:

$$x_\gamma := \alpha(s_\gamma) \quad \text{and} \quad y_\gamma := \beta(t_\gamma).$$

Let $L_\gamma := d(x_\gamma, y_\gamma)$. By the triangle inequality and from the definition of s_0 , we have

$$L_\gamma \leq d(\mathfrak{o}, x_\gamma) + d(\mathfrak{o}, y_\gamma) \leq \|x_\gamma\| + \sup_{y \in \beta} \|y\| \leq \|x_\gamma\| + \frac{r_0}{2} \leq \frac{3\|x_\gamma\|}{2}.$$

Therefore

$$2\|x_\gamma\| - L_\gamma \geq 2\|x_\gamma\| - \frac{3\|x_\gamma\|}{2} \geq \frac{\|x_\gamma\|}{2} \geq \frac{r_0}{2} \geq \text{diam}(\beta),$$

which we rearrange as

$$(4) \quad \|x_\gamma\| \geq \frac{1}{2}(L_\gamma + \text{diam}(\beta)).$$

Since $\beta(t_\gamma)$ is the closest point in β to x_γ , Lemma 3.6 implies that the subsegment

$$\beta[a, t_\gamma] \cup [\beta(t_\gamma), \alpha(s_\gamma)]$$

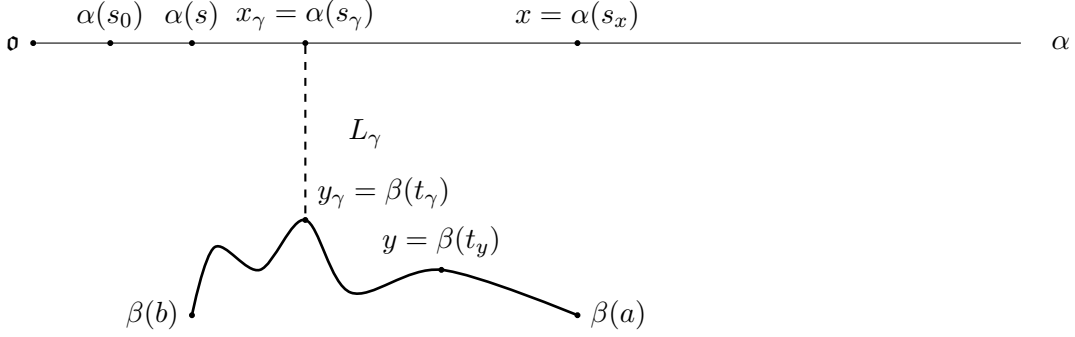


FIGURE 6. proof of Lemma 3.7.

of γ is a $(3\mathbf{q}, 3\mathbf{Q})$ -quasi-geodesic. Also, since $\alpha(s_\gamma)$ is the closest point in $\alpha[s, \infty)$ to y_γ , Lemma 3.6 implies that the subsegment

$$[\beta(t_\gamma), \alpha(s_\gamma)] \cup \alpha[s_\gamma, \infty)$$

of γ is a $(3\mathbf{q}, 3\mathbf{Q})$ -quasi-geodesic. It remains to consider a pair of points $y = \beta(t_y)$ and $x = \alpha(s_x)$ with $t_y \in [a, t_\gamma]$ and $s_x \in \alpha[s_\gamma, \infty)$. The required upper-bound for $d(x, y)$ follows from the fact that γ is a concatenation of geodesics and quasi-geodesics. We now find a lower bound for $d(x, y)$:

$$\begin{aligned}
(\text{Definition of } r_0) \quad d(x, y) &\geq \|x\| - \|y\| = (\|x\| - \|x_\gamma\|) + \|x_\gamma\| - \frac{1}{2}r_0 \\
(\text{Since } s_x \geq s_0) &\geq (\|x\| - \|x_\gamma\|) + \frac{1}{2}\|x_\gamma\| \\
(\text{Equation 4}) &\geq (\|x\| - \|x_\gamma\|) + \frac{1}{4}(L_\gamma + \text{diam}(\beta)) \\
&\geq (\|x_\gamma\| - \|x_\gamma\|) + \frac{1}{4}(L_\gamma + d(y, y_\gamma)) \\
(\alpha \text{ and } \beta \text{ are } (\mathbf{q}, \mathbf{Q})\text{-quasi-geodesics}) &\geq \frac{1}{\mathbf{q}}(s_x - s_\gamma) - \mathbf{Q} + \frac{1}{4}L_\gamma + \frac{1}{4\mathbf{q}}(t_\gamma - t_y) - \frac{\mathbf{Q}}{4} \\
&\geq \frac{(s_x - s_\gamma) + L_\gamma + (t_\gamma - t_y)}{4\mathbf{q}} - \frac{5\mathbf{Q}}{4}.
\end{aligned}$$

Thus γ is a $(4\mathbf{q}, 3\mathbf{Q})$ -quasi-geodesic ray. \square

Lemma 3.8 (Surgery Lemma). *(Lemma 4.3 [QRT19]) For every $\mathbf{q}, \mathbf{Q}, r > 0$ there exists $R > 0$ such that the following holds. Let γ be a geodesic ray of length at least R and α be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray. Assume that $d_X(\gamma_r, \alpha) \leq r/2$. Then, there exists a $(9\mathbf{q}, \mathbf{Q})$ -quasi-geodesic ray γ' so that*

$$\alpha|_{r/2} = \gamma'|_{r/2}$$

and the portion of γ' outside of $B(\mathfrak{o}, R)$ is the same as γ .

3.3. κ -weakly Morse rays. As in [QRT19], we also define a different notion of sublinearly Morse which more closely matches the usual definition of Morse.

Definition 3.9. Let $Z \subseteq X$ be a closed set, and let κ be a concave sublinear function. We say that Z is κ -weakly Morse if there exists a proper function $m_Z : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for any (\mathbf{q}, \mathbf{Q}) -quasi-geodesic $\gamma : [s, t] \rightarrow X$ with endpoints on Z ,

$$\gamma([s, t]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathbf{q}, \mathbf{Q}))$$

The function m_Z will be called a κ -weakly Morse gauge of Z .

As in the setting of CAT(0) metric spaces, we show that the two notions are equivalent.

Proposition 3.10. *Let $\alpha : [0, \infty) \rightarrow X$ be a quasi-geodesic ray. Then α is κ -Morse if and only if it is κ -weakly Morse.*

Proof. First assume that α is κ -Morse. Let $\beta : [t_0, t_3] \rightarrow X$ be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic with endpoints on α . We need to check that β stays in a κ -neighborhood of α (see Definition 3.2). Making \mathbf{q} and \mathbf{Q} larger if necessary, we can assume α is also a (\mathbf{q}, \mathbf{Q}) quasi-geodesic. We will find a $(12\mathbf{q}, 9\mathbf{Q})$ -quasi-geodesic ray γ' that is eventually equal to α and covers a substantial portion of β , that is, the size of what remains is κ -small. Since α is κ -Morse, γ' is contained in $\mathcal{N}_\kappa(\alpha, m_\alpha(12\mathbf{q}, 9\mathbf{Q}))$. Hence, the same is true for β for a larger κ -neighborhood.

Choose $s_0 \in [0, \infty)$ as in Lemma 3.7 for β and α . Let $\beta(t_2)$ in β and $\alpha(s)$ in $\alpha[s_0, \infty)$ be points such that $[\beta(t_2), \alpha(s)]$ is a segment that realizes the distance between β and $\alpha[s_0, \infty)$. Define γ as a concatenation of the following segments:

$$\gamma = \beta[t_0, t_2] \cup [\beta(t_2), \alpha(s)] \cup \alpha[s, \infty).$$

Then, by Lemma 3.7, γ is a $(4\mathbf{q}, 3\mathbf{Q})$ -quasi-geodesic.

Now, we apply Lemma 3.6 to γ and the base point \mathbf{o} . That is, let \mathbf{o}_γ be a closest point in γ to \mathbf{o} . Taking s_0 large enough, we know that \mathbf{o}_γ is either on the segment $\beta[t_0, t_2]$ or the segment $[\beta(t_2), \alpha(s)]$. We define $t_1 \in [t_0, t_2]$ such that, in the first case $\mathbf{o}_\gamma = \beta(t_1)$ and in the second case $t_1 = t_2$. Let

$$\gamma' = [\mathbf{o}, \mathbf{o}_\gamma] \cup [\mathbf{o}_\gamma, \alpha(s)]_\gamma \cup \alpha[s, \infty).$$

By Lemma 3.6, γ' is a $(12\mathbf{q}, 9\mathbf{Q})$ -quasi-geodesic.

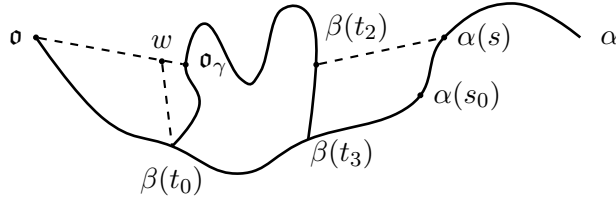


FIGURE 7. The path $[\beta(t_0), \mathbf{o}_\gamma]_\beta \cup [\mathbf{o}_\gamma, w]$ is a quasi-geodesic. Hence the distance $d(\beta(t_0), w)$ is a coarse upper bound for the distance between any point in $[\beta(t_0), \mathbf{o}_\gamma]$ and α .

Let w be a closest point in γ' to $\beta(t_0)$. By Lemma 3.4,

$$d(\beta(t_0), w) \leq 2m_\alpha(12\mathbf{q}, 9\mathbf{Q}) \cdot \kappa(\beta(t_0)).$$

Note that γ' is contained in the union of segments $[\mathbf{o}, \mathbf{o}_\gamma]$, $\beta[t_0, t_2]$ and $[\beta(t_2), \alpha(s)]$ and w can be in either of these segments. But in every case, there is a $(12\mathbf{q}, 9\mathbf{Q})$ -quasi-geodesic γ'' connecting $\beta(t_0)$ to w that contains $\beta[t_0, t_1]$. Namely,

if	$w \in [\mathbf{o}, \mathbf{o}_\gamma]$	we define	$\gamma'' = \beta[t_0, t_1] \cup [\mathbf{o}_\gamma, w]$
if	$w \in \beta[t_0, t_2]$	we define	$\gamma'' = [\beta(t_0), w]_\beta,$
and if	$w \in [\beta(t_2), \alpha(s)]$	we define	$\gamma'' = \beta[t_0, t_2] \cup [\beta(t_2), w].$

The fact that γ'' is a $(12\mathbf{q}, 9\mathbf{Q})$ -quasi-geodesic follows again from Lemma 3.6 and Lemma 3.7.

Then, there are constants c_1, c_2 and c_3 depending only on \mathbf{q} and \mathbf{Q} such that, for any point $u \in \gamma''$, we have

$$d(\beta(t_0), u) \leq c_1 d(\beta(t_0), w) + c_2 \leq c_3 m_\alpha(12\mathbf{q}, 9\mathbf{Q}) \cdot \kappa(\beta(t_0)),$$

which also implies $\kappa(u) \asymp \kappa(\beta(t_0))$ (see [QRT19, Lemma 3.2]). Since, γ'' contains $\beta[t_0, t_1]$, we have

$$\beta[t_0, t_1] \subset \mathcal{N}_\kappa(\alpha, c_4 m_\alpha(12\mathfrak{q}, 9\mathfrak{Q}))$$

for some c_4 depending only of \mathfrak{q} and \mathfrak{Q} .

But $\beta[t_1, t_2]$ is a subset of γ' and by the definition of κ -Morse

$$\beta[t_1, t_2] \subset \gamma' \subset \mathcal{N}_\kappa(\alpha, m_\alpha(12\mathfrak{q}, 9\mathfrak{Q})).$$

Hence, $\beta[t_0, t_2]$ is contained in a κ -neighborhood of α .

A similar proof shows that the segment $\beta[t_2, t_3]$ also stays in a κ -neighborhood of α . This finishes the proof of the first direction.

To see the other direction, assume α is a κ -weakly Morse quasi-geodesic ray with κ -weakly Morse gauge m'_α . For a given r and a sub-linear function κ' , choose $R > 0$ such that

$$(5) \quad R > 2\kappa'(R) + r.$$

For a $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic γ , let $s \geq 0$ be such that

$$d(\gamma_R, \alpha(s)) \leq \kappa'(R).$$

Then

$$\|\alpha(s)\| \geq \|\gamma_R\| - \kappa'(R) = R - \kappa'(R).$$

Let $\gamma(t)$ be the closest point in γ to $\alpha(s)$, By Lemma 3.6, the concatenation

$$\beta = \gamma[0, t] \cup [\gamma(t), \alpha(s)]$$

is a $(3\mathfrak{q}, 3\mathfrak{Q})$ -quasi-geodesic with end points on α . Therefore, by definition of κ -weakly Morse, $\beta \subset \mathcal{N}_\kappa(\alpha, m'_\alpha(3\mathfrak{q}, 3\mathfrak{Q}))$. We now show that β contains $\gamma|_r$. That is, if $\gamma_r = \gamma(t_0)$, we show that $t \geq t_0$. Assume, for contradiction, that $t < t_0$ which implies $\|\gamma(t)\| \leq r$. Then,

$$\kappa'(R) \geq d(\gamma_R, \alpha(s)) \geq d(\gamma(t), \alpha(s)) \geq \|\alpha(s)\| - \|\gamma(t)\| \geq (R - \kappa'(R)) - r.$$

Solving for R we get, $R \leq 2\kappa'(R) + r$ which contradicts (5). The contradiction implies that β contains $\gamma|_r$ and hence

$$\gamma|_r \subset \mathcal{N}_\kappa(\alpha, m'_\alpha(3\mathfrak{q}, 3\mathfrak{Q})).$$

That is, α is κ -Morse with the Morse gauge $m_\alpha(\mathfrak{q}, \mathfrak{Q}) = m'_\alpha(3\mathfrak{q}, 3\mathfrak{Q})$. \square

4. THE TOPOLOGY ON THE κ -MORSE BOUNDARY

We denote as $\partial_\kappa X$ the set of all equivalence classes of κ -Morse quasi-geodesic rays. In this section we will define a topology on $\partial_\kappa X$ to make it into a topological space. Even more, we will construct a bordification $X \cup \partial_\kappa X$ of X . Recall that, for topological spaces $X \subseteq Y$, we say Y is a *bordification* of X if X is a topological subspace of Y and X is open and dense in Y . Hence, we define a topology on $X \cup \partial_\kappa X$ that make it a bordification of X . We will also show that both X and $X \cup \partial_\kappa X$ have good topological properties.

4.1. Bordification. Let β be a κ -Morse quasi-geodesic ray based at \mathfrak{o} and let m_β be a Morse gauge for β .

Definition 4.1. We define the set $\mathcal{U}(\beta, r) \subseteq X \cup \partial_\kappa X$ as follows.

- An equivalence class $\mathfrak{a} \in \partial_\kappa X$ belongs to $\mathcal{U}(\beta, r)$ if, for any $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray $\alpha \in \mathfrak{a}$, where $m_\beta(\mathfrak{q}, \mathfrak{Q})$ is small compared to r (in the sense of Equation 2), we have the inclusion

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathfrak{q}, \mathfrak{Q})).$$

- A point $p \in X$ belongs to $\mathcal{U}(\beta, r)$ if $d_X(\mathfrak{o}, p) \geq r$ and, for every $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic α between \mathfrak{o} and p where $m_\beta(\mathfrak{q}, \mathfrak{Q})$ is small compared to r , we have

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathfrak{q}, \mathfrak{Q})).$$

We denote $\mathcal{U}(\beta, r) \cap \partial_\kappa X$ by $\partial\mathcal{U}(\beta, r)$.

We now verify the following basic properties of $\mathcal{U}(\beta, r)$.

Lemma 4.2. *Let β be a quasi-geodesic ray which belongs to the class \mathbf{b} . Then:*

- (1) *There exists a (not necessarily unique) geodesic ray in the class \mathbf{b} ;*
- (2) *The class \mathbf{b} belongs to $\mathcal{U}(\beta, r)$ for any $r > 0$;*
- (3) $\bigcap_{r>0} \mathcal{U}(\beta, r) = \{\mathbf{b}\}$;
- (4) *If $\beta_1 \sim \beta_2$ are two κ -Morse quasi-geodesic rays, for any $r_1 > 0$, there exists $r_2 > 0$ such that*

$$\mathcal{U}(\beta_1, r_1) \supseteq \mathcal{U}(\beta_2, r_2).$$

Proof. (1) Consider the geodesic segment β_n connecting \mathfrak{o} and $\beta(n)$ for $n = 1, 2, 3, \dots$. The sequence of geodesic segments has a convergent subsequence converging to a geodesic ray β' by Arzelá-Ascoli. Since β is κ -Morse, it is κ -weakly Morse and, for $m'(\cdot, \cdot)$ as in Proposition 3.10, we have

$$\beta_n \subset \mathcal{N}_\kappa(\beta, m'_\beta(1, 0)).$$

Therefore the same is true for β' , hence β' belongs to the class \mathbf{b} .

- (2) If β' is a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray which belongs to \mathbf{b} , then by Lemma 3.4

$$\beta' \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q}))$$

hence also

$$\beta'|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q}))$$

for any $r > 0$, as needed.

- (3) Let $\mathbf{c} \in \partial_\kappa X$ be a class which belongs to $\mathcal{U}(\beta, r)$ for any r , let $(\mathbf{q}', \mathbf{Q}')$ be the quasi-geodesic constants of β , and let $\gamma \in \mathbf{c}$ be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray. Let β' be a geodesic ray based at \mathfrak{o} with $\beta' \sim \beta$.

Take $y \in \gamma$ such that $\|y\| = r$. Then, by assumption,

$$\gamma|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q})),$$

for any r . Hence, if p is a nearest point projection of y onto β , we have

$$d_X(y, \beta) = d_X(y, p) \leq m_\beta(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r).$$

Moreover, by Lemma 2.2, we have $\|p\| \leq 2\|y\| = 2r$ and, by Lemma 3.4 (i), we have

$$\begin{aligned} d_X(y, \beta') &\leq d_X(y, p) + d_X(p, \beta') \\ &\leq m_\beta(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r) + 2m_{\beta'}(\mathbf{q}', \mathbf{Q}') \cdot \kappa(r). \end{aligned}$$

Setting

$$\tilde{m}_{\beta'}(\mathbf{q}, \mathbf{Q}) := m_\beta(\mathbf{q}, \mathbf{Q}) + 2m_{\beta'}(\mathbf{q}', \mathbf{Q}'),$$

we get,

$$\gamma|_r \subseteq \mathcal{N}_\kappa(\beta', \tilde{m}_{\beta'}(\mathbf{q}, \mathbf{Q})),$$

for any $r > 0$.

Let now p_r be a nearest point projection of γ_r to β' . Then

$$|d_X(\mathfrak{o}, p_r) - r| = |d_X(\mathfrak{o}, p_r) - d_X(\mathfrak{o}, \gamma_r)| \leq d_X(\gamma_r, p_r) \leq \tilde{m}_{\beta'}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r).$$

Since β' is geodesic, we have

$$d_X(p_r, \beta'_r) = |d_X(\mathfrak{o}, p_r) - r| \leq m_{\beta'}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r)$$

and

$$d_X(\gamma_r, \beta'_r) \leq d_X(\gamma_r, p_r) + d_X(p_r, \beta'_r) \leq 2\tilde{m}_{\beta'}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r).$$

But $\kappa(r)$ is sublinear, therefore,

$$\lim_{r \rightarrow \infty} \frac{d_X(\gamma_r, \beta'_r)}{r} = 0$$

which implies $\gamma \in \mathbf{b}$ and $\mathbf{c} = \mathbf{b}$. Finally, since $p \in \mathcal{U}(\beta, r)$ implies $d_X(\mathbf{o}, p) \geq r$, the intersection $\bigcap_{r>0} \mathcal{U}(\beta, r)$ does not contain any point of X .

(4) Let β_1 be a $(\mathbf{q}_1, \mathbf{Q}_1)$ -quasi-geodesic ray, β_2 a $(\mathbf{q}_2, \mathbf{Q}_2)$ -quasi-geodesic ray with $\beta_1 \sim \beta_2$, and let $r_1 > 0$. For $r > 0$, let $\mathbf{a} \in \mathcal{U}(\beta_2, r)$, and pick $\alpha \in \mathbf{a}$ be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray such that $m_{\beta_2}(\mathbf{q}, \mathbf{Q})$ is small compared to r . By definition of $\mathcal{U}(\beta_2, r)$ we have

$$d_X(\alpha_r, \beta_2) \leq m_{\beta_2}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r).$$

Let p_r be a nearest point projection of α_r to β_2 . By Lemma 2.2,

$$\|p_r\| \leq 2r.$$

Moreover, by Lemma 3.4 (i),

$$d_X(p_r, \beta_1) \leq \kappa(\|p_r\|)m_{\beta_1}(\mathbf{q}_2, \mathbf{Q}_2) \leq 2m_{\beta_1}(\mathbf{q}_2, \mathbf{Q}_2) \cdot \kappa(r)$$

hence

$$(6) \quad d_X(\alpha_r, \beta_1) \leq m_{\beta_2}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r) + 2m_{\beta_1}(\mathbf{q}_2, \mathbf{Q}_2) \cdot \kappa(r).$$

Now, if we take

$$\kappa'(r) := m_{\beta_2}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r) + 2m_{\beta_1}(\mathbf{q}_2, \mathbf{Q}_2) \cdot \kappa(r),$$

by Definition 3.2, there exists r_2 such that (6) for $r = r_2$ implies

$$\alpha|_{r_1} \subseteq \mathcal{N}_\kappa(\beta_1, m_{\beta_1}(\mathbf{q}, \mathbf{Q}))$$

hence $\mathbf{a} \in \mathcal{U}(\beta_1, r_1)$, as required. \square

We now verify that a sequence of points of X that sublinearly tracks a quasi-geodesic ray γ , converges to the class of γ in $\partial_\kappa X$.

Lemma 4.3. *Let $\gamma \in \mathbf{c}$ be a κ -Morse quasi-geodesic ray based at $\mathbf{o} \in X$ and let $(x_n) \subseteq X$ be a sequence of points with $\|x_n\| \rightarrow \infty$. Moreover, suppose that there exists a constant $C > 0$ such that*

$$(7) \quad d_X(x_n, \gamma) \leq C \cdot \kappa(\|x_n\|)$$

for all n . Then the sequence (x_n) converges to \mathbf{c} in the topology of $X \cup \partial_\kappa X$.

Proof. In order to show the claim, we need to prove that for any quasi-geodesic ray $\beta \in \mathbf{c}$ and any $r > 0$ there exists n_0 such that for all $n \geq n_0$ we have

$$x_n \in \mathcal{U}(\beta, r).$$

Equivalently, we need to show that, for any r and any (\mathbf{q}, \mathbf{Q}) -quasi-geodesic segment α joining \mathbf{o} and x_n with $m_\beta(\mathbf{q}, \mathbf{Q})$ small with respect to r , we have

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q})).$$

Let p_n be a nearest point projection of x_n onto γ ; by Lemma 2.2, $\|p_n\| \leq 2\|x_n\|$. Now, by (7) and Lemma 3.4,

$$\begin{aligned} d_X(x_n, \beta) &\leq d_X(x_n, p_n) + d_X(p_n, \beta) \\ &\leq C \cdot \kappa(\|x_n\|) + 2m_\beta(\mathbf{q}', \mathbf{Q}') \cdot \kappa(\|x_n\|) \end{aligned}$$

where $(\mathbf{q}', \mathbf{Q}')$ are the quasi-geodesic constants of γ . Note moreover that β is κ -Morse by Lemma 3.4. Hence, consider the sublinear function

$$\kappa'(r) := (C + 2m_\beta(\mathbf{q}', \mathbf{Q}')) \cdot \kappa(r)$$

and apply the definition of κ -Morse, obtaining R such that if $d_X(x_n, \beta) \leq \kappa'(R)$ then $\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q}))$. Thus, if we choose n_0 so that $\|x_n\| \geq R$ for all $n \geq n_0$, the definition of κ -Morse implies the following as needed,

$$\alpha|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q})). \quad \square$$

We now show that the sets $\mathcal{U}(\beta, r)$ and $\partial\mathcal{U}(\beta, r)$ can be used as a neighborhood basis to define topologies on $X \cup \partial_\kappa X$ and $\partial_\kappa X$.

Definition 4.4. Given an equivalence class \mathbf{b} , we define the set $\mathcal{B}(\mathbf{b})$ as the set of subsets $\mathcal{V} \subseteq X \cup \partial_\kappa X$ such that there exists $\beta \in \mathbf{b}$ and $r > 0$ for which $\mathcal{U}(\beta, r) \subseteq \mathcal{V}$. Let $\partial\mathcal{B}(\mathbf{b})$ be the set of subsets of $\partial_\kappa X$ of the form $\mathcal{V} \cap \partial_\kappa X$ where $\mathcal{V} \in \mathcal{B}(\mathbf{b})$, equivalently a set in $\partial\mathcal{B}(\mathbf{b})$ contains a set of the form $\partial\mathcal{U}(\beta, r)$. Also, for $x \in X$, define $\mathcal{B}(x)$ to be the set of subsets \mathcal{V} of X such that \mathcal{V} contains a ball $B(x, r)$ of radius r centered at x .

Lemma 4.5. For every $\mathbf{b} \in \partial_\kappa X$, the set $\mathcal{B}(\mathbf{b})$ satisfies the following properties:

- (i) Every subset of $X \cup \partial_\kappa X$ which contains a set belonging to $\mathcal{B}(\mathbf{b})$ itself belongs to $\mathcal{B}(\mathbf{b})$;
- (ii) Every finite intersection of sets of $\mathcal{B}(\mathbf{b})$ belongs to $\mathcal{B}(\mathbf{b})$;
- (iii) The element \mathbf{b} is in every set of $\mathcal{B}(\mathbf{b})$;
- (iv) If $\mathcal{V} \in \mathcal{B}(\mathbf{b})$ then there is $\mathcal{W} \in \mathcal{B}(\mathbf{b})$ such that, for every $\mathbf{a} \in \mathcal{W}$, we have $\mathcal{V} \in \mathcal{B}(\mathbf{a})$.

Furthermore, the same is true for subsets of $\partial\mathcal{B}(\mathbf{b})$ and $\mathcal{B}(x)$.

Proof. We prove the Lemma for $\mathcal{B}(\mathbf{b})$. The proof for $\partial\mathcal{B}(\mathbf{b})$ is identical. The proof for $\mathcal{B}(x)$ is immediate from the fact that the open balls in X define a neighborhood basis for X .

(i) This is immediate from the definition of $\mathcal{B}(\mathbf{b})$.

(ii) It is enough to show that, for $\beta_1, \dots, \beta_k \in \mathbf{b}$ and $r_1, \dots, r_k > 0$, the intersection

$$\mathcal{U}(\beta_1, r_1) \cap \mathcal{U}(\beta_2, r_2) \cap \dots \cap \mathcal{U}(\beta_k, r_k),$$

belongs to $\mathcal{B}(\mathbf{b})$. By Lemma 4.2 (4), for any $i = 1, \dots, k$ there exists R_i such that

$$\mathcal{U}(\beta_i, r_i) \supseteq \mathcal{U}(\beta_1, R_i).$$

Thus, if we set

$$r := \max_{1 \leq i \leq k} \{R_i\} \quad \text{we have} \quad \bigcap_{i=1}^k \mathcal{U}(\beta_i, r_i) \supseteq \mathcal{U}(\beta_1, r)$$

and hence the intersection belongs to $\mathcal{B}(\mathbf{b})$.

(iii) Established by Lemma 4.2 (2).

(iv) We need to prove the following claim.

Claim 4.6. For any $\mathcal{U}(\beta, r)$, there exists r' (usually larger than r) such that if $\mathbf{a} \in \mathcal{U}(\beta, r')$ then there exists r'' (depending on α and r' but not on β) such that $\mathcal{U}(\alpha, r'') \subseteq \mathcal{U}(\beta, r)$ for some α representative of \mathbf{a} .

In particular, we will prove the claim for a geodesic representative $\alpha \in \mathbf{a}$, whose existence is established by Lemma 4.2 (1). We adapt here the proof from [QRT19].

Let us pick a κ -Morse quasi-geodesic ray β and $r > 0$. Let

$$M := \sup_{m_\beta(\mathbf{q}, \mathbf{Q}) \leq r} m_\beta(9\mathbf{q}, \mathbf{Q})$$

and let r' be such that

- (1) $r' > 2r$,
- (2) $M \leq \frac{r'}{2\kappa(r')}$,
- (3) $r' > R(\beta, r, M\kappa)$.

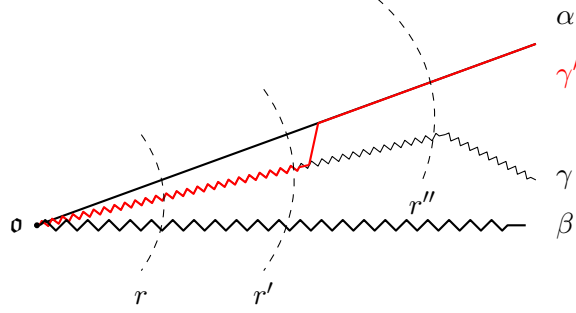


FIGURE 8. The proof of Claim 4.6.

Let α be a geodesic representative of $\mathbf{a} \in \mathcal{U}(\beta, r')$. Choose r'' such that,

$$r'' \geq 2r' \quad \text{and} \quad \sup_{m_\beta(\mathbf{q}, \mathbf{Q}) \leq r} m_\alpha(\mathbf{q}, \mathbf{Q}) \leq \frac{r''}{4\kappa(r'')}.$$

Now consider $\mathbf{c} \in \mathcal{U}(\alpha, r'')$. Let $\gamma \in \mathbf{c}$ be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray, with $m_\beta(\mathbf{q}, \mathbf{Q})$ small compared to r . By the choice of r'' above and by Lemma 3.1,

$$d_X(\alpha_{r''}, \gamma) \leq 2m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r'') \leq \frac{r''}{2}.$$

We apply Lemma 3.8, with radius being r'' , to modify γ to a $(9\mathbf{q}, \mathbf{Q})$ -quasi-geodesic ray $\gamma' \in \mathbf{a}$. Since $r' \leq r''/2$, we have $\gamma|_{r'} = \gamma'|_{r'}$.

Also, $\gamma' \in \mathbf{a} \in \mathcal{U}(\beta, r')$ and $m_\beta(9\mathbf{q}, \mathbf{Q})$ is small compared to r' (by point (2) above), therefore

$$\gamma|_{r'} = \gamma'|_{r'} \subset \mathcal{N}_\kappa(\beta, m_\beta(9\mathbf{q}, \mathbf{Q})).$$

Hence, by the choice of r' (point (3) above) and Definition 3.2, we obtain

$$\gamma|_r \subseteq \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q})).$$

This holds for every $\gamma \in \mathbf{c}$ with $m_\beta(\mathbf{q}, \mathbf{Q})$ small compared to r , thus $\mathbf{c} \in \mathcal{U}(\beta, r)$. And this argument holds for every $\mathbf{c} \in \mathcal{U}(\alpha, r'')$, therefore $\mathcal{U}(\alpha, r'') \subset \mathcal{U}(\beta, r)$. \square

These properties for $\mathcal{B}(x)$, $\mathcal{B}(\mathbf{b})$ and $\partial\mathcal{B}(\mathbf{b})$ are characteristic of the set of neighborhoods of \mathbf{b} , as stated in the following proposition.

Proposition 4.7 ([Bou98], Proposition 2). *Let Y be a set. If to each element $y \in Y$ there corresponds a set $\mathcal{B}(y)$ of subsets of Y such that properties (i) to (iv) from Lemma 4.5 above are satisfied, then there is a unique topological structure on Y such that for each $y \in Y$, $\mathcal{B}(y)$ is the set of neighborhoods of y in this topology.*

Thus we now use the sets $\partial\mathcal{B}(b)$ to equip $\partial_\kappa X$ with a topological structure and use the sets $\mathcal{B}(b)$ and $\mathcal{B}(x)$ to equip $X \cup \partial_\kappa X$ with a topological structure. Note that, since neighborhoods in $\partial_\kappa X$ are intersections of neighborhoods of $X \cup \partial_\kappa X$ with $\partial_\kappa X$, we have that the inclusion $\partial_\kappa X \subset X \cup \partial_\kappa X$ is a topological embedding and $X \cup \partial_\kappa X$ is a bordification of X .

Recall that a set is open if it contains a neighborhood of each of its points. Thus, a set $\mathcal{W} \subseteq \partial_\kappa X$ is open if for every $\mathbf{b} \in \mathcal{W}$ there is $\beta \in \mathbf{b}$ and $r > 0$ such that $\partial\mathcal{U}(\beta, r) \subset \mathcal{W}$. A set $\mathcal{W} \subset X \cup \partial_\kappa X$ is open if its intersection with both X and $\partial_\kappa X$ is open.

4.2. Metrizable. We now establish the metrizable of the space $\partial_\kappa X$. To begin with, we need the following property of the topology:

Lemma 4.8. *For each κ -Morse quasi-geodesic ray β and $r > 0$, there exists a radius $r' > 0$ such that for any point $\mathbf{a} \in \partial_\kappa X$ there exists $r'' > 0$ (depending only on \mathbf{a} and r' and not on β) such that for every geodesic representative $\alpha_0 \in \mathbf{a}$*

$$\mathcal{U}(\alpha_0, r'') \cap \mathcal{U}(\beta, r') \neq \emptyset \quad \implies \quad \mathbf{a} \in \partial\mathcal{U}(\beta, r).$$

Similarly, for $x \in X$, let $B(x, 1)$ be the ball of radius 1 centered at x . Then

$$B(x, 1) \cap \mathcal{U}(\beta, r') \neq \emptyset \quad \implies \quad x \in \mathcal{U}(\beta, r).$$

Proof. This will be done using the Surgery Lemma 3.8. Pick a κ -Morse quasi-geodesic ray β and $r > 0$. Let $\mathcal{Q} := \{(\mathbf{q}, \mathbf{Q}) : m_\beta(\mathbf{q}, \mathbf{Q}) \leq \frac{r}{2\kappa(r)}\}$, which is bounded by properness. Set

$$M := \sup_{(\mathbf{q}, \mathbf{Q}) \in \mathcal{Q}} m_\beta(9\mathbf{q}, \mathbf{Q} + 1),$$

and $r' := R(\beta, r, M\kappa)$. Let $\mathbf{a} \in \partial_\kappa X$.

By Corollary 3.5, there exists a constant $u > 0$ such that, for any geodesic ray $\star \in \mathbf{a}$ and any $(\mathbf{q}, \mathbf{Q}) \in \mathcal{Q}$ we have

$$m_\star(1, 0) + 3m_\star(\mathbf{q}, \mathbf{Q}) \leq u.$$

Let R be the radius given in Lemma 3.8 associated to \mathbf{q}, \mathbf{Q} and $2r'$, and let r'' be large enough so that

$$r'' \geq \max\{2u \cdot \kappa(r''), 2r', R\}.$$

Let α_0 be a geodesic ray in \mathbf{a} . By assumption, there is a point \mathbf{c} inside the intersection

$$\mathbf{c} \in \mathcal{U}(\alpha_0, r'') \cap \mathcal{U}(\beta, r').$$

If $\mathbf{c} \in \partial_\kappa X$, let $\gamma \in \mathbf{c}$ be a geodesic ray in this class and if $\mathbf{c} \in X$, let γ be a geodesic ray connecting \mathbf{o} to \mathbf{c} . In either case, $\gamma_{r''}$ is well defined since, in the second case, $d_X(\mathbf{o}, \mathbf{c}) \geq r''$. Let $\alpha \in \mathbf{a}$ be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray with $m_\beta(\mathbf{q}, \mathbf{Q})$ small compared to r . To conclude $\mathbf{a} \in \partial\mathcal{U}(\beta, r)$ we need to show that $\alpha|_r \subset \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q}))$.

Since $\mathbf{c} \in \mathcal{U}(\alpha_0, r'')$,

$$d_X(\gamma_{r''}, \alpha_0) \leq m_{\alpha_0}(1, 0) \cdot \kappa(r'').$$

Let $p \in \pi_{\alpha_0}(\gamma_{r''})$. By definition of u and r'' , we have

$$\|p\| \leq r'' + m_{\alpha_0}(1, 0) \cdot \kappa(r'') \leq \frac{3}{2}r''.$$

Therefore, Lemma 3.4 (ii) implies

$$d_X(p, \alpha) \leq 2m_{\alpha_0}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(p) \leq 3m_{\alpha_0}(\mathbf{q}, \mathbf{Q}) \cdot \kappa(r'').$$

Hence,

$$d_X(\gamma_{r''}, \alpha) \leq d_X(\gamma_{r''}, p) + d_X(p, \alpha) \leq u \cdot \kappa(r'') \leq r''/2.$$

We can now apply the Surgery lemma (Lemma 3.8) to α and γ with radius $2r'$ to obtain a $(9\mathbf{q}, \mathbf{Q})$ -quasi-geodesic ray γ' that is either in the class \mathbf{c} if $\mathbf{c} \in \partial_\kappa X$ or ends in \mathbf{c} if $\mathbf{c} \in X$ where $\gamma'|_{r'} = \alpha|_{r'}$. Since $\mathbf{c} \in \mathcal{U}(\beta, r')$, we have

$$\alpha|_{r'} = \gamma'|_{r'} \subset \mathcal{N}_\kappa(\beta, m_\beta(9\mathbf{q}, \mathbf{Q})).$$

Observing that $\alpha|_{r'}$ is (\mathbf{q}, \mathbf{Q}) -quasi-geodesic and letting $\kappa' = m_\beta(9\mathbf{q}, \mathbf{Q}) \cdot \kappa$, the definition of r' and κ -Morse implies that

$$\alpha|_r \subset \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q})),$$

hence $\mathbf{a} \in \mathcal{U}(\beta, r)$.

To see the second assertion, assume $y \in B(x, 1) \cap \mathcal{U}(\beta, r')$. Let α be a (\mathbf{q}, \mathbf{Q}) -quasi-geodesic ray ending at x where $m_\beta(\mathbf{q}, \mathbf{Q})$ is small compared to r . Let γ be a quasi-geodesic ray that is identical to α but at the last point in sent to y instead of x . Then γ is $(\mathbf{q}, \mathbf{Q} + 1)$ -quasi-geodesic. The definition of r' and κ -Morse implies that

$$\alpha|_r = \gamma|_r \subset \mathcal{N}_\kappa(\beta, m_\beta(\mathbf{q}, \mathbf{Q})),$$

hence $x \in \mathcal{U}(\beta, r)$. □

Our method for establishing metrizability is via the following criterion.

Theorem 4.9 (Theorem 3, [Fri37]). *Assume, for every point \mathbf{b} of a topological space, there exists a monotonically decreasing sequence $\mathcal{V}_1(\mathbf{b}), \mathcal{V}_2(\mathbf{b}), \dots, \mathcal{V}_i(\mathbf{b}), \dots$ of neighborhoods whose intersection is \mathbf{b} and such that the following holds: For every point \mathbf{b} of the neighborhood space and every integer i , there exists an integer $j = j(\mathbf{b}, i) > i$ such that if \mathbf{a} is a point for which $\mathcal{V}_j(\mathbf{a})$ and $\mathcal{V}_j(\mathbf{b})$ have a point in common then $\mathcal{V}_j(\mathbf{a}) \subset \mathcal{V}_i(\mathbf{b})$. Then the space is homeomorphic to a metric space.*

We check this condition for $\partial_\kappa X$, using as neighborhoods $\mathcal{V}_i(\mathbf{b})$ the sets $\partial\mathcal{U}(\beta, r)$ previously defined.

Theorem 4.10. *The space $\partial_\kappa X$ is metrizable.*

Proof. Our goal is to construct, for any $i \in \mathbb{N}$ and $\mathbf{b} \in \partial_\kappa X$, neighborhoods $\mathcal{V}_i(\mathbf{b})$ which satisfy the conditions of Theorem 4.9.

Recall that, given a κ -Morse quasi-geodesic ray β and $r > 0$, we can define r' as

$$r'(\beta, r) := R(\beta, r, M\kappa),$$

as in the proof of Lemma 4.8. Let $\mathbf{a} \in \partial_\kappa X$ and pick a geodesic representative $\alpha_0 \in \mathbf{a}$. Note that both in Claim 4.6 and in Lemma 4.8, r'' does not depend on β or r , but it depends on α_0 , r' and on

$$\sup_{m_\beta(\mathbf{q}, \mathbf{Q}) \leq r} m_{\alpha_0}(\mathbf{q}, \mathbf{Q}).$$

Since $\mathbf{q}, \mathbf{Q} \leq m_\beta(\mathbf{q}, \mathbf{Q}) \leq r \leq r'$, the maximum value of $m_{\alpha_0}(\mathbf{q}, \mathbf{Q})$ can be bounded in terms of α_0 , r' , without referring to β or r . Hence, we can consider the function $r''(\alpha_0, r')$ such that both Claim 4.6 and Lemma 4.8 hold.

For $i \in \mathbb{N}$ define

$$\mathcal{V}_i(\mathbf{a}) := \partial\mathcal{U}(\alpha_0, r_i(\mathbf{a})), \quad \text{where} \quad r_i(\mathbf{a}) := \max(i, r''(\alpha_0, i)).$$

Also, given \mathbf{b} and i , we define $\rho_i(\mathbf{b}) := r'(\beta_0, r_i(\mathbf{b}))$, and

$$j = j(\mathbf{b}, i) := \left\lceil r'(\beta_0, \rho_i(\mathbf{b})) \right\rceil,$$

where β_0 is a geodesic representative in \mathbf{b} . Assume $\mathcal{V}_j(\mathbf{a})$ and $\mathcal{V}_j(\mathbf{b})$ have a point in common, that is,

$$\partial\mathcal{U}(\alpha_0, r_j(\mathbf{a})) \cap \partial\mathcal{U}(\beta_0, r_j(\mathbf{b})) \neq \emptyset.$$

Then, since $r_j(\mathbf{a}) \geq r''(\alpha_0, j)$ and $r_j(\mathbf{b}) \geq j$, by Lemma 4.8 we have

$$\mathbf{a} \in \partial\mathcal{U}(\beta_0, \rho_i(\mathbf{b})).$$

Now, Claim 4.6 implies

$$\partial\mathcal{U}(\alpha_0, r''(\alpha_0, \rho_i(\mathbf{b}))) \subset \partial\mathcal{U}(\beta_0, r_i(\mathbf{b})).$$

But $r_j(\mathbf{a}) = \max(j, r''(\alpha_0, j))$, thus

$$r_j(\mathbf{a}) \geq r''(\alpha_0, r'(\beta_0, \rho_i(\mathbf{b}))) \geq r''(\alpha_0, \rho_i(\mathbf{b})).$$

Therefore,

$$\partial\mathcal{U}(\alpha_0, r_j(\mathbf{a})) \subset \partial\mathcal{U}(\beta_0, r_i(\mathbf{b})),$$

which is to say $\mathcal{V}_j(\mathbf{a}) \subset \mathcal{V}_i(\mathbf{b})$. The theorem now follows from Theorem 4.9. \square

Similarly, we have

Theorem 4.11. *The space $X \cup \partial_\kappa X$ is metrizable.*

Proof. For $i \in \mathbb{N}$ and $\mathbf{a} \in \partial_\kappa X$, let $r_i(\mathbf{a})$ be as in the proof of Theorem 4.10 and let

$$\mathcal{V}_i(\mathbf{a}) := \mathcal{U}(\alpha_0, r_i(\mathbf{a})).$$

For a point $x \in X$, we define $\mathcal{V}_i(x) := B(x, \frac{1}{i})$, the ball of radius $\frac{1}{i}$ centered around x . Since Lemma 4.8 holds for $\mathcal{U}(\alpha_0, r_i(\mathbf{a}))$, the same proof as above works to check the conditions of Theorem 4.9 for any point $\mathbf{b} \in \partial_\kappa X$.

For $x \in X$, we define $j(x, i) := 3i$. Then, if

$$\mathcal{V}_j(x) \cap \mathcal{V}_j(y) \neq \emptyset,$$

there is a point $z \in B(x, \frac{1}{3i}) \cap B(y, \frac{1}{3i})$ and, by the triangle inequality,

$$\mathcal{V}_j(y) = B(y, \frac{1}{3i}) \subset B(x, \frac{1}{i}).$$

Also, if $\mathcal{V}_j(x) \cap \mathcal{V}_j(\mathbf{b}) \neq \emptyset$, for $x \in X$ and $\mathbf{b} \in \partial_\kappa X$, then $B(x, 1) \cap \mathcal{U}(\beta, r_j(\mathbf{b})) \neq \emptyset$. By the definition of $j(\mathbf{b}, i)$ and the second part of Lemma 4.8, this implies that

$$\mathcal{V}_i(x) \subset B(x, 1) \subset \mathcal{U}(\beta, r_i(\mathbf{b})).$$

Again, the theorem follows from Theorem 4.9. \square

We are now ready to establish the quasi-isometric invariance of $\partial_\kappa X$.

Theorem 4.12. *Consider proper geodesic metric spaces X and Y , let $\Phi: X \rightarrow Y$ be a (k, K) -quasi-isometry and let κ be a concave sublinear function. Then Φ induces a homeomorphism $\Phi^*: \partial_\kappa X \rightarrow \partial_\kappa Y$ where, for $\mathbf{b} \in \partial_\kappa X$ and $\beta \in \mathbf{b}$,*

$$\Phi^*(\mathbf{b}) = [\Phi \circ \beta],$$

where $[\cdot]$ denotes the equivalence class of a quasi-geodesic ray.

The proof is identical to the proof of Theorem 5.1 in [QRT19].

4.3. The union of $\partial_\kappa X$. We note that topologies of different sublinear boundaries are compatible.

Proposition 4.13 (Proposition 4.10, [QRT19]). *Let κ and κ' be sublinear functions such that, for some $M > 0$*

$$\kappa'(t) \leq M \cdot \kappa(t), \quad \forall t > 0.$$

Then, $\partial_{\kappa'} X \subset \partial_\kappa X$ as a subspace with the subspace topology.

The proof is identical to the proof of Proposition 4.10 from [QRT19] and is skipped. In view of this proposition, we can define the *sublinearly Morse boundary* of X as

$$\partial X := \bigcup_{\kappa} \partial_\kappa X$$

which is the space of equivalence classes (up to sublinear fellow traveling) of all sublinearly Morse quasi-geodesic rays in X .

Remark 4.14. An open neighborhood \mathcal{V} of a point $\mathbf{b} \in \partial X$ can be described as follows: assume $\mathbf{b} \in \partial_\kappa X$ for some κ and choose a quasi-geodesic ray $\beta \in \mathbf{b}$ and a radius $r > 0$. Let $\mathcal{U}_\kappa(\beta, r)$ be the neighborhood of \mathbf{b} in $(X \cup \partial_\kappa X)$ and let \mathcal{V} be the closure of $\mathcal{U}_\kappa(\beta, r)$ in $(X \cup \partial X)$. That is, a point in $\mathcal{V} \cap \partial X$ is a class of κ' -sublinearly Morse quasi-geodesic rays for some κ' different from κ that are eventually contained in $\mathcal{U}_\kappa(\beta, r)$. The intersection $\mathcal{V} \cap X$ equals $X \cap \mathcal{U}_\kappa(\beta, r)$.

Similar arguments as in Theorem 4.10 and Theorem 4.11 can be used to show that ∂X and $(X \cup \partial X)$ are also metrizable. We skip these, for the sake of brevity.

5. GENERAL PROJECTIONS AND WEAKLY SUBLINEARLY CONTRACTING SETS

In order to deal with several applications to proper geodesic spaces (in particular, our applications to mapping class groups and relatively hyperbolic groups we shall see next), the usual notion of nearest point projection may be ill-suited; for instance, it is well-known that nearest point projection to a closed subset of a general (e.g., not hyperbolic) metric space need not be, even coarsely, well-defined.

Thus, we now introduce a more general notion of projection, which we call κ -*projection*, where we allow an additive error which is controlled by a sublinear function κ .

Let us denote as $\mathcal{P}(Z)$ the set of subsets of Z , and let us use the notation $\kappa(x) := \kappa(\|x\|)$.

Definition 5.1. Let (X, d_X) be a proper geodesic metric space and $Z \subseteq X$ a closed subset, and let κ be a concave sublinear function. A map $\pi_Z: X \rightarrow \mathcal{P}(Z)$ is a κ -*projection* if there exist constants D_1, D_2 , depending only on Z and κ , such that for any points $x \in X$ and $z \in Z$,

$$\text{diam}_X(\{z\} \cup \pi_Z(x)) \leq D_1 \cdot d_X(x, z) + D_2 \cdot \kappa(x).$$

A κ -projection differs from a nearest point projection by a uniform multiplicative error and a sublinear additive error. In particular, the nearest point projection is a κ -projection. Indeed, for $z \in Z$ and $w \in \pi_Z(x)$, we have

$$d(z, w) \leq d(z, x) + d(x, w) \leq 2d(z, x).$$

Lemma 5.2. *Given a closed set Z , we have for any $x \in X$*

$$\text{diam}_X(\{x\} \cup \pi_Z(x)) \leq (D_1 + 1) \cdot d_X(x, Z) + D_2 \cdot \kappa(x).$$

Proof. Let $z \in Z$ be a point that realizes $d_X(x, Z)$. Then, by triangle inequality and applying Definition 5.1, we obtain

$$\text{diam}_X(\{x\} \cup \pi_Z(x)) \leq d_X(x, z) + \text{diam}_X(\{z\} \cup \pi_Z(x)) \leq (D_1 + 1) \cdot d_X(x, Z) + D_2 \cdot \kappa(x). \quad \square$$

We now formulate a general definition of κ -*weakly contracting* with respect to a κ -projection π_Z .

Definition 5.3 (κ -weakly contracting). For a closed subspace Z of a metric space (X, d_X) and a κ -projection π_Z onto Z , we say Z is κ -*weakly contracting* with respect to π_Z if there are constants C_1, C_2 , depending only on Z and κ , such that, for every $x, y \in X$

$$d_X(x, y) \leq C_1 \cdot d_X(x, Z) \implies \text{diam}_X(\pi_Z(x) \cup \pi_Z(y)) \leq C_2 \cdot \kappa(x).$$

In the special case that π_Z is the nearest point projection and $C_1 = 1$, this property was called κ -*contracting* in [QRT19]. It was shown in [QRT19] that, in the setting of CAT(0) spaces, this is stronger than the κ -Morse condition.

With respect to any projection we prove the following analogous statement of [QRT19, Theorem 3.14]:

Theorem 5.4 (κ -weakly contracting implies sublinearly Morse). *Let κ be a concave sublinear function and let Z be a closed subspace of X . Let π_Z be a κ -projection onto Z and suppose that Z is κ -weakly contracting with respect to π_Z . Then, there is a function $m_Z: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that, for every*

constant $r > 0$ and every sublinear function κ' , there is an $R = R(Z, r, \kappa') > 0$ where the following holds: Let $\eta: [0, \infty) \rightarrow X$ be a $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray so that $m_Z(\mathfrak{q}, \mathfrak{Q})$ is small compared to r , let t_r be the first time $\|\eta(t_r)\| = r$ and let t_R be the first time $\|\eta(t_R)\| = R$. Then

$$d_X(\eta(t_R), Z) \leq \kappa'(R) \implies \eta([0, t_r]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathfrak{q}, \mathfrak{Q})).$$

The proof of this result is similar to the one in [QRT19], so we will postpone it to the appendix. Moreover, in the appendix we shall prove the following equivalence between κ -weakly contracting and κ -Morse (with a possibly different sublinear function) for any given closed set.

Theorem 5.5. *Let (X, \mathfrak{o}) be a proper geodesic metric space with a fixed base point and let α be a quasi-geodesic ray in X . Let π be any κ -projection from X to α in the sense of Definition 5.1. Then the following hold:*

- (1) *If α is κ -weakly contracting with respect to π , then it is κ -Morse;*
- (2) *The quasi-geodesic α is κ -Morse if and only if it is κ -weakly Morse;*
- (3) *If α is κ -weakly Morse, then it is κ' -weakly contracting with respect to π for some sublinear function κ' .*

6. THE POISSON BOUNDARY

We now show a general criterion (Theorem 6.2) for the κ -Morse boundary of a group to be identified with its Poisson boundary.

Random walks. Let G be a locally compact, second countable group, with left Haar measure m , and let μ be a Borel probability measure on G , which we assume to *spread-out*, i.e. such that there exists n for which μ^n is not singular w.r.t. m . Given μ , we consider the *step space* $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$, whose elements we denote as (g_n) . The *random walk driven by μ* is the G -valued stochastic process (w_n) , where for each n we define the product

$$w_n := g_1 g_2 \dots g_n.$$

We denote as (Ω, \mathbb{P}) the *path space*, i.e. the space of sequences (w_n) , where \mathbb{P} is the measure induced by pushing forward the measure $\mu^{\mathbb{N}}$ from the step space. Elements of Ω are called *sample paths* and will be also denoted as ω . Finally, let $T: \Omega \rightarrow \Omega$ be the left shift on the path space.

Background on boundaries. Let us recall some fundamental definitions from the boundary theory of random walks. We refer to [Kai00] for more details. Let (B, \mathcal{A}) be a measurable space on which G acts by measurable isomorphisms; a measure ν on B is μ -*stationary* if $\nu = \int_G g_* \nu d\mu(g)$, and in that case the pair (B, ν) is called a (G, μ) -*space*. Recall that a μ -*boundary* is a measurable (G, μ) -space (B, ν) such that there exists a T -invariant, measurable map $\mathbf{bnd}: (\Omega, \mathbb{P}) \rightarrow (B, \nu)$, called the *boundary map*.

Moreover, a function $f: G \rightarrow \mathbb{R}$ is μ -*harmonic* if $f(g) = \int_G f(gh) d\mu(h)$ for any $g \in G$. We denote by $H^\infty(G, \mu)$ the space of bounded, μ -harmonic functions. One says a μ -boundary is the *Poisson boundary* of (G, μ) if the map

$$\Phi: H^\infty(G, \mu) \rightarrow L^\infty(B, \nu)$$

given by $\Phi(f)(g) := \int_B f dg_* \nu$ is a bijection. The Poisson boundary (B, ν) is the maximal μ -boundary, in the sense that for any other μ -boundary (B', ν') there exists a G -equivariant, measurable map $p: (B, \nu) \rightarrow (B', \nu')$.

Finally, a metric d on G is *temperate* if there exists C such that

$$m(\{g \in G : d(1, g) \leq R\}) \leq C e^{CR}$$

for any $R > 0$. A measure μ has *finite first moment* with respect to d if $\int_G d(1, g) d\mu(g) < +\infty$.

We will use the ray approximation criterion from [Kai00] for the Poisson boundary (for this precise version, see [FT18]).

Theorem 6.1. *Let G be a locally compact, second countable group equipped with a temperate metric d , and let μ be a spread-out probability measure on G with finite first moment with respect to d . Let (B, λ) be a μ -boundary, and suppose that there exist maps $\pi_n : B \rightarrow G$ for any $n \in \mathbb{N}$ such that for almost every sample path $\omega = (w_n)$ we have*

$$(8) \quad \lim_{n \rightarrow \infty} \frac{d(w_n, \pi_n(\mathbf{bnd}(\omega)))}{n} = 0.$$

Then (B, λ) is the Poisson boundary of (G, μ) .

The κ -Morse boundary is the Poisson boundary. We now apply this criterion to identify the Poisson boundary with the κ -Morse boundary. The following result was obtained in collaboration with Ilya Gekhtman.

Theorem 6.2. *Let G be a finitely generated group, and let (X, d_X) be a Cayley graph of G . Let μ be a probability measure on G with finite first moment with respect to d_X , such that the semigroup generated by the support of μ is a non-amenable group. Let κ be a concave sublinear function, and suppose that for almost every sample path $\omega = (w_n)$, there exists a κ -Morse geodesic ray γ_ω such that*

$$(9) \quad \lim_{n \rightarrow \infty} \frac{d_X(w_n, \gamma_\omega)}{n} = 0.$$

Then almost every sample path converges to a point in $\partial_\kappa X$, and moreover the space $(\partial_\kappa X, \nu)$, where ν is the hitting measure for the random walk, is a model for the Poisson boundary of (G, μ) .

Proof. By the subadditive ergodic theorem and finite first moment, the limit

$$\ell := \lim_{n \rightarrow \infty} \frac{d_X(\mathbf{o}, w_n)}{n}$$

exists almost surely and is constant, and $\ell > 0$ since the group generated by the support of μ is non-amenable (see [Woe00, Theorem 8.14 and Corollary 12.5]).

By Lemma 4.3 and Eq. (9), almost every sequence converges to $[\gamma_\omega] \in \partial_\kappa X$. Thus, we can define $\mathbf{bnd} : \Omega \rightarrow \partial_\kappa X$ as

$$\mathbf{bnd}(\omega) := \lim_{n \rightarrow \infty} w_n \in \partial_\kappa X,$$

which is T -invariant by definition. Moreover, \mathbf{bnd} is measurable, since it is a pointwise limit of the measurable functions w_n with values in the space $X \cup \partial_\kappa X$, which is metrizable by Theorem 4.11. Since G is finitely generated, any word metric d_X on it is temperate.

Finally, by Eq. (9), almost every sample path sublinearly tracks a κ -Morse quasi-geodesic ray. Hence, let us define $\pi_n : \partial_\kappa X \rightarrow G$ as $\pi_n(\xi) := \alpha_{r_n}$ where α is a geodesic representative of the class of $\xi \in \partial_\kappa X$, and $r_n := \lfloor \ell n \rfloor$.

Now, let $\omega \in \Omega$ and $\gamma = \gamma_\omega$, and let p_n be a nearest point projection of w_n onto γ . By (9), we have for almost every $\omega \in \Omega$

$$\lim_{n \rightarrow \infty} \frac{d_X(w_n, p_n)}{n} \rightarrow 0,$$

hence also $\frac{\|p_n\|}{n} \rightarrow \ell$. Since p_n and γ_{r_n} lie on the same geodesic, this implies

$$(10) \quad \frac{d_X(w_n, \gamma_{r_n})}{n} \leq \frac{d_X(w_n, p_n)}{n} + \frac{d_X(p_n, \gamma_{r_n})}{n}$$

$$(11) \quad \leq \frac{d_X(w_n, p_n)}{n} + \frac{|\|p_n\| - r_n|}{n} \rightarrow 0 + \ell - \ell = 0$$

as $n \rightarrow \infty$. Finally, we obtain

$$\frac{d_X(w_n, \pi_n(\mathbf{bnd}(\omega)))}{n} = \frac{d_X(w_n, \alpha_{r_n})}{n} \leq \frac{d_X(w_n, \gamma_{r_n})}{n} + \frac{d_X(\gamma_{r_n}, \alpha_{r_n})}{n},$$

and the first term tends to 0 because of (10), while the second term tends to 0 since $\alpha \sim \gamma$. Hence, by Theorem 6.1, $(\partial_\kappa X, \nu)$ is a model for the Poisson boundary of (G, μ) . \square

7. BOUNDARIES OF MAPPING CLASS GROUPS

In this section, we show that for an appropriate choice of κ , the κ -Morse boundary of any mapping class group $G = \text{Map}(S)$ can function as a topological model for the Poisson boundary of the pair (G, μ) , where μ is any finitely-supported non-elementary measure.

We need to show that a generic sample path of such a random walk sublinearly tracks a κ -Morse quasi-geodesic ray. We will do so but showing that, in fact, the limiting quasi-geodesic ray is κ -weakly contracting.

7.1. Background on mapping class groups. Let S be a surface of finite hyperbolic type, let $g(S)$ be its genus and $b(S)$ the number of its boundary components. Let $\text{Map}(S)$ denote the mapping class group of S equipped with a word metric d_w associated to a finite generating set. That is, we are in the setting where $(X, d_X) = (\text{Map}(S), d_w)$.

Ending laminations. Let $\mathcal{C}(S)$ denote the curve graph of S (see [MM00] for definition and details). The curve graph is known to be δ -hyperbolic [MM00]. By [Kla98], the Gromov boundary of $\mathcal{C}(S)$ can be identified with the space of ending laminations $\mathcal{EL}(S)$, that is, the space of minimal filling laminations after forgetting the measure.

Subsurface projections. By a subsurface Y we always mean a connected π_1 -injective subsurface of S . For any subsurface Y let $\mathcal{C}(Y)$ denote the curve graph of Y . Let ∂Y denote the multi-curve consisting of all boundary components of Y . There is a projection map $\pi_Y: \mathcal{C}(S) \rightarrow \mathcal{C}(Y)$ defined on a subset of $\mathcal{C}(S)$ consisting of curves that intersect Y . This is essentially a map that sends a curve $\alpha \in \mathcal{C}(S)$ to a set of curves in Y obtained from surgery between α and ∂Y . (Again, see [MM00] for details). The set $\pi_Y(\alpha)$ has a uniformly bounded diameter in $\mathcal{C}(Y)$, independent of α or Y .

We can extend this projection to a map $\pi_Y: \text{Map}(S) \rightarrow \mathcal{C}(Y)$ as follows. Consider a set θ of curves on S that fill S . For example, following [MM00], we can assume θ is the union of a pants decomposition and a set of *dual curves*, one transverse to each curve in the pants decomposition. Then for $x \in \text{Map}(S)$, define

$$\pi_Y(x) := \bigcup_{\alpha \in \theta} \pi_Y(x(\alpha)).$$

Again, the set $\pi_Y(x)$ has a uniformly bounded diameter in $\mathcal{C}(Y)$. For, $x, y \in \text{Map}(S)$ define

$$d_Y(x, y) := \text{diam}_{\mathcal{C}(Y)}(\pi_Y(x) \cup \pi_Y(y)).$$

In particular,

$$d_S(x, y) := \text{diam}_{\mathcal{C}(S)}(x(\theta), y(\theta)).$$

Also, when Y is an annulus with core curve α , we often use $D_{\alpha, K}(x, y)$ instead of $d_Y(x, y)$.

In the discussion above, x and y can be replaced with an ending lamination $\xi \in \mathcal{EL}(S)$ since ξ has non-trivial projection to every subsurface and $\pi_Y(\xi)$ is always well-defined. That is, we define

$$d_Y(x, \xi) := \text{diam}_{\mathcal{C}(Y)}(\pi_Y(x), \pi_Y(\xi)).$$

The distance formula. In [MM00], it was shown that the word metric on $\text{Map}(S)$ can be estimated up to uniform additive and multiplicative constants by these subsurface projection distances. To simplify the exposition, we adopt the following notation. We fix S and a generating set for $\text{Map}(S)$, and we say a constant M is *uniform* if it depends only on the topology of S and the generating set. For two quantities \mathbf{A} and \mathbf{B} , we write $\mathbf{A} \prec \mathbf{B}$ if there is a uniform constant M such that

$$\mathbf{A} \leq M \cdot \mathbf{B} + M.$$

We write $\mathbf{A} \asymp \mathbf{B}$ if $\mathbf{A} \prec \mathbf{B}$ and $\mathbf{B} \prec \mathbf{A}$ and we use the notation $O(\mathbf{A})$ for a quantity that has an upper bound of $M \cdot \mathbf{A}$. Also, recall that, for $K > 0$,

$$\lfloor \mathbf{A} \rfloor_K := \begin{cases} x & x \geq K \\ 0 & x < K. \end{cases}$$

The following inequality is useful: if $K \geq 2C \geq 0$, then

$$\lfloor a + C \rfloor_K \leq 2 \lfloor a \rfloor_{K/2} \quad \forall a \geq 0.$$

Now the Masur-Minsky distance formula can be stated as follows: for any K sufficiently large, for any $x, y \in \text{Map}(S)$ we have:

$$(12) \quad d_w(x, y) \asymp \sum_{Y \subseteq S} \lfloor d_Y(x, y) \rfloor_K,$$

where the implicit constant depends on K . Given $K \geq 0$, for any $x, y \in \text{Map}(S) \cup \mathcal{EL}(S)$ and α a simple closed curve, let us denote

$$D_{\alpha, K}(x, y) := \sum_{\alpha \cap Y = \emptyset} \lfloor d_Y(x, y) \rfloor_K.$$

The hierarchy of geodesics. To every pair of points $x, y \in \text{Map}(S)$ one can associate a *hierarchy of geodesics* connecting $x(\theta)$ to $y(\theta)$ [MM00, Theorem 4.6]. The hierarchy $H = H(x, y)$ consists of a geodesic $[x, y]_S$ in $\mathcal{C}(S)$ connecting $x(\theta)$ to $y(\theta)$ and other geodesics $[x, y]_Y$ in various curve graphs $\mathcal{C}(Y)$, where $[x, y]_Y$ is essentially a geodesic connecting $\pi_Y(x)$ to $\pi_Y(y)$. Hence we write $H = \{[x, y]_Y\}$. Besides S , other subsurfaces that appear in H are described as follows: for every curve α in $[x, y]_S$, we include every component of $S - \alpha$ that is not a pair of pants and the annulus A_α (the annulus whose core curve is α). Also, if a subsurface Y appears in H , then for every β that appears in $[x, y]_Y$, we also include every component of $Y - \beta$ that is not a pair of pants and the annulus A_β . The length $|H|$ is defined to be the sum of the lengths $|[x, y]_Y|$ of geodesics $[x, y]_Y$. By [MM00, Theorem 3.1], there exists K , depending only on the topology of S , such that for every subsurface Y , if $d_Y(x, y) \geq K$ then Y is included in H . Furthermore, by [MM00, Theorems 6.10, 6.12 and 7.1] we have

$$(13) \quad d_w(x, y) \asymp |H(x, y)| := \sum_{Y \text{ in } H} |[x, y]_Y|.$$

A *resolution* $\mathcal{G}(x, y)$ of a hierarchy $H(x, y)$ is a uniform quasi-geodesic in $\text{Map}(S)$ connecting x to y where, for any subsurface Y , the projection of $\mathcal{G}(x, y)$ to $\mathcal{C}(Y)$ is contained in a uniformly bounded neighborhood of the geodesic segment $[x, y]_Y$.

We can also replace x or y with a point $\xi \in \mathcal{EL}$. We start with a (tight) geodesic $[x, \xi]_S$ in $\mathcal{C}(S)$ and build $H(x, \xi)$ the same as before replacing, for every subsurface Y , $\pi_Y(y(\theta))$ with $\pi_Y(\xi)$. The resolution $\mathcal{G}(x, \xi)$ of $H(x, \xi)$ is then a uniform quasi-geodesic in $\text{Map}(S)$ starting from x such that the *shadow* of $\mathcal{G}(x, \xi)$ in $\mathcal{C}(S)$ (namely, the set $\{z(\theta) : z \in \mathcal{G}(x, \xi)\}$) converges to ξ .

We use the hierarchy paths to show:

Proposition 7.1. *Let $p := 3g(S) - 3 + b(S)$ be the complexity of S . For any $x, y \in X$, assume that $d_Y(x, y) \leq E$ for all $Y \subsetneq S$ and some $E > 1$. Then we have*

$$d_w(x, y) \prec d_S(x, y) \cdot E^p.$$

Proof. This is essentially contained in [MM00]. We sketch the proof here and refer the reader to [MM00] for definitions and details. In view of Equation (13), we need to show

$$|H(x, y)| \prec d_S(x, y) \cdot E^p.$$

The restriction of $H(x, y)$ to a subsurface Y is again a hierarchy which we denote with $H_Y(x, y)$. We check the Proposition inductively. When S is $S_{1,1}$ or $S_{0,4}$, we have $p = 1$ and for every curve α in S , $S - \alpha$ does not have any complementary component that is not a pair of pants. Also, by assumption, for every $\alpha \in [x, y]_S$, we have $|[x, y]_\alpha| \prec D_{\alpha, K}(x, y) \leq E$. Therefore, $|H(x, y)| \prec E \cdot |[x, y]_S| \prec E \cdot d_S(x, y) \leq E$, as required.

Now let S be a larger surface and assume, by induction, that for every subsurface Y , the hierarchy $H_Y(x, y)$ satisfies $|H_Y(x, y)| \prec d_Y(x, y) \cdot E^{p-1}$. We have

$$|H(x, y)| \prec \sum_{\alpha \in [x, y]_S} \left(|[x, y]_\alpha| + \sum_{Y \subset S - \alpha} |H_Y(x, y)| \right) \prec |[x, y]_S| \cdot (E + 2d_Y(x, y) \cdot E^{p-1}).$$

But $d_Y(x, y) \leq E$ and $[x, y]_S \prec d_S(x, y)$, thus $|H(x, y)| \prec d_S(x, y) \cdot E^p$. \square

Projections in mapping class groups. Here, we recall the construction of the center of a triangle in $\text{Map}(S)$ according to Eskin-Masur-Rafi [EMR17]. For $x \in \text{Map}(S)$ and a subsurface Y , we denote $\pi_Y(x(\theta))$ simply by x_Y . Also, as before, for $x, y \in \text{Map}(S)$, the geodesic segment in $\mathcal{C}(Y)$ connecting x_Y and y_Y is denoted by $[x, y]_Y$. For any subsurface Y , the curve graph $\mathcal{C}(Y)$ is δ -hyperbolic for some uniform constant δ . Thus, for any three points $x, y, z \in \text{Map}(S)$ and every subsurface Y , there exists a point $\text{ctr}_Y(x, y, z)$ in $\mathcal{C}(Y)$ that is δ -close to all three geodesic segments $[x, y]_Y$, $[x, z]_Y$ and $[y, z]_Y$. We refer to $\text{ctr}_Y(x, y, z)$ as the *center* of the triple $x_Y, y_Y, z_Y \in \mathcal{C}(Y)$.

It was shown in [EMR17] that there is an element $\eta \in \text{Map}(S)$ that projects near the center of x_Y, y_Y, z_Y for every subsurface Y . More precisely:

Lemma 7.2 ([EMR17, Lemma 4.11]). *There exists a constant D such that the following holds. For any $x, y, z \in \text{Map}(S)$, there exists a point $\eta \in \text{Map}(S)$ such that, for any subsurface $Y \subseteq S$, we have*

$$d_Y(\eta_Y, \text{ctr}_Y(x, y, z)) \leq D.$$

We call η the center of x, y and z and we denote it by $\text{ctr}(x, y, z)$.

Note that, as always, we can replace each of x, y, z with an ending lamination $\xi \in \mathcal{PML}$. That is, $\text{ctr}(x, y, \xi)$ is a well-defined element of $\text{Map}(S)$. From now on, we will denote as \mathfrak{o} the identity element in $\text{Map}(S)$, which will function as base point.

Definition 7.3. Let D be given from Lemma 7.2, and let $\xi \in \mathcal{EL}$ be an ending lamination on S . We define a D -cloud of a ray in the direction of ξ to be

$$\mathcal{Z}(\mathfrak{o}, \xi) := \left\{ z \in \text{Map}(S) \mid d_{\mathcal{C}(Y)}(z_Y, [\mathfrak{o}, \xi]_Y) \leq D \quad \forall Y \right\}.$$

By construction, the resolution $\mathcal{G}(\mathfrak{o}, \xi)$ of the hierarchy $H(\mathfrak{o}, \xi)$ is contained in $\mathcal{Z}(\mathfrak{o}, \xi)$. Fixing $\xi \in \mathcal{EL}$, we define a projection map

$$\Pi_\xi: \text{Map}(S) \rightarrow \mathcal{Z}(\mathfrak{o}, \xi) \quad \text{where} \quad \Pi_\xi(x) := \text{ctr}(\mathfrak{o}, x, \xi), \quad x \in \text{Map}(S).$$

We now check that Π_ξ satisfies the usual properties of a projection; in particular, it is a κ -projection according to Definition 5.1.

Lemma 7.4. *For any $\xi \in \mathcal{EL}$, the map Π_ξ is coarsely Lipschitz with respect to d_w . Furthermore, if $x \in \mathcal{Z}(\mathfrak{o}, \xi)$, then $d_w(x, \Pi_\xi(x))$ is uniformly bounded. As a consequence, Π_ξ is a κ -projection.*

Proof. Consider points $x, x' \in \text{Map}(S)$ where $d_w(x, x') \leq 1$. Then $x(\theta)$ and $x'(\theta)$ have a uniformly bounded intersection number, which implies that there exists a uniform constant $C_1 > 0$ such that

$$\forall Y, \quad d_Y(x_Y, x'_Y) \leq C_1.$$

Let $\eta := \text{ctr}(\mathfrak{o}, x, \xi)$ and $\eta' := \text{ctr}(\mathfrak{o}, x', \xi)$. Since $\mathcal{C}(Y)$ is hyperbolic, the dependence of η_Y on x_Y is Lipschitz, that is, there exists a uniform constant $C_2 > 0$ such that

$$\forall Y \subseteq S, \quad d_Y(\eta_Y, \eta'_Y) \leq C_2.$$

Now, Proposition 7.1 implies that

$$d_w(\Pi_\xi(x), \Pi_\xi(x')) \prec (C_2)^{p+1}$$

which means Π_ξ is coarsely Lipschitz. Similarly, if $x \in \mathcal{Z}(\mathfrak{o}, \xi)$ then for $\eta = \text{ctr}(\mathfrak{o}, x, \xi)$ we have $d_Y(x_Y, \eta_Y) \leq C_2$ for all subsurfaces Y and hence, $d_Y(x, \Pi_\xi(x)) \prec (C_2)^{p+1}$. \square

This projection has the following desirable property as shown by Duchin-Rafi [DR09].

Theorem 7.5 ([DR09], Theorem 4.2). *There exist constants B_1, B_2 depending on the topology of the surface S and D such that, for $x, y \in \text{Map}(S)$,*

$$d_w(x, \mathcal{Z}(\mathfrak{o}, \xi)) \geq B_1 \cdot d_w(x, y) \quad \implies \quad d_S(\Pi_\xi(x), \Pi_\xi(y)) \leq B_2.$$

In [DR09], the theorem is proven under the assumption that the geodesic (or cloud) is cobounded. However, the result holds in general: we will see in Proposition 8.5 a detailed proof for relatively hyperbolic groups, which can be easily adapted to mapping class groups.

Logarithmic projections. We now consider the set of points in \mathcal{EL} that have *logarithmically bounded projection* to all subsurfaces. Given a proper subsurface $Y \subsetneq S$, let ∂Y denote the multi-curve of boundary components of Y and define

$$\|Y\|_S := d_S(\theta, \partial Y).$$

Similarly, for $x \in \text{Map}(S)$, define

$$\|x\|_S := d_S(\theta, x(\theta)).$$

We now choose a constant $K \geq 4D$ and large enough that Eqns. (12) and (13) hold.

Definition 7.6. For a constant $c > 0$, let \mathcal{L}_c be the set of points $\xi \in \mathcal{EL}$ such that

$$(14) \quad D_{\alpha, K}(\mathfrak{o}, \xi) \leq c \cdot \log d_S(\mathfrak{o}, \alpha)$$

for every simple closed curve α in S .

Proposition 7.7. *For any $\xi \in \mathcal{L}_c$, the set $\mathcal{Z}(\mathfrak{o}, \xi)$ is κ -weakly contracting, where $\kappa(r) = \log(r)$. Furthermore, any resolution $\mathcal{G}(\mathfrak{o}, \xi)$ of the hierarchy $H(\mathfrak{o}, \xi)$ is also κ -Morse.*

Proof. In this proof, we use the notations \prec_c and O_c to mean that the implicit constants additionally depend on c . Let $\mathcal{Z} = \mathcal{Z}(\mathfrak{o}, \xi)$. Given $x, x' \in X$ where

$$B_1 \cdot d_w(x, x') < d_w(x, \mathcal{Z}),$$

let $y = \Pi_\xi(x)$, $y' = \Pi_\xi(x')$. We claim that, for every simple closed curve α ,

$$D_{\alpha, 2K}(y, y') \prec_c \log \|x\|_S.$$

Since $\mathcal{C}(S)$ is hyperbolic, nearest point projection in $\mathcal{C}(S)$ is coarsely distance decreasing, hence $\|y\|_S \prec \|x\|_S$. Also, by Theorem 7.5

$$(15) \quad d_S(y, y') \leq B_2$$

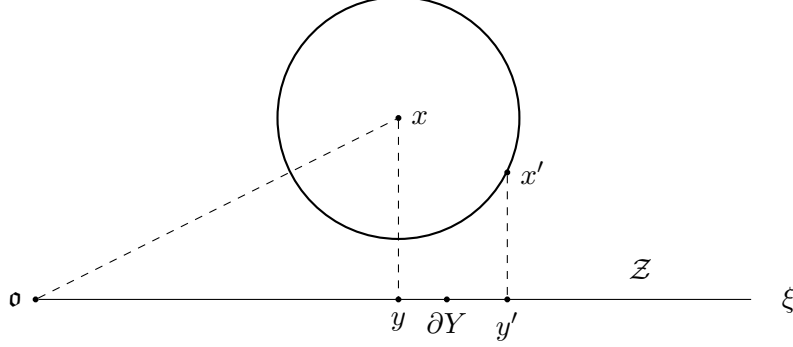


FIGURE 9. The projections of x and x' to \mathcal{Z} are κ -close.

therefore, $\|y'\|_S \prec \|x\|_S$. Which means, for every curve α in the geodesic segment $[y, y']_S$ in $\mathcal{C}(S)$ we have $d_S(o, \alpha) \prec \|x\|_S$. The Bounded Geodesic Image Theorem [MM00, Theorem 3.1] implies that if $d_Y(y, y')$ is large then $d_{\mathcal{C}(S)}([y, y']_S, \partial Y) \prec 1$, hence

$$\|Y\|_S \prec \|x\|_S.$$

By the definition of Π_ξ , y_Y and y'_Y are D -close to the geodesic segment $[o, \xi]_Y$ in $\mathcal{C}(Y)$. Now, by summing over all Y disjoint from α , and using the definition of \mathcal{L}_c ,

$$D_{\alpha, 2K}(y, y') = \sum_{Y \cap \alpha = \emptyset} [d_Y(y, y')]_{2K} \prec \sum_{Y \cap \alpha = \emptyset} [d_Y(o, \xi)]_K \prec_c \log d_S(o, \alpha) \prec \log \|x\|_S.$$

Now, we apply Eq. (13), recalling that every subsurface in the hierarchy is disjoint from some curve α in $[y, y']_S$, hence only the curves in $[y, y']_S$ contribute to the sum. Hence, in view of Equation (15), there are at most B_2 of them, so we get

$$(16) \quad |H(y, y')| \prec \sum_{\alpha \in [y, y']_S} \sum_{Y \cap \alpha = \emptyset} [d_Y(y, y')]_{2K} \prec \sum_{\alpha \in [y, y']_S} D_{\alpha, 2K}(y, y') \prec_c \log \|x\|_S$$

thus by Eq. (12)

$$d_w(y, y') \asymp |H(y, y')| \prec_c \log \|x\|_S.$$

Now, by Theorem 5.5, $\mathcal{Z}(o, \xi)$ is κ -Morse. Let $m_{\mathcal{Z}}$ be the associated Morse gauge for $\mathcal{Z}(o, \xi)$.

Now we show $\mathcal{G}(o, \xi)$ is also κ -Morse. Assume κ' and $r > 0$ be given (see Definition 3.2) and, using the fact that $\mathcal{Z}(o, \xi)$ is κ -Morse, let R be a radius such that, for any (q, Q) -quasi-geodesic ray β in $\text{Map}(S)$ with $m_{\mathcal{Z}}(q, Q)$ small compared to r , we have

$$d_w(\beta_R, \mathcal{Z}(o, \xi)) \leq \kappa'(R) \quad \implies \quad \beta|_r \subset \mathcal{N}(\mathcal{Z}(o, \xi), m_{\mathcal{Z}}(q, Q)).$$

Also, assume

$$d_w(\beta_R, \mathcal{G}(o, \xi)) \leq \kappa'(R).$$

We need to show that every $x \in \beta|_r$ is close to $\mathcal{G}(o, \xi)$.

Since $\mathcal{G}(o, \xi) \subset \mathcal{Z}(o, \xi)$ we can still conclude that there is a point $y \in \mathcal{Z}(o, \xi)$ with

$$d_w(x, y) \leq m_{\mathcal{Z}}(q, Q) \cdot \kappa(x).$$

In fact y can be taken to be $\Pi_\xi(x)$ and hence $\|y\|_S \prec \|x\|_S$. Let z be a point in $\mathcal{G}(o, \xi)$ where $d_S(z_S, y_S) \leq D$ (such a point exists since the shadow of $\mathcal{G}(o, \xi)$ to $\mathcal{C}(S)$ is the geodesic ray $[o, \xi]_S$). Since $y, z \in \mathcal{Z}(o, \xi)$, we have for every curve α on $[y, z]_S$ that

$$\sum_{Y \cap \alpha = \emptyset} [d_Y(y, z)]_{2K} \leq \sum_{Y \cap \alpha = \emptyset} [d_Y(o, \xi) + 2D]_{2K} \prec \sum_{Y \cap \alpha = \emptyset} [d_Y(o, \xi)]_K \prec_c \log d_S(o, \alpha) \prec \log \|x\|_S.$$

Therefore, by summing over all α as in (16), we have

$$d_w(y, z) \prec_c \log \|x\|_S \prec \kappa(x).$$

And hence,

$$d_w(x, z) \leq d_w(x, y) + d_w(y, z) \prec_c m_{\mathcal{Z}}(q, Q) \cdot \kappa(x).$$

We have shown

$$\beta|_r \subset \mathcal{N}\left(\mathcal{G}(\mathfrak{o}, \xi), O_c(m_{\mathcal{Z}}(q, Q))\right).$$

That is, $\mathcal{G}(\mathfrak{o}, \xi)$ is κ -Morse with a Morse gauge $m_{\mathcal{G}} = O_c(m_{\mathcal{Z}})$. \square

7.2. Convergence to the κ -Morse boundary. Let μ be a probability measure on $\text{Map}(S)$. We say that μ is *non-elementary* if the semigroup generated by its support contains two pseudo-Anosov elements with disjoint fixed sets in PMF .

Let us recall some useful facts on random walks on the mapping class group.

Theorem 7.8. *Let μ be a finitely supported, non-elementary probability measure on $\text{Map}(S)$. Then:*

- (1) *For almost every sample path $\omega = (w_n)$, the sequence (w_n) converges to a point ξ_ω in the Gromov boundary of $\mathcal{C}(S)$, which is $\mathcal{EL}(S)$.*
- (2) *Moreover, there exists $\ell > 0, c < 1$ such that*

$$\mathbb{P}(d_S(\mathfrak{o}, w_n) \geq \ell n) \geq 1 - c^n$$

for any n .

- (3) *Further, for any $k > 0$ there exists $C > 0$ such that*

$$\mathbb{P}(d_S(w_n, \gamma_\omega) \geq C \log n) \leq Cn^{-k}$$

for any n , where $\gamma_\omega = [\mathfrak{o}, \xi_\omega]_S$.

Claim (2) is proven by Maher ([Mah10], [Mah12]), while (1) and (3) are proven in [MT18]. By adapting the proof of [QRT19, Theorem A.17] (inspired by [ST19] and [Sis17, Lemma 4.4]), we obtain for the mapping class group:

Theorem 7.9. *Let μ be a finitely supported, non-elementary probability measure on $\text{Map}(S)$. Then for any $k > 0$ and for any K sufficiently large there exists $C > 0$ such that for all n we have*

$$\mathbb{P}\left(\sup_{\alpha} D_{\alpha, K}(\mathfrak{o}, w_n) \geq C \log n\right) \leq Cn^{-k},$$

where the supremum is taken over all simple closed curves α in S . As a consequence, for almost every sample path there exists $C > 0$ such that for all n

$$\sup_{\alpha} D_{\alpha, K}(\mathfrak{o}, w_n) \leq C \log n.$$

Proof. The idea of the proof is that in order to make progress with respect to $D_{\alpha, K}$, the sample path must project close to α in the curve graph $\mathcal{C}(S)$. However, linear progress with exponential decay implies that the sample path can stay close to the projection of α only for a time of order $\log n$, which completes the proof.

Let us see the details. By the bounded geodesic image theorem, there exists B (independent of Y) such that if ∂Y is far from the geodesic segment $[x, y]$ in $\mathcal{C}(S)$, namely

$$d_{\mathcal{C}(S)}(\partial Y, [x, y]) \geq 2,$$

then $d_Y(x, y) \leq B$ is uniformly bounded.

By linear progress with exponential decay (Theorem 7.8 (2)), replacing n by $A \log n$ we obtain for any $A > 0$ and $n \geq e^{2/\ell A}$

$$(17) \quad \mathbb{P}(d_S(1, w_{A \log n}) \leq 2) \leq n^{A \log c}.$$

Consider now the path of vertices $(w_i)_{i \leq n}$ in $\text{Map}(S)$, and suppose that for a curve α we have $D_{\alpha, K}(1, w_n) \geq C \log n$. Let

$$\begin{aligned} i_1 &:= \min\{0 \leq i \leq n : d_S(w_i, \alpha) \leq 1\}, \\ i_2 &:= \max\{0 \leq i \leq n : d_S(w_i, \alpha) \leq 1\}, \end{aligned}$$

and

$$D := \max\{d_w(1, g) : g \in \text{supp } \mu\}.$$

Then, for any subsurface Y disjoint from α ,

$$d_Y(1, w_n) \leq d_Y(1, w_{i_1}) + d_Y(w_{i_1}, w_{i_2}) + d_Y(w_{i_2}, w_n) \leq d_Y(w_{i_1}, w_{i_2}) + 2B$$

hence for $K \geq 4B$

$$\lfloor d_Y(1, w_n) \rfloor_K \leq \lfloor d_Y(w_{i_1}, w_{i_2}) + 2B \rfloor_K \leq 2 \lfloor d_Y(w_{i_1}, w_{i_2}) \rfloor_{K/2}$$

and, summing over Y and using the distance formula, there exists C_1 such that

$$D_{\alpha, K}(1, w_n) = \sum_{\alpha \cap Y = \emptyset} \lfloor d_Y(1, w_n) \rfloor_K \leq 2 \sum_{\alpha \cap Y = \emptyset} \lfloor d_Y(w_{i_1}, w_{i_2}) \rfloor_{K/2} \leq 2C_1 d_w(w_{i_1}, w_{i_2}) + C_1$$

thus

$$C \log n \leq D_{\alpha, K}(1, w_n) \leq 2C_1 d_w(w_{i_1}, w_{i_2}) + C_1 \leq 2C_1 D(i_2 - i_1) + C_1$$

hence, for n large enough,

$$|i_1 - i_2| \geq \frac{C \log n}{4C_1 D}.$$

Hence

$$\mathbb{P}(\exists \alpha : D_{\alpha, K}(1, w_n) \geq C \log n) \leq \mathbb{P}\left(\exists i_1 \leq i_2 \leq n, i_2 - i_1 \geq \frac{C}{4C_1 D} \log n : d_S(w_{i_1}, w_{i_2}) \leq 2\right)$$

and by (17) this is bounded above by

$$\leq n^2 \cdot n^{-\frac{C \log(1/c)}{4C_1 D}}$$

which tends to 0 for $n \rightarrow \infty$ as long as $C > \frac{8C_1 D}{\log(1/c)}$.

The second claim follows immediately from the first one for $k = 2$ by Borel-Cantelli. \square

We now prove show that almost every sample path converges to a point in the κ -Morse boundary of the mapping class group, where $\kappa(r) = \log(r)$.

Theorem 7.10. *Let μ be a finitely supported, non-elementary probability measure on $\text{Map}(S)$, and let $\kappa(r) := \log(r)$. Then:*

- (1) *Almost every sample path $\omega = (w_n)$ converges to a point in the κ -Morse boundary with respect to the topology of $X \cup \partial_\kappa X$;*
- (2) *Moreover, for almost every sample path there exists a κ -Morse geodesic ray \mathcal{G}_ω in $\text{Map}(S)$ such that*

$$\limsup_{n \rightarrow \infty} \frac{d(w_n, \mathcal{G}_\omega)}{\log n} < +\infty.$$

Proof. By Theorem 7.8, for almost every sample path, $w_n \theta$ converges to a point ξ_ω on the boundary of $\mathcal{C}(S)$ (with respect to the topology on $\mathcal{C}(S) \cup \mathcal{EL}(S)$).

Let $\mathcal{G} = \mathcal{G}(\mathfrak{o}, \xi_\omega)$ be a resolution of a hierarchy towards ξ_ω and let $\gamma_\omega := [\mathfrak{o}, \xi_\omega]_S$ be the shadow of \mathcal{G} , which is a geodesic ray in $\mathcal{C}(S)$ starting from θ and limiting to ξ_ω .

By Proposition 7.7, in order to prove that \mathcal{G} is κ -Morse, it is sufficient to prove that there is a constant c such that

$$(18) \quad D_{\alpha, K}(\mathfrak{o}, \xi_\omega) \leq c \log d_S(\mathfrak{o}, \alpha)$$

for any simple closed curve α .

Since the drift of the random walk is positive with exponential decay (Theorem 7.8 (2)), we have by the Markov property that there exists $0 < C_0 < 1$ such that

$$(19) \quad \mathbb{P}(d_S(w_n, w_{n+m}) \leq \ell m) = \mathbb{P}(d_S(\mathfrak{o}, w_m) \leq \ell m) \leq (C_0)^m \quad \forall n, m \geq 1$$

with $\ell > 0$ a constant larger than the drift of the random walk. By setting $m = A \log n$, we get

$$\mathbb{P}(d_S(w_n, w_{n+A \log n}) \leq \ell A \log n) \leq n^{A \log C_0} \quad \forall n$$

hence, if A is sufficiently large, we have

$$(20) \quad \mathbb{P}(d_S(w_n, w_{n+A \log n}) \leq \ell A \log n) \leq n^{-2} \quad \forall n.$$

By Theorem 7.8 (3), there exists $C_1 > 0$ so that

$$(21) \quad \mathbb{P}(d_S(w_n, p_n) \geq C_1 \log n) \leq C_1 n^{-2} \quad \forall n.$$

Finally, by Theorem 7.9, there exists $C_2 > 0$ so that

$$(22) \quad \mathbb{P}\left(\sup_{\alpha} D_{\alpha, K/4}(\mathfrak{o}, w_n) \geq C_2 \log n\right) \leq C_2 n^{-2} \quad \forall n.$$

Now, the complement of the union of all events expressed by (19), (20), (21), (22) has measure at least $1 - C_3 n^{-2}$ for some new C_3 . We will consider from now on a sample path in such a set.

Step 1. We claim that there exists $C_4 > 0$ such that

$$(23) \quad \mathbb{P}(d_w(w_n, c_n) \geq C_4 \log n) \leq C_4 n^{-2}$$

for all n .

Proof. Let us pick a subsurface Y . By the bounded geodesic image theorem, there exists B (independent of Y) such that if ∂Y is far from the geodesic segment $[w_n, p_n]$ in $\mathcal{C}(S)$, namely

$$d_{\mathcal{C}(S)}(\partial Y, [w_n, p_n]) \geq B,$$

then $d_Y(w_n, c_n) \leq B$ is uniformly bounded.

On the other hand, if $d_{\mathcal{C}(S)}(\partial Y, [w_n, p_n]) \leq B$, let us denote by q_1 a nearest point projection of ∂Y onto $[w_n, p_n]$ and by q_2 a nearest point projection of ∂Y onto $[w_{n+m}, p_{n+m}]$. Then

$$\begin{aligned} d_S(\partial Y, [w_{n+m}, p_{n+m}]) &= d_S(\partial Y, q_2) \geq d_S(w_n, w_{n+m}) - d_S(\partial Y, q_1) - d_S(q_1, w_n) - d_S(w_{n+m}, q_2) \\ &\geq d_S(w_n, w_{n+m}) - d_S(\partial Y, q_1) - d_S(w_n, p_n) - d_S(w_{n+m}, p_{n+m}) \end{aligned}$$

and, if we set $m = A \log n$ with $A > \frac{C_1}{\ell}$, we obtain

$$\geq \ell A \log n - B - C_1 \log n - C_1 \log(n + A \log n) > B$$

for n sufficiently large (independently of Y and ω). Hence, by the bounded geodesic image theorem,

$$d_Y(w_{n+m}, c_{n+m}) \leq B.$$

Now, since projection in a δ -hyperbolic space is coarsely distance-decreasing,

$$d_Y(c_n, c_{n+m}) \leq d_Y(w_n, w_{n+m}) + B_1.$$

where the constant B_1 is independent of Y . By putting these estimates together and by the triangle inequality we obtain

$$\begin{aligned} d_Y(w_n, c_n) &\leq d_Y(w_n, w_{n+m}) + d_Y(w_{n+m}, c_{n+m}) + d_Y(c_{n+m}, c_n) \\ &\leq 2d_Y(w_n, w_{n+m}) + B + B_1 \end{aligned}$$

hence

$$\left[d_Y(w_n, c_n) \right]_K \leq \left[2d_Y(w_n, w_{n+m}) + B + B_1 \right]_K \leq 4 \left[d_Y(w_n, w_{n+m}) \right]_{K/4}.$$

Hence, by the distance formula, there exists a constant C_5 such that

$$\begin{aligned}
d_w(w_n, c_n) &\leq C_5 \sum_{Y \subseteq S} [d_Y(w_n, c_n)]_K + C_5 \\
&\leq 4C_5 \sum_{Y \subseteq S} [d_Y(w_n, w_{n+m})]_{K/4} + C_5 \\
&\leq 4C_5(C_5 d_w(w_n, w_{n+m}) + C_5) + C_5 \\
&\leq 4C_5^2 AD \cdot \log n + (4C_5^2 + C_5)
\end{aligned}$$

completing the proof of the claim. \square

Step 2. We show that there exists a constant C_6 such that for any n we have

$$(24) \quad \mathbb{P}(\sup_{\alpha} D_{\alpha, K/2}(\mathfrak{o}, c_n) \geq C_6 \log d_S(\mathfrak{o}, p_n) + C_6) \leq C_6 n^{-2}.$$

Proof. First note that by the triangle inequality if $d_S(\mathfrak{o}, w_n) \geq \ell n$ and $d_S(w_n, p_n) \leq C_1 \log n$ then

$$\begin{aligned}
d_S(\mathfrak{o}, p_n) &\geq d_S(\mathfrak{o}, w_n) - d_S(w_n, p_n) \\
&\geq \ell n - C_1 \log n \geq \ell n/2
\end{aligned}$$

for n sufficiently large, hence by (19) and (21) there exists C_7 for which

$$\mathbb{P}(d_S(\mathfrak{o}, p_n) \geq \ell n/2) \geq 1 - C_7 n^{-2} \quad \forall n.$$

Then, since p_n is the closest point projection of w_n onto $[\mathfrak{o}, \xi_\omega]$ in the δ -hyperbolic space $\mathcal{C}(Y)$,

$$d_Y(\mathfrak{o}, c_n) \leq d_Y(\mathfrak{o}, w_n) + B_1.$$

Then, by summing over Y , for $K \geq 4B_1$,

$$\begin{aligned}
D_{\alpha, K/2}(\mathfrak{o}, c_n) &= \sum_{Y \cap \alpha = \emptyset} [d_Y(\mathfrak{o}, c_n)]_{K/2} \leq \sum_{Y \cap \alpha = \emptyset} [d_Y(\mathfrak{o}, w_n) + B_1]_{K/2} \\
&\leq \sum_{Y \cap \alpha = \emptyset} 2[d_Y(\mathfrak{o}, w_n)]_{K/4} \\
&\leq 2D_{\alpha, K/4}(\mathfrak{o}, w_n)
\end{aligned}$$

and by Equation (22)

$$\begin{aligned}
&\leq 2C_2 \log n \\
&\leq 2C_2 \log d_S(\mathfrak{o}, p_n) + 2C_2 \log(2/\ell) \\
&\leq C_6 \log d_S(\mathfrak{o}, p_n) + C_6
\end{aligned}$$

for the appropriate choice of C_6 . \square

Step 3. For almost every ω , there exists $C_8(\omega)$ such that,

$$(25) \quad D_{\alpha, K}(\mathfrak{o}, \xi_\omega) \leq C_8(\omega) \log d_S(\mathfrak{o}, \alpha)$$

for every simple closed curve α on S .

Proof. By Step 2. and Borel-Cantelli, for almost every ω there exists $n_0 = n_0(\omega)$ such that

$$(26) \quad \sup_{\alpha} D_{\alpha, K/2}(\mathfrak{o}, c_n) \leq C_6 \log d_S(\mathfrak{o}, p_n) + C_6$$

for any $n \geq n_0$.

Now, observe that the random walk is finitely supported, so $d_S(w_n, w_{n+1})$ is uniformly bounded, hence, since nearest point projection is coarsely distance decreasing, there exists $D_1 > 0$ such that

$$d_S(p_n, p_{n+1}) \leq D_1$$

is also uniformly bounded.

Now, let α be a simple closed curve, and Y a subsurface disjoint from α . By the bounded geodesic image theorem, $d_Y(\mathfrak{o}, \xi_\omega) \leq B$ unless α lies in a B -neighborhood of $[\mathfrak{o}, \xi_\omega]_S$.

Note that by minimality, $d_S(\mathfrak{o}, p_n) - D_1 \leq d_S(\mathfrak{o}, p_{n-1}) \leq d_S(\mathfrak{o}, \alpha) + B \leq d_S(\mathfrak{o}, p_n)$, hence $d_S(\mathfrak{o}, p_n) \leq d_S(\mathfrak{o}, \alpha) + D_1 + B$.

Moreover, by the bounded geodesic image theorem,

$$|d_Y(\mathfrak{o}, \xi_\omega) - d_Y(\mathfrak{o}, c_n)| \leq B$$

since the projection of $[p_n, \xi]_S$ to $\mathcal{C}(Y)$ is uniformly bounded. Hence by (26), if $n \geq n_0(\omega)$ then

$$d_Y(\mathfrak{o}, \xi_\omega) \leq d_Y(\mathfrak{o}, c_n) + B$$

so for $K \geq 2B$ we obtain

$$\begin{aligned} D_{\alpha, K}(\mathfrak{o}, \xi_\omega) &= \sum_{Y \cap \alpha = \emptyset} [d_Y(\mathfrak{o}, \xi_\omega)]_K \leq 2 \sum_{Y \cap \alpha = \emptyset} [d_Y(\mathfrak{o}, c_n)]_{K/2} \\ &= 2D_{\alpha, K/2}(\mathfrak{o}, c_n) \\ &\leq 2C_6 \log d_S(\mathfrak{o}, p_n) + 2C_6 \\ &\leq 2C_6 \log(d_S(\mathfrak{o}, \alpha) + D_1 + B) + 2C_6. \end{aligned}$$

On the other hand, if $n(Y, \omega) \leq n_0(\omega)$ then $d_S(\mathfrak{o}, \alpha) \leq d_S(\mathfrak{o}, p_n) \leq d_S(\mathfrak{o}, p_{n_0(\omega)})$, and there are at most finitely many such subsurfaces Y for which the projection of $[\mathfrak{o}, \xi_\omega]_S$ onto $\mathcal{C}(Y)$ is large. Hence

$$\sup_{Y: n(Y, \omega) \leq n_0} d_Y(\mathfrak{o}, \xi_\omega) < +\infty,$$

thus the claim follows by adjusting the constant to a constant $C_8(\omega)$, which depends on ω , to take into account the initial part. \square

Let us now prove the second claim, namely the tracking estimate between the sample path and the geodesic ray. From that and Lemma 4.3, it follows that almost every sample path converges in the topology of $X \cup \partial_\kappa X$.

For a given sample path $\omega = (w_n)$, let $\xi \in \mathcal{EL}(S)$ be the ending lamination the path (w_n) is converging to, and let $c_n := \text{ctr}(\mathfrak{o}, w_n, \xi)$. Note that by construction the projection of c_n onto $\mathcal{C}(S)$ is within uniformly bounded distance of p_n . By Step 1, there exists $C_4 > 0$ such that

$$(27) \quad \mathbb{P}(d_w(w_n, c_n) \geq C_4 \log n) \leq C_4 n^{-2}$$

for any n . As a consequence, the Borel-Cantelli lemma implies that for almost every $\omega \in \Omega$ there exists n_0 such that

$$(28) \quad d_w(w_n, c_n) \leq C_4 \log n$$

for all $n \geq n_0$. The claim follows by noting that $d_w(w_n, \mathcal{G}) \leq d_w(w_n, c_n) + O(1)$. \square

We now complete the proof of Theorem C by identifying the κ -Morse boundary with the Poisson boundary.

Theorem 7.11. *Let μ be a non-elementary, finitely supported measure on $G = \text{Map}(S)$. Then for $\kappa(r) := \log(r)$, the κ -Morse boundary is a model for the Poisson boundary of (G, μ) .*

Proof. By Theorem 7.10, almost every sample path sublinearly tracks a κ -Morse geodesic ray, with $\kappa(r) = \log(r)$, so we can apply Theorem 6.2. \square

8. RELATIVELY HYPERBOLIC GROUPS

8.1. Background. Let G be a finitely generated group, and let P_1, P_2, \dots, P_k be a set of subgroups of G which we call *peripheral subgroups*. Fix a finite generating set S . Let (G, d_G) denote the Cayley graph of G with respect to S , equipped with the word metric. Following [Sis13], we denote by $(\widehat{G}, d_{\widehat{G}})$ the metric graph obtained from the Cayley graph of G by adding an edge between every pair of distinct vertices contained in a left coset P of a peripheral subgroup. Setting up this way, G and \widehat{G} have the same vertex set, so any set in G can also be considered as a set in \widehat{G} . However, for $x, y \in G$, $d_{\widehat{G}}(x, y) \leq d_G(x, y)$. In this Section, $[x, y]$ will denote the geodesic segment between x, y in the metric $d_{\widehat{G}}$ and $\llbracket x, y \rrbracket$ the geodesic segment in d_G .

Definition 8.1. A group G is *relatively hyperbolic*, relative to peripheral subgroups P_1, P_2, \dots, P_k , if the graph \widehat{G} has the properties:

- It is δ -hyperbolic;
- It is *fine*: for each integer n , every edge belongs to only finitely many simple cycles of length n .

The collection of all cosets of peripheral subgroups is denoted by \mathcal{P} . We shall denote as $\mathcal{N}_D(P)$ the D -neighborhood of P in the metric d_G .

Moreover, we say a relatively hyperbolic group is *non-elementary* if it is infinite, not virtually cyclic, and each P_i is infinite and with infinite index in G . We now fix a non-elementary relatively hyperbolic group G and a generating set. We shall use the symbols \prec and \asymp as before, where the implicit constants depend only on G and the generating set; we shall write \asymp_K if the implicit constants additionally depend on K . By definition of relative hyperbolicity, the graph \widehat{G} is δ -hyperbolic for some $\delta > 0$. Moreover, for $P \in \mathcal{P}$ we let $\pi_P : G \rightarrow P$ be a nearest point projection to P , and denote

$$d_P(x, y) := d_G(\pi_P(x), \pi_P(y)).$$

We need some properties of relatively hyperbolic groups from [Sis13], [Sis17]. A *lift* of a geodesic ray in \widehat{G} is a path in G obtained by substituting edges labeled by an element of some P_i , and possibly their endpoints, with a geodesic in the corresponding left coset. Given D, R and a geodesic segment γ in G , we define the set $\text{deep}_{D,R}(\gamma)$ as the set of points p of γ that belong to some subgeodesic $\llbracket x_1, y_1 \rrbracket$ of γ with endpoints in $\mathcal{N}_D(P)$ for some $P \in \mathcal{P}$ and with $d_G(x_1, p) > R, d_G(y_1, p) > R$. A point which does not belong to $\text{deep}_{D,R}(\gamma)$ is called a *transition point*.

Proposition 8.2. *Let G be a relatively hyperbolic group, and fix a generating set. Then we have:*

- (1) (Coarse lifting property, [Sis13, Proposition 1.14]) *There exist uniform constants $q_0, Q_0 > 0$ such that if α is a geodesic in \widehat{G} , then its lifts are (q_0, Q_0) -quasi-geodesics in G .*
- (2) (Distance formula, [Sis13, Theorem 0.1]) *There exists K_0 such that, for any $K \geq K_0$ and every pair of points $x, y \in G$,*

$$d_G(x, y) \asymp_K \sum_{P \in \mathcal{P}} [d_P(x, y)]_K + d_{\widehat{G}}(x, y).$$

- (3) (Bounded geodesic image theorem, [Sis13, Lemma 1.15]) *There exists $L_0 \geq 0$ such that, if $d_P(x, y) \geq L_0$ for some $P \in \mathcal{P}$, all geodesics in \widehat{G} connecting x to y contain an edge in P . Moreover, for every $\mathbf{q}, \mathbf{Q} > 0$ there is $D = D(\mathbf{q}, \mathbf{Q})$ such that every (\mathbf{q}, \mathbf{Q}) -quasi-geodesics in G connecting x and y intersect the balls of radius D , $B_D(\pi_P(x))$ and $B_D(\pi_P(y))$. Let $D_0 = D(1, 0)$ be the constant associated to geodesics.*
- (4) (Deep components, [Sis17, Lemma 3.3]) *There exist D, t, R such that for any $x, y \in G$, the set $\text{deep}_{D,R}(\llbracket x, y \rrbracket)$ is contained in a disjoint union of subgeodesics of $\llbracket x, y \rrbracket$, each contained*

in $\mathcal{N}_{tD}(P)$ for some $P \in \mathcal{P}$. We call each subgeodesic a deep component of $\llbracket x, y \rrbracket$ along P . Moreover, if $d_P(x, y) \geq L_0$, then $\llbracket x, y \rrbracket$ contains a deep component along P .

Let $\mathfrak{o} \in G$ be the vertex representing the identity element and consider an infinite geodesic ray γ in (G, d_G) starting at \mathfrak{o} . For $x \in G$, define $\|x\|_{\widehat{G}} := d_{\widehat{G}}(\mathfrak{o}, x)$. Also, for $P \in \mathcal{P}$, define $\|P\|_{\widehat{G}} := d_{\widehat{G}}(\mathfrak{o}, P)$. We will include in our κ -Morse boundary the geodesic rays which have excursion in each peripheral set bounded by a multiple of κ of its \widehat{G} -norm. To be precise, we have the following.

Definition 8.3. Let D_0 be given by Proposition 8.2 (3). We say that γ has κ -excursion with respect to \mathcal{P} if there exists a constant E_γ such that, for each $P \in \mathcal{P}$, we have

$$\text{diam}_G(\gamma \cap \mathcal{N}_{D_0}(P)) \leq E_\gamma \cdot \kappa(\|P\|_{\widehat{G}}),$$

where $\mathcal{N}_{D_0}(P)$ is the D_0 -neighborhood of P in (G, d_G) . That is, the amount of time γ stays near P grows sublinearly with $\|P\|_{\widehat{G}}$.

Our goal is to prove that if γ has κ -excursion, then it is κ -Morse. In fact, we will first show that γ is κ -weakly contracting (see Definition 5.3) and then use Theorem 5.4 to conclude that γ is κ -Morse.

Let γ be a geodesic ray in G , and let us define $\pi_\gamma: G \rightarrow \gamma$ to be a nearest point projection onto γ in the metric $d_{\widehat{G}}$. Note that this is well-defined up to bounded distance in \widehat{G} , but we will fix one such choice for the remainder of this Section.

By hyperbolicity of \widehat{G} , there exist L_1, R_0 (depending on δ) such that the following holds (see e.g. [Mah10, Proposition 3.4]): for any $x, y \in G$, if $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L_1$, the geodesic $[x, y]$ in \widehat{G} and the broken geodesic $\gamma' = [x, \pi_\gamma(x)] \cup [\pi_\gamma(x), \pi_\gamma(y)] \cup [\pi_\gamma(y), y]$ lie within a R_0 -neighborhood of each other in the metric $d_{\widehat{G}}$. Moreover, any geodesic segment $[x, \bar{x}]$ and $[y, \bar{y}]$, with $[\bar{x}, \bar{y}]$ a subsegment of $[\pi_\gamma(x), \pi_\gamma(y)]$, belongs to an R_0 -neighborhood of γ' .

We denote as $\mathcal{P}_{\gamma, x}$ the set of $P \in \mathcal{P}$ such that P (which has diameter 1 in \widehat{G}) lies within $d_{\widehat{G}}$ -distance $R_1 := 1 + 4R_0$ of $\pi_\gamma(x)$. Now, define a projection $\Pi_\gamma: G \rightarrow \gamma$ by

$$\Pi_\gamma(x) := \bigcup_{P \in \mathcal{P}_{\gamma, x}} (\gamma \cap \mathcal{N}_{D_0}(P))$$

if $\mathcal{P}_{\gamma, x} \neq \emptyset$, and $\Pi_\gamma(x) := \pi_\gamma(x)$ otherwise. As usual, the image of Π_γ is a subset of γ that could have a large diameter.

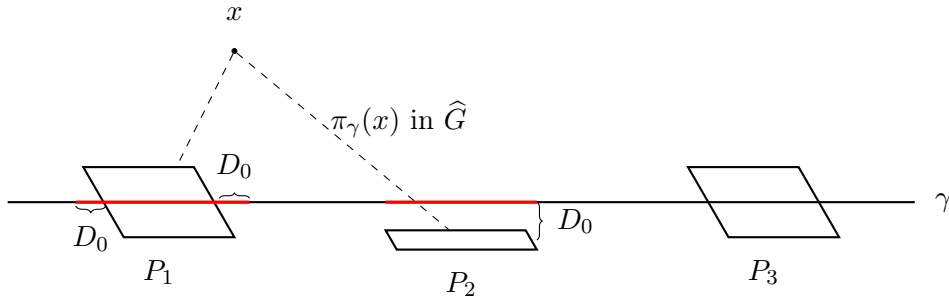


FIGURE 10. Definition of $\Pi_\gamma(x)$. Suppose P_1, P_2 and P_3 are peripheral sets that are within distance D_0 of γ in G , and that, out of the three, $P_1, P_2 \in \mathcal{P}_{\gamma, x}$; then we consider the union of intersections of γ with the D_0 -neighborhoods of P_1 and P_2 , which are illustrated in red.

Finally, given $x, y \in G$, we define $c_{\gamma, y}(x)$ to be the point of γ closest to $\pi_\gamma(x)$ such that the segment $\llbracket \pi_\gamma(x), c_{\gamma, y}(x) \rrbracket$ contains all deep components of all P in $\mathcal{P}_{\gamma, x}$.

Lemma 8.4. *There exists $L > 0$ such that the following holds. Let $x, y \in G$, and let γ be a geodesic ray based at \mathfrak{o} , and suppose that $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L$. Let $x_\gamma := c_{\gamma, y}(x)$.*

(1) *Then we have*

$$d_{\widehat{G}}(x, x_\gamma) \leq d_{\widehat{G}}(x, y) + L.$$

(2) *For any $P \in \mathcal{P}$, if $d_P(x, x_\gamma) \geq L$, we have*

$$d_P(x, x_\gamma) \leq d_P(x, y) + L.$$

Proof. Let $y_\gamma := \pi_\gamma(y)$.

(1) By the choice of L_1, R_0 , if $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L_1$, the geodesic $\gamma_1 := [x, y]$ in \widehat{G} and the broken geodesic

$$\gamma_2 := [x, \pi_\gamma(x)] \cup [\pi_\gamma(x), \pi_\gamma(y)] \cup [\pi_\gamma(y), y]$$

lie in a R_0 -neighborhood of each other for the metric $d_{\widehat{G}}$. Let p_1, p_2 be nearest point projections (in \widehat{G}), respectively, of x_γ and y_γ onto $[x, y]$. This implies

$$(29) \quad d_{\widehat{G}}(x, y) \geq d_{\widehat{G}}(x, p_1) \geq d_{\widehat{G}}(x, x_\gamma) - R_0$$

which proves (1), as long as $L \geq R_0$.

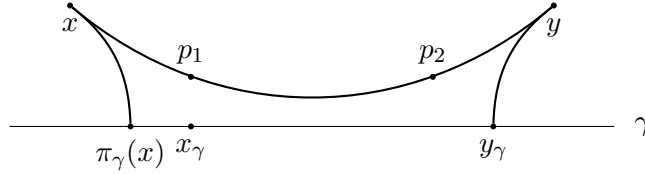


FIGURE 11. A thin quadrilateral in \widehat{G} .

(2) To prove (2), suppose that $d_P(x, x_\gamma) \geq 2L_0$ is large, where L_0 is given by Proposition 8.2 (3). First, we claim that $d_P(x_\gamma, y_\gamma) \leq L_0$. By the triangle inequality,

$$d_P(x, x_\gamma) \leq d_P(x, \pi_\gamma(x)) + d_P(\pi_\gamma(x), x_\gamma)$$

hence there are two cases: either $d_P(x, \pi_\gamma(x)) \geq L_0$ or $d_P(\pi_\gamma(x), x_\gamma) \geq L_0$.

If $d_P(\pi_\gamma(x), x_\gamma) \geq L_0$, then by Proposition 8.2 (4), the geodesic segment $[[\pi_\gamma(x), x_\gamma]]$ in G contains a deep component along P . Then, since x_γ is a transition point and deep components are disjoint, the segment $[[x_\gamma, y_\gamma]]$ in G has no deep component along P . This implies by Proposition 8.2 (4) that $d_P(x_\gamma, y_\gamma) \leq L_0$, as claimed.

Otherwise, we can assume that $d_P(\pi_\gamma(x), x_\gamma) \leq L_0$ and $d_P(x, \pi_\gamma(x)) \geq L_0$. First, by the bounded geodesic image theorem, the geodesic segment $[x, \pi_\gamma(x)]$ in \widehat{G} contains an edge in P ; let $p \in P$ be a vertex of this edge. Now, by contradiction, suppose that $d_P(x_\gamma, y_\gamma) \geq L_0$; then, by the bounded geodesic image theorem, $[x_\gamma, y_\gamma]$ also contains an edge in P . Thus, there exists $p' \in [x_\gamma, y_\gamma] \cap P$ with $d_{\widehat{G}}(p, p') \leq 1$. Thus, using that γ_1 and γ_2 lie in a R_0 -neighborhood of each other,

$$d_{\widehat{G}}(p, \pi_\gamma(x)) \leq d_{\widehat{G}}(p, p') + 4R_0 \leq 1 + 4R_0 = R_1.$$

Hence, P belongs to $\mathcal{P}_{\gamma, x}$, which, as above, implies $d_P(x_\gamma, y_\gamma) \leq L_0$.

We now claim that

$$d_{\widehat{G}}([x, x_\gamma], [y, y_\gamma]) \geq 2,$$

which again by Proposition 8.2 (3) implies $d_P(y, y_\gamma) \leq L_0$. Indeed, let q_1 be a point in $[x, x_\gamma]$ and q_2 be a point in $[y, y_\gamma]$; by hyperbolicity, $d_{\widehat{G}}(q_1, [x, p_1]) \leq R_0$ and also $d_{\widehat{G}}(q_2, [y, p_2]) \leq R_0$; hence,

$$d_{\widehat{G}}(q_1, q_2) \geq d_{\widehat{G}}(p_1, p_2) - 2R_0 \geq d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) - 2R_2 \geq L - 2R_2 \geq 2$$

where $R_2 = 2R_0 + D_0 + R_1$, provided that we choose $L \geq 2R_2 + 2$.

Finally, by the triangle inequality

$$\begin{aligned} d_P(x, y) &\geq d_P(x, x_\gamma) - d_P(x_\gamma, y_\gamma) - d_P(y_\gamma, y) \\ &\geq d_P(x, x_\gamma) - 2L_0. \end{aligned}$$

Thus, if we choose $L := \max\{L_1, R_0, 2R_2 + 2, 2L_0\}$, both (1) and (2) hold. \square

Proposition 8.5. *Let γ be a geodesic ray with κ -excursion. Then, the map Π_γ defined above is a κ -projection map. Furthermore, there exist $D_1 < 1, D_2 > 1$ such that for any two points $x, y \in G$ we have*

$$d_G(x, y) \leq D_1 \cdot d_G(x, \gamma) \quad \implies \quad d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \leq D_2.$$

Proof. We now fix L as given by Lemma 8.4, and we start by contradiction, by assuming that $d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \geq L$. By Lemma 8.4, $d_{\widehat{G}}(x, x_\gamma) \prec d_{\widehat{G}}(x, y)$, and $d_P(x, x_\gamma) \prec d_P(x, y)$ whenever $d_P(x, x_\gamma)$ is large enough. Now, applying the Distance formula (Proposition 8.2 (2)) to the pair of points (x, y) we have:

$$\begin{aligned} d_G(x, y) &\asymp \sum_{P \in \mathcal{P}} [d_P(x, y)]_L + d_{\widehat{G}}(x, y) \\ &\succ \sum_{P \in \mathcal{P}} [d_P(x, x_\gamma)]_L + d_{\widehat{G}}(x, x_\gamma) - O(\delta) \quad \text{by Eq. (29)} \\ &\asymp d_G(x, x_\gamma). \end{aligned}$$

That is to say, there exists $D_1 = D_1(L, \delta)$ such that

$$d_G(x, y) \geq D_1 \cdot d_G(x, x_\gamma),$$

which is a contradiction since $x_\gamma \in \gamma$. Therefore, setting $D_2 = L$ yields

$$d_G(x, y) \leq D_1 \cdot d_G(x, \gamma) \quad \implies \quad d_{\widehat{G}}(\pi_\gamma(x), \pi_\gamma(y)) \leq D_2. \quad \square$$

We now show that every κ -excursion geodesic ray is κ -weakly contracting.

Proposition 8.6. *Every κ -excursion geodesic ray $\gamma \in (G, d_G)$ is κ -weakly contracting. That is to say, there exists $D_1 < 1, D_2 > 1$ such that for any two points $x, y \in G$ we have*

$$d_G(x, y) \leq D_1 \cdot d_G(x, \gamma) \quad \implies \quad \text{diam}_G(\Pi_\gamma(x) \cup \Pi_\gamma(y)) \leq D_2 \cdot \kappa(x).$$

As a consequence, every κ -excursion geodesic ray is κ -Morse.

Proof. By Proposition 8.5, $d_{\widehat{G}}(x_\gamma, y_\gamma) \leq D_2$, so there are only boundedly many P intersecting $[x_\gamma, y_\gamma]$ in \widehat{G} ; let \mathcal{P}_0 denote the set of such P . By definition of κ -excursion, for each $P \in \mathcal{P}_0$, $d_P(x_\gamma, y_\gamma)$ is bounded above by $\kappa(\|P\|_{\widehat{G}})$.

We claim that for all $P \in \mathcal{P}_0$ we have $\|P\|_{\widehat{G}} \prec \|x\|$, hence also $\kappa(\|P\|_{\widehat{G}}) \prec \kappa(x)$. Indeed, since P intersects $[x_\gamma, y_\gamma]$ and $d_{\widehat{G}}(x_\gamma, y_\gamma)$ is bounded,

$$\|P\|_{\widehat{G}} = d_{\widehat{G}}(\mathfrak{o}, P) \prec d_{\widehat{G}}(\mathfrak{o}, x_\gamma);$$

then, since nearest point projection in the δ -hyperbolic space \widehat{G} is coarsely distance decreasing, $d_{\widehat{G}}(\mathfrak{o}, x_\gamma) \prec d_{\widehat{G}}(\mathfrak{o}, x)$, and finally

$$d_{\widehat{G}}(\mathfrak{o}, x) \prec d_G(\mathfrak{o}, x)$$

since the inclusion $G \rightarrow \widehat{G}$ is Lipschitz.

Thus, the claim together with the previous estimates and the distance formula yields

$$d_G(x_\gamma, y_\gamma) \asymp \sum_{P \in \mathcal{P}} [d_P(x_\gamma, y_\gamma)]_L \prec \sum_{P \in \mathcal{P}_0} \kappa(\|P\|_{\widehat{G}}) \prec D_2 \cdot \kappa(x).$$

Finally, by Theorem 5.4, a κ -weakly contracting geodesic ray is κ -Morse. \square

We now show that the map Π_γ is a κ -projection.

Proposition 8.7. *Let γ be a geodesic ray with κ -excursion. Then, the map Π_γ defined above is a κ -projection map.*

Proof. Let $x \in X$ and $z \in \gamma$, and let $x_\gamma := c_{\gamma,z}(x)$. Then, if $d_{\widehat{G}}(x_\gamma, z) \geq L$, we have, as in the proof of Lemma 8.4, that $d_{\widehat{G}}(x_\gamma, z) \prec d_{\widehat{G}}(x, z)$ and $d_P(x_\gamma, z) \prec d_P(x, z)$ for any $P \in \mathcal{P}$, hence by the Distance formula (Proposition 8.2 (2))

$$d_G(x_\gamma, z) \prec d_G(x, z).$$

On the other hand, if $d_{\widehat{G}}(x_\gamma, z) \leq L$, then, as in the proof of Proposition 8.6,

$$d_G(x_\gamma, z) \prec \kappa(\|x_\gamma\|_{\widehat{G}}) \prec \kappa(\|x\|_{\widehat{G}}).$$

Moreover, since γ has κ -excursion, we have $\text{diam}_G(\Pi_\gamma(x)) \prec E_\gamma \cdot \kappa(\|x_\gamma\|_{\widehat{G}}) \prec E_\gamma \cdot \kappa(\|x\|_{\widehat{G}})$. Thus, we obtain

$$\text{diam}_G(\Pi_\gamma(x) \cup \{z\}) \leq C_1 \cdot d_G(x, z) + C_2 \cdot \kappa(x),$$

where C_1, C_2 depend only on γ , completing the proof. \square

The following is our main result on relatively hyperbolic groups.

Theorem 8.8. *Let μ be a finitely supported probability measure on a relatively hyperbolic group G , and let S be a finite generating set. Let $\kappa(r) := \log r$, and let $\partial_\kappa X$ be the κ -Morse boundary of the Cayley graph X of G with respect to S . Then:*

- (1) *Almost every sample path (w_n) converges to a point in $\partial_\kappa X$;*
- (2) *The pair $(\partial_\kappa X, \nu)$, where ν is the hitting measure of the random walk on $\partial_\kappa X$, is a model for the Poisson boundary of (G, μ) .*

Proof. The proof is fairly similar to the proof of Theorem 7.10 for the mapping class group, after replacing subsurfaces Y by peripherals $P \in \mathcal{P}$, d_Y by d_P , and the curve complex $\mathcal{C}(S)$ by \widehat{G} . Note, however, that one difference is that there need not be a hyperbolic space analogous to $\mathcal{C}(Y)$ for each $P \in \mathcal{P}$; moreover, the concept of *center* is not well-defined. Essentially, the only step where the proof as written does not immediately generalize is the proof of Eq. (??), where we do *not* know that nearest point projections are distance decreasing (not even coarsely). We shall give an alternative proof of this point, not using the hyperbolicity of $\mathcal{C}(Y)$.

By Theorem 7.8, almost every sample path $\omega = (w_n)$ converges to a point ξ_ω in the Gromov boundary of \widehat{G} . Consider a geodesic ray in \widehat{G} joining the base point \mathfrak{o} and ξ_ω , and let $\gamma = \gamma_\omega$ be a lift to G .

Let $c_n := c_{\gamma, w_{2n}}(w_n)$, following the notation before Lemma 8.4. We replace Step 1 in the proof of Theorem 7.10 with the following.

Step 1. We claim that there exists $C > 0$ such that

$$(30) \quad \mathbb{P}(\sup_P d_P(w_n, c_n) \geq C \log n) \leq Cn^{-2}$$

for all n .

Proof. Since the drift of the random walk is positive with exponential decay (Theorem 7.8 (2)), we have by the Markov property that there exists $0 < C_0 < 1$ such that

$$(31) \quad \mathbb{P}(d_{\widehat{G}}(w_n, w_{2n}) \leq \ell n) = \mathbb{P}(d_{\widehat{G}}(\mathfrak{o}, w_n) \leq \ell n) \leq (C_0)^n \quad \forall n$$

where $\ell > 0$ is the drift of the random walk. By Theorem 7.8 (3), and recalling that c_n projects close to, there exists $C_1 > 0$ so that

$$(32) \quad \mathbb{P}(d_{\widehat{G}}(w_n, c_n) \geq C_1 \log n) \leq C_1 n^{-2} \quad \forall n.$$

If a sample path lies in the complement of the union of the events expressed by (31), (32), we have

$$\begin{aligned} d_{\widehat{G}}(c_n, c_{2n}) &\geq d_{\widehat{G}}(w_n, w_{2n}) - d_{\widehat{G}}(w_n, c_n) - d_{\widehat{G}}(w_{2n}, c_{2n}) \\ &\geq \ell n - C_1 \log n - C_2 \log(2n) \geq L \end{aligned}$$

where L is given by Lemma 8.4, hence, by Lemma 8.4, we have

$$(33) \quad d_P(w_n, c_n) \leq d_P(w_n, w_{2n}) + L$$

for any $P \in \mathcal{P}$. Moreover, by [Sis17, Lemma 4.4], there exists $C_2 > 0$ for which

$$(34) \quad \mathbb{P}(\sup_P d_P(w_n, w_{2n}) \geq C_2 \log n) = \mathbb{P}(\sup_P d_Y(\mathfrak{o}, w_n) \geq C_2 \log n) \leq C_2 n^{-2} \quad \forall n.$$

hence, by combining Eq. (33) and (34) we obtain (30). \square

Then, we proceed exactly as in Theorem 7.10 (Steps 2 and 3), proving that for almost every $\omega \in \Omega$, there is a constant c such that

$$(35) \quad \sup_{P \in \mathcal{P}} \text{diam}_G(\gamma \cap \mathcal{N}_{D_0}(P)) \leq c \log d_{\widehat{G}}(\mathfrak{o}, P).$$

Hence, by Proposition 8.6, the geodesic ray γ_ω has κ -excursion, hence it is κ -Morse. This shows (1).

Finally, (2) follows by Theorem 6.1 using Theorem 6.2. \square

APPENDIX A. GENERAL PROJECTION AND WEAKLY κ -CONTRACTING PROPERTY

We begin this appendix by proving the following, announced in Section 5.

Theorem A.1 (κ -weakly contracting implies sublinearly Morse). *Let κ be a concave sublinear function and let Z be a closed subspace of X . Let π_Z be a κ -projection onto Z and suppose that Z is κ -weakly contracting with respect to π_Z . Then, there is a function $m_Z: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that, for every constant $r > 0$ and every sublinear function κ' , there is an $R = R(Z, r, \kappa') > 0$ where the following holds: Let $\eta: [0, \infty) \rightarrow X$ be a $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray so that $m_Z(\mathfrak{q}, \mathfrak{Q})$ is small compared to r , let t_r be the first time $\|\eta(t_r)\| = r$ and let t_R be the first time $\|\eta(t_R)\| = R$. Then*

$$d_X(\eta(t_R), Z) \leq \kappa'(R) \implies \eta([0, t_r]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathfrak{q}, \mathfrak{Q})).$$

Proof. Let C_1, C_2, D_1, D_2 be the constants which appear in the definitions of κ -projection and κ -weakly contracting (Definition 5.1 and Definition 5.3). Note that the condition of being κ -weakly contracting becomes weaker as C_1 gets smaller, hence we can assume that $C_1 \leq 1/2$. We first set

$$(36) \quad m_0 := \max \left\{ \frac{\mathfrak{q}(C_2 + \mathfrak{q} + 1) + \mathfrak{Q}}{C_1}, \frac{2C_2(D_1 + 1)}{(\mathfrak{q} - 1)}, \mathfrak{Q} \right\}, \quad m_1 := \mathfrak{q}(C_2 + 1)(D_1 + 1).$$

Claim A.2. Consider a time interval $[s, s']$ during which η is outside of $\mathcal{N}_\kappa(Z, m_0)$. Then there exists a constant \mathfrak{A} depending only on $\{C_1, C_2, D_1, D_2, \mathfrak{q}, \mathfrak{Q}\}$, such that

$$(37) \quad |s' - s| \leq m_1(d_X(\eta(s), Z) + d_X(\eta(s'), Z)) + \mathfrak{A} \cdot \kappa(\eta(s')).$$

Proof of Claim A.2. Let

$$s = t_0 < t_1 < t_2 < \dots < t_\ell = s'$$

be a sequence of times such that, for $i = 0, \dots, \ell - 2$, we have t_{i+1} is a first time after t_i where

$$(38) \quad d_X(\eta(t_i), \eta(t_{i+1})) = C_1 d_X(\eta(t_i), Z) \quad \text{and} \quad d_X(\eta(t_{\ell-1}), \eta(t_\ell)) \leq C_1 d_X(\eta(t_{\ell-1}), Z).$$

To simplify the notation, we define

$$\eta_i := \eta(t_i), \quad r_i := \|\eta(t_i)\|$$

and moreover, we pick some $\pi_i \in \pi_Z(\eta_i)$ and let

$$d_i^\pi := d_X(\eta_i, \pi_i), \quad d_i := d_X(\eta_i, Z).$$

Note that, by assumption

$$(39) \quad d_i^\pi \geq d_i = d_X(\eta_i, Z) \geq m_0 \cdot \kappa(r_i).$$

Claim A.3. We have the inequality $d_{\ell-1}^\pi \leq 2(D_1 + 1)d_\ell^\pi + D_2 \cdot \kappa(\eta_{\ell-1})$.

Proof of Claim A.3. By the triangle inequality and Eq. (38),

$$\begin{aligned} d_X(\eta_{\ell-1}, Z) &\leq d_X(\eta_{\ell-1}, \eta_\ell) + d_X(\eta_\ell, Z) \\ &\leq C_1 d_X(\eta_{\ell-1}, Z) + d_X(\eta_\ell, Z) \end{aligned}$$

hence, using $C_1 \leq 1/2$,

$$d_X(\eta_{\ell-1}, Z) \leq \frac{1}{1 - C_1} d_X(\eta_\ell, Z) \leq 2d_X(\eta_\ell, Z).$$

Thus, by Lemma 5.2,

$$d_{\ell-1}^\pi \leq (D_1 + 1)d_X(\eta_{\ell-1}, Z) + D_2 \cdot \kappa(\eta_{\ell-1})$$

and by the above equation

$$\begin{aligned} &\leq 2(D_1 + 1)d_X(\eta_\ell, Z) + D_2 \cdot \kappa(\eta_{\ell-1}) \\ &\leq 2(D_1 + 1)d_\ell^\pi + D_2 \cdot \kappa(\eta_{\ell-1}). \end{aligned} \quad \blacksquare$$

Now, since Z is κ -weakly contracting, by Definition 5.3 we get

$$d_X(\pi_0, \pi_\ell) \leq \sum_{i=0}^{\ell-1} d_X(\pi_i, \pi_{i+1}) \leq \sum_{i=0}^{\ell-1} C_2 \cdot \kappa(r_i).$$

But η is $(\mathfrak{q}, \mathbb{Q})$ -quasi-geodesic, hence,

$$(40) \quad \begin{aligned} |s' - s| &\leq \mathfrak{q} d_X(\eta_0, \eta_\ell) + \mathbb{Q} \\ &\leq \mathfrak{q} (d_0^\pi + d_X(\pi_0, \pi_\ell) + d_\ell^\pi) + \mathbb{Q} \\ &\leq \mathfrak{q} C_2 \left(\sum_{i=0}^{\ell-1} \kappa(r_i) \right) + \mathfrak{q} (d_0^\pi + d_\ell^\pi) + \mathbb{Q}. \end{aligned}$$

On the other hand,

$$|s' - s| = \sum_{i=0}^{\ell-1} |t_{i+1} - t_i| \geq \frac{1}{\mathfrak{q}} \sum_{i=0}^{\ell-1} (d_X(\eta_i, \eta_{i+1}) - \mathbb{Q}).$$

Meanwhile, for $i = 0, \dots, \ell - 2$ we have $d_X(\eta_i, \eta_{i+1}) = C_1 d_X(\eta_i, Z)$. Furthermore, we have by triangle inequality,

$$d_X(\eta_{\ell-1}, \eta_\ell) + d_\ell^\pi + d_X(\pi_{\ell-1}, \pi_\ell) \geq d_{\ell-1}^\pi \geq d_X(\eta_{\ell-1}, Z),$$

which gives

$$d_X(\eta_{\ell-1}, \eta_\ell) \geq d_X(\eta_{\ell-1}, Z) - d_\ell^\pi - C_2 \cdot \kappa(r_{\ell-1}).$$

Hence, together with Equation (39) and using $C_1 \leq 1$ we have

$$\begin{aligned}
|s' - s| &\geq \frac{1}{\mathfrak{q}} \sum_{i=0}^{\ell-1} (C_1 d_X(\eta_i, Z) - \mathfrak{Q}) - \frac{d_\ell^\pi + C_2 \cdot \kappa(r_{\ell-1})}{\mathfrak{q}} \\
&\geq \frac{1}{\mathfrak{q}} \sum_{i=0}^{\ell-1} (C_1 m_0 \cdot \kappa(r_i) - \mathfrak{Q}) - \frac{d_\ell^\pi}{\mathfrak{q}} - C_2 \frac{\kappa(r_{\ell-1})}{\mathfrak{q}} && \text{By Equation (39)} \\
&\geq \frac{1}{\mathfrak{q}} \sum_{i=0}^{\ell-1} (C_1 m_0 \cdot \kappa(r_i) - \mathfrak{Q} \cdot \kappa(r_i)) - \frac{d_\ell^\pi}{\mathfrak{q}} - C_2 \frac{\kappa(r_{\ell-1})}{\mathfrak{q}} && \kappa(t) \geq 1 \text{ for all } t \\
&\geq \left(\frac{C_1 m_0 - \mathfrak{Q}}{\mathfrak{q}} \right) \sum_{i=0}^{\ell-1} \kappa(r_i) - \frac{d_\ell^\pi}{\mathfrak{q}} - C_2 \frac{\kappa(r_{\ell-1})}{\mathfrak{q}}.
\end{aligned}$$

Combining the above inequality with Equation (40) we get

$$\begin{aligned}
(41) \quad \mathfrak{q}(d_0^\pi + d_\ell^\pi) + \mathfrak{Q} + \frac{d_\ell^\pi}{\mathfrak{q}} + C_2 \frac{\kappa(r_{\ell-1})}{\mathfrak{q}} &\geq \left(\frac{C_1 m_0 - \mathfrak{Q}}{\mathfrak{q}} - \mathfrak{q} C_2 \right) \sum_{i=0}^{\ell-1} \kappa(r_i) \\
&\geq (\mathfrak{q} + 1) \sum_{i=0}^{\ell-1} \kappa(r_i),
\end{aligned}$$

where in the last step we plugged in the definition of m_0 from (36).

By (39) we also have

$$\mathfrak{q}(d_0^\pi + d_\ell^\pi) + \mathfrak{Q} + \frac{d_\ell^\pi}{\mathfrak{q}} + C_2 \frac{\kappa(r_{\ell-1})}{\mathfrak{q}} \leq \mathfrak{q}(d_0^\pi + d_\ell^\pi) + \mathfrak{Q} + \frac{d_\ell^\pi}{\mathfrak{q}} + C_2 \frac{d_{\ell-1}^\pi}{m_0 \mathfrak{q}}$$

By the expression of m_0 in (36), $Q \leq m_0$ and by Equation (39), $m_0 \leq d_0^\pi$. Thus we have $Q \leq d_0^\pi$ and again by plugging in m_0 and using Claim A.3, we obtain

$$\mathfrak{q}(d_0^\pi + d_\ell^\pi) + \mathfrak{Q} + \frac{d_\ell^\pi}{\mathfrak{q}} + C_2 \frac{d_{\ell-1}^\pi}{m_0 \mathfrak{q}} \leq (\mathfrak{q} + 1)(d_0^\pi + d_\ell^\pi) + \frac{C_2 D_2}{m_0 \mathfrak{q}} \cdot \kappa(\eta_{\ell-1}).$$

Plugging this inequality into Equation (41), we get

$$\begin{aligned}
\sum_{i=0}^{\ell-1} \kappa(r_i) &\leq d_0^\pi + d_\ell^\pi + \frac{C_2 D_2}{m_0 \mathfrak{q}(\mathfrak{q} + 1)} \cdot \kappa(\eta_{\ell-1}) \\
&\leq (D_1 + 1)(d_0 + d_\ell) + D_2(\kappa(\eta_0) + \kappa(\eta_\ell)) + \frac{C_2 D_2}{m_0 \mathfrak{q}(\mathfrak{q} + 1)} \cdot \kappa(\eta_{\ell-1})
\end{aligned}$$

where we recall $d_i = d_X(\eta_i, Z)$, and the last inequality comes from Lemma 5.2 (note the difference between d_i and $d_i^\pi = d_X(\eta_i, \pi_i)$).

Lastly, we claim that since η is a quasi-geodesic ray, there is a constant C_3 , related to $\mathfrak{q}, \mathfrak{Q}$, such that $\|\eta\|_i \leq C_3 \cdot \|\eta(s')\| + C_3$ for $i = 0, \dots, \ell$, hence also $\kappa(\eta_i) \leq 2C_3 \cdot \kappa(\eta(s')) + \kappa(2C_3)$; thus, to shorten the preceding expression, let \mathfrak{A} be a constant, depending on $\{C_1, C_2, D_1, D_2, \mathfrak{q}, \mathfrak{Q}, \kappa\}$, such that

$$\mathfrak{q}(C_2 + 1)D_2(\kappa(\eta_0) + \kappa(\eta_\ell)) + \mathfrak{Q} + \frac{C_2^2 D_2}{m_0(\mathfrak{q} + 1)} \cdot \kappa(\eta_{\ell-1}) \leq \mathfrak{A} \cdot \kappa(\eta(s')).$$

By Equation (40) and the definition $m_1 = \mathfrak{q}(C_2 + 1)(D_1 + 1)$ from (36),

$$|s' - s| \leq m_1(d_0 + d_\ell) + \mathfrak{A} \cdot \kappa(\eta(s')).$$

This proves Claim A.2. ■

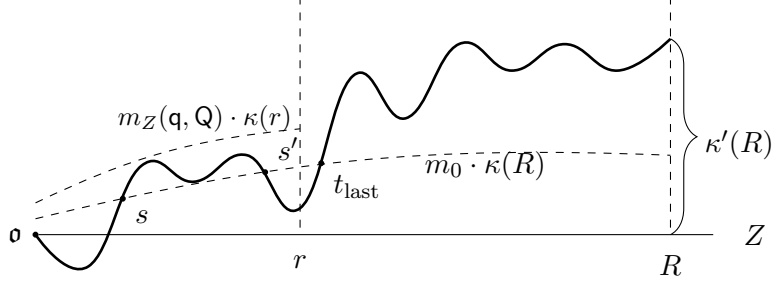


FIGURE 12. The proof of Theorem A.1.

Now let t_{last} be the last time η is in $\mathcal{N}_\kappa(Z, m_0)$ and consider the quasi-geodesic path $\eta([t_{\text{last}}, t_R])$. Since this path is outside of $\mathcal{N}_\kappa(Z, m_0)$, we can use Equation (37) to get

$$|t_R - t_{\text{last}}| \leq m_1(d_X(\eta(t_{\text{last}}), Z) + d_X(\eta(t_R), Z)) + \mathfrak{A} \cdot \kappa(R).$$

But

$$\begin{aligned} d_X(\eta(t_{\text{last}}), Z) &\leq m_0 \cdot \kappa(\eta(t_{\text{last}})) && \text{by the choice of } t_{\text{last}} \\ &\leq m_0 \cdot \kappa(R) && \text{since } \kappa \text{ is monotone} \end{aligned}$$

and we have by assumption $d_X(\eta(t_R), Z) \leq \kappa'(R)$. Therefore,

$$|t_R - t_{\text{last}}| \leq m_0 m_1 \cdot \kappa(R) + m_1 \cdot \kappa'(R) + \mathfrak{A} \cdot \kappa(R).$$

Since η is $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic, we obtain $R = d_X(\eta(0), \eta(t_R)) \leq \mathfrak{q}t_R + \mathfrak{Q}$, hence

$$t_R \geq \frac{R - \mathfrak{Q}}{\mathfrak{q}}.$$

Since m_0 and m_1 are given and κ and κ' are sublinear, there is a value of R depending on $m_0, m_1, r, \mathfrak{A}, \kappa$ and κ' such that

$$m_0 \cdot m_1 \cdot \kappa(R) + m_1 \cdot \kappa'(R) + \mathfrak{A} \cdot \kappa(R) \leq \frac{R - \mathfrak{Q}}{\mathfrak{q}} - r.$$

For any such R , we then have

$$t_{\text{last}} \geq t_R - \frac{R - \mathfrak{Q}}{\mathfrak{q}} + r \geq r.$$

We show that $\eta([0, t_{\text{last}}])$ stays in a larger κ -neighborhood of Z . Consider any other subinterval $[s, s'] \subset [0, t_{\text{last}}]$ where η exits $\mathcal{N}_\kappa(Z, m_0)$. By taking $[s, s']$ as large as possible, we can assume $\eta(s), \eta(s') \in \mathcal{N}_\kappa(Z, m_0)$. In this case,

$$d_X(\eta(s), Z) \leq m_0 \cdot \kappa(\eta(s)) \quad \text{and} \quad d_X(\eta(s'), Z) \leq m_0 \cdot \kappa(\eta(s')).$$

Again applying Equation (37), we get

$$(42) \quad |s' - s| \leq m_0 m_1 \cdot (\kappa(\eta(s)) + \kappa(\eta(s'))) + \mathfrak{A} \cdot \kappa(\eta(s'))$$

and thus

$$\begin{aligned} d_X(\eta(s'), \eta(s)) &\leq \mathfrak{q} m_0 m_1 \cdot (\kappa(\eta(s)) + \kappa(\eta(s'))) + \mathfrak{q} \mathfrak{A} \cdot \kappa(\eta(s')) + \mathfrak{Q} \\ &\leq (2\mathfrak{q} m_0 m_1 + \mathfrak{q} \mathfrak{A} + \mathfrak{Q}) \cdot \max(\kappa(\eta(s)), \kappa(\eta(s'))). \end{aligned}$$

Applying the Sublinear Estimation Lemma ([QRT19, Lemma 3.2]), we obtain

$$\kappa(\eta(s')) \leq m_2 \cdot \kappa(\eta(s))$$

for some m_2 depending on \mathfrak{q} , \mathfrak{Q} and κ . Therefore, by plugging this inequality back into (42), we have for any $t \in [s, s']$

$$(43) \quad |t - s| \leq (m_0 m_1 (1 + m_2) + \mathfrak{A} m_2) \cdot \kappa(\eta(s)) = m_3 \cdot \kappa(\eta(s)).$$

with $m_3 = (m_0 m_1 (1 + m_2) + \mathfrak{A} m_2)$. As before, this implies,

$$d_X(\eta(t), \eta(s)) \leq \mathfrak{q} m_3 \cdot \kappa(\eta(s)) + \mathfrak{Q} \leq (\mathfrak{q} m_3 + \mathfrak{Q}) \cdot \kappa(\eta(s)).$$

Applying [QRT19, Lemma 3.2] again, we have

$$(44) \quad \kappa(\eta(s)) \leq m_4 \cdot \kappa(\eta(t)),$$

for some m_4 depending on \mathfrak{q} , \mathfrak{Q} and κ .

Now, for any $t \in [s, s']$ we have

$$\begin{aligned} d_X(\eta(t), Z) &\leq d_X(\eta(t), \eta(s)) + r_0 \\ &\leq \mathfrak{q} |t - s| + \mathfrak{Q} + m_0 \cdot \kappa(\eta(s)) \\ \text{(Equation (43))} \quad &\leq (\mathfrak{q} m_3 + \mathfrak{Q} + m_0) \cdot \kappa(\eta(s)) \\ \text{(Equation (44))} \quad &\leq (\mathfrak{q} m_3 + \mathfrak{Q} + m_0) m_4 \cdot \kappa(\eta(t)). \end{aligned}$$

Now setting

$$(45) \quad m_Z(\mathfrak{q}, \mathfrak{Q}) = (\mathfrak{q} m_3 + \mathfrak{Q} + m_0) m_4$$

we have the inclusion

$$\eta([s, s']) \subset \mathcal{N}_\kappa(Z, m_Z(\mathfrak{q}, \mathfrak{Q})) \quad \text{and hence} \quad \eta([0, t_{\text{last}}]) \subset \mathcal{N}_\kappa(Z, m_Z(\mathfrak{q}, \mathfrak{Q})).$$

The R we have chosen depends on the value of \mathfrak{q} and \mathfrak{Q} . However, the assumption that $m_Z(\mathfrak{q}, \mathfrak{Q})$ is small compared to r (see Equation (2)) gives an upper bound for the values of \mathfrak{q} and \mathfrak{Q} . Hence, we can choose R to be the radius associated to the largest possible value for \mathfrak{q} and the largest possible value for \mathfrak{Q} . This finishes the proof.

Note that, the assumption that $m_Z(\mathfrak{q}, \mathfrak{Q})$ is small compared to r is not really needed here and any upper bound on the values of \mathfrak{q} and \mathfrak{Q} would suffice. But this is the assumption we will have later on and hence it is natural to state the theorem this way. \square

Next, we show that being κ -Morse implies being κ' -weakly contracting, with respect to κ -projection maps, for a sublinear function κ' . Note that κ' is not assumed to be the same function as κ . This is parallel to Theorem 3.8 in [QRT19], where we show that in a CAT(0) space being κ -contracting is equivalent to being κ -Morse (with the same κ); an identical statement cannot hold in general for proper geodesic spaces, as evidenced by the following example.

Example A.4. We give here a folklore example of a geodesic ray in proper metric space which is 1-Morse but not 1-contracting (see also the related [ACGH17, Example 3.4]). The points x_i form loops with the geodesic ray γ such that each path going through x_i represents a detour that is locally an $(i, 0)$ quasi-geodesic segment. This geodesic has the following properties:

- γ is 1-Morse, as any $(\mathfrak{q}, \mathfrak{Q})$ -quasi-geodesic ray must lie in a \mathfrak{q} -neighborhood of γ (in particular, it only goes through finitely many loops);
- γ is not 1-contracting, but it is \sqrt{i} -contracting, as $\|x_i\| \gtrsim i^2$ and, if π_γ is the nearest point projection to γ , we have $\text{diam}(\pi_\gamma(x_i)) = i$.

Thus it makes sense for us to prove in general that a κ -Morse set is κ' -weakly contracting for some sublinear function κ' .

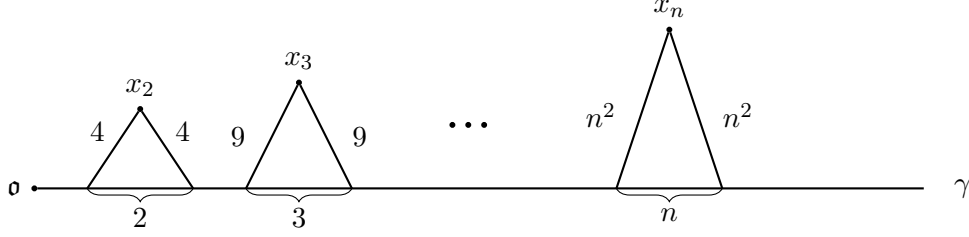


FIGURE 13. A 1-Morse geodesic ray which is \sqrt{t} -contracting.

To begin with, a set Z is Morse if it is κ -Morse for $\kappa = 1$, and its Morse gauge is denoted by $m_Z(\mathbf{q}, \mathbf{Q})$. We say that a closed set Z is ρ -radius-contracting if there exists a function ρ such that, for each ball B of radius r that is disjoint from Z , the nearest point projection of B to Z is a set whose diameter is bounded above by $\rho(r)$. We present Proposition 4.2 in [ACGH17] here with notation adapted to that of this paper. Furthermore, we take into account that in the setting of this paper, nearest point projection is nonempty:

Proposition A.5 (Proposition 4.2 [ACGH17]). *Let Z be a closed subspace of a geodesic metric space X . Suppose Z is Morse with Morse gauge $m_Z(\mathbf{q}, \mathbf{Q})$. Then there is a sublinear function ρ , depending on $m_Z(\mathbf{q}, \mathbf{Q})$, such that the nearest point projection of balls of radius r (disjoint from Z) onto Z is bounded above by $\rho(r)$. Specifically, ρ is obtained as follows:*

$$\rho(r) := \sup_s \left\{ s \leq 4r \text{ and } s \leq 18 m_Z \left(\frac{12r}{s}, 0 \right) \right\}.$$

Given a point $x \in X$ and a sublinearly Morse set $Z \subseteq X$, first we note that if a ball $B(x, r)$ is disjoint from Z , then

$$r \leq d_X(x, Z) \leq d_X(x, \mathbf{o}) = \|x\|.$$

We call a set Z *sublinearly weakly contracting* if it is κ -weakly contracting for some sublinear function κ .

Proposition A.6. *Let α be a quasi-geodesic ray in a proper geodesic space X that is κ -weakly Morse. Then there exists a sublinear function κ' such that α is κ' -weakly contracting.*

Proof. It suffices to prove the statement for nearest point projections, since by Lemma 5.2, the uniform multiplicative error and the sublinear additive error will not contradict the conclusion. Consider a ball $B(x, r)$, disjoint from α , and centered at x with radius r . Observe first of all that $B(x, r)$ is inside the ball of radius $\|x\| + r \leq 2\|x\|$. Thus the distance between any point $y \in B(x, r)$ and a point in its nearest point projection $y_1 \in \pi_\alpha^{(\text{near})}(y)$ is also bounded above by $2\|x\|$. We have by triangle inequality

$$\|y_1\| \leq 4\|x\|.$$

That is to say, given a ball $B(x, r)$, we only have to consider its projection to $\alpha \cap B(\mathbf{o}, 4\|x\|)$. By Proposition 3.10, since α is κ -Morse, it is also κ -weakly Morse in the sense of Definition 3.9. Let m_α be the κ -weakly Morse gauge of α . Thus the set $\alpha \cap B(\mathbf{o}, 4\|x\|)$ is Morse with its Morse gauge $m_\alpha(\mathbf{q}, \mathbf{Q}) \cdot \kappa(4x)$. Now let s denote the function that measures the diameter of the projection of disjoint ball $B(x, r)$ to α . By Proposition A.5,

$$s \leq 18 m_\alpha \left(\frac{12r}{s}, 0 \right) \cdot \kappa(4x) \quad \text{by definition of } \kappa\text{-Morse.}$$

Since $r \leq \|x\|$, we have

$$s \leq 18 m_\alpha \left(\frac{12\|x\|}{s}, 0 \right) \cdot \kappa(4x).$$

Suppose s is not a sublinear function of $\|x\|$, that is to say, as $\|x\| \rightarrow \infty$, there exists a sequence of disjoint balls $\{B_i\}$ with projections $\{s_i\}$ such that there exists a positive number c such that

$$\lim_{i \rightarrow \infty} \frac{s_i}{\|x\|_i} \geq c$$

then for every ϵ , there exists N such that for all $j > N$,

$$\begin{aligned} s_j &\leq 18 m_\alpha \left(\frac{12\|x\|_j}{s_j}, 0 \right) \cdot \kappa(4x_j) \\ &\leq 18 m_\alpha \left(12 \left(\frac{1}{c} - \epsilon \right), 0 \right) \cdot \kappa(4x_j). \end{aligned}$$

That is to say, s is bounded above by a sublinear function of $\|x\|$, which means s itself is a sublinear function of $\|x\|$, which is contrary to our assumption. Therefore, there does not exist such a sequence of balls and thus s is a sublinear function of $\|x\|$, which we denote as $\kappa'(x)$.

Lastly, by Lemma 5.2, the same claim holds for all κ -projection maps. \square

To summarize, we have proven the equivalence between κ -weakly contracting, κ -weakly Morse and κ -Morse (for a possibly different κ) for any quasi-geodesic:

Theorem A.7. *Let (X, \mathfrak{o}) be a proper geodesic metric space with a fixed base point and let α be a quasi-geodesic ray in X . Let π be any κ -projection from X to α in the sense of Definition 5.1. Then the following hold:*

- (1) *If α is κ -weakly contracting with respect to π , then it is κ -Morse;*
- (2) *The quasi-geodesic α is κ -Morse if and only if it is κ -weakly Morse;*
- (3) *If α is κ -weakly Morse, then it is κ' -weakly contracting with respect to π for some sublinear function κ' .*

Proof. If α is κ -weakly contracting, then by Theorem 5.4 it is κ -Morse. The second assertion is the statement of Proposition 3.10. Finally, if α is κ -Morse, by Proposition A.6 there exists a sublinear function κ' for which α is κ' -weakly contracting. \square

REFERENCES

- [AK11] Y. Algom-Kfir, *Strongly contracting geodesics in outer space*, *Geom. Topol.* 15 (2011), no. 4, 2181–2233.
- [ACGH17] G. Arzhantseva, C. Cashen, D. Gruber, and D. Hume, *Characterizations of Morse quasi-geodesics via superlinear divergence and sublinear contraction*, *Doc. Math.* 22 (2017), 1193–1224.
- [ACT15] G. Arzhantseva, C. Cashen and J. Tao, *Growth tight actions*, *Pacific J. Math.* 278 (2015), no. 1, 1–49.
- [Beh06] J. Behrstock, *Asymptotic geometry of the mapping class group and Teichmüller space*, *Geom. Topol.* 10 (2006), 1523–1578.
- [BHS17] J. Behrstock, M. Hagen and A. Sisto, *Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups*, *Geom. Topol.* 21 (2017), no. 3, 1731–1804.
- [BF09] M. Bestvina and K. Fujiwara, *A characterization of higher rank symmetric spaces via bounded cohomology*, *Geom. Funct. Anal.* 19 (2009), no. 1, 11–40.
- [Bou98] N. Bourbaki, *General topology. Chapters 1-4*, Translated from the French. Reprint of the 1989 English translation. *Elements of Mathematics* (Berlin), Springer-Verlag, Berlin, 1998. vii+437.
- [CM15] D. Calegari and J. Maher, *Statistics and compression of scl*, *Ergod. Th. Dynam. Sys.* (2015), 35, 64–110.
- [Cas16] C. Cashen, *Quasi-isometries need not induce homeomorphisms of contracting boundaries with the Gromov product topology*, *Anal. Geom. Metr. Spaces* 4 (2016), no. 1, 278–281.
- [CM19] C. Cashen and J. Mackay, *A metrizable topology on the contracting boundary of a group*, *Trans. Amer. Math. Soc.* 372 (2019), no. 3, 1555–1600.
- [CS15] R. Charney and H. Sultan, *Contracting boundaries of CAT(0) spaces*, *J. Topol.* 8 (2015), no. 1, 93–117.
- [CD60] G. Choquet and J. Deny, *Sur l'équation de convolution $\mu = \mu * \sigma$* , *C. R. Math. Acad. Sci. Paris* 250 (1960), 799–801.
- [Cor18] M. Cordes, *Morse boundaries of proper geodesic spaces*, *Groups Geom. Dyn.* 11 (2017), no. 4, 1281–1306.

- [CDG20] M. Cordes, M. Dussaule, and I. Gekhtman, *An embedding of the Morse boundary in the Martin boundary*, preprint arXiv:2004.14624.
- [CK00] C. B. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, *Topology* 39 (2000), no. 3, 549–556.
- [Dru00] C. Drutu, *Quasi-isometric classification of non-uniform latticed in semisimple groups of higher rank*, *Geom. Funct. Anal.* 10 (2000), 327–388.
- [DR09] M. Duchin and K. Rafi, *Divergence of Geodesics in Teichmüller Space and the Mapping Class Group*, *Geom. Funct. Anal.* 19 (2009), 722–742.
- [DM60] E. B. Dynkin and M. B. Maljutov, *Random walk on groups with a finite number of generators*, *Dokl. Akad. Nauk SSSR* 137 (1961), 1042–1045.
- [EFW12] A. Eskin, D. Fisher and K. Whyte, *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs*, *Ann. of Math. (2)* 176 (2012), no. 1, 221–260.
- [EFW13] A. Eskin, D. Fisher and K. Whyte, *Coarse differentiation of quasi-isometries II: Rigidity for Sol and lamplighter groups*, *Ann. of Math. (2)* 177 (2013), no. 3, 869–910.
- [EMR18] A. Eskin, H. Masur and K. Rafi, *Rigidity of Teichmüller space*, *Geom. Topol.* 22 (2018), no. 7, 4259–4306.
- [EMR17] A. Eskin, H. Masur and K. Rafi, *Large scale rank of Teichmüller space*, *Duke Math. J.* 166 (2017), no. 8, 1517–1572.
- [Far98] B. Farb, *Relatively hyperbolic groups*, *Geom. Funct. Anal.* 8 (1998), 810–840.
- [FM98] B. Farb and H. Masur, *Superrigidity and mapping class groups*, *Topology* 37 (1998), 1169–1176.
- [Flo80] W. Floyd, *Group completions and limit sets of Kleinian groups*, *Invent. Math.* 57 (1980), 205–218.
- [FT18] B. Forghani and G. Tiozzo, *Shannon’s theorem for locally compact groups*, preprint arXiv:1812.07292.
- [Fri37] A. Frink, *Distance functions and the metrization problem*, *Bull. Amer. Math. Soc.* 43 (1937), no. 2, 133–142.
- [Fur63] H. Furstenberg, *A Poisson Formula for Semi-Simple Lie Groups*, *Ann. of Math. (2)* 77 (1963), no. 2, 335–386.
- [GM12] F. Gautero and F. Mathéus, *Poisson boundary of groups acting on \mathbb{R} -trees*, *Israel J. Math.* 191 (2012), no. 2, 585–646.
- [Gro87] M. Gromov, *Hyperbolic groups*, in S. Gersten (ed.), *Essays in group theory*, Mathematical Sciences Research Institute Publications 8, Springer, New York, 1987, 75–263.
- [GQR20] I. Gekhtman, Y. Qing and K. Rafi, *QI-invariant model of Poisson boundaries of CAT(0) groups*, in preparation.
- [Kai94] V. Kaimanovich, *The Poisson boundary of hyperbolic groups*, *C. R. Acad. Sci. Paris* 318 (1994), no. 1, 59–64.
- [Kai96] V. Kaimanovich, *Boundaries of invariant Markov operators: The identification problem*, in M. Pollicott and K. Schmidt (Eds.), *Ergodic Theory and \mathbb{Z}^d Actions* (London Mathematical Society Lecture Note Series, pp. 127–176). Cambridge: Cambridge University Press.
- [Kai00] V. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*, *Ann. of Math. (2)* 152 (2000), no. 3, 659–692.
- [KM96] V. Kaimanovich and H. Masur, *The Poisson boundary of the mapping class group*, *Invent. Math.* 125 (1996), no. 2, 221–264.
- [KM98] V. Kaimanovich and H. Masur, *The Poisson boundary of Teichmüller space*, *J. Funct. Anal.* 156 (1998), no. 2, 301–332.
- [Kar11] A. Kar, *Asymptotically CAT(0) groups*, *Publ. Mat.* 55 (2011), no. 1, 67–91.
- [KM99] A. Karlsson and G. Margulis, *A multiplicative ergodic theorem and nonpositively curved spaces*, *Comm. Math. Phys.* 208 (1999), no. 1, 107–123.
- [KN04] A. Karlsson and G. Noskov, *Some groups having only elementary actions on metric spaces with hyperbolic boundaries*, *Geom. Dedicata* 104 (2004), 119–137.
- [Kla98] E. Klarreich, *The Boundary at Infinity of the Curve Complex and the Relative Teichmüller Space*, preprint arXiv:1803.10339.
- [Mah10] J. Maher, *Linear progress in the complex of curves*, *Trans. Amer. Math. Soc.* 362 (2010), no. 6, 2963–2991.
- [Mah12] J. Maher, *Exponential decay in the mapping class group*, *J. Lond. Math. Soc. (2)* 86 (2012), no. 2, 366–386.
- [MT18] J. Maher and G. Tiozzo, *Random walks on weakly hyperbolic groups*, *J. Reine Angew. Math.* 742, (2018), 187–239.
- [MM00] H. Masur and Y. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, *Geom. Funct. Anal.* 10 (2000), no. 4, 902–974.
- [MS20] P. Mathieu and A. Sisto, *Deviation inequalities for random walks*, *Duke Math. J.* 169 (2020), no. 5, 961–1036.
- [Mor24] H. M. Morse, *A fundamental class of geodesics on any closed surface of genus greater than one*, *Trans. Amer. Math. Soc.* 26 (1924), 25–60.
- [Qin16] Y. Qing, *Geometry of Right-Angled Coxeter Groups on the Croke-Kleiner Spaces*, *Geom. Dedicata* 183 (2016), no. 1, 113–122.

- [QRT19] Y. Qing, K. Rafi and G. Tiozzo, *Sublinearly Morse boundary I: CAT(0) spaces*, preprint arXiv:1909.02096
- [QT19] Y. Qing and G. Tiozzo, *Excursions of generic geodesics in right-angled Artin groups and graph products*, Int. Math. Res. Not. IMRN (2019), rnz294.
- [RV18] K. Rafi and Y. Verberne, *Geodesics in the mapping class groups*, preprint arXiv:1810.12489.
- [Ser83] C. Series, *Martin boundaries of random walks on Fuchsian groups*, Israel J. Math. 44 (1983), 221–242.
- [Sis13] A. Sisto, *Projections and relative hyperbolicity*, Enseign. Math. (2) 59 (2013), no. 1-2, 165–181.
- [Sis17] A. Sisto, *Tracking rates of random walks*, Israel J. Math. 220 (2017), 1–28.
- [Sis18] A. Sisto, *Contracting elements and random walks*, J. Reine Angew. Math. 742 (2018), 79–114.
- [ST19] A. Sisto and S. Taylor, *Largest projections for random walks and shortest curves for random mapping tori*, Math. Res. Lett. 26 (2019) no. 1, 293–321.
- [Tio15] G. Tiozzo, *Sublinear deviation between geodesics and sample paths*, Duke Math. J. 164 (2015), no. 3, 511–539.
- [Woe00] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2000.
- [Yan20] W. Yang, *Genericity of contracting elements in groups*, Math. Ann. 376 (2020), 823–861.

DEPARTMENT OF MATHEMATICS, FUDAN UNIVERSITY, CHINA
Email address: `yulan.qing@gmail.com`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON, CANADA
Email address: `rafi@math.toronto.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON, CANADA
Email address: `tiozzo@math.toronto.edu`