# THURSTON MEASURE IS STATIONARY WITH EXPONENTIAL TEICHMÜLLER MOMENT

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Abstract. In 1967, Furstenberg proved that every lattice in a connected semisimple Lie group with finite center admits a probability measure whose stationary measure on the associated Furstenberg boundary belongs to the Lebesgue measure class. We prove an analogue of this theorem for the action of the mapping class group on Teichmüller space. Every regular probability measure in the Thurston measure class on the space of projective measured foliations is the stationary measure of a non-elementary random walk with a finite exponential moment in the Teichmüller metric.

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## 1. Introduction

A $G$–space is a metric space with a continuous action of a group $G$. A useful first example of a $G$–space is the Poincaré disk: the group $G = \mathrm{PSL}_2(\mathbb{R})$ acts on the hyperbolic disk by isometries, and the circle at infinity records the directions in which points of the disk can escape. The action extends naturally to this boundary circle. Thus the boundary circle is itself a compact $G$–space. This picture has a broad generalization. For a semisimple Lie group, the canonical boundary is the Furstenberg

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boundary $G/P$, where $P$ is a minimal parabolic subgroup; it plays, in higher rank, the role played by the ideal circle for the Poincaré disk.

**Boundaries and random walks: Lie groups.** Furstenberg's theory explains how the boundary $G/P$ enters probability. Let $\mu$ be a probability measure on $G$, or on a lattice $\Gamma < G$, and let

$$g_n = h_1 \cdots h_n$$

be the associated random walk. Under suitable nondegeneracy hypotheses, a sample path determines an almost sure limiting flag $\xi \in G/P$. If $\nu$ denotes the law of $\xi$, then $\nu$ is $\mu$–stationary:

$$\nu = \mu * \nu = \int g_* \nu \, d\mu(g).$$

This identity reflects the Markov property. After the first step, the remaining tail of the walk has the same distribution as a fresh copy of the walk, while the endpoint at infinity is translated by the first increment. Thus $\nu$ can be viewed as the hitting measure of the random walk on the boundary: it records the distribution of the point at infinity hit, in the limit, by a typical sample path.

**Boundaries and random walks: mapping class groups.** The same picture has a mapping-class-group analogue. Let $S$ be an oriented surface of finite type, let $\mathrm{Mod} = \mathrm{Mod}(S)$ be its mapping class group, and let $\mathcal{T} = \mathcal{T}(S)$ be its Teichmüller space. The Thurston compactification of $\mathcal{T}$ has boundary $\mathcal{PMF}$, the space of projective measured foliations. The action of Mod on $\mathcal{T}$ extends continuously to an action on $\mathcal{PMF}$.

Fix a basepoint $Y \in \mathcal{T}$ and let

$$\gamma_n = h_1 \cdots h_n$$

be the random walk on Mod driven by $\mu_\Gamma$. Kaimanovich and Masur, in [KM94, KM96], proved that, assuming that the semigroup generated by $\mathrm{supp}(\mu_\Gamma)$ is non-elementary, the orbit points $\gamma_n Y$ converge almost surely to a projective measured foliation; moreover, the limiting foliation is almost surely uniquely ergodic. If $\nu$ denotes the law of this limit, then $\nu$ is $\mu_\Gamma$–stationary:

$$\mu_\Gamma * \nu = \sum_{\gamma \in \mathrm{Mod}} \mu_\Gamma(\gamma) \, \gamma_* \nu = \nu. \tag{1.1}$$

In this setting, $\nu$ is the unique non-atomic $\mu_\Gamma$–stationary measure on $\mathcal{PMF}$.

Thus $\nu$ is the hitting measure of the random walk on the Thurston boundary.

**Inverse Theorems.** For lattices in semisimple Lie groups, Furstenberg also gives an inverse statement. Let $\Gamma$ be a lattice in $G$. Then there are probability measures $\mu$ on $\Gamma$ with finite first moment for displacement in the associated symmetric space, for which the resulting stationary measure is equivalent to the natural smooth measure on $G/P$ [Fur67, Fur71]. This result, which goes back to 1967, played a key role in later developments. In particular, it was used by Furstenberg to prove an early (pre-Mostow) rigidity theorem for lattices, and later it was used by Margulis in the first proof of the Margulis superrigidity theorem.

In this paper, we are considering the analogue of Furstenberg's inverse theorem in the Teichmüller space setting. Instead of starting with a random walk and asking

what boundary measure it produces, we start with a measure $\nu$ in the Thurston measure class on $\mathcal{PMF}$ and ask whether one can find a probability measure $\mu_\Gamma$ on Mod such that

$$\nu = \mu_\Gamma * \nu.$$

In other words, we ask whether a prescribed Thurston-class measure can be realized as a stationary measure for a random walk on the mapping class group.

**Main theorem.** We exclude the once-punctured torus throughout. This is the classical flat-torus case of Furstenberg's theory [Fur63, Fur71].

We work with the following regular representatives of the Thurston measure class. Let $\rho : \mathcal{MF} \to \mathbb{R}_{\geq 0}$ be continuous, homogeneous of degree one, positive away from 0, and piecewise smooth in train-track coordinates. Here and below, piecewise smoothness is understood on $\mathcal{MF} \setminus \{0\}$: in each train-track coordinate domain there is a locally finite polyhedral subdivision on whose pieces $\rho$ is smooth up to the faces, meaning that each restriction extends smoothly to a neighborhood of the closed piece. Together with continuity, this implies that $\rho$ is locally Lipschitz away from 0. The Thurston measure on $\mathcal{MF}$ can be disintegrated using the level sets of $\rho$, and this gives a probability measure $\nu_\rho$ on $\mathcal{PMF}$.

**Theorem 1.1.** *Let $\nu = \nu_\rho$ be a probability measure on $\mathcal{PMF}$ in the Thurston measure class, where $\rho$ is continuous, homogeneous of degree one, positive away from 0, and piecewise smooth in train-track coordinates. Then there exist $a > 0$ and a probability measure $\mu_\Gamma$ on* Mod *such that the semigroup generated by the support of $\mu_\Gamma$ is non-elementary and*

$$\nu = \mu_\Gamma * \nu.$$

*Moreover, $\nu$ is the unique non-atomic $\mu_\Gamma$–stationary probability measure on $\mathcal{PMF}$. In addition $\mu_\Gamma$ has a small exponential moment: for every $P \in \mathcal{T}$,*

$$\sum_{\gamma \in \mathrm{Mod}} \mu_\Gamma(\gamma)\, e^{a\, d_{\mathcal{T}}(P, \gamma P)} < \infty.$$

*Remark* 1.2 (Regularity and limits). The piecewise smoothness assumption on $\rho$ is used to obtain uniform spike and Lipschitz estimates for the Radon–Nikodym derivatives of the mapping class group action on $\mathcal{PMF}$. This assumption is not expected to be essential. Suppose $\nu_n = \nu_{\rho_n}$ are measures of the above regular type, $\nu_n \to \nu$ weakly, and the measures $\mu_n$ constructed from

$$\nu_n = \mu_n * \nu_n$$

satisfy a uniform exponential Teichmüller moment bound. Then the sequence $\mu_n$ is tight on the countable group Mod. Passing to a subsequence, we get $\mu_n \to \mu$, and by testing against continuous functions on $\mathcal{PMF}$ and using continuity of the Mod-action, we obtain

$$\nu = \mu * \nu.$$

Thus the construction extends to limits of regular densities whenever the required moment bounds are uniform.

*Remark* 1.3. In fact, our proof proves the following more general statement. Let

$$\nu_1 = \nu_{\rho_1} \qquad \text{and} \qquad \nu_2 = \nu_{\rho_2}$$

be two probability measures on $\mathcal{PMF}$ in the Thurston measure class, where $\rho_1$ and $\rho_2$ are continuous, homogeneous of degree one, positive away from 0, and piecewise smooth in train-track coordinates. Then, there exists a probability measure $\mu_\Gamma$ on Mod such that

$$\nu_1 = \mu_\Gamma * \nu_2.$$

Moreover, $\mu_\Gamma$ has finite exponential moment with respect to the Teichmüller metric.

**Moment assumptions and singularity.** The moment condition in Theorem 1.1 is stated using the Teichmüller metric, rather than a word metric on Mod, and this distinction is essential. For many natural random walks on Mod, the resulting hitting measure on $\mathcal{PMF}$ is known to be singular with respect to the Thurston measure class. Gadre proved this for finitely supported measures [Gad14], and Gadre–Maher–Tiozzo proved more generally that, for a non-elementary random walk with finite first moment in a word metric, the hitting measure on $\mathcal{PMF}$ is singular with respect to the Lebesgue, equivalently Thurston, measure class [GMT17].

Thus the conclusion of Theorem 1.1 cannot hold under a finite first word-metric moment assumption. The theorem is close to optimal in this respect. Fix a basepoint $Y \in \mathcal{T}$. Word length is at most exponentially distorted relative to the orbit metric induced by the Teichmüller metric: there are constants $A, B > 0$ such that

$$|\gamma|_{\text{word}} \leq A \exp\big(B\, d_{\mathcal{T}}(Y, \gamma Y)\big).$$

Consequently, any exponential moment in the Teichmüller metric strong enough to dominate this distortion would imply finite first moment in the word metric, and hence would force singularity of the hitting measure. The random walks constructed here must therefore have heavy tails in every word metric. The point of Theorem 1.1 is that this obstruction does not prevent a small exponential moment in the Teichmüller metric.

**Relation to Lyapunov exponents.** Another motivation comes from the study of Lyapunov exponents. Eskin and Matheus used Furstenberg's inverse theorem for lattices in $SL(2, \mathbb{R})$ to give a coding-free simplicity criterion for the Lyapunov exponents of the Kontsevich–Zorich cocycle over Teichmüller curves [EM15]. The random walk constructed by Furstenberg's inverse theorem has Lyapunov exponents that are identical (up to scaling) to those of the Teichmüller geodesic flow (this was an observation of Alex Furman).

Theorem 1.1 is designed to provide an analogous source of random walks in the mapping class group setting. It can be plugged into the proof of [EM15] to give another proof of the simplicity of the odd and even parts of the Lyapunov spectrum for the principal stratum of quadratic differentials. The corresponding simplicity result for Abelian differentials is due to Avila and Viana [AV07]. For quadratic differentials, the plus and minus spectra are known to be simple in all components [BDGGS24]. See also the references in [BDGGS24] for a more complete history. A broader boundary-theoretic approach to Lyapunov spectra developed in [BF25] also recovers the result for the principal stratum of quadratic differentials.

**Historical remarks and prior work.** Furstenberg's inverse theorem dates to 1967. Since then, many analogues of statements from homogeneous dynamics have been

proved in the Teichmüller setting, but Furstenberg's inverse theorem has been a notable holdout. The original (highly non-trivial) proof is based on a discretization of Brownian motion. In the homogeneous setting, it is clear that the hitting measure for Brownian motion is in the Lebesgue measure class. Furstenberg then develops an iterative argument for approximating Brownian motion using a measure supported on the lattice $\Gamma$. This approach was considered by Kaimanovich and Masur in [KM94, Section 3]. They study invariant Markov chains on Teichmüller space and show that, under their regularity and positive recurrence assumptions, these can be discretized to random walks on the mapping class group without changing the hitting measure. However, this does not yield Theorem 1.1 since it is not clear whether the hitting measure of their process belongs to the Thurston measure class.

**Outline of the proof.** Our approach is strongly influenced by the work of Connell and Muchnik [CM07a, CM07b]. They studied the corresponding inverse problem for groups acting on hyperbolic spaces. In [CM07a], they show that, under their geometric hypotheses, a continuous quasiconformal measure on the limit set can be realized as a stationary measure for a random walk on the group, provided the radial limit set has full measure. In the CAT($-1$) cocompact case, for a Lipschitz quasiconformal measure the resulting random walk has finite first moment and the boundary with this measure is the Poisson boundary. In [CM07b], they extend the construction to measures coming from generalized Gibbs states.

The Connell–Muchnik argument is based on a positive-basis construction using boundary functions with controlled decay, called spikes. Hyperbolicity supplies sharp estimates for these spikes: a spike is large on a small boundary ball and decays at a controlled rate away from that ball.

In our setting, the spikes are the Radon–Nikodym derivatives $H_\rho$ on the Thurston boundary defined in (2.9). Our task is to find nonnegative coefficients $\mu_\Gamma(\gamma)$ satisfying

$$\sum_{\gamma\in\mathrm{Mod}} \mu_\Gamma(\gamma) H_\rho(\gamma, \boldsymbol{\cdot}) = \mathbb{1}$$

$\nu_\rho$–almost everywhere. By (2.10), this is precisely the stationarity equation, so such a positive decomposition gives the desired random walk. For the pseudo-Anosov elements $\gamma$ constructed below, the function $H_\rho(\gamma, \boldsymbol{\cdot})$ has a peak near the attracting foliation. The peak becomes narrower as the translation length grows. With enough spike centers, the coarse estimates in Theorem 11.1(a) already let us approximate $\mathbb{1}$ by positive sums of spikes. We subtract one such sum, then approximate the positive remainder using narrower spikes.

However, to obtain even a finite first moment, one needs finer estimates than hyperbolicity alone provides. Connell and Muchnik call these $Q$-spike estimates; we formulate the version needed here in Section 12. The main technical question is how narrow the next set of spikes must be after each subtraction. Without a quantitative answer, the translation lengths needed at successive steps could grow too quickly to obtain a finite first moment. The Lipschitz estimate in Theorem 11.1(b) provides this control. It bounds the variation of the remainder and hence the scale needed for the next approximation.

We use extremal length to control the size of $H_\rho$. To estimate its variation, we use homogeneous train-track coordinates in which the mapping class action on measured foliations is linear (Lemma 3.8).

The difficulty is obtaining uniform control near measured foliations that are endpoints of Teichmüller geodesics with long excursions into the thin part. To address this, we work at times when geodesics have good statistics (Definition 7.2). At such a time the differential lies in a fixed compact subset of the principal stratum in moduli space, and the geodesic spends most of every backward interval ending at that time in this thick part. We also need a boundary metric adapted to this control. The usual separation time of two rays may occur during a thin excursion. We instead use the last time at which the rays are still close and both have good statistics, as in (9.1). From these times we construct the Gromov metric $d_G$ (see Section 9). Choosing the metric this way is a key part of the argument as it gives a notion of boundary distance for which the needed size and Lipschitz estimates hold.

We also need enough pseudo-Anosov elements with these statistical properties. We obtain them by combining the horospherical large-deviation and mixing estimates of Appendix A (Proposition A.30) with the closing argument in Proposition 7.4. These elements have controlled translation lengths. Proposition 10.1 shows that the small balls around their attracting foliations, where the corresponding spikes are large, cover each boundary point at a large proportion of the first $n$ approximation scales, except for a set of $\nu_\rho$–measure exponentially small in $n$.

In Sections 13–14, we use these families to approximate the constant function 1 by positive sums of spikes. The Lipschitz estimates allow the translation lengths to grow only linearly with the step number, while the frequency of coverage just described makes the integral of the positive remainder decrease exponentially. The resulting coefficients satisfy the stationarity equation and define a probability measure with a small exponential Teichmüller moment.

**Order of choosing the constants.** To say that a geodesic has good statistics, we have to decide what part of the principal stratum we will call thick and what proportion of each backward interval the geodesic may spend outside it. The less time we allow outside, the larger the thick part may have to be. The contraction rate in Corollary 4.9 depends on this choice of thick part, and we use that rate to define the boundary metric and estimate how the spikes vary. We therefore fix the allowed proportion first. The order of choices below ensures that the later estimates work with that proportion.

The surface $S$, base point $P$ in Teichmüller space and prescribed measure $\nu_\rho$ on $\mathcal{PMF}$ are fixed throughout, and $w = \dim(\mathcal{MF}) = 6g - 6 + 2n$.

(1) *Charts and closing accuracy.* Fix the chart system and the Mod–compact set $K_0$ containing all backward chart rays, as in Theorem 5.1. This fixes $\mathsf{D}_{\mathrm{chart}}$, $\mathsf{r}_{\mathrm{chart}}$, $\mathsf{D}_{\mathrm{temp}}$, and the change-of-charts constant $\mathsf{k}$ of Lemma 5.2. The closing lemma (Proposition 7.1) approximates a sufficiently recurrent segment that nearly returns to its starting point in moduli space by a periodic orbit. Its constants $\epsilon_{\mathrm{clos}}, C_{\mathrm{clos}}$ depend only on $K_0$. Choose the allowed return error $\epsilon$ as in the proof of Proposition 7.4, small enough that closing preserves the required inclusions between the nested chart domains. This choice precedes the return-time bounds and the larger compact sets below. The bound $\mathsf{D}_0$ on

the distance from $P$ to the resulting pseudo-Anosov axes also depends only on these fixed geometric data.

(2) *Time outside compact sets.* Fix once and for all

$$\eta_2 = \frac{1}{4}, \qquad 0 < \eta_1 < \min\left\{\frac{1}{100}, \frac{1}{512w}\right\}, \qquad \eta = 256w\eta_1,$$

as required in (11.1). Here $\eta_1$ bounds the proportion of time outside $K_1$ on each backward segment, $\eta_2$ is the larger tolerance for time outside $K_2$ used in defining the Gromov metric, and $\eta$ is the error in the spike exponents. The notation $\eta = \eta_1/2$ used locally in the proof of Proposition 7.4 denotes its input to the appendix, not this spike error. These choices do not depend on $K_1, K_2$ or on the later contraction rate $\beta$.

(3) *Return data and $K_1$.* Apply Proposition A.30 with tolerance $\eta_1/2$ to obtain a Mod–compact set $K$, a sampling interval $\tau$, an exponential decay rate $\kappa$ for the exceptional measure, and a tolerance $\theta_0$ for the measure omitted from the return set. Set $\theta = \theta_0/2$. With $\epsilon$ fixed, Claim 7.5 gives a set $\Omega_\theta$ of normalized Masur–Veech measure at least $1 - \theta$ whose points flow into the required neighborhoods of the initial chart points within a bounded time $\rho_\theta = \rho_1$. Choose $K_1$ as in the proof of Proposition 7.4, containing the specified AGY neighborhoods of $K_0$, the initial chart sets, and $\{g_t q : q \in K,\ 0 \le t \le \rho_1\}$. Next choose the approximation error $\delta_\theta$ small enough to remain within the closing accuracy after flowing for up to $\rho_1$, and obtain the lower bound $\tau_0' > \rho_1$ on the sampling interval used later. The closing lemma now gives the required total time $R_1$ in $K_1$; this threshold affects only how many initial indices are omitted. None of these choices changes $\epsilon$.

(4) *$K_2$ and the metric.* Choose $K_2$ by Definition 8.4, large enough to accommodate the time shifts bounded by $\mathsf{D}_0$, $\mathsf{D}_{\mathrm{chart}}$, and $\mathsf{D}_{\mathrm{temp}}$ and the AGY perturbations of radius $1/100$ that arise when comparing rays through $K_1$. With $K_2$ fixed, Corollary 4.9 gives the contraction rate $\beta = \beta_{K_2}$ and multiplicative constant $C_{\mathrm{AGY}} = C_{K_2}$. We then choose the radius $\mathsf{r}_1$ and minimum time $\mathsf{R}_1$ used to compare the AGY and Gromov distances. In particular, (9.10) uses the convergence threshold from Proposition 8.2 and the positive margin $\eta_2 - \eta_1$. This $\mathsf{R}_1$ is distinct from the closing threshold $R_1$.

(5) *Spike estimates.* The multiplicative bounds and additive time delays in the geometric and Gauss–Manin estimates of §11 may now depend on the fixed compact sets. In the Lipschitz proof, first choose the accuracies and delays that keep the projections inside their chart domains ((11.30)–(11.31)), and only then the step size in (11.32). The coefficients 8 and 16 multiplying the time errors in Proposition 11.3 remain numerical, so the compact-dependent constants do not force a smaller choice of $\eta_1$. The approximation argument uses the volume exponent $W$ and the error exponent $\Xi$ given by

$$W = \frac{w}{\beta}, \qquad \Xi = \frac{2\eta}{\beta}.$$

The required inequality $\Xi < W$ follows from $\Xi/W = 512\eta_1 < 1$. No condition comparing $\eta$ with $\beta$ is needed.

(6) *Iteration and moment exponent.* Fix the spike and volume bounds, the bound $\mathsf{B}$ on the number of balls covering any point at one stage, and the bound $\mathsf{W}_0 = e^{\beta(\rho_1+2)}$ on their radius ratios. Proposition 13.1 then gives the required separation factor $\mathsf{m}$ between successive scales and a lower bound $\epsilon_0$ on the fraction subtracted from the current remainder on the covered set. Only now choose $\tau' \in \tau\mathbb{N}$ above the recurrence, metric, spike, and scale-separation thresholds listed in §14, and set $R_j = j\tau'$. The frequency of coverage and this subtraction make the integrals of the positive remainders decay at a rate $0 < \kappa_1 < 1$. Finally choose $0 < a < -\log(\kappa_1)/\tau'$, so that this decay outweighs the factor $e^{aj\tau'}$ in the exponential moment. These final choices may depend on $\rho$, but require no further change to $\eta_1$ or to $K_0, K_1, K_2$.

**Use of AI tools.** This project began as joint work with Maryam Mirzakhani around 2008. The overall strategy and main ideas of the proof have been in place for many years, but completing the manuscript took much longer.

We used OpenAI's Codex to help finish the manuscript. It assisted with routine editing, reference searches, and checking and completing technical details, including repairing gaps in proofs and checking how the constants depend on one another. It also helped simplify some existing proofs, for example in Section 6. We take full responsibility for the mathematical content of the paper.

## 2. Preliminaries

In this section, we recall the notation and some facts about Teichmüller space and measured foliations that we will use. We also rewrite the stationarity equation (1.1) as (2.10), which asks us to express the constant function 1 as a nonnegative sum of the functions defined in (2.9). The rest of the paper constructs these coefficients with the required exponential moment bound.

Let $S$ be a surface of finite hyperbolic type, i.e., a surface of genus $g$ with $n$ punctures such that

$$3g - 3 + n > 0.$$

We also assume $(g, n) \neq (1, 1)$. Let $\mathcal{T}(S) = \mathcal{T}_{g,n}$ be the Teichmüller space of $S$. When the surface $S$ is fixed, we simply write $\mathcal{T}$ for Teichmüller space. Points in $\mathcal{T}$ correspond to marked Riemann surfaces $(\phi_X \colon S \to X)$, where $\phi_X$ is defined up to homotopy. However, we often simply write $X \in \mathcal{T}$, keeping the marking map implicit. For two points $(\phi_X \colon S \to X)$ and $(\phi_Y \colon S \to Y)$ in $\mathcal{T}$, the map $\phi_X \circ \phi_Y^{-1} \colon Y \to X$ is referred to as the difference of markings map, again defined up to isotopy (see [Hub06, IT92] for background on Teichmüller space).

Let $\mathcal{QT} = \mathcal{QT}(S)$ be the space of nonzero meromorphic quadratic differentials on $S$, where we always assume that quadratic differentials have at most simple poles. We denote a quadratic differential by $q$, or, when emphasizing the underlying Riemann surface $X$, by $(X, q)$. We denote the space of unit-area quadratic differentials by $\mathcal{Q}_1\mathcal{T}$.

Let $\mathcal{PQT}(S) \subset \mathcal{QT}(S)$ denote the principal stratum: the stratum of quadratic differentials whose zeros are simple and which have a simple pole at every marked point. Let $\mathcal{PQ}_1\mathcal{T}$ be the principal stratum in $\mathcal{Q}_1\mathcal{T}$ (see [Str84] for background on quadratic differentials).

Let $\mathrm{Mod} = \mathrm{Mod}(S)$ be the mapping class group of $S$ (see [FM12] for background on the mapping class group). The group Mod naturally acts on all of the above spaces by changing the marking map. For example, for $(\phi_X \colon S \to X) \in \mathcal{T}$ and $\gamma \in \mathrm{Mod}$,

$$\gamma(\phi_X \colon S \to X) = (\phi_X \circ \gamma^{-1} \colon S \to X).$$

We denote the quotients of $\mathcal{T}$, $\mathcal{QT}$, and $\mathcal{PQT}$ by Mod, respectively, as $\mathcal{M}$, $\mathcal{QM}$, and $\mathcal{PQM}$. Similarly, $\mathcal{Q}_1\mathcal{M}$ and $\mathcal{PQ}_1\mathcal{M}$ denote the quotients of $\mathcal{Q}_1\mathcal{T}$ and $\mathcal{PQ}_1\mathcal{T}$ by Mod. Abusing notation, we denote all such quotient maps by $\pi$. For example, if $(X, q) \in \mathcal{PQ}_1\mathcal{T}$, then

$$\pi(X) \in \mathcal{M} \qquad \text{and} \qquad \pi(X, q) = \pi(q) \in \mathcal{PQ}_1\mathcal{M}.$$

**Definition 2.1.** A set $K \subset \mathcal{PQ}_1\mathcal{T}$ is called Mod–*compact* if it is Mod–invariant and $\pi(K)$ is compact.

**Measured foliations.** Let $\mathcal{MF} = \mathcal{MF}(S)$ denote the space of measured foliations on $S$, defined up to homotopy and Whitehead moves. For $F, F' \in \mathcal{MF}$, let

$$\mathrm{i}(F, F')$$

denote the intersection number between the two measured foliations (see [FLP79] for background on measured foliations).

For $c > 0$, let $c \cdot F$ denote the measured foliation obtained by scaling the transverse measure of $F$ by a factor of $c$. The space of projectivized measured foliations is

$$\mathcal{PMF} = (\mathcal{MF} \setminus \{0\})/\mathbb{R}_{>0}.$$

For $F \in \mathcal{MF} \setminus \{0\}$, the notation $[F] \in \mathcal{PMF}$ represents the projective class of $F$.

Given a quadratic differential $(X, q)$, one obtains two transverse measured foliations, called the horizontal and vertical foliations, by looking at the local square-root coordinate of $q$. Concretely, away from the zeros of $q$ there is a local coordinate chart from $X$ to $\mathbb{C}$ such that $q = dz^2$. In this coordinate, the horizontal leaves are given by

$$\mathrm{Im}\,(z) = \text{constant}$$

and have transverse measure $\left|\mathrm{d}(\mathrm{Im}\, z)\right|$, while the vertical leaves are given by

$$\mathrm{Re}\,(z) = \text{constant}$$

and have transverse measure $\left|\mathrm{d}(\mathrm{Re}\, z)\right|$. Define the maps

$$F_{\pm} \colon \mathcal{QT} \to \mathcal{MF}$$

by assigning to a quadratic differential $q$ its vertical measured foliation $F_+(q)$ and its horizontal measured foliation $F_-(q)$.

This process can often be reversed. We say measured foliations $F$ and $F'$ fill $S$ if for every nonzero measured foliation $G \in \mathcal{MF} - \{0\}$,

$$\mathrm{i}(F, G) + \mathrm{i}(F', G) > 0.$$

For $F \in \mathcal{MF}$, let $\mathcal{PMF}(F)$ be the set of points $x \in \mathcal{PMF}$ such that $F$ and any, equivalently every, representative of $x$ fill $S$. Let

$$\mathcal{NF} = \left\{(F, F') \in \mathcal{MF} \times \mathcal{MF} : \exists\, G \in \mathcal{MF} - \{0\} \text{ such that } \mathrm{i}(F, G) + \mathrm{i}(F', G) = 0\right\}$$

be the set of pairs of measured foliations that do not fill $S$.

**Theorem 2.2** (Gardiner and Masur [GM91]). *The map*

$$F_+ \times F_- \colon \mathcal{QT} \longrightarrow \mathcal{MF} \times \mathcal{MF} - \mathcal{NF}$$

*is a homeomorphism. That is, for $F, F' \in \mathcal{MF}$, if $F$ and $F'$ fill the surface $S$, then there exists a unique quadratic differential $q$ such that*

$$F_+(q) = F \qquad \text{and} \qquad F_-(q) = F'.$$

*We denote this quadratic differential by $Q(F, F')$.*

The area of the differential determined by a filling pair is

$$\operatorname{area}\big(Q(G,F)\big) = \mathrm{i}(G,F).$$

Thus, for $x = [G] \in \mathcal{PMF}(F)$, the unique unit-area differential with horizontal foliation $F$ and projective vertical foliation $x$ is

$$Q_1(x,F) = Q\left(\frac{G}{\mathrm{i}(G,F)}, F\right).$$

This does not depend on the representative $G$ of $x$. In this notation, the first foliation is vertical, the second is horizontal, and the subscript 1 indicates unit area.

Hubbard and Masur [HM79] showed that a point $X \in \mathcal{T}$ and a nonzero measured foliation $F$ determine a unique quadratic differential on $X$ with vertical foliation $F$, and a unique one with horizontal foliation $F$. We denote them by $Q(F,X)$ and $Q(X,F)$, respectively. Their unit-area rescalings depend only on $x = [F]$ and are denoted by $Q_1(x,X)$ and $Q_1(X,x)$.

**Extremal length and Teichmüller distance.** By a curve, we always mean a free homotopy class of non-trivial, non-peripheral, simple closed curves. Given a quadratic differential $(X,q)$, there are several notions of length one can associate to a curve $\alpha$. The extremal length $\operatorname{Ext}_X(\alpha)$ of a curve $\alpha$ is defined to be

$$\operatorname{Ext}_X(\alpha) = \sup_\sigma \frac{\operatorname{length}_\sigma(\alpha)^2}{\operatorname{area}(\sigma)},$$

where the supremum is over all metrics $\sigma$ in the conformal class of $X$.

The uniformization theorem implies that there is a unique hyperbolic metric on $X$. Let $\operatorname{Hyp}_X(\alpha)$ denote the hyperbolic length of $\alpha$ in $X$, representing the length of the geodesic representative of $\alpha$ in this metric. Also, $|q|$ defines a singular Euclidean metric on $X$. In this metric, every curve has a unique geodesic representative except when there is a family of geodesic representatives foliating a flat cylinder. We denote the length of the geodesic representative of $\alpha$ in this metric by $\operatorname{Flat}_q(\alpha)$.

For weighted curves, flat and hyperbolic length scale linearly with the weight, while extremal length scales quadratically. Weighted curves are dense in $\mathcal{MF}$, and these functions extend continuously to measured foliations.

**Theorem 2.3** (Corollary 28 in [DLR10]). *The length functions*

$$\operatorname{Flat}\colon \mathcal{QT} \times \mathcal{MF} \to \mathbb{R}_{\geq 0}, \qquad \operatorname{Hyp},\ \operatorname{Ext}\colon \mathcal{T} \times \mathcal{MF} \to \mathbb{R}_{\geq 0}$$

*are continuous with respect to all parameters.*

For a fixed $\epsilon > 0$, let $\mathcal{M}_{\geq\epsilon} \subset \mathcal{M}$ denote the space of Riemann surfaces where, for every $X \in \mathcal{M}_{\geq\epsilon}$ and every curve $\alpha$, we have $\mathrm{Hyp}_X(\alpha) \geq \epsilon$. We refer to the set

$$\mathcal{T}_{\geq\epsilon} = \pi^{-1}(\mathcal{M}_{\geq\epsilon})$$

as the *thick part* of Teichmüller space. Also, let $\mathcal{Q}_1\mathcal{M}_{\geq\epsilon}$ be the restriction of $\mathcal{Q}_1\mathcal{M}$ to $\mathcal{M}_{\geq\epsilon}$ and let $\mathcal{Q}_1\mathcal{T}_{\geq\epsilon}$ be the restriction of $\mathcal{Q}_1\mathcal{T}$ to $\mathcal{T}_{\geq\epsilon}$. Note that $\mathcal{M}_{\geq\epsilon}$ and $\mathcal{Q}_1\mathcal{M}_{\geq\epsilon}$ are compact. Compactness, continuity, homogeneity, and Mod–equivariance give the following comparison:

**Proposition 2.4.** *For every $(X, q) \in \mathcal{Q}_1\mathcal{T}_{\geq\epsilon}$ and every curve $\alpha$, we have*

$$\sqrt{\mathrm{Ext}_X(\alpha)} \asymp \mathrm{Hyp}_X(\alpha) \asymp \mathrm{Flat}_q(\alpha),$$

*where the implied constants depend only on $\epsilon$.*

Teichmüller space can be equipped with several natural Mod–invariant metrics. We work with the usual Teichmüller metric. We will use Kerckhoff's formula [Ker80]:

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \sup_{\alpha} \log \frac{\mathrm{Ext}_Y(\alpha)}{\mathrm{Ext}_X(\alpha)}, \tag{2.1}$$

where the supremum is taken over all curves $\alpha$.

**Teichmüller geodesics.** The Teichmüller geodesic flow

$$g_t \colon \mathcal{QT} \to \mathcal{QT}, \qquad t \in \mathbb{R},$$

is defined by

$$g_t(q) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot q = Q\big(e^t \cdot F_+(q),\, e^{-t} \cdot F_-(q)\big).$$

A Teichmüller geodesic is the trajectory of a quadratic differential under this flow. To simplify notation, for $q \in \mathcal{QT}$, we write $g_t(q)$ as $q(t)$. In particular, $q = q(0)$.

We call $[F_+(q)]$ and $[F_-(q)]$ the forward and backward endpoints of $q(t)$. Note that these names do not assert that the underlying Teichmüller geodesic converges in the Thurston compactification of Teichmüller space. For a quadratic differential $(X, q)$, its area equals the extremal length of either of its measured foliations. In particular, if $(X, q) \in \mathcal{Q}_1\mathcal{T}$, then

$$\mathrm{Ext}_X(F_+(q)) = \mathrm{Ext}_X(F_-(q)) = 1.$$

Since extremal length is quadratic under scaling,

$$\mathrm{Ext}_X(cF) = c^2\, \mathrm{Ext}_X(F),$$

for unit-area quadratic differentials $q$ and $(X_t, q_t) = g_t(q)$ we have

$$\mathrm{Ext}_{X_t}(e^{-t} \cdot F_-(q)) = 1 \qquad \Longrightarrow \qquad \mathrm{Ext}_{X_t}(F_-(q)) = e^{2t}, \tag{2.2}$$

and

$$\mathrm{Ext}_{X_t}(e^{t} \cdot F_+(q)) = 1 \qquad \Longrightarrow \qquad \mathrm{Ext}_{X_t}(F_+(q)) = e^{-2t}. \tag{2.3}$$

Thus, along a Teichmüller geodesic, the extremal length of $F_-(q)$, the horizontal foliation, increases exponentially, while the extremal length of $F_+(q)$, the vertical foliation, decreases exponentially.

Sometimes, we describe a Teichmüller geodesic using its horizontal and vertical foliations. For any $F \in \mathcal{MF}$ and $x \in \mathcal{PMF}(F)$, define

$$x_F \colon \mathbb{R} \to \mathcal{Q}_1\mathcal{T}$$

as the Teichmüller geodesic given by

$$x_F(t) = Q_1\big(x, e^{-t} \cdot F\big) = g_t\big(Q_1(x, F)\big). \tag{2.4}$$

**Horospheres.** In $\mathcal{Q}_1\mathcal{T}$, a horosphere refers to the set of quadratic differentials whose vertical or horizontal foliation is a given measured foliation. Hence, there are two types of horospheres. For $F \in \mathcal{MF}$, define $\mathrm{Hor}_+(F)$ as the set of area-one quadratic differentials $p \in \mathcal{Q}_1\mathcal{T}$ such that

$$F_+(p) = F,$$

and define $\mathrm{Hor}_-(F)$ as the set of $p \in \mathcal{Q}_1\mathcal{T}$ such that

$$F_-(p) = F.$$

When no sign is indicated, we write

$$\mathrm{Hor}(F) = \mathrm{Hor}_-(F).$$

Under the geodesic flow,

$$x_F(t) \in \mathrm{Hor}_-(e^{-t}F).$$

We may also define a horosphere using a quadratic differential. For $q \in \mathcal{Q}_1\mathcal{T}$, set

$$\mathrm{Hor}_+(q) = \mathrm{Hor}_+(F_+(q)) \qquad \text{and} \qquad \mathrm{Hor}_-(q) = \mathrm{Hor}_-(F_-(q)).$$

We call $\mathrm{Hor}_+(q)$ the stable horosphere and $\mathrm{Hor}_-(q)$ the unstable horosphere. Fixing only the projective vertical foliation gives the corresponding center-stable leaf. Alternatively, for $x \in \mathcal{PMF}$ and $X \in \mathcal{T}$, define

$$\mathrm{Hor}_+(x, X) = \mathrm{Hor}_+\big(Q_1(x, X)\big) \qquad \text{and} \qquad \mathrm{Hor}_-(X, x) = \mathrm{Hor}_-\big(Q_1(X, x)\big).$$

**The $\mathrm{GL}^+(2,\mathbb{R})$-action.** The space $\mathcal{QT}$ carries a natural $\mathrm{GL}^+(2,\mathbb{R})$–action. Its positive diagonal subgroup

$$D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b > 0 \right\} \subset \mathrm{GL}^+(2,\mathbb{R}) \tag{2.5}$$

acts by scaling the vertical and horizontal foliations separately:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot Q(G, F) = Q(aG, bF).$$

We denote the $D$–orbit of a set $E \subset \mathcal{QT}$ by $DE$. For $q \in \mathcal{Q}_1\mathcal{T}$,

$$D\,\mathrm{Hor}_\pm(q) = \{p \in \mathcal{QT} : [F_\pm(p)] = [F_\pm(q)]\}. \tag{2.6}$$

Thus the area and the scale of the fixed foliation may vary, while its projective class stays fixed.

**The Thurston measure class.** The Thurston measure $\mu_{\mathrm{Th}}$ on $\mathcal{MF}$ is invariant under the mapping class group and homogeneous of degree

$$w = \dim(\mathcal{MF}) = 6g - 6 + 2n. \tag{2.7}$$

Thus scaling a measurable set by $c > 0$ multiplies its measure by $c^w$. We refer to [PH92, SB01] for the construction of $\mu_{\mathrm{Th}}$.

Let $\rho\colon \mathcal{MF} \to \mathbb{R}_{\geq 0}$ be a continuous function that is positive away from 0 and is homogeneous of degree one:

$$\rho(cF) = c\,\rho(F) \qquad (c > 0).$$

Write

$$S_\rho = \{F \in \mathcal{MF} : \rho(F) = 1\}, \qquad B_\rho = \{F \in \mathcal{MF} : \rho(F) \leq 1\}.$$

The set $B_\rho$ is compact and contains a neighborhood of the origin.

The function $\rho$ defines a probability measure on $\mathcal{PMF}$ as follows. For a measurable set $E \subset \mathcal{PMF}$, let

$$\mathbb{R}_+E = \{F \in \mathcal{MF} \setminus \{0\} : [F] \in E\}$$

be the cone over $E$ and define

$$\widetilde{\nu}_\rho(E) = \mu_{\mathrm{Th}}(\mathbb{R}_+E \cap B_\rho).$$

Then set

$$m_\rho = \widetilde{\nu}_\rho(\mathcal{PMF}) = \mu_{\mathrm{Th}}(B_\rho), \qquad \text{and} \qquad \nu_\rho = \frac{1}{m_\rho}\widetilde{\nu}_\rho.$$

Since $0 < m_\rho < \infty$, this defines a probability measure on $\mathcal{PMF}$. Homogeneity gives the polar-coordinate formula (recall the definition of $w$ from (2.7))

$$d\mu_{\mathrm{Th}} = m_\rho\, w\, r^{w-1}\, dr\, d\nu_\rho, \qquad r = \rho(F). \tag{2.8}$$

Any two such functions $\rho_1, \rho_2$ have a ratio bounded above and below by positive constants: the ratio is well defined and continuous on the compact space $\mathcal{PMF}$. The associated measures are therefore equivalent. We call their common measure class the *Thurston measure class on* $\mathcal{PMF}$.

A natural example for such a function is $\rho(F) = \sqrt{\mathrm{Ext}_P(F)}$ for a fixed $P \in \mathcal{T}$. To control the variation of our spike functions, we fix $\rho$ to be piecewise smooth in train-track coordinates, with the regularity convention stated before Theorem 1.1: the subdivision is locally finite and each restriction extends smoothly across its faces. In particular, $\rho$ is locally Lipschitz on $\mathcal{MF} \setminus \{0\}$, and $S_\rho$ is the piecewise smooth boundary of $B_\rho$.

**The stationarity equation.** Recall that the pushforward of $\nu_\rho$ by a mapping class $\gamma$ is defined as:

$$(\gamma_*\nu_\rho)(E) = \nu_\rho(\gamma^{-1}E).$$

By Mod–invariance of $\mu_{\mathrm{Th}}$, this equals

$$\frac{1}{m_\rho}\,\mu_{\mathrm{Th}}\big(\{F \in \mathbb{R}_+E : \rho(\gamma^{-1}F) \leq 1\}\big).$$

On the ray through $F \in S_\rho$, the radial cutoff is now $r \le \rho(\gamma^{-1}F)^{-1}$. Integrating the polar density (2.8) up to this cutoff we get that, for $x = [F]$ and any nonzero representative $F$,

$$H_\rho(\gamma, x) = \frac{d\,(\gamma_*\nu_\rho)}{d\nu_\rho}(x) = \left(\frac{\rho(F)}{\rho(\gamma^{-1}F)}\right)^w. \tag{2.9}$$

The ratio is unchanged when $F$ is rescaled. These functions satisfy the cocycle identity

$$H_\rho(\gamma_1\gamma_2, x) = H_\rho(\gamma_1, x)\, H_\rho(\gamma_2, \gamma_1^{-1}x).$$

For example, if $\gamma F = \lambda F$ with $\lambda > 0$, then

$$H_\rho(\gamma, [F]) = \lambda^w.$$

Section 11 gives the estimates that allow us to use suitable functions $H_\rho(\gamma, \cdot)$ as spikes. We often suppress $\rho$ and write $H(\gamma, x)$.

Since $\gamma_*\nu_\rho$ is a probability measure, each of these densities has integral one:

$$\int_{\mathcal{PMF}} H_\rho(\gamma, x)\, d\nu_\rho(x) = 1.$$

For a probability measure $\mu_\Gamma$ on Mod, stationarity (1.1) is therefore equivalent to

$$\sum_{\gamma \in \mathrm{Mod}} \mu_\Gamma(\gamma) H_\rho(\gamma, \cdot) = \mathbb{1} \tag{2.10}$$

in $L^1(\nu_\rho)$, or equivalently $\nu_\rho$–almost everywhere. Conversely, if nonnegative coefficients $\mu_\Gamma(\gamma)$ satisfy this identity, integrating and applying Tonelli's theorem gives

$$\sum_{\gamma \in \mathrm{Mod}} \mu_\Gamma(\gamma) = 1.$$

Thus our task is to write $\mathbb{1}$ as a nonnegative sum of these densities, with coefficients satisfying the required exponential moment bound.

## 3. Period Coordinates and the Gauss–Manin Connection

Period coordinates describe nearby differentials by complex vectors. Their real and imaginary parts give local coordinates for the vertical and horizontal measured foliations. We explain this description, including when these foliations have saddle connections, and show how Gauss–Manin transport compares the resulting coordinates along a path. We begin with Abelian differentials, where the coordinates are obtained by integration.

**Period coordinates for Abelian differentials.** Let $\Sigma$ be a closed oriented surface. An Abelian differential $(Y, \omega)$ consists of a Riemann surface $Y$ homeomorphic to $\Sigma$ and a nonzero holomorphic 1–form $\omega$. Let

$$Z = \{z_1, \dots, z_m\}$$

consist of its zeros together with any regular marked points we wish to retain. The stratum $\Omega\mathcal{M}(\kappa)$ fixes the orders $\kappa = (k_1, \dots, k_m)$ at these points, with $k_j = 0$ for a regular marked point. If $\omega$ has no zeros, we retain one regular marked point for the triangulation below.

Away from $Z$, integrating $\omega$ gives flat coordinates whose transition maps are translations. For an oriented saddle connection $\delta$, its relative period is

$$\mathrm{Per}(\omega, \delta) = \int_\delta \omega.$$

Its absolute value is the length of $\delta$. The transverse measures of the vertical and horizontal foliations across this straight edge are, respectively, the absolute values of its real and imaginary parts.

Choose a triangulation by saddle connections with vertices in $Z$. Each oriented edge has a relative period, and the three periods around an oriented triangle sum to zero. Nearby edge vectors satisfying these relations and the edge identifications give new triangles. For sufficiently small changes, all triangles remain nondegenerate with the same orientation, and gluing them in the same pattern gives a nearby differential in the same stratum. This is the geometric reason that periods provide local coordinates.

The full list of edge vectors is redundant. To remove this redundancy, we use relative homology:

$$H_1(Y, Z; \mathbb{Z}).$$

Its elements are represented by relative 1–cycles, that is, chains whose boundary lies in $Z$. A basis consists of absolute cycles around the handles, together with arcs joining one point of $Z$ to the others. Fix a small enough marked neighborhood $\mathcal{V}(\omega)$ of $(Y, \omega)$, so that these cycles and arcs can be followed on nearby surfaces $(Y', \omega') \in \mathcal{V}(\omega)$. For a basis $\{\delta_1, \ldots, \delta_N\}$ of $H_1(Y, Z; \mathbb{Z})$, the period coordinates are

$$(Y', \omega') \longmapsto \left( \int_{\delta_1} \omega', \ldots, \int_{\delta_N} \omega' \right) \in \mathbb{C}^N.$$

Equivalently, the periods define a linear functional on relative homology, hence a relative cohomology class. Using the local marking to identify the homology groups, we write the period map as

$$\mathcal{V}(\omega) \longrightarrow H^1(Y, Z; \mathbb{C}), \qquad \omega' \longmapsto \left[ \delta \mapsto \int_\delta \omega' \right].$$

When $\mathcal{V}(\omega)$ is sufficiently small, this gives complex coordinates onto an open subset of $H^1(Y, Z; \mathbb{C})$. The triangle relations are the cocycle equations for this cohomology class, and the gluing construction gives the local inverse. The same triangulation persists throughout a smaller neighborhood, where both the periods and their inverse vary continuously. These charts give $\Omega\mathcal{M}(\kappa)$ the structure of an orbifold (see [Zor06, Section 3.4]) (when $(Y, \omega)$ has symmetries, we can pass to the finite quotient by automorphisms).

**Period coordinates for quadratic differentials.** The flat coordinates of a quadratic differential $(X, q)$ have transition maps $z \mapsto \pm z + c$. Thus integrating the local forms $dz = \sqrt{q}$ on $X$ does not in general give a cohomology class.

To remedy this, we consider the *orienting double cover*

$$\pi : \widetilde{X} \to X$$

of $(X, q)$. Here we fill in the punctures of $X$ and include the points over them in $\widetilde{X}$. Let $\mathcal{Z}(q)$ denote the set of singularities of $q$, that is, its zeros and poles. Continuing a square root around an odd-order singularity reverses its sign. In the principal

stratum, the zeros and poles are all simple, so the orienting double cover is branched exactly over $\mathcal{Z}(q)$. By abuse of notation, we also write $\mathcal{Z}(q) \subset \widetilde{X}$ for the ramification locus, which is naturally identified with the set of singularities of $q$.

The pullback $\tilde{q} = \pi^* q$ has orientable horizontal and vertical foliations. Therefore $\tilde{q}$ is the square of a holomorphic Abelian differential, well-defined up to sign. We write

$$\tilde{q} = \omega_q^2.$$

The points over simple zeros of $q$ are double zeros of $\omega_q$, while the points over simple poles are regular marked points. Indeed, writing $z = w^2$ at a branch point transforms the local models $z\,dz^2$ and $dz^2/z$ into $4w^4\,dw^2$ and $4\,dw^2$, respectively. We keep all points of $\mathcal{Z}(q)$ as marked points in the relative cohomology, so that saddle connections ending at poles are included. Since the deck involution $i : \widetilde{X} \to \widetilde{X}$ sends $\omega_q$ to $-\omega_q$, the corresponding period classes are odd for the deck involution.

After choosing the sign of $\omega_q$ and choosing a local marking of the orienting double cover, period coordinates give a local map

$$\mathcal{V}(q) \hookrightarrow H^1(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}), \qquad q' \longmapsto \left[\gamma \mapsto \int_\gamma \omega_{q'}\right],$$

for some neighborhood $\mathcal{V}(q)$ of $q$ in the marked principal stratum $\mathcal{PQT}(S)$. Changing the sign of the square root multiplies all coordinates by $-1$.

**Proposition 3.1.** *Let $(X, q) \in \mathcal{PQT}(S)$, where $S$ is a surface of genus $g$ with $n$ marked points, equivalently punctures. Denote by $\widetilde{X}$ the orienting double cover of $(X, q)$. Then $\widetilde{X}$ is a closed surface of genus*

$$\widetilde{g} \;=\; 4g \;-\; 3 \;+\; n.$$

*Let $i : \widetilde{X} \to \widetilde{X}$ be the deck involution associated to this double cover. Then the induced map on cohomology*

$$i^* : H^1(\widetilde{X}; \mathbb{C}) \;\longrightarrow\; H^1(\widetilde{X}; \mathbb{C})$$

*splits $H^1(\widetilde{X}; \mathbb{C})$ into a direct sum of $+1$ and $-1$ eigenspaces,*

$$H^1(\widetilde{X}; \mathbb{C}) = H^1_{\text{even}}(\widetilde{X}; \mathbb{C}) \oplus H^1_{\text{odd}}(\widetilde{X}; \mathbb{C}).$$

*The* odd *part $H^1_{\text{odd}}(\widetilde{X}; \mathbb{C})$, the $-1$ eigenspace, has complex dimension*

$$\dim_{\mathbb{C}}\big(H^1_{\text{odd}}(\widetilde{X}; \mathbb{C})\big) = 6g \;-\; 6 \;+\; 2n.$$

*Proof.* There are $4g - 4 + n$ simple zeros and $n$ simple poles, hence $b = 4g - 4 + 2n$ branch points. Riemann–Hurwitz gives

$$2\widetilde{g} - 2 = 2(2g - 2) + b = 8g - 8 + 2n,$$

which proves the genus formula.

The invariant subspace is $H^1_{\text{even}}(\widetilde{X}; \mathbb{C}) = \pi^* H^1(X; \mathbb{C})$: pullback classes are invariant, and invariant classes descend to the quotient $\widetilde{X}/i = X$. Its dimension is therefore $2g$. Since the total dimension is $2\widetilde{g}$, the odd part has dimension

$$2\widetilde{g} - 2g = 6g - 6 + 2n. \qquad \square$$

Similarly, the relative cohomology splits into even and odd parts:

$$H^1(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}) = H^1_{\text{even}}(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}) \oplus H^1_{\text{odd}}(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}).$$

The long exact sequence of the pair $(\widetilde{X}, \mathcal{Z}(q))$ gives

$$0 \longrightarrow H^0(\widetilde{X}; \mathbb{C}) \longrightarrow H^0(\mathcal{Z}(q); \mathbb{C}) \xrightarrow{\delta} H^1(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}) \xrightarrow{r} H^1(\widetilde{X}; \mathbb{C}) \longrightarrow 0. \tag{3.1}$$

This sequence is equivariant with respect to the deck involution. Since the deck involution fixes every point of $\mathcal{Z}(q)$, it acts trivially on $H^0(\mathcal{Z}(q); \mathbb{C})$. Hence the image of $\delta$ lies in the even part of $H^1(\widetilde{X}, \mathcal{Z}(q); \mathbb{C})$. Taking the odd part of (3.1), we obtain an isomorphism

$$r : H^1_{\text{odd}}(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}) \longrightarrow H^1_{\text{odd}}(\widetilde{X}; \mathbb{C}).$$

Thus, for the principal stratum, the odd relative period coordinates may be viewed canonically as absolute odd cohomology classes. This uses that every marked singularity is a branch point; in an arbitrary stratum the relative and absolute odd spaces can differ. We will also use the real vector space

$$L(q) := H^1_{\text{odd}}(\widetilde{X}; \mathbb{R}) \cong H^1_{\text{odd}}(\widetilde{X}, \mathcal{Z}(q); \mathbb{R}). \tag{3.2}$$

The relative description lets us evaluate these classes on saddle connections. The absolute description lets us use the Hodge norm and cup product on the closed orienting cover.

**Theorem 3.2** (Masur [Mas82])**.** *For any $(X, q) \in \mathcal{PQT}(S)$, there exists a neighborhood $\mathcal{V}(q)$ on which a local marking of the orienting covers gives a period coordinate map*

$$\Phi_q \colon \mathcal{V}(q) \to H^1_{\text{odd}}(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}) \cong H^1_{\text{odd}}(\widetilde{X}; \mathbb{C}),$$

*well-defined up to multiplication by $-1$, that is a homeomorphism onto an open subset of the indicated cohomology space.*

Here the remaining sign is the choice of square root. As in the Abelian case, a triangulation persists on a small neighborhood and provides a common local marking for the period construction. This gives continuous coordinates and inverse throughout that neighborhood. The marked preimages of poles are retained in this construction. See also [Dum15, Theorem 5.4] for the period-coordinate description in the closed-surface case.

**Gauss–Manin transport.** To compare periods at different surfaces, we must identify their cohomology groups. Gauss–Manin transport does this by following cycles with the surface and keeping the value of a transported class on each such cycle constant.

First consider marked Riemann surfaces $Y_1, Y_2 \in \mathcal{T}(\Sigma)$, with markings $\phi_{Y_j} : \Sigma \to Y_j$. Their difference-of-markings map is

$$f_{12} = \phi_{Y_2} \circ \phi_{Y_1}^{-1} : Y_1 \to Y_2,$$

defined up to isotopy. The induced transport is

$$\mathrm{GM}_{Y_1,Y_2} : H^1(Y_1; \mathbb{C}) \to H^1(Y_2; \mathbb{C}), \qquad \mathrm{GM}_{Y_1,Y_2}(\alpha) = (f_{12}^{-1})^* \alpha.$$

Equivalently, it is characterized by

$$\int_{(f_{12})_*\delta} \mathrm{GM}_{Y_1,Y_2}(\alpha) = \int_\delta \alpha \qquad \text{for every } \delta \in H_1(Y_1;\mathbb{Z}).$$

If a finite set $\mathcal{Z} \subset \Sigma$ is marked as well and if the marking maps are taken up to homotopy rel $\mathcal{Z}$, the same construction gives transport between $H^1(Y_j, \phi_{Y_j}(\mathcal{Z});\mathbb{C})$. These maps are also defined over $\mathbb{R}$.

The markings make transport independent of the path in Teichmüller space. In moduli space however, a loop can return a cycle to a different homology class. This is called *monodromy*.

For the orienting covers, a marking of $X$ by $S$ does not determine a marking of $\widetilde{X}$ relative to $\mathcal{Z}(q)$. The zeros of $q$ are not part of the marking of $X$, and choosing a lift also leaves a deck-involution ambiguity. Thus transport on odd cohomology must keep track of a path in $\mathcal{PQT}(S)$.

For a path $\xi(t) = (X_t, q_t)$, $0 \le t \le 1$, in $\mathcal{PQT}(S)$ from $(X,q)$ to $(Y,p)$, write $i_t$ for the deck involution of $\widetilde{X}_t$. On a sufficiently short subinterval, a triangulation used for period coordinates persists. Matching the corresponding lifted triangles identifies nearby covers, carries their ramification points along, and commutes with the deck involutions. Subdivide $[0,1]$ into finitely many such subintervals and compose these local identifications, matching them at the subdivision points and starting with the identity. This gives a continuous family of homeomorphisms

$$\widetilde{f}_t : \widetilde{X} \longrightarrow \widetilde{X}_t, \qquad \widetilde{f}_0 = \mathrm{Id}, \qquad \widetilde{f}_t \circ i_0 = i_t \circ \widetilde{f}_t.$$

This is what it means to trivialize the family along $\xi$. The endpoint map $\widetilde{f}_\xi = \widetilde{f}_1$ respects the deck involutions, so its inverse pullback preserves odd cohomology:

$$\mathrm{GM}_\xi = (\widetilde{f}_\xi^{-1})^* : H^1_{\mathrm{odd}}(\widetilde{X};\mathbb{C}) \to H^1_{\mathrm{odd}}(\widetilde{Y};\mathbb{C}).$$

**Lemma 3.3** (Gauss–Manin map is well-defined)**.** *For a path $\xi$ connecting $(X,q)$ to $(Y,p)$ in $\mathcal{PQT}(S)$, the map*

$$\mathrm{GM}_\xi \colon\ H^1_{\mathrm{odd}}(\widetilde{X};\mathbb{C}) \to H^1_{\mathrm{odd}}(\widetilde{Y};\mathbb{C})$$

*is well-defined by Gauss–Manin transport along $\xi$. It depends only on the homotopy class of $\xi$ in $\mathcal{PQT}(S)$ relative to the endpoints. Furthermore, $\mathrm{GM}_\xi$ is defined over $\mathbb{R}$ and commutes with complex conjugation.*

*Proof.* Two choices of the family $\widetilde{f}_t$ above differ by a continuous family of self-homeomorphisms of $\widetilde{X}$ starting at the identity. Their endpoint maps are therefore isotopic and induce the same map on cohomology. The same argument over a homotopy of paths proves homotopy invariance. Pullback by a homeomorphism is defined over $\mathbb{R}$, which also gives compatibility with complex conjugation. □

For example, consider the loop $q_t = e^{2\pi i t} q$, $0 \le t \le 1$. The underlying surface $X$ stays fixed. Away from the singularities, the orienting cover consists of the two choices of a square root $\zeta$ with $\zeta^2 = q_t$. Continuing these choices along the loop sends $\zeta$ to $e^{\pi i t}\zeta$. At $t = 1$ this is the deck involution, so transport around the loop is $-\mathrm{Id}$ on odd cohomology. Thus even a loop over one fixed base surface can have nontrivial monodromy.

Within a contractible neighborhood, transport identifies the period vectors of the same differential in different charts. This is the content of the next lemma. The period vector of the differential itself can change as the differential moves; it need not be a parallel class.

**Lemma 3.4.** *Let $\xi$ be a path in $\mathcal{PQT}(S)$ connecting $(X,q)$ to $(Y,p)$, and suppose there is a contractible set $\mathcal{U} \subset \mathcal{PQT}(S)$ containing $\xi([0,1])$ as well as the period-coordinate neighborhoods $\mathcal{V}(q)$ and $\mathcal{V}(p)$. Consider the maps*

$$\Phi_q \colon \mathcal{V}(q) \to H^1_{\mathrm{odd}}(\widetilde{X};\mathbb{C}), \qquad \Phi_p \colon \mathcal{V}(p) \to H^1_{\mathrm{odd}}(\widetilde{Y};\mathbb{C}),$$

*and the Gauss–Manin transport*

$$\mathrm{GM}_\xi \colon H^1_{\mathrm{odd}}(\widetilde{X};\mathbb{C}) \to H^1_{\mathrm{odd}}(\widetilde{Y};\mathbb{C}).$$

*After choosing the signs of $\Phi_q$ and $\Phi_p$ consistently over $\mathcal{U}$, we have, on the overlap $\mathcal{V}(q) \cap \mathcal{V}(p)$,*

$$\mathrm{GM}_\xi \circ \Phi_q = \Phi_p.$$

*Proof.* Let $\widetilde{\mathcal{U}}$ be the family of orienting double covers of points in $\mathcal{U}$. Since $\mathcal{U}$ is contractible, we may choose a lift of this family to the marked Abelian stratum. This choice identifies all odd cohomology groups over $\mathcal{U}$ with a single fixed vector space. Under this identification, the two period-coordinate maps $\Phi_q$ and $\Phi_p$ are given by the same period vector, and the Gauss–Manin map $\mathrm{GM}_\xi$ is the identity on the fixed vector space. The claim follows. □

**Measured foliations and the local product structure.** To control the variation of $H_\rho$ in (2.9), we will need to compare nearby measured foliations and their images under a mapping class in linear coordinates. Train-track charts provide such coordinates on $\mathcal{MF}$, identified with the space of measured laminations. A foliation in such a chart is represented by nonnegative branch weights satisfying the switch equations: at each switch, the total weight on one side equals that on the other. These charts define the standard piecewise-linear structure; see Penner–Harer [PH92, Chapters 1 and 3]. In suitable source and target charts, the mapping class action is linear on these weights. We now relate these coordinates to period coordinates.

Near a principal differential $q$, the real and imaginary edge periods record the transverse measures of the vertical and horizontal foliations. We choose a saddle-connection triangulation with no horizontal or vertical edge, so that the corresponding train-track weights are linear functions of the real and imaginary periods throughout a small neighborhood. The two foliations can then be varied independently, giving the local product structure in Proposition 3.5. Fixing the horizontal foliation, for example, gives real period coordinates for the vertical foliation. This identification will let us use Gauss–Manin transport to compare foliation coordinates along paths. The size of the neighborhood $\mathcal{V}(q)$ may depend on $q$.

**Proposition 3.5.** *At every $q_0 \in \mathcal{PQT}(S)$ there is a saddle-connection triangulation with no horizontal or vertical edge. It persists in a period neighborhood*

$$\mathcal{V}(q_0) = \Phi^{-1}(U + iV),$$

*where $U, V$ are open convex subsets of a fixed real odd cohomology space $L$. There are fixed train-track coordinate maps $f, h$ such that*

$$F_+\big(\Phi^{-1}(u+iv)\big) = f(u), \qquad F_-\big(\Phi^{-1}(u+iv)\big) = h(v).$$

*Their train-track weights are linear functions of $u$ and $v$, respectively. In particular, with $\mathcal{V}_+(q_0) = f(U)$ and $\mathcal{V}_-(q_0) = h(V)$, the foliation map is a homeomorphism*

$$\mathcal{V}(q_0) \longrightarrow \mathcal{V}_+(q_0) \times \mathcal{V}_-(q_0).$$

*Proof.* Suppose first that $\Delta$ is a saddle-connection triangulation with no horizontal or vertical edge. We explain how it gives the product coordinates, then construct such a triangulation.

The triangles of $\Delta$ persist in a small period neighborhood of $q_0$. Shrink it until the real and imaginary parts of every edge period remain nonzero with fixed signs. In each triangle, the largest absolute real edge period is the sum of the other two. Place a trivalent switch in the triangle, with its large branch dual to the edge with the largest weight and its two small branches dual to the other edges, as in Figure 1. Gluing dual branches across edges gives a train track $\tau_+$ carrying the vertical foliation; the branch crossing an edge has weight equal to the absolute value of that edge's real period. The horizontal construction gives $\tau_-$. A simple zero gives a complementary trigon, and a simple pole gives a punctured monogon. Thus these are maximal train tracks, and the weights of the two foliations are strictly positive.

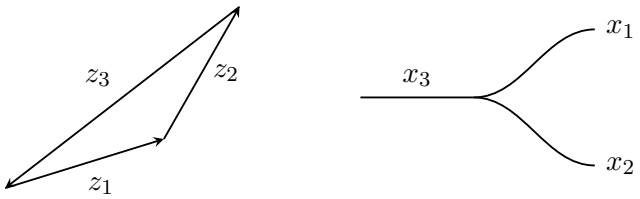


FIGURE 1. From triangle periods to switch weights: $x_j = |\mathrm{Re}\, z_j|$, with $\mathrm{Re}\, z_1, \mathrm{Re}\, z_2 > 0$ and $\mathrm{Re}\, z_3 < 0$.

Lift $\Delta$ to the orienting cover and orient one lift of each edge so that its real period is positive. An odd relative cohomology class is represented by an odd cellular 1-cocycle on this triangulation: there are no relative 0-cochains, since all vertices belong to the relative set. Its cocycle equation on each triangle is exactly the switch equation for $\tau_+$. Conversely, switch weights assign values to all lifted edges, with the required signs, and give such a cocycle. This defines a fixed linear isomorphism

$$E_+ : L \longrightarrow W(\tau_+),$$

where $W(\tau_+)$ is the real vector space of switch weights. We used the identification of odd relative and absolute cohomology from (3.1); every zero and pole in the principal stratum is a branch point. Repeated edges in a triangle are counted with their multiplicities. The imaginary periods give $E_-$ and $\tau_-$ in the same way.

Positive weights in these maximal track charts give local coordinates on $\mathcal{MF}$. Choose a small convex product $U + iV$ inside the period neighborhood. The two fixed linear maps $E_+$ and $E_-$ give the asserted maps $f, h$. Their images are open, and

both maps are homeomorphisms onto their images. This gives the claimed coordinates from $\Delta$.

It remains to choose $\Delta$. We follow [Dum15, Lemma 5.7]. We avoid both coordinate directions and include surfaces with punctures. First rescale $q_0$ to area one; rescaling does not change its saddle connections or their directions. There are only finitely many horizontal and vertical saddle connections on $q_0$, since each starts along one of finitely many prongs. By [KW04, Theorem 1.2(iii)], choose a rotation $r_\theta q_0$ whose forward orbit in moduli space has compact closure inside its stratum, and such that neither original coordinate direction is contracted by $g_t r_\theta$. The theorem applies to punctured surfaces, including one-prong singularities at simple poles. Each of the finitely many original horizontal and vertical saddle connections has length tending to infinity under this flow.

Choose local marked lifts of the compact orbit closure. At each lift, choose a triangulation by saddle connections, with vertices at the zeros and poles, that persists nearby in the stratum. A finite cover by smaller such neighborhoods provides a number $M$ such that every surface in the orbit closure has one of these triangulations with all edge lengths at most $M$. For large $t$, none of the original horizontal or vertical saddle connections can be an edge of that triangulation on $g_t r_\theta q_0$. Pull it back by the affine map, and undo the initial area normalization, to obtain the required triangulation $\Delta$ of $q_0$. The parameters $\theta$, $t$, and $M$ are chosen for this $q_0$; the size of the resulting product neighborhood may depend on $q_0$. □

It follows in particular that, for $p_1, p_2 \in \mathcal{V}(q)$,

$$F_+(p_1) = F_+(p_2) \quad \Longleftrightarrow \quad \operatorname{Re} \Phi_q(p_1) = \operatorname{Re} \Phi_q(p_2). \tag{3.3}$$

The corresponding equivalence holds for the horizontal foliation and the imaginary periods. These are local assertions. Along a path they apply by continuation through successive product boxes; they do not require one product box to contain the entire path.

**Homogeneous foliation coordinates and their transport.** We now assign a real vector to a measured foliation, keeping track of its scale. To do this near $F_+(q)$, fix the horizontal foliation $F_-(q)$ and take the real period of the resulting differential. For $G \in \mathcal{V}_+(q)$, define

$$\Psi_q(G) = \operatorname{Re} \Phi_q\big(Q(G, F_-(q))\big).$$

In the preceding coordinates this is $f^{-1}(G)$. We extend it homogeneously by setting

$$\Psi_q : \mathbb{R}_+ \mathcal{V}_+(q) \longrightarrow L(q), \qquad \Psi_q(cG) = c\, f^{-1}(G).$$

The extension is well-defined because the train-track weights and their inverse linear map both respect positive scaling. Its domain is the cone over the local foliation neighborhood.

**Lemma 3.6.** *For $c > 0$ and $F \in \mathbb{R}_+ \mathcal{V}_+(q)$,*

$$\Psi_q(cF) = c\Psi_q(F).$$

*For $s > 0$, identify the orienting covers of $q$ and $s^2 q$ canonically and choose compatible period signs. Then the maps $\Psi_q$ and $\Psi_{s^2 q}$ agree as germs near their common projective vertical foliation: they agree on a common conical neighborhood of that ray.*

*Proof.* The first assertion follows from the homogeneous extension. Scalar multiplication by $s^2$ multiplies periods and both measured foliations by $s$, so $\Psi_{s^2q}(sG) = s\Psi_q(G)$ near $F_+(q)$. The first assertion then gives $\Psi_{s^2q}(G) = \Psi_q(G)$ on a common neighborhood of that ray. □

To describe these coordinates on a unit-area horosphere, first write the period of $q$ as $a + ib$ and set

$$A(u, v) = \frac{1}{2}\int_{\widetilde{X}} u \wedge v \qquad (u, v \in L(q)).$$

The area form on the orienting cover is $\operatorname{Re}\omega_q \wedge \operatorname{Im}\omega_q$. Since the cover has twice the area of $q$, we have

$$\operatorname{area}(q) = A(a, b).$$

In coordinates continued along the Teichmüller flow, the period becomes $e^t a + ie^{-t}b$. This describes the stretching and contraction while also showing that area is preserved.

Now suppose $q$ has area one. Fixing its horizontal foliation fixes $b$, so a small neighborhood in its unit-area horosphere has affine coordinates $a + v + ib$ with $A(v, b) = 0$. For a nonzero measured foliation $F$ such that $p = Q_1([F], F_-(q))$ lies in this neighborhood, the normalization in Section 2 gives $F_+(p) = F/\mathrm{i}(F, F_-(q))$. Homogeneity therefore gives

$$\Psi_q(F) = \mathrm{i}(F, F_-(q)) \operatorname{Re}\Phi_q(p).$$

The point $p$ records the projective class $[F]$; the intersection factor restores the scale of $F$. The same formula defines an extension whenever these affine leaf coordinates continue injectively to a larger domain. Lemma 4.3 will supply such a domain of uniform intrinsic size. We shrink the original local period box so that its homogeneous coordinate is the restriction of that extension.

We next compare the vectors representing a fixed measured foliation as the base differential moves. The straight paths in the leaf charts specify how these coordinates are continued. Requiring those paths to vary continuously makes their continuation agree with Gauss–Manin transport along the base path.

**Lemma 3.7.** *Let $\xi : [0, 1] \to \mathcal{PQ}_1\mathcal{T}$ be a continuous path from $q$ to $p$, and let $F \neq 0$. Suppose that*

$$p_t = Q_1([F], F_-(\xi(t)))$$

*belongs to the affine leaf-coordinate domain based at $\xi(t)$ used to define $\Psi_{\xi(t)}(F)$. Suppose also that the straight leaf paths from $\xi(t)$ to $p_t$ vary continuously with $t$. With signs chosen consistently along these paths,*

$$\mathrm{GM}_\xi\big(\Psi_q(F)\big) = \Psi_p(F).$$

*Proof.* Parametrize the straight leaf path from $\xi(t)$ to $p_t$ by $H(t, s)$, $0 \le s \le 1$. Thus $H(t, 0) = \xi(t)$ and $H(t, 1) = p_t$. By assumption this is a continuous homotopy in the principal stratum; its sides $H(0, s)$ and $H(1, s)$ are the leaf paths based at $q$ and $p$. Write $c_t = \mathrm{i}(F, F_-(\xi(t)))$. By the definition of the homogeneous coordinate, transport along the leaf path identifies $\Psi_{\xi(t)}(F)$ with $c_t \operatorname{Re}\omega_{p_t}$.

The vertical foliation of $p_t$ is $F/c_t$. Consequently, the path $c_t^2 p_t$ has the fixed vertical foliation $F$. In each local product box its real period is constant, by (3.3). Continuing this identity through finitely many boxes shows that $c_t \operatorname{Re}\omega_{p_t}$ is parallel

along $p_t$. Scalar multiplication gives the canonical identification of the covers used here. Finally, homotopy invariance of Gauss–Manin transport along the boundary of this homotopy identifies the endpoint coordinates as claimed. □

The construction following Lemma 4.3 supplies uniform leaf domains with the continuous paths required here. In applications it will therefore suffice to place the points $p_t$ in those domains.

Finally, recall that $\mathrm{Mod}(S)$ acts by changing markings, with the convention fixed in Section 2. For $\gamma \in \mathrm{Mod}(S)$, the induced linear map $\gamma^* : L(q) \to L(\gamma q)$ is defined as follows: for each $v \in L(q) = H^1_{\mathrm{odd}}(\widetilde{X}; \mathbb{R})$ and each homology class $h \in H_1(\widetilde{\gamma X}; \mathbb{R})$, set

$$\big(\gamma^* v\big)(h) = v\big(\gamma_*^{-1} h\big).$$

The map $\gamma^*$ is compatible with the Gauss–Manin map in the following sense:

**Lemma 3.8.** *For $q \in \mathcal{PQT}(S)$, $\gamma \in \mathrm{Mod}(S)$, and $F \in \mathbb{R}_+\mathcal{V}_+(q)$, the homogeneous local coordinates satisfy*

$$\gamma^*\big(\Psi_q(F)\big) = \Psi_{\gamma q}(\gamma F).$$

*Here the local charts are taken to be translates, and the period signs are chosen compatibly. The same identity holds on the extended unit-area domains constructed after Lemma 4.3. If a path $\xi$ from $q$ to $\gamma q$ satisfies the hypotheses of Lemma 3.7 for $\gamma F$, then also*

$$\gamma^*\big(\Psi_q(F)\big) = \Psi_{\gamma q}(\gamma F) = \mathrm{GM}_\xi\big(\Psi_q(\gamma F)\big).$$

*Proof.* Changing the marking sends $Q(G, F_-(q))$ to $Q(\gamma G, F_-(\gamma q))$ and sends its period vector to $\gamma^*$ of that vector. Taking real parts proves the local identity, and homogeneity extends it to the cone $\mathbb{R}_+\mathcal{V}_+(q)$. The affine leaf continuation, its norm domain, and intersection number are likewise equivariant, so the same proof applies on the extended domain. The final equality is Lemma 3.7 applied to $\gamma F$. □

## 4. The Avila-Gouëzel-Yoccoz metric

We use the Avila–Gouëzel–Yoccoz norm to measure changes in period coordinates. It gives uniform charts on small horospherical neighborhoods and controls projection to a common horosphere. We also record how horospherical distances change under the Teichmüller flow, with strict contraction when the orbit spends enough time in a compact set.

The norm was introduced in [AGY06] and its further properties were developed in [AG13]. We work on the marked principal stratum; the construction is $\mathrm{Mod}(S)$–invariant and descends to moduli space. By Theorem 3.2, for $(X, q) \in \mathcal{PQT}$ we have

$$T_q\mathcal{PQT} \cong H^1_{\mathrm{odd}}(\widetilde{X}, \mathcal{Z}(q); \mathbb{C}).$$

For a tangent vector $\omega$ in this space, define

$$\|\omega\|_q = \sup_{\substack{\gamma \subset (\widetilde{X}, \tilde{q}) \\ \gamma \text{ a saddle connection}}} \frac{\big|\omega([\gamma])\big|}{\mathrm{Flat}_{\tilde{q}}(\gamma)}. \tag{4.1}$$

Here the supremum runs over all saddle connections $\gamma$ of the flat surface $(\widetilde{X}, \tilde{q})$; equivalently, the denominator is $\big|\int_\gamma \omega_q\big|$. The points of $\mathcal{Z}(q)$ are retained as marked

endpoints of saddle connections. In particular, this includes the preimages of simple poles, which are regular points of $\omega_q$ on the orienting cover. The norm is continuous [AGY06, Propositions 2.10–2.11].

The Abelian strata in [AGY06, Section 2.2.1] allow regular marked points, so they include these marked preimages of poles. For $q \in \mathcal{PQ}_1\mathcal{T}$, the cover has area 2. To apply their unit-area results, rescale its Abelian differential by $1/\sqrt{2}$. Scaling the tangent vector by the same factor leaves the ratio in (4.1) unchanged. We use their estimates on the locus carrying the orienting involution, restricted to odd cohomology.

For a piecewise $C^1$ path $\xi\colon [0,1] \to \mathcal{PQ}_1\mathcal{T}$, its length is

$$\ell(\xi) = \int_0^1 \|\dot{\xi}(t)\|_{\xi(t)}\, dt. \tag{4.2}$$

The *ambient AGY-metric* on $\mathcal{PQ}_1\mathcal{T}$ is

$$D_{\mathrm{AGY}}(q,q') = \inf\big\{\ell(\xi) : \xi(0) = q,\ \xi(1) = q',\ \xi([0,1]) \subset \mathcal{PQ}_1\mathcal{T}\big\}.$$

Here ambient paths lie in the unit-area principal stratum, while horospherical paths will be constrained to a single horosphere. Completeness follows from [AGY06, Corollary 2.13] on the marked orienting cover. The locus on which the fixed covering involution acts by $-1$ is closed; local period charts give convergence in its intrinsic length metric as well. Its moduli quotient is a complete, locally compact length space, hence is proper. In particular, closed neighborhoods of Mod–compact sets are again Mod–compact.

**Notation 4.1.** For $q \in \mathcal{PQ}_1\mathcal{T}$ and $r > 0$, let

$$B_{\mathrm{AGY}}(q,r) = \big\{p \in \mathcal{PQ}_1\mathcal{T} :\ D_{\mathrm{AGY}}(p,q) < r\big\}$$

be the ambient *AGY*–ball of radius $r$ around $q$. Also, for a set $E \subset \mathcal{PQ}_1\mathcal{T}$, let $N_{\mathrm{AGY}}(E,r)$ be the $r$–neighborhood of $E$ in the ambient *AGY*–metric.

**AGY norm for $L(q)$.** The same norm restricts to the real space $L(q)$ from (3.2). Recall the area pairing

$$A(u,v) = \frac{1}{2}\int_{\widetilde{X}} u \wedge v \qquad (u,v \in L(q)).$$

We will need to control the change of area when periods are varied. The following comparison gives a bound independent of the differential. Here $\|u\|_{H,q}$ is the $L^2$ norm of the harmonic representative of the absolute class $u$ on the compact orienting cover $\widetilde{X}$.

**Lemma 4.2.** *For $q \in \mathcal{PQ}_1\mathcal{T}$ and $u \in L(q)$, we have*

$$\|u\|_{H,q} \le \sqrt{2}\,\|u\|_q. \tag{4.3}$$

*Consequently,*

$$|A(u,v)| \le \|u\|_q\,\|v\|_q \qquad (u,v \in L(q)). \tag{4.4}$$

*Proof.* The periods of $u$ define a function on the lifts of the marked points. We extend it to a Lipschitz function on the universal cover. Its differential will represent $u$ with controlled energy.

Let $Y$ be the universal cover of $\widetilde{X}$ with its lifted flat metric, let $\widehat{\Sigma}$ be the set of lifts of $\mathcal{Z}(q)$, and fix $z_0 \in \widehat{\Sigma}$. For $z \in \widehat{\Sigma}$, define $f(z)$ by evaluating $u$ on the projection of

a path from $z_0$ to $z$. This is independent of the path because $Y$ is simply connected. A minimizing flat geodesic between two points of $\widehat{\Sigma}$ splits into finitely many saddle connections: the lifted flat metric is proper and its marked set is locally finite. The definition of the AGY norm, with $N = \|u\|_q$, therefore gives

$$|f(z) - f(z')| \le N\, d_Y(z, z').$$

For a deck transformation $h$ of $Y$, we also have $f(hz) = f(z) + u(h)$, where $u(h)$ is the absolute period of $u$.

The extension

$$\widetilde{f}(y) = \inf_{z \in \widehat{\Sigma}} \big( f(z) + N\, d_Y(y, z) \big)$$

is finite, is $N$–Lipschitz, and agrees with $f$ on $\widehat{\Sigma}$. Reindexing the infimum gives $\widetilde{f}(hy) = \widetilde{f}(y) + u(h)$. Hence $d\widetilde{f}$ descends to a weakly closed real one-form on $\widetilde{X}$ representing the absolute image of $u$, with pointwise flat norm at most $N$ almost everywhere. Since the orienting cover has area 2, its energy is at most $2N^2$. Energy of one-forms on a surface is conformally invariant, and the harmonic representative minimizes energy, also among weakly closed $L^2$ representatives by smooth approximation. This proves (4.3).

The Hodge Cauchy–Schwarz inequality now gives

$$|A(u, v)| \le \frac{1}{2} \|u\|_{H,q} \|v\|_{H,q} \le \|u\|_q \|v\|_q. \qquad \square$$

If the period of $q \in \mathcal{PQ}_1\mathcal{T}$ is $a + ib$, then $\|a\|_q, \|b\|_q \le 1$ directly from (4.1). Since $A(a, b) = 1$, the lemma gives

$$1 = A(a, b) \le \|a\|_q \|b\|_q \le 1.$$

Thus $\|a\|_q = \|b\|_q = 1$. We will use this normalization when comparing foliation coordinates along the flow.

We also use the following norm comparison along a path. If $\xi$ is a piecewise $C^1$ path in $\mathcal{PQ}_1\mathcal{T}$, of length $L$, and $v$ is an odd complex cohomology class at its initial point, then

$$e^{-L} \|v\|_{\xi(0)} \le \|\mathrm{GM}_\xi(v)\|_{\xi(1)} \le e^{L} \|v\|_{\xi(0)}. \tag{4.5}$$

This is [AG13, Proposition 5.5], restricted to the odd subspace as above.

**The metric on horospheres.** Suppose $x \in \mathcal{PQ}_1\mathcal{T}$ and $y \in \mathrm{Hor}(x) \cap \mathcal{PQ}_1\mathcal{T}$. The *horospherical AGY-metric* is

$$d_{\mathrm{AGY}}(x, y) = \inf \big\{ \ell(\xi) : \xi(0) = x,\ \xi(1) = y,\ \xi(t) \in \mathrm{Hor}(x) \cap \mathcal{PQ}_1\mathcal{T}, 0 \le t \le 1 \big\}.$$

The infimum is over piecewise $C^1$ paths, with length defined by (4.2). This is the metric denoted by $d_{W^u}$ in [AG13, Section 5.1].

We use the same definition and notation on $\mathrm{Hor}_+(x) \cap \mathcal{PQ}_1\mathcal{T}$.

We now continue affine coordinates a uniform distance along a single horosphere. Write the period of $q \in \mathcal{PQ}_1\mathcal{T}$ as $a + ib$, and put

$$E_q = \{ v \in L(q) : A(v, b) = 0,\ \|v\|_q < 1/2 \}.$$

The condition $A(v, b) = 0$ makes the segment $a + tv + ib$ stay in the area-one hyperplane.

**Lemma 4.3.** *For every $q \in \mathcal{PQ}_1\mathcal{T}$, continuation of the periods $a+tv+ib$, $0 \le t \le 1$, defines an injective coordinate map*

$$\mathrm{Exp}_q : E_q \longrightarrow \mathrm{Hor}(q) \cap \mathcal{PQ}_1\mathcal{T}.$$

*Its image contains every $p \in \mathrm{Hor}(q) \cap \mathcal{PQ}_1\mathcal{T}$ with $d_{\mathrm{AGY}}(q,p) \le 1/50$.*

*Proof.* The radial continuation is the one in [AG13, Proposition 5.3], restricted to the odd real subspace and the area-one hyperplane. Here is the length estimate that ensures it exists. If $\ell(t)$ is the length of the radial lift, then (4.5) gives

$$\ell'(t) \le e^{\ell(t)}\|v\|_q, \qquad \ell(t) \le -\log(1 - t\|v\|_q).$$

Completeness continues the lift to $t = 1$, and the local product description in Proposition 3.5 keeps its horizontal foliation equal to $F_-(q)$. Covering a lifted segment by finitely many period charts also gives continuous dependence on $q, v, t$, with compatible local markings.

We will use local inverses for this family before proving global injectivity. Fix a radial segment, and transport its endpoint period chart back to $L(q)$. In these coordinates the endpoint period is $a(q) + v + ib(q)$. The horizontal-coordinate maps at the two ends agree near $b(q)$: nearby radial lifts preserve the horizontal foliation, and Proposition 3.5 identifies that foliation with the imaginary period. Thus, for a nearby pair $(q', p')$ with the same horizontal foliation and area one, the imaginary period of $p'$ is $b(q')$, and its inverse vector is its real period minus $a(q')$. This vector satisfies $A(v', b(q')) = 0$ and remains in the open norm ball of radius $1/2$. The inverse depends continuously on the pair. This argument uses local period charts, without a smoothness assumption on foliation coordinates.

First suppose that $H = F_-(q)$ has the principal singularity type and no saddle connections. Every differential with horizontal foliation a positive multiple of $H$ is then principal. Hubbard–Masur gives $D\,\mathrm{Hor}_-(H) \cong \mathcal{T}(S) \times \mathbb{R}_{>0}$. If $r$ in this set has area $B$ and horizontal foliation $cH$, the normalization

$$r \longmapsto \begin{bmatrix} c/B & 0 \\ 0 & 1/c \end{bmatrix} r$$

together with the parameters $B, c$, identifies this set with $\mathrm{Hor}_-(H) \times \mathbb{R}^2_{>0}$. Hence $\mathrm{Hor}_-(H)$ is contractible, and every loop in it has trivial Gauss–Manin transport with the square-root sign continued consistently. Two radial paths with the same endpoint therefore have the same endpoint period in $L(q)$, so their displacement vectors agree.

For an arbitrary $q$, suppose that distinct $v, w \in E_q$ have the same image $p$. Choose disjoint local inverse branches about $(q, v)$ and $(q, w)$ as above. Take $q_n \to q$ with horizontal foliations having no saddle connections; small rotations avoiding the countably many saddle-connection directions suffice. Set

$$p_n = Q_1\big([F_+(p)], F_-(q_n)\big).$$

One local product neighborhood at $p$ shows that $p_n$ is principal for large $n$ and tends to $p$. The pair $(q_n, p_n)$ then lies in the domains of both inverse branches. This gives distinct vectors in $E_{q_n}$ with the same image $p_n$, contradicting the generic case. Thus $\mathrm{Exp}_q$ is injective at every center.

Finally, let $d_{\mathrm{AGY}}(q,p) \le 1/50$. Choose an intrinsic path from $q$ to $p$ of length less than $1/40$ and continue its periods from $q$. Its imaginary period is $b$, and its real

displacement $u(s)$ satisfies

$$\|u(s)\|_q \le e^{\ell(s)} - 1 < e^{1/40} - 1 < 1/2$$

by (4.5). Local uniqueness of period lifts identifies the path with $\mathrm{Exp}_q(u(s))$. In particular, $p$ belongs to the image. □

These charts give the larger domains for $\Psi_q$ promised in Section 3. For $F \neq 0$ such that $p = Q_1([F], F_-(q)) = \mathrm{Exp}_q(v)$, define

$$\Psi_q(F) = \mathrm{i}(F, F_-(q))\,(a + v). \tag{4.6}$$

The factor restores the scale of $F$, since $F_+(p) = F/\mathrm{i}(F, F_-(q))$. This agrees with the local homogeneous coordinate after its original neighborhood is shrunk.

The joint local inverses in the proof show that the image of the family of charts is open in the space of pairs with the same horizontal foliation and area one. Consequently the set of pairs $(q, F)$ on which (4.6) is defined is open. On this set, $\Psi_q(F)$ and the radial paths $t \mapsto \mathrm{Exp}_q(tv)$ depend continuously on $(q, F)$ in local markings. In particular, these paths satisfy the continuity requirement in Lemma 3.7. All the constructions commute with changing markings. Reversing the square-root sign reverses every period vector, so they also respect the sign convention in Lemma 3.8.

We will also use the stable version of the construction. Rotating periods by $a+ib \mapsto -b+ia$ preserves the AGY norm and exchanges the two foliations. Thus, if $A(a, u) = 0$ and $\|u\|_q < 1/2$, the segment $a + i(b + tu)$ continues in $\mathrm{Hor}_+(q) \cap \mathcal{PQ}_1\mathcal{T}$, has length at most $-\log(1 - \|u\|_q)$, and depends continuously on its parameters with the same local inverse properties.

**Proposition 4.4.** *Let $q \in \mathcal{PQ}_1\mathcal{T}$, and suppose $p_1, p_2 \in \mathrm{Hor}(q) \cap \mathcal{PQ}_1\mathcal{T}$ satisfy $d_{\mathrm{AGY}}(q, p_i) \le 1/200$ for $i = 1, 2$. Put $F_i = F_+(p_i)$. Then*

$$\frac{1}{2} d_{\mathrm{AGY}}(p_1, p_2) \le \|\Psi_q(F_1) - \Psi_q(F_2)\|_q \le 2 d_{\mathrm{AGY}}(p_1, p_2).$$

*Proof.* Use Lemma 4.3 to write $p_i = \mathrm{Exp}_q(v_i)$. Since the two points have area one and the same horizontal foliation as $q$, their foliation vectors are $a+v_i$ in these coordinates. Norms throughout the affine domain are comparable to the norm at $q$ by a factor of two: the radial paths have length less than $\log 2$, so this follows from (4.5).

The straight segment from $v_1$ to $v_2$ lies in the convex set $E_q$. Its image has length at most $2\|v_1 - v_2\|_q$, which gives the first inequality. Conversely, a nearly minimizing path from $p_1$ to $p_2$ has length less than $1/100 + \varepsilon$. Every point on it is within $3/200 + \varepsilon < 1/50$ of $q$, for small $\varepsilon$, so the entire path lies in this one leaf chart. Integrating its period velocity and letting the length error tend to zero gives $\|v_1 - v_2\|_q \le 2 d_{\mathrm{AGY}}(p_1, p_2)$. □

**The AGY metric and the Teichmüller geodesic flow.** Teichmüller geodesics are 1–Lipschitz paths in the AGY metric [AG13, Lemma 5.1]: for $q \in \mathcal{PQ}_1\mathcal{T}$ and $t \in \mathbb{R}$,

$$D_{\mathrm{AGY}}(g_t(q), q) \le |t|. \tag{4.7}$$

On an unstable horosphere, backward flow does not increase distances. The forward flow has the following upper bound [AG13, Lemma 5.2].

**Theorem 4.5.** *For $F \in \mathcal{MF}$ and $x, y \in \mathcal{PMF}(F)$ with $x_F(0), y_F(0) \in \mathcal{PQ}_1\mathcal{T}$, and for $s \le t$, we have*

$$d_{\mathrm{AGY}}\big(x_F(s), y_F(s)\big) \le d_{\mathrm{AGY}}\big(x_F(t), y_F(t)\big) \le e^{2(t-s)} d_{\mathrm{AGY}}\big(x_F(s), y_F(s)\big). \tag{4.8}$$

*Proof.* For $h \ge 0$, the flow multiplies an unstable tangent vector by $e^h$, while the length of each saddle connection changes by a factor between $e^{-h}$ and $e^h$. The definition of the AGY norm therefore gives

$$\|v\|_q \le \|Dg_h(v)\|_{g_h q} \le e^{2h}\|v\|_q \qquad (v \in T_q \operatorname{Hor}(q)).$$

Integrating along paths in the horosphere, and using that the flow maps these paths bijectively to paths in the image horosphere, proves the two distance inequalities. □

Gauss–Manin transport keeps a cohomology vector fixed while the norm changes. For the vector representing the vertical foliation, homogeneity gives an exact formula.

**Lemma 4.6.** *Let $q \in \mathcal{PQ}_1\mathcal{T}$, $\tau \ge 0$, and let $\xi$ be the segment $q(t) = g_t(q)$, $0 \le t \le \tau$. For $\mathbf{v} \in L(q)$,*

$$e^{-\tau}\|\mathbf{v}\|_q \le \|\mathrm{GM}_\xi(\mathbf{v})\|_{q(\tau)} \le e^{\tau}\|\mathbf{v}\|_q. \tag{4.9}$$

*If $[F] = [F_+(q)]$, then*

$$\|\Psi_q(F)\|_q = e^{\tau}\|\Psi_{q(\tau)}(F)\|_{q(\tau)}. \tag{4.10}$$

*Proof.* Under the flow, saddle connections correspond bijectively and their lengths change by factors between $e^{-\tau}$ and $e^{\tau}$. Their evaluations on the transported class are unchanged, proving (4.9). For (4.10), write $F = cF_+(q) = ce^{-\tau}F_+(q(\tau))$. The real period has AGY norm one throughout $\mathcal{PQ}_1\mathcal{T}$, as observed after Lemma 4.2. Homogeneity of $\Psi$ therefore gives $\|\Psi_q(F)\|_q = c$ and $\|\Psi_{q(\tau)}(F)\|_{q(\tau)} = ce^{-\tau}$. □

Time spent in a compact set gives strict contraction. We first record the tangent-vector estimate from [AG13, Proposition 4.3].

**Proposition 4.7.** *For every* Mod*–compact set $K' \subset \mathcal{PQ}_1\mathcal{T}$, there is $T = T(K') > 0$ such that, if $x \in \mathcal{PQ}_1\mathcal{T}$ and $t \ge 0$ satisfy*

$$\big|\{s \in [0,t] : g_{-s}x \in K'\}\big| \ge T, \tag{4.11}$$

*then*

$$\|Dg_{-t}v\|_{g_{-t}x} \le \frac{1}{2}\|v\|_x \qquad (v \in T_x \operatorname{Hor}(x)).$$

*Here $T_x \operatorname{Hor}(x)$ is the real period subspace $\{u \in L(x) : A(u, \mathit{Im}\,\Phi_x(x)) = 0\}$.*

*Proof.* The cited statement assumes that both endpoints lie in the compact set. Trim the interval to its first and last visits to $K'$. These visits are attained because $K'$ is closed (Definition 2.1), and trimming removes no time spent in $K'$. Apply the cited contraction estimate on this interval. Backward flow does not increase the unstable tangent norm on either omitted piece, by the calculation above. □

To pass from tangent vectors to distances, we allow the points of a short connecting path to visit a slightly larger compact set.

**Corollary 4.8.** *For every* Mod*–compact set* $K \subset \mathcal{PQ}_1\mathcal{T}$*, there is* $T = T(K) > 0$ *such that the following holds. Suppose* $x \in K$*,* $y \in \mathrm{Hor}(x) \cap \mathcal{PQ}_1\mathcal{T}$*, and* $d_{\mathrm{AGY}}(x, y) \le \frac{1}{100}$*. If* $t > 0$ *and*

$$\big|\{s \in [0, t] : g_{-s}x \in K\}\big| \ge T, \tag{4.12}$$

*then*

$$d_{\mathrm{AGY}}(g_{-t}x, g_{-t}y) \le \frac{1}{2} d_{\mathrm{AGY}}(x, y).$$

*Proof.* Let $K'$ be the closed 1/100–neighborhood of $K$ in the ambient AGY metric. This is Mod–compact by properness of the quotient metric. Choose $T = T(K')$ from Proposition 4.7.

First suppose $d_{\mathrm{AGY}}(x, y) < 1/100$. Choose a piecewise $C^1$ path $\gamma$ in $\mathrm{Hor}(x) \cap \mathcal{PQ}_1\mathcal{T}$ from $x$ to $y$, of length less than $d_{\mathrm{AGY}}(x, y) + \varepsilon < 1/100$. Backward nonexpansion implies that every point of $g_{-s}\gamma$ is within 1/100 of $g_{-s}x$. Whenever $g_{-s}x \in K$, all these points therefore lie in $K'$. Apply Proposition 4.7 to the tangent vectors along $\gamma$ and integrate:

$$d_{\mathrm{AGY}}(g_{-t}x, g_{-t}y) \le \ell(g_{-t}\gamma) \le \frac{1}{2}\ell(\gamma) < \frac{1}{2}\big(d_{\mathrm{AGY}}(x, y) + \varepsilon\big).$$

Let $\varepsilon$ tend to zero. If $d_{\mathrm{AGY}}(x, y) = 1/100$, truncating nearly minimizing paths gives points $y_j$ with $d_{\mathrm{AGY}}(x, y_j) < 1/100$ and $d_{\mathrm{AGY}}(y_j, y) \to 0$. Backward nonexpansion also gives $d_{\mathrm{AGY}}(g_{-t}y_j, g_{-t}y) \to 0$, so the same estimate follows by passage to the limit. □

Repeated halving gives exponential contraction when the orbit spends a fixed proportion of its time in $K$.

**Corollary 4.9.** *For every* Mod*–compact set* $K \subset \mathcal{PQ}_1\mathcal{T}$*, there exist constants* $0 < \beta_K < 1$ *and* $C_K \ge 1$*, depending only on* $K$*, with the following property. Let* $F \in \mathcal{MF}$ *and* $x, y \in \mathcal{PMF}(F)$ *with* $x_F(0), y_F(0) \in \mathcal{PQ}_1\mathcal{T}$*. If* $s \le t$*,*

$$d_{\mathrm{AGY}}(x_F(t), y_F(t)) \le \tfrac{1}{100}, \qquad \big|\{u \in [s, t] : x_F(u) \in K\}\big| \ge \frac{1}{2}(t - s),$$

*then*

$$d_{\mathrm{AGY}}\big(x_F(s), y_F(s)\big) \le C_K e^{-\beta_K(t-s)} d_{\mathrm{AGY}}\big(x_F(t), y_F(t)\big). \tag{4.13}$$

*Proof.* Let $T = T(K)$ be given by Corollary 4.8, and put

$$m(u) = \big|\{v \in [s, u] : x_F(v) \in K\}\big|, \qquad M = m(t), \qquad N = \lfloor M/T \rfloor.$$

For $1 \le j \le N$, let $s_j$ be the first time at which $m(s_j) = jT$, and put $s_0 = s$. Continuity of $m$ gives these times. Since $K$ is closed, minimality of $s_j$ implies $x_F(s_j) \in K$: otherwise $m$ would be constant on a neighborhood of $s_j$. Each interval $[s_{j-1}, s_j]$ contains $T$ units of time in $K$.

Starting at time $t$, apply backward nonexpansion down to $s_N$ and Corollary 4.8 on each of the $N$ full intervals. All distances remain at most 1/100. With no full interval, use nonexpansion on $[s, t]$. Since $N \ge M/T - 1$, in either case

$$\begin{aligned} d_{\mathrm{AGY}}(x_F(s), y_F(s)) &\le 2e^{-(\log 2)M/T} d_{\mathrm{AGY}}(x_F(t), y_F(t)) \\ &\le 2e^{-(\log 2)(t-s)/(2T)} d_{\mathrm{AGY}}(x_F(t), y_F(t)). \end{aligned}$$

Thus, for example, $C_K = 2$ and $\beta_K = \min\{1/2, (\log 2)/(2T)\}$ satisfy the assertion. □

For points farther apart, each full block of compact time removes a fixed amount of distance, as long as the final distance is at least 1/100.

**Corollary 4.10.** *Let $K \subset \mathcal{PQ}_1\mathcal{T}$ be* Mod*–compact, and let $T = T(K)$ be as in Corollary 4.8. Suppose $x \in \mathcal{PQ}_1\mathcal{T}$, $y \in \mathrm{Hor}(x) \cap \mathcal{PQ}_1\mathcal{T}$, and $t \geq 0$ satisfy*

$$d_{\mathrm{AGY}}(g_{-t}x, g_{-t}y) \geq \tfrac{1}{100}, \qquad \big|\{s \in [0,t] : g_{-s}x \in K\}\big| \geq 2T.$$

*Then*

$$d_{\mathrm{AGY}}(g_{-t}x, g_{-t}y) \leq d_{\mathrm{AGY}}(x,y) - \frac{1}{400T}\big|\{s \in [0,t] : g_{-s}x \in K\}\big|.$$

*Proof.* We may assume $d_{\mathrm{AGY}}(x,y) < \infty$, since otherwise the conclusion is immediate. Put

$$m(u) = \big|\{s \in [0,u] : g_{-s}x \in K\}\big|, \qquad M = m(t), \qquad N = \lfloor M/T \rfloor.$$

Let $t_0$ be the first visit to $K$, and for $1 \leq j \leq N$ let $t_j$ be the first time with $m(t_j) = jT$. These times exist, and $g_{-t_j}x \in K$ by closedness of $K$ and the same minimality argument as above. Each interval $[t_j, t_{j+1}]$ contains $T$ units of compact time. Write $x_j = g_{-t_j}x$ and $y_j = g_{-t_j}y$.

Backward nonexpansion and the hypothesis at time $t$ give $d_{\mathrm{AGY}}(x_j, y_j) \geq 1/100$. For $0 \leq j < N$, take a path in $\mathrm{Hor}(x_j) \cap \mathcal{PQ}_1\mathcal{T}$ from $x_j$ to $y_j$ of length less than $d_{\mathrm{AGY}}(x_j, y_j) + \varepsilon$, and let $z_j$ be its point at length 1/100. Since $d_{\mathrm{AGY}}(x_j, z_j) \leq 1/100$, Corollary 4.8 sends these two endpoints to distance at most 1/200 over the next block. The rest of the path does not increase in length. Therefore

$$d_{\mathrm{AGY}}(x_{j+1}, y_{j+1}) \leq \frac{1}{200} + d_{\mathrm{AGY}}(x_j, y_j) + \varepsilon - \frac{1}{100}.$$

Letting $\varepsilon$ tend to zero gives a decrease of 1/200 on each block. The initial and final omitted pieces are nonexpanding, so

$$d_{\mathrm{AGY}}(g_{-t}x, g_{-t}y) \leq d_{\mathrm{AGY}}(x,y) - \frac{N}{200} \leq d_{\mathrm{AGY}}(x,y) - \frac{M}{400T},$$

where $N \geq M/T - 1 \geq M/(2T)$ because $M \geq 2T$. □

**Lipschitz projection.** We now estimate projection along the center-stable foliation. Given a quadratic differential $q \in \mathcal{PQ}_1\mathcal{T}$, let $\mathcal{Q}_q \subset \mathcal{PQ}_1\mathcal{T}$ be the set of all quadratic differentials $q'$ such that $F_+(q')$ and $F_-(q)$ fill $S$. There is a map

$$\pi_q^{\mathrm{cs}} \colon \mathcal{Q}_q \to \mathrm{Hor}(q),$$

defined by

$$\pi_q^{\mathrm{cs}}(q') = Q_1\big([F_+(q')], F_-(q)\big).$$

Recall that $\mathrm{Hor}(q) = \mathrm{Hor}_-(q)$. Thus $\pi_q^{\mathrm{cs}}(q')$ is the unique point in $\mathrm{Hor}(q)$ with the same projective vertical foliation as $q'$. In other words, $\pi_q^{\mathrm{cs}}(q')$ is the unique point in the intersection of the center-stable foliation of $q'$ with the unstable foliation of $q$; see Figure 2. Recall that $B_{\mathrm{AGY}}(q', r) \subset \mathcal{PQ}_1\mathcal{T}$ is the ball of radius $r$ in the $D_{\mathrm{AGY}}$–metric centered at $q'$. The following estimate also ensures that this projection is defined on the stated neighborhood and remains in the principal stratum.

**Theorem 4.11.** *For $q, q' \in \mathcal{PQ}_1\mathcal{T}$, assume $D_{\mathrm{AGY}}(q, q') \leq \frac{1}{50}$. Then the projection $\pi_q^{\mathrm{cs}}$ is defined on $B_{\mathrm{AGY}}(q', \frac{1}{50})$, its image lies in $\mathrm{Hor}(q) \cap \mathcal{PQ}_1\mathcal{T}$, and*

$$d_{\mathrm{AGY}}\big(\pi_q^{\mathrm{cs}}(q_1), \pi_q^{\mathrm{cs}}(q_2)\big) \leq 4 D_{\mathrm{AGY}}(q_1, q_2) \qquad \textit{for } q_1, q_2 \in B_{\mathrm{AGY}}(q', \tfrac{1}{50}).$$

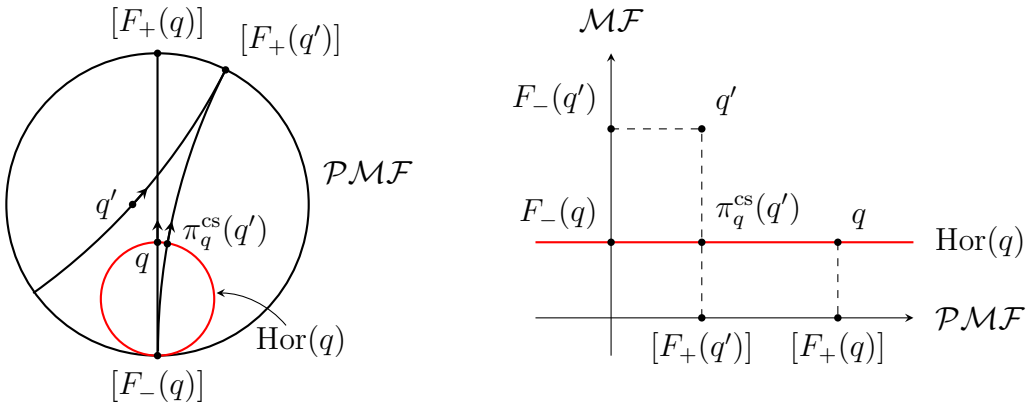


FIGURE 2. The map $\pi_q^{\mathrm{cs}}$.

*Proof.* In local period coordinates, the projection keeps the real period up to the factor needed to restore area one, and replaces the imaginary period by that of $q$. We first show that this operation can be continued along a short path. This proves that the projected differential exists in the principal stratum, including when one of its foliations has saddle connections. We then estimate its derivative.

Let $\kappa$ be a piecewise $C^1$ path in $\mathcal{PQ}_1\mathcal{T}$ starting at $q$, of length less than $\log(11/10)$. Transport the odd cohomology groups along $\kappa$, and write the period at $r_s = \kappa(s)$ as $a_s + ib_s$. Put $b = b_0$. If $L_s$ is the length of the initial segment of $\kappa$ ending at $r_s$, then (4.5), applied to the imaginary part of its velocity, gives

$$\delta_s := \|b - b_s\|_{r_s} \le \int_0^s e^{L_s - L_u} \|\dot\kappa(u)\|_{r_u}\, du = e^{L_s} - 1 < \frac{1}{10}.$$

By Lemma 4.2,

$$h_s := A(a_s, b), \qquad |h_s - 1| \le \delta_s.$$

The candidate for the projected period is therefore $a_s/h_s + ib$.

We construct this point in two steps: move along a stable leaf, then flow to the required normalization. Set

$$u_s = \frac{b}{h_s} - b_s.$$

Then $A(a_s, u_s) = 0$, and

$$\|u_s\|_{r_s} \le \frac{\|b - b_s\|_{r_s} + |1 - h_s|\|b_s\|_{r_s}}{h_s} \le \frac{2\delta_s}{1 - \delta_s} < \frac{1}{2}.$$

The stable construction following Lemma 4.3 therefore gives a principal differential with period $a_s + ib/h_s$. Flow for time $-\log h_s$ to obtain the period $a_s/h_s + ib$. Parametrize this concatenation by $H_s : [0,1] \to \mathcal{PQ}_1\mathcal{T}$. Its length satisfies

$$\ell(H_s) \le -\log(1 - \|u_s\|_{r_s}) + |\log h_s| \le -\log(1 - 3\delta_s),$$

where we used $|\log h_s| \le -\log(1 - \delta_s)$.

These connecting paths depend continuously on $s$ and form a homotopy from $\kappa$ to the path of their endpoints. Transport over the parameter square gives compatible period coordinates throughout this family. Each connecting path preserves the projective vertical foliation $[F_+(r_s)]$. The endpoint path has constant imaginary period

$b$ and starts at $q$, so its horizontal foliation is $F_-(q)$, by Proposition 3.5. These local foliation identities hold also at saddle connections. Uniqueness of the differential determined by its two foliations now gives

$$H_s(1) = Q_1\big([F_+(r_s)], F_-(q)\big) = \pi_q^{\mathrm{cs}}(r_s).$$

In particular, the foliations fill, the projected differential is principal, and it is independent of the path chosen from $q$. The same uniqueness allows us to forget the auxiliary markings of zeros and orienting covers.

We next estimate the derivative, using the local period formula $a/h + ib$. At $r = r_s$, write $a = a_s$, $h = h_s$, and $\delta = \delta_s$. For a tangent vector $\xi = u + iv$ to the unit-area locus, the derivative of the projection has real period

$$w = \frac{u}{h} - \frac{a\,A(u,b)}{h^2}.$$

Since $\|a\|_r = 1$, $\|b\|_r \le 1 + \delta$, and $\|u\|_r \le \|\xi\|_r$, Lemma 4.2 gives

$$\|w\|_r \le \frac{2}{(1-\delta)^2}\|\xi\|_r.$$

Transport along $H_s$ and (4.5) then give

$$\|D\pi_q^{\mathrm{cs}}(\xi)\|_{\pi_q^{\mathrm{cs}}(r)} \le \frac{2}{(1-\delta)^2(1-3\delta)}\|\xi\|_r \le \frac{2000}{567}\|\xi\|_r < 4\|\xi\|_r.$$

This is an intrinsic horospherical tangent norm, since the image has fixed horizontal foliation and area one.

Finally, let $q_1, q_2 \in B_{\mathrm{AGY}}(q', \frac{1}{50})$. Join $q$ to $q_1$ by a path of length arbitrarily close to $D_{\mathrm{AGY}}(q, q_1)$, and then join $q_1$ to $q_2$ by a path of length arbitrarily close to $D_{\mathrm{AGY}}(q_1, q_2)$. Their total length can be chosen less than $4/50 + \varepsilon < \log(11/10)$. The preceding construction and derivative bound apply along this whole concatenation, even if the second path leaves the stated ball. Integrating along the second path, then letting its length error tend to zero, proves the required inequality. □

Projection immediately compares the ambient and intrinsic metrics for nearby points on the same horosphere.

**Lemma 4.12.** *For $x \in \mathcal{PQ}_1\mathcal{T}$ and $y \in \mathrm{Hor}(x) \cap \mathcal{PQ}_1\mathcal{T}$ with $D_{\mathrm{AGY}}(x,y) \le \frac{1}{100}$, we have*

$$D_{\mathrm{AGY}}(x,y) \le d_{\mathrm{AGY}}(x,y) \le 4D_{\mathrm{AGY}}(x,y).$$

*Proof.* The first inequality follows from the definitions. For the second, apply Theorem 4.11 with $q = q' = x$. Both points lie in $B_{\mathrm{AGY}}(x, \frac{1}{50})$ and are fixed by the projection. □

The same comparison holds on stable horospheres, by the rotation $q \mapsto -q$.

## 5. A system of charts for $\mathcal{PMF}$

To compare points of $\mathcal{PMF}$, we will represent them on a common horosphere and use the coordinates and estimates of the preceding sections. We need finitely many charts such that every triple lies in an inner set and the outer sets have small intrinsic AGY diameter after this normalization. The nested sets leave room for the perturbations used later, and a partition of unity will combine the chart distances

into a global metric. The linear coordinates are on the corresponding cones in $\mathcal{MF}$; their projectivizations give coordinates on $\mathcal{PMF}$.

Choose a $\mathrm{Mod}(S)$–compact set $K_0 \subset \mathcal{PQ}_1\mathcal{T}$ whose quotient $\pi(K_0) \subset \mathcal{PQ}_1\mathcal{M}$ is connected and has more than $\frac{2}{3}$ of the total Masur–Veech volume. We will enlarge $K_0$ during this construction and then fix it.

A *chart* $\alpha$ consists of a horosphere $\mathrm{Hor}(F_\alpha)$, where $F_\alpha$ is a stable foliation of a pseudo-Anosov map $\phi_\alpha$ with axis in $K_0$, and open sets

$$U_\alpha \subset V_\alpha^{(1)} \subset V_\alpha^{(2)} \subset V_\alpha^{(3)} \subset V_\alpha^{(4)} \subset W_\alpha \subset \mathcal{PMF}(F_\alpha).$$

The closure of each set is contained in the next, including $\overline{W}_\alpha \subset \mathcal{PMF}(F_\alpha)$. We choose the axes in the principal stratum. Their stable foliations have the principal singularity type and no saddle connections, so $\mathrm{Hor}(F_\alpha)$ lies in $\mathcal{PQ}_1\mathcal{T}$.

For $x \in W_\alpha$ and $t \in \mathbb{R}$, write

$$x^\alpha(t) := x_{F_\alpha}(t) = Q_1(x, e^{-t}F_\alpha).$$

For $E \subset W_\alpha$, put $E_\alpha(t) = \{x^\alpha(t) : x \in E\}$, abbreviating $(U_\alpha)_\alpha(t)$ by $U_\alpha(t)$, and similarly for $V_\alpha^{(i)}(t)$ and $W_\alpha(t)$. We require

$$d_{\mathrm{AGY}}\big(x^\alpha(0), y^\alpha(0)\big) \le \frac{1}{200} \qquad (x, y \in W_\alpha). \tag{5.1}$$

Here $U_\alpha$ is the covering set, and $V_\alpha^{(1)}$ will contain the closed-orbit centers constructed in Proposition 7.4. The sets $V_\alpha^{(2)} \subset V_\alpha^{(3)}$ allow small perturbations while keeping a uniform lower bound for the partition function $\theta_\alpha$, constructed below. Its support lies in $V_\alpha^{(4)}$, leaving room inside the full chart $W_\alpha$ to compare nearby points.

We first choose the covering sets. There is a pseudo-Anosov map whose axis lies in the principal stratum; this follows, for example, from [EMR19, Theorem 1.2]. Its repelling foliation has infinitely many distinct images under $\mathrm{Mod}(S)$. Thus, given $x_1, x_2, x_3 \in \mathcal{PMF}$, a conjugate $\phi$ has repelling foliation outside this triple. Choose open sets $U_\phi, Z_\phi$ with disjoint closures such that

$$x_1, x_2, x_3, [F_+(\phi)] \in U_\phi, \qquad [F_-(\phi)] \in Z_\phi.$$

The sets $U_\phi^3$ cover the compact space $\mathcal{PMF}^3$. Take a finite subcover, indexed by $\mathcal{A}$, and denote its maps and sets by $\phi_\alpha$ and $U_\alpha$. In particular, every triple lies in some $U_\alpha$.

We now choose the normalizations so that the outer sets are small in the intrinsic AGY metric. For each $\alpha$, choose $q'_\alpha$ on the axis of $\phi_\alpha$ and put $F'_\alpha = F_-(q'_\alpha)$. Choose an open neighborhood $W'_\alpha$ of $[F_+(\phi_\alpha)]$ such that

$$\overline{W}'_\alpha \subset \left\{ x \in \mathcal{PMF}(F'_\alpha) : d_{\mathrm{AGY}}\big(x_{F'_\alpha}(0), q'_\alpha\big) < \frac{1}{400} \right\}, \qquad \mathbb{R}_+\overline{W}'_\alpha \subset \mathbb{R}_+\mathcal{V}_+(q'_\alpha),$$

where $\mathcal{V}_+$ is the neighborhood from Proposition 3.5. For $E \subset \mathcal{PMF}$, the cone $\mathbb{R}_+E$ consists of the measured foliations whose projective classes lie in $E$; for $E \subset \mathcal{MF}$, it consists of positive multiples of elements of $E$. Since $\overline{U}_\alpha$ avoids the repelling fixed point, north–south dynamics gives a power $k$ such that $\phi_\alpha^k(\overline{U}_\alpha) \subset W'_\alpha$. Set

$$q_\alpha = \phi_\alpha^{-k} q'_\alpha, \qquad F_\alpha = \phi_\alpha^{-k} F'_\alpha, \qquad W_\alpha = \phi_\alpha^{-k} W'_\alpha,$$

and take the product box at $q_\alpha$ to be the translate of the one at $q'_\alpha$. Equivariance of $Q_1$ gives

$$x_{F_\alpha}(0) = \phi_\alpha^{-k}\big((\phi_\alpha^k x)_{F'_\alpha}(0)\big).$$

The mapping class action is an AGY isometry. Hence $\overline{U}_\alpha \subset W_\alpha$, the cone containment is preserved, and every point of $Q_1(\overline{W}_\alpha, F_\alpha)$ is within $1/400$ of $q_\alpha$ in the intrinsic metric. The triangle inequality proves (5.1).

Insert the four open sets $V_\alpha^{(1)}, \dots, V_\alpha^{(4)}$ between $\overline{U}_\alpha$ and $W_\alpha$, with the compact containments specified above. Choose continuous functions $\chi_\alpha \colon \mathcal{PMF} \to [0,1]$ equal to one on $\overline{V}_\alpha^{(3)}$ and supported in $V_\alpha^{(4)}$, and set

$$\theta_\alpha(x) = \frac{\chi_\alpha(x)}{\sum_{\beta\in\mathcal{A}} \chi_\beta(x)}.$$

Since the $U_\alpha$ cover $\mathcal{PMF}$, the denominator lies between 1 and $|\mathcal{A}|$. Thus the $\theta_\alpha$ form a partition of unity, are supported in $V_\alpha^{(4)}$, and are at least $1/|\mathcal{A}|$ on $V_\alpha^{(3)}$.

By backward nonexpansion (Theorem 4.5), $W_\alpha(-s)$ remains within intrinsic distance $1/400$ of $q_\alpha(-s)$ for every $s \geq 0$. The finitely many axes have compact images in moduli space. Properness of the ambient quotient metric therefore gives a $\mathrm{Mod}(S)$–compact neighborhood containing all these sets. Enlarge $K_0$ to contain this neighborhood, adjoining finitely many connecting paths in the quotient if necessary to keep $\pi(K_0)$ connected. This finishes the choice of $K_0$.

The finite union of the lifted closures $Q_1(\overline{W}_\alpha, F_\alpha)$ is compact in the marked principal stratum. Its projection to $\mathcal{T}$ is therefore bounded relative to $P$, giving a constant $\mathsf{D}_{\mathrm{chart}}$ for both inequalities below. For each consecutive pair between $U_\alpha$ and $W_\alpha$, the lifted inner closure is compact inside the next lifted open set. Its intrinsic AGY distance from the complement is positive. Choose $\mathsf{r}_{\mathrm{chart}} > 0$ strictly smaller than the minimum of these finitely many distances. This gives the following properties.

**Theorem 5.1.** *There is a finite set of charts $\mathcal{A}$, with data*

$$\alpha = \big\{(X_\alpha, q_\alpha), U_\alpha, V_\alpha^{(1)}, V_\alpha^{(2)}, V_\alpha^{(3)}, V_\alpha^{(4)}, W_\alpha, \theta_\alpha\big\},$$

*and constants $\mathsf{D}_{\mathrm{chart}} > 0$ and $\mathsf{r}_{\mathrm{chart}} > 0$ satisfying:*

*(a) For $\alpha, \alpha' \in \mathcal{A}$ and $x \in W_\alpha \cap W_{\alpha'}$,*

$$d_{\mathcal{T}}\big(x^\alpha(0), x^{\alpha'}(0)\big) \leq \mathsf{D}_{\mathrm{chart}}, \qquad d_{\mathcal{T}}\big(x^\alpha(0), P\big) \leq \mathsf{D}_{\mathrm{chart}}.$$

*(b) The sets*

$$U_\alpha \subset V_\alpha^{(1)} \subset V_\alpha^{(2)} \subset V_\alpha^{(3)} \subset V_\alpha^{(4)} \subset W_\alpha \subset \mathcal{PMF}(F_\alpha)$$

*are open, with the closure of each contained in the next. Every three points of $\mathcal{PMF}$ lie in some $U_\alpha$.*

*(c) For any $x, y \in W_\alpha$, (5.1) holds.*

*(d) We have*

$$\mathbb{R}_+\overline{W}_\alpha \subset \mathbb{R}_+\mathcal{V}_+(q_\alpha), \tag{5.2}$$

*where $\mathcal{V}_+$ is as in Proposition 3.5.*

*(e) The continuous functions* $\theta_\alpha\colon \mathcal{PMF} \to [0,1]$ *form a partition of unity with* $\operatorname{supp}(\theta_\alpha) \subset V_\alpha^{(4)}$*, and*

$$\theta_\alpha(x) \geq \frac{1}{|\mathcal{A}|} \qquad (x \in V_\alpha^{(3)}).$$

*(f) For any consecutive pair* $A, B$ *in the chain* $U_\alpha, V_\alpha^{(1)}, \dots, V_\alpha^{(4)}, W_\alpha$*, if* $x \in A$*,* $y \in \mathcal{PMF}(F_\alpha)$*, and*

$$d_{\mathrm{AGY}}\big(x_{F_\alpha}(0), y_{F_\alpha}(0)\big) \leq \mathsf{r}_{\mathrm{chart}},$$

*then* $y \in B$*.*

*(g) For every* $\alpha \in \mathcal{A}$ *and* $s > 0$*,* $W_\alpha(-s) \subset K_0$*.*

We fix these charts and $K_0$ for the rest of the paper. A *uniform constant* is one depending only on the topology of $S$, the chosen charts, and the fixed basepoint $P$.

Changing charts amounts to changing the horizontal foliation and renormalizing the area. When both points already belong to both charts, the projection theorem gives the comparison directly. Indeed, let $x, y \in W_\alpha \cap W_{\alpha'}$ and $s, s' \in \mathbb{R}$, and assume

$$D_{\mathrm{AGY}}\big(x^\alpha(s), x^{\alpha'}(s')\big) \leq \frac{1}{100}, \qquad d_{\mathrm{AGY}}\big(x^\alpha(s), y^\alpha(s)\big) \leq \frac{1}{100}.$$

Then

$$d_{\mathrm{AGY}}\big(x^{\alpha'}(s'), y^{\alpha'}(s')\big) \leq 4\, d_{\mathrm{AGY}}\big(x^\alpha(s), y^\alpha(s)\big).$$

To see this, apply Theorem 4.11 with target $q = x^{\alpha'}(s')$ and input-ball center $x^\alpha(s)$. The target is within $1/100$ of the center, and both inputs lie strictly inside the center's ambient $1/50$ ball. Their projections are $x^{\alpha'}(s')$ and $y^{\alpha'}(s')$, so the estimate follows from the Lipschitz constant 4.

The next lemma also allows us to put a nearby point into the second chart. We first compare the two representatives of a common projective foliation and align their measured vertical foliations.

**Lemma 5.2.** *There are uniform constants* $\mathsf{k}$ *and* $\mathsf{D}_{\mathrm{temp}}$ *such that, for* $\alpha, \alpha' \in \mathcal{A}$*, the following hold.*

*(a) If* $x \in W_\alpha \cap W_{\alpha'}$*, then for every* $t \geq 0$*,*

$$d_{\mathcal{T}}\big(x^\alpha(t), x^{\alpha'}(t)\big) \leq \mathsf{D}_{\mathrm{temp}}. \tag{5.3}$$

*(b) For* $x \in W_\alpha \cap W_{\alpha'}$*, there is a time* $t_0 = t_0(x, \alpha, \alpha')$ *with* $|t_0| \leq \mathsf{D}_{\mathrm{chart}}$ *such that*

$$F_+\big(x^{\alpha'}(t + t_0)\big) = F_+\big(x^\alpha(t)\big) \qquad (t \in \mathbb{R}).$$

*(c) Let* $x \in W_\alpha \cap V_{\alpha'}^{(4)}$*,* $y \in W_\alpha$*, and let* $t_0$ *be as in* (b)*. Suppose that, for some* $t \geq 0$*,*

$$d_{\mathrm{AGY}}\big(x^\alpha(t), x^{\alpha'}(t + t_0)\big) \leq \frac{1}{100}, \qquad d_{\mathrm{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) \leq \frac{1}{100\mathsf{k}}.$$

*Then* $y \in W_{\alpha'}$ *and*

$$d_{\mathrm{AGY}}\big(x^{\alpha'}(t), y^{\alpha'}(t)\big) < \frac{1}{100}.$$

*Moreover, if* $d_{\mathrm{AGY}}\big(x^\alpha(t), x^{\alpha'}(t + t_0)\big) \leq 1/500$*, then, for every* $z, w \in \{x, y\}$*,*

$$D_{\mathrm{AGY}}\big(z^\alpha(t), w^{\alpha'}(t + t_0)\big) < \frac{1}{100}.$$

In part (c), the first intrinsic distance is in the common stable horosphere supplied by (b). The other intrinsic distances are in unstable horospheres; the final mixed distance is ambient.

*Proof.* For part (a), the initial points lie in a fixed compact subset of $\mathcal{T}$ and have distance at most $\mathsf{D}_{\text{chart}}$. The rays have the same projective vertical foliation. Theorem 7.1 and Remark 7.2 of [Raf14] give uniform fellow travelling. To pass from initial distance at most 1 to $\mathsf{D}_{\text{chart}}$, subdivide a connecting geodesic into boundedly many segments of length at most 1; the subdivision points remain in a fixed thick part. This gives $\mathsf{D}_{\text{temp}}$.

For part (b), put $F = F_+(x^\alpha(0))$ and $F' = F_+(x^{\alpha'}(0))$, and write $F = e^{t_0}F'$. Let $X, X'$ be the underlying surfaces. Unit area gives $\operatorname{Ext}_X(F) = \operatorname{Ext}_{X'}(F') = 1$, so Kerckhoff's formula gives

$$e^{-2\mathsf{D}_{\text{chart}}} \le \operatorname{Ext}_X(F') = e^{-2t_0} \le e^{2\mathsf{D}_{\text{chart}}}.$$

Thus $|t_0| \le \mathsf{D}_{\text{chart}}$. Since the flow multiplies the vertical foliation by $e^t$, the asserted identity holds for all real $t$.

For part (c), choose $\mathsf{k} \ge 8$ so that

$$\frac{4e^{2\mathsf{D}_{\text{chart}}}}{100\mathsf{k}} < \min\left\{\mathsf{r}_{\text{chart}}, \frac{1}{100}\right\}.$$

We project the nearby point to the second horosphere, check its chart membership at time zero, and then estimate at time $t$. Set $u = t + t_0$ and $q = x^{\alpha'}(u)$. Apply Theorem 4.11 with target $q$ and input-ball center $x^\alpha(t)$. The hypotheses put the target within $1/100$ of the center and both inputs strictly inside the center's ambient $1/50$ ball. Thus $p = \pi_q^{\text{cs}}(y^\alpha(t))$ exists, and $\pi_q^{\text{cs}}(x^\alpha(t)) = q$ gives

$$d_{\text{AGY}}(q,p) \le 4\, d_{\text{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) \le \frac{4}{100\mathsf{k}}.$$

The projected point has $[F_+(p)] = y$ and $F_-(p) = e^{-u}F_{\alpha'}$. In particular, $y \in \mathcal{PMF}(F_{\alpha'})$. Flowing $q, p$ by time $-u$ gives $q_0 = x^{\alpha'}(0)$ and $p_0 = Q_1(y, F_{\alpha'})$. Since $u \ge -\mathsf{D}_{\text{chart}}$, backward nonexpansion and the forward bound in (4.8) give

$$d_{\text{AGY}}(q_0,p_0) \le e^{2\mathsf{D}_{\text{chart}}} d_{\text{AGY}}(q,p) \le \frac{4e^{2\mathsf{D}_{\text{chart}}}}{100\mathsf{k}} < \mathsf{r}_{\text{chart}}.$$

Now $x \in V^{(4)}_{\alpha'}$, so Theorem 5.1(f) puts $y$ in $W_{\alpha'}$. We may therefore write $p = y^{\alpha'}(u)$. To estimate at time $t$, apply (4.8) between $u$ and $t$:

$$d_{\text{AGY}}\big(x^{\alpha'}(t), y^{\alpha'}(t)\big) \le e^{2|t_0|} d_{\text{AGY}}(q,p) \le \frac{4e^{2\mathsf{D}_{\text{chart}}}}{100\mathsf{k}} < \frac{1}{100}.$$

For the additional assertion, use the triangle inequality through $x^\alpha(t)$ and $q$. For every $z, w \in \{x, y\}$,

$$D_{\text{AGY}}\big(z^\alpha(t), w^{\alpha'}(u)\big) \le \frac{1}{100\mathsf{k}} + \frac{1}{500} + \frac{4}{100\mathsf{k}} < \frac{1}{100},$$

since $\mathsf{k} \ge 8$. □

## 6. Norm estimates for the Gauss–Manin map

The idea of this section is to measure a change of foliation by the change it makes in intersection numbers. Extremal length makes this quantity vary at most exponentially in Teichmüller distance. On compact subsets of the principal stratum, it is comparable to the Hodge norm of the corresponding change in periods. Comparing at the two endpoints will give the required bound on Gauss–Manin transport.

This is the same principle as Forni's estimate for the variation of the Hodge norm along the Teichmüller flow [For02, Lemma 2.1′]: control the logarithmic change of a norm to bound the growth of transported vectors. Here we use a norm of intersection functions, whose variation follows directly from extremal length. A fixed projective foliation along the path will identify the vectors being compared at its endpoints.

**The estimate.** For a compact Riemann surface $Z$, the Hodge norm of $\alpha \in H^1(Z;\mathbb{C})$ is

$$\|\alpha\|_{H,Z}^2 = \int_Z \alpha_{\mathrm{harm}} \wedge \star_Z \overline{\alpha_{\mathrm{harm}}}, \tag{6.1}$$

where $\alpha_{\mathrm{harm}}$ is its harmonic representative. For $q = (X,q) \in \mathcal{PQT}(S)$, we write $\|\cdot\|_{H,q}$ for this norm on the odd cohomology of the compactified orienting cover $\widetilde{X}$, and on its real subspace $L(q)$.

On marked compact surfaces, the marking-induced map $\mathrm{GM}_{Z,W} : H^1(Z;\mathbb{C}) \to H^1(W;\mathbb{C})$ satisfies

$$\max\{\|\mathrm{GM}_{Z,W}\|_H, \|\mathrm{GM}_{Z,W}^{-1}\|_H\} \le e^{d_{\mathcal{T}}(Z,W)}.$$

Indeed, pulling a harmonic form across an extremal quasiconformal map increases its $L^2$ norm by at most the square root of the dilatation, $e^{d_{\mathcal{T}}(Z,W)}$, and taking the harmonic representative can only decrease that norm. Apply this also to the inverse map. For orienting covers, this bounds transport by the distance between the covers. We need the distance between the base surfaces $X, Y$. The restriction to a fixed projective foliation is essential: a general path can have nontrivial monodromy even with equal endpoints.

A measured foliation is *generic* if it has the singularity type of the principal stratum and every arc between singularities has positive intersection with it, unless the arc is homotopic rel endpoints into a singularity. A differential with a generic vertical or horizontal foliation lies in the principal stratum.

**Proposition 6.1.** *For every* $\mathrm{Mod}(S)$*–compact set* $K \subset \mathcal{PQ}_1\mathcal{T}(S)$ *there is* $B_K \ge 0$ *with the following property. Let* $F$ *be generic and let*

$$q = (X,q), \quad p = (Y,p) \in K \cap D\,\mathrm{Hor}_{\pm}(F).$$

*For every path* $\xi$ *in* $D\,\mathrm{Hor}_{\pm}(F)$ *from* $q$ *to* $p$*,*

$$\log \max\{\|\mathrm{GM}_\xi\|_H, \|\mathrm{GM}_\xi^{-1}\|_H\} \le d_{\mathcal{T}}(X,Y) + B_K. \tag{6.2}$$

*The constant is independent of* $F$ *and* $\xi$*. Moreover,* $\mathrm{GM}_\xi$ *is independent of the path in* $D\,\mathrm{Hor}_{\pm}(F)$ *with these endpoints.*

Only the endpoints must lie in $K$. The coefficient of distance is 1; all dependence on the compact set is in the additive constant.

**A norm from intersection numbers.** View a measured foliation $G$ through its intersection function $J \mapsto \mathrm{i}(G, J)$. For $X \in \mathcal{T}(S)$, measure the difference of two such functions by

$$\delta_X(G, H) = \sup_{J \in \mathcal{MF} \setminus \{0\}} \frac{|\mathrm{i}(G, J) - \mathrm{i}(H, J)|}{\sqrt{\mathrm{Ext}_X(J)}}. \tag{6.3}$$

This is a weighted supremum norm of the difference of the intersection functions. Equivalently, we take the supremum over the compact set $\{J : \mathrm{Ext}_X(J) = 1\}$. Intersection numbers separate measured foliations, so $\delta_X$ is a metric on $\mathcal{MF}$, with $\delta_X(cG, cH) = c\delta_X(G, H)$ for $c > 0$.

The advantage of this norm is its simple dependence on $X$. Kerckhoff's formula bounds the change in every denominator and gives

$$e^{-d_{\mathcal{T}}(X,Y)} \delta_X(G, H) \le \delta_Y(G, H) \le e^{d_{\mathcal{T}}(X,Y)} \delta_X(G, H). \tag{6.4}$$

Thus, for fixed $G \neq H$, the function $X \mapsto \log \delta_X(G, H)$ is 1–Lipschitz. Along a unit-speed Teichmüller geodesic $X(t)$, this says $|\frac{d}{dt} \log \delta_{X(t)}(G, H)| \le 1$ almost everywhere. We use the finite comparison (6.4), which requires no differentiation in the foliation variable. We also have mapping class equivariance: $\delta_{\gamma X}(\gamma G, \gamma H) = \delta_X(G, H)$.

It remains to compare this distance with the size of a change in periods. Recall from Proposition 3.5 that a triangulation transverse to both foliations gives a product box $\mathcal{B} = \Phi^{-1}(U + iV)$ with fixed train-track maps $f, h$:

$$F_+\big(\Phi^{-1}(u + iv)\big) = f(u), \qquad F_-\big(\Phi^{-1}(u + iv)\big) = h(v). \tag{6.5}$$

Here $U, V$ are open convex subsets of a fixed real odd cohomology space $L$, and the track weights are linear in $u, v$. In this trivialization, $\Psi_q = f^{-1}$ near $F_+(q)$, with consistent signs. These boxes exist even when the foliations have saddle connections.

**Lemma 6.2.** *For every* $\mathrm{Mod}(S)$*–compact* $K \subset \mathcal{PQ}_1\mathcal{T}(S)$ *there is* $C_K \ge 1$ *such that, at each* $q = (X, q) \in K$*, one can choose a neighborhood* $\mathcal{U}_q$ *of* $F_+(q)$ *on which*

$$C_K^{-1} \|\Psi_q(G) - \Psi_q(H)\|_{H,q} \le \delta_X(G, H) \le C_K \|\Psi_q(G) - \Psi_q(H)\|_{H,q} \tag{6.6}$$

*for all* $G, H \in \mathcal{U}_q$*. The same constant works near any positive multiple of* $F_+(q)$*, using the homogeneous extension of* $\Psi_q$*.*

*Proof.* Work on a smaller product box with compact closure in $\mathcal{B}$, and choose a Euclidean norm $|\cdot|$ on $L$. In this proof, $\prec$ and $\succ$ allow constants depending on the chosen box, but uniform on its smaller closure. Continuity of the Hodge norm gives

$$\|z\|_{H,q} \succ |z|. \tag{6.7}$$

We prove the lower bound using finitely many fixed test foliations, and the upper bound by controlling all normalized tests.

For the lower bound, consider the constant pairing $\Omega(z, w) = \frac{1}{2} \int_{\widetilde{X}} z \wedge w$ on $L$. It is nondegenerate: the even and odd eigenspaces of the deck involution are orthogonal for cup product, which is nondegenerate on the full cohomology. The area identity is

$$\mathrm{i}(f(u), h(v)) = \mathrm{area}\big(\Phi^{-1}(u + iv)\big) = \Omega(u, v). \tag{6.8}$$

The factor $1/2$ accounts for the degree of the orienting cover. Choose a basis $v_1, \dots, v_N$ of $L$ in the open set $V$, and put $H_j = h(v_j)$. These foliations detect every real-period

direction. For $G = f(u)$ and $H = f(u')$ in the smaller box,

$$|u - u'| \prec \max_j |\Omega(u - u', v_j)| = \max_j |\mathrm{i}(G, H_j) - \mathrm{i}(H, H_j)| \prec \delta_X(G, H).$$

The last comparison is uniform because the finitely many $\mathrm{Ext}_X(H_j)$ are bounded as $X$ ranges over the compact projection of the smaller box. Together with (6.7), this gives the lower bound. The mixed differentials with foliations $(G, H_j)$ are used only for the area identity; extremal lengths are evaluated on $X$.

For the upper bound, use the Dehn–Thurston metric $d_{\mathrm{DT}}$ of Luo and Stong, and put $N(J) = d_{\mathrm{DT}}(J, 0)$. This metric is the sum, over coordinate pairs, of

$$d([x, t], [x', t']) = \min\{|x - x'| + |t - t'|,\ |x + x'| + |t + t'|\},$$

where the brackets mean simultaneous identification of both signs. Their intersection estimate, extended to measured foliations, is

$$|\mathrm{i}(G, J) - \mathrm{i}(H, J)| \le 3N(J)\, d_{\mathrm{DT}}(G, H). \tag{6.9}$$

See [LS04, p. 2, inequality (1.1), and Section 6]. For punctured surfaces, restrict their surface-with-boundary coordinates to the closed-curve subspace, with zero boundary coordinates. Thus no tests by proper arcs are needed.

Changes from train-track to Dehn–Thurston coordinates are piecewise linear [PH92, Section 2.6]. Finitely many linear pieces suffice on our smaller compact coordinate closure, so $d_{\mathrm{DT}}(f(u), f(u')) \prec |u - u'|$. The tests with $\mathrm{Ext}_X(J) = 1$ also lie in a fixed compact subset of $\mathcal{MF}$: choose $X_*$ and $R$ with $d_{\mathcal{T}}(X, X_*) \le R$ on the smaller box, and use $\mathrm{Ext}_{X_*}(J) \le e^{2R}$. Their $N(J)$ are therefore uniformly bounded. Taking the supremum in (6.9) gives

$$\delta_X(G, H) \prec |u - u'| \asymp \|\Psi_q(G) - \Psi_q(H)\|_{H,q},$$

as required.

Choose finitely many such smaller boxes around marked lifts covering $K/\operatorname{Mod}(S)$, and take the maximum of their comparison constants and 1. Equivariance of all the constructions gives $C_K$ on $K$. This also explains uniformity near saddle connections: at each limiting differential the triangulation is chosen transverse to both foliations, so its edge signs and linear coordinate maps stay fixed nearby. None of the bounds divides by a transverse measure tending to zero. The compact set lies inside the principal stratum, so zeros do not collide. Finally, homogeneity of $\delta_X$ and Lemma 3.6 give the same constant near each positive multiple of $F_+(q)$. The sizes of these neighborhoods need not be uniform. □

**From foliation coordinates to transport.** The comparison just proved contains the analytic part of the argument. We only need to identify the same small change of foliation at the two endpoints. This is where the path's fixed projective foliation is used.

**Lemma 6.3.** *Let $\xi : [0, 1] \to \mathcal{PQT}(S)$ be a path with $F_+(\xi(t)) = a_t F$, where $a_t > 0$ and $F \neq 0$ is fixed. Put $q = \xi(0)$ and $p = \xi(1)$. With endpoint period signs chosen consistently along $\xi$, there is an open neighborhood $\mathcal{W}$ of $F$ in $\mathcal{MF}$ such that*

$$\mathrm{GM}_\xi\big(\Psi_q(G)\big) = \Psi_p(G) \qquad (G \in \mathcal{W}). \tag{6.10}$$

*The endpoint maps are the homogeneous local coordinates near the ray of $F$.*

*Proof.* Replace $\xi(t)$ by $a_t^{-2}\xi(t)$ to make its vertical foliation equal to $F$. Scalar multiplication leaves the orienting cover canonically unchanged, and Lemma 3.6 identifies the homogeneous coordinate maps. It suffices to prove the assertion on this normalized path.

Subdivide the path into finitely many subpaths lying in product boxes from Proposition 3.5. For $G$ near $F$, (6.5) makes its coordinate $f^{-1}(G)$ constant in the flat trivialization along each subpath. At a junction $r$, the section $G \mapsto Q(G, F_-(r))$ lies in both boxes for $G$ near $F$; hence their coordinate maps agree under the Gauss–Manin change of trivialization. Choose signs successively and intersect these finitely many neighborhoods of $F$. Composing the local identities proves (6.10). □

The neighborhood may depend on the whole path. Its size will not enter the estimate, since a neighborhood of zero suffices to test a linear map. Without a fixed projective foliation, these local identities need not compare the same foliation at both endpoints.

*Proof of Proposition 6.1.* First suppose $\xi \subset D\operatorname{Hor}_+(F)$. Let $C$ be the comparison constant from Lemma 6.2 for $K \cup (-K)$; using this compact set also covers the horizontal case below. Shrink $\mathcal{W}$ from Lemma 6.3 so that both endpoint comparisons hold there. The homogeneous assertion allows this even when $F_+(q)$ and $F_+(p)$ are different multiples of $F$.

For $G \in \mathcal{W}$, put $z = \Psi_q(G) - \Psi_q(F)$. Then

$$\begin{aligned}\|\mathrm{GM}_\xi z\|_{H,p} &= \|\Psi_p(G) - \Psi_p(F)\|_{H,p} \\ &\le C\,\delta_Y(G,F) \le Ce^{d_{\mathcal{T}}(X,Y)}\delta_X(G,F) \\ &\le C^2 e^{d_{\mathcal{T}}(X,Y)}\|z\|_{H,q}.\end{aligned}$$

These $z$ fill a neighborhood of zero in $L(q)$, so linearity gives the bound for every vector. Reversing $\xi$ gives the inverse bound. Both extend to complex cohomology because transport is real and $\|z+iw\|_H^2 = \|z\|_H^2 + \|w\|_H^2$ for real $z, w$.

For $D\operatorname{Hor}_-(F)$, apply this argument to $-\xi$. The rotation $q \mapsto -q$ interchanges the two foliations and identifies the conformal orienting covers by Hodge isometries that intertwine transport. Thus (6.2) holds with $B_K = 2\log C$. Finally, Hubbard–Masur identifies $D\operatorname{Hor}_+(F)$ with $\mathcal{T}(S) \times \mathbb{R}_{>0}$ by $(X, c) \mapsto c^2 Q(F, X)$. Genericity puts this contractible leaf entirely in the principal stratum, so homotopy invariance (Lemma 3.3) gives path independence. The horizontal case is the same. □

**The estimate in the AGY norm.** The Hodge and AGY norms on the real spaces $L(q)$ are continuous and $\mathrm{Mod}(S)$–invariant, so they are uniformly comparable on a $\mathrm{Mod}(S)$–compact subset $K \subset \mathcal{PQ}_1\mathcal{T}$. Choose $A_K \ge 1$ with

$$A_K^{-1}\|z\|_{H,q} \le \|z\|_{\mathrm{AGY},q} \le A_K\|z\|_{H,q} \qquad (q \in K).$$

Under the hypotheses of Proposition 6.1,

$$\max\{\|\mathrm{GM}_\xi\|_{\mathrm{AGY}}, \|\mathrm{GM}_\xi^{-1}\|_{\mathrm{AGY}}\} \le A_K^2 e^{B_K} e^{d_{\mathcal{T}}(X,Y)}. \tag{6.11}$$

Changing norms affects only the prefactor. In particular, $d_{\mathcal{T}}(X,Y) \le ct + L$ gives the bound $A_K^2 e^{B_K+L} e^{ct}$, with the coefficient $c$ unchanged.

## 7. Good geodesics

We construct pseudo-Anosov maps whose attracting foliations will be the centers of the spikes. Their periods lie near prescribed times $R_n = n\tau'$. The appendix supplies many times with good backward statistics. We add a bounded return segment and apply the closing lemma, then project the resulting periodic orbit to the original chart. We keep track of the statistics and the approximation error because both enter the spike estimates.

Recall that $K \subset \mathcal{PQ}_1\mathcal{T}$ is Mod–*compact* if it is $\mathrm{Mod}(S)$–invariant and its moduli quotient is compact; in particular, $K$ is closed. We use $\mathrm{axis}(\gamma)$ for both the Teichmüller axis of a pseudo-Anosov map and its oriented lift to the space of unit-area differentials. On this lift, $g_{\ell(\gamma)}z = \gamma z$.

**Proposition 7.1** (Closing Lemma)**.** *There are constants $\epsilon_{\mathrm{clos}} > 0$ and $C_{\mathrm{clos}} > 1$ depending on $K_0$ with the following property. For every* Mod–*compact set $K_1 \subset \mathcal{PQ}_1\mathcal{T}$, there is $R_1 > 0$ depending on $K_0$, $\epsilon_{\mathrm{clos}}$, and $K_1$ such that, if $0 < \epsilon < \epsilon_{\mathrm{clos}}$, $q \in K_0$, and $t > 0$ satisfy*

$$D_{\mathrm{AGY}}\big(\pi(q), \pi(q(t))\big) < \epsilon, \qquad \big|\{s \in [0,t] : q(s) \in K_1\}\big| \geq R_1,$$

*then there is a pseudo-Anosov map $\gamma$ with axis in the principal stratum such that*

$$|\ell(\gamma) - t| \leq C_{\mathrm{clos}}\epsilon, \qquad D_{\mathrm{AGY}}\big(q(s), \mathrm{axis}(\gamma)\big) \leq C_{\mathrm{clos}}\epsilon \quad (0 \leq s \leq t).$$

*Proof.* We adapt the closing argument of [EMR19, Lemma 8.1 and Remark 8.2] to the AGY metric. Fix local product neighborhoods and their comparison constants over $K_0$; these determine $\epsilon_{\mathrm{clos}}$ and $C_{\mathrm{clos}}$. Stable forward and unstable backward nonexpansion (Theorem 4.5) keep nearby comparison paths in a fixed-radius compact thickening of $K_1$ whenever the original orbit visits $K_1$. By Proposition 4.7, sufficiently large total time there gives the contraction required by the fixed-point argument. This determines $R_1$. The construction bounds the displacement and period correction by $C_{\mathrm{clos}}\epsilon$, and nonexpansion gives the tracking bound. Thus $K_1$ affects the occupation threshold, not the closing accuracy or the tracking constant. □

Good statistics requires control on every interval ending at the point under consideration, however far into the past it begins.

**Definition 7.2.** For a Mod–compact set $K$ and $0 < \eta < 1/2$, a differential $q \in \mathcal{PQ}_1\mathcal{T}$ has $(K,\eta)$–good statistics if $q \in K$ and, for every $s < 0$,

$$\big|\{u \in [s,0] : q(u) \in K\}\big| \geq (1-\eta)|s|. \tag{7.1}$$

We use the same definition for open neighborhoods of Mod–compact sets. For rays starting in our charts, the entire negative ray lies in $K_0$, so only the positive-time part needs checking.

**Lemma 7.3.** *Suppose $0 < \eta < 1/2$, $K \supset K_0$ is* Mod–*compact, $q \in U_\alpha(0)$, and $R > 0$. If, for every $0 < s < R$,*

$$\big|\{u \in [s,R] : q(u) \in K\}\big| \geq (1-\eta)(R-s), \tag{7.2}$$

*then $q(R)$ has $(K,\eta)$–good statistics.*

*Proof.* Letting $s \downarrow 0$ gives (7.2) at $s = 0$. Letting $s \uparrow R$ gives points of $K$ approaching $q(R)$, so closedness gives $q(R) \in K$. For $u < 0$, Theorem 5.1(g) gives $q(u) \in K_0$. Adding this negative-time interval proves (7.2) for every $s < 0$, hence the assertion. □

The next proposition records both the geometry of the periodic orbits and how often they approximate rays from each chart. All uniform constants depend only on the fixed geometric data of Theorem 5.1.

**Proposition 7.4.** *There are uniform constants $\mathsf{D}_0, \mathsf{D}_1$ such that, for every $0 < \eta_1 < 1/2$, there are a* Mod*–compact set $K_1 \supset K_0$, constants $\rho_1, \tau > 0$, $\tau_0' > \rho_1$, and $0 < \kappa_0 < 1$ depending on $\eta_1$ with the following properties. Fix $\tau' \in \tau\mathbb{N}$ with $\tau' > \tau_0'$, and set $R_n = n\tau'$. For each chart $\alpha$ and all sufficiently large $n$ depending on $\eta_1$, there is a set $\Gamma_\alpha(R_n) \subset \mathrm{Mod}(S)$ satisfying* (a)–(f) *below. Parts* (a)–(e) *hold for each $\gamma \in \Gamma_\alpha(R_n)$; part* (f) *concerns the whole family.*

(a) *The map $\gamma$ is pseudo-Anosov, and its axis passes within $d_{\mathcal{T}}$-distance $\mathsf{D}_0$ of $P$.*

(b) *Its translation length satisfies*

$$R_n - 1 \le \ell(\gamma) \le R_n + \rho_1 + 1.$$

(c) *Write $a_\gamma = [F_+(\gamma)]$. Then*

$$D_{\mathrm{AGY}}\big(a_\gamma^\alpha(t), \mathrm{axis}(\gamma)\big) \le \frac{1}{100} \qquad (t \ge 0).$$

(c') *We have*

$$d_{\mathcal{T}}\big(a_\gamma^\alpha(\ell(\gamma)), \gamma(P)\big) \le \mathsf{D}_1.$$

(d) *We have $a_\gamma \in V_\alpha^{(1)}$, and $a_\gamma^\alpha(\ell(\gamma))$ has $(K_1, \eta_1)$–good statistics.*

(e) *Set*

$$G_\gamma = \{u \in [0, \ell(\gamma)] : a_\gamma^\alpha(u) \text{ has } (K_1, \eta_1)\text{–good statistics}\}.$$

*For every $0 \le s < \ell(\gamma)$,*

$$\big|[s, \ell(\gamma)] \setminus G_\gamma\big| \le \eta_1\big(\ell(\gamma) - s\big).$$

(f) *Let $\mathsf{k}$ be as in Lemma 5.2, and define*

$$E_\alpha(R_n) = \big\{q \in U_\alpha(0) : \text{there exists } \gamma \in \Gamma_\alpha(R_n) \text{ with } d_{\mathrm{AGY}}\big(q(\ell(\gamma)), a_\gamma^\alpha(\ell(\gamma))\big) < \tfrac{1}{100\mathsf{k}}\big\}.$$

*At the excluded initial indices, take both $\Gamma_\alpha(R_j)$ and $E_\alpha(R_j)$ to be empty. Put*

$$k_n(q) = \#\{1 \le j \le n : q \in E_\alpha(R_j)\}.$$

*For all sufficiently large $n$ depending on $\eta_1$ and $\tau'$,*

$$\mu\big(\{q \in U_\alpha(0) : k_n(q) \le (1 - \eta_1)n\}\big) \le \kappa_0^n \mu(U_\alpha(0)).$$

*Here $\mu$ is the conditional Masur–Veech volume on $\mathrm{Hor}(F_\alpha)$, normalized as in Section 9.1.*

*Proof. Closing accuracy and return data.* First choose the accuracy needed for the projections and the radius in part (f). With the constants of Proposition 7.1, Theorem 5.1, and Lemma 5.2, fix $\epsilon > 0$ such that

$$\epsilon < \frac{\epsilon_{\mathrm{clos}}}{2}, \qquad C_{\mathrm{clos}}\epsilon < \min\left\{\frac{1}{2000}, \frac{1}{800\mathsf{k}}, \frac{\mathsf{r}_{\mathrm{chart}}}{4}\right\}.$$

This choice uses only the fixed geometric data. Choose finite sets $\widehat{\mathcal{B}}(\alpha) \subset U_\alpha(0)$ whose ambient $\epsilon/4$–balls cover $U_\alpha(0)$, and put $\widehat{\mathcal{B}} = \bigcup_\alpha \widehat{\mathcal{B}}(\alpha)$.

Now fix $\eta = \eta_1/2$. Proposition A.30 gives $K \supset K_0$, $\tau$, $\kappa$, and $\theta_0$; set $\theta = \theta_0/2$. Write $\lambda = \mu_{\mathrm{MV}}$ for the normalized Masur–Veech probability measure on $\mathcal{PQ}_1\mathcal{M}$.

**Claim 7.5.** *For every $\theta > 0$, there are $\rho_\theta > 0$ and a compact set $\Omega_\theta \subset \mathcal{PQ}_1\mathcal{M}$ with $\lambda(\Omega_\theta) \geq 1 - \theta$ such that, for every $p \in \Omega_\theta$ and $z \in \widehat{\mathcal{B}}$, the segment $\{p(t) : 0 \leq t \leq \rho_\theta\}$ meets $\pi(B_{\mathrm{AGY}}(z, \epsilon/4))$.*

*Proof.* Each target is open and has positive measure. Ergodicity implies that almost every forward orbit meets all these finitely many targets. Thus the measure of the set that does so by time $R$ tends to one as $R \to \infty$. Choose $\rho_\theta$ so that this measure exceeds $1 - \theta$, then take a compact subset of measure at least $1 - \theta$. □

Put $\rho_1 = \rho_\theta$ and use $\Omega_\theta$ also for its full lift to $\mathcal{PQ}_1\mathcal{T}$. Write $N_r$ for ambient AGY neighborhoods. Define

$$\mathcal{K}_{\mathrm{aux}} = N_2\big(\{p(t) : p \in K,\ 0 \leq t \leq \rho_1\}\big),$$
$$K_1 = \left\{q \in \mathcal{PQ}_1\mathcal{T} : \mathrm{sys}(q) \geq \inf_{p \in \mathcal{K}_{\mathrm{aux}}} \mathrm{sys}(p)\right\},$$

where sys is the shortest saddle-connection length. The swept set is compact modulo Mod, and its closed AGY neighborhood is also compact modulo Mod. Hence the infimum is positive and $K_1$ is Mod–compact. By Theorem 5.1(g) and closedness, $U_\alpha(0) \subset K_0 \subset K$. Thus $K_1$ contains the ambient 2–neighborhoods of $K_0$ and of every $U_\alpha(0)$.

The AGY norm gives $D_{\mathrm{AGY}}(g_t p, g_t p') \leq e^{2|t|} D_{\mathrm{AGY}}(p, p')$: the numerator in each norm ratio increases by at most $e^{|t|}$, and its denominator decreases by at most that factor; integrate along ambient paths. Choose $0 < \delta_\theta < \min\{1/400, \epsilon e^{-2\rho_1}/8\}$. Then

$$D_{\mathrm{AGY}}(p, p') \leq 2\delta_\theta, \quad |t| \leq \rho_1 \quad \Longrightarrow \quad D_{\mathrm{AGY}}(p(t), p'(t)) \leq \epsilon/4. \tag{7.3}$$

Only now choose $\tau_0'$ from Proposition A.30, enlarged so that $\tau_0' > \rho_1$. Its measure hypothesis holds because $\lambda(\Omega_\theta/\,\mathrm{Mod}) \geq 1 - \theta$. Thus $K_1$ precedes $\delta_\theta$, and the later data affect only time thresholds. In particular, $\epsilon$ is unchanged.

*Closing the orbit segments.* A point $q \in U_\alpha(0)$ is $(K, \eta, \delta, \Omega)$–generic at $R$ (Definition A.29) if

$$q(R) \in N_{2\delta}(\Omega) \quad \text{and has } (K, \eta)\text{–good statistics}, \tag{7.4}$$

and, for every $0 \leq s < R$,

$$\big|\{u \in (s, R) : q(u) \text{ has } (K, \eta)\text{–good statistics}\}\big| \geq (1 - \eta)(R - s). \tag{7.5}$$

Let $\mathcal{G}_\alpha(R)$ be the set of points in $U_\alpha(0)$ that are $(N_{\delta_\theta}(K), \eta, \delta_\theta, \Omega_\theta)$–generic at $R$. Take the closing threshold $R_1$ for $K_1$, and choose an integer $j_0 \geq 1$ with $(1 - \eta) j_0 \tau_0' > R_1$. This choice works for every $\tau' \in \tau\mathbb{N}$ above $\tau_0'$.

Fix $j \geq j_0$ and $q \in \mathcal{G}_\alpha(R_j)$. Choose $z_q \in \widehat{\mathcal{B}}$ with $D_{\mathrm{AGY}}(q, z_q) < \epsilon/4$ and, by (7.4), $p' \in \Omega_\theta$ with $D_{\mathrm{AGY}}(q(R_j), p') < 2\delta_\theta$. Claim 7.5 gives a return time $0 \leq t_{\mathrm{ret}} \leq \rho_1$ with $\pi(p'(t_{\mathrm{ret}})) \in \pi(B_{\mathrm{AGY}}(z_q, \epsilon/4))$. Together with (7.3), the three errors give

$$D_{\mathrm{AGY}}\big(\pi(q), \pi(q(R_j + t_{\mathrm{ret}}))\big) \leq \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon.$$

Good statistics at $R_j$ supply at least $(1-\eta)R_j > R_1$ time in $K_1$. The closing lemma therefore gives a pseudo-Anosov map $\gamma_q$. Set

$$\Gamma_\alpha(R_j) = \{\gamma_q : q \in \mathcal{G}_\alpha(R_j)\},$$

and take $\Gamma_\alpha(R_j) = E_\alpha(R_j) = \varnothing$ for $j < j_0$.

*Geometry of the closed orbit.* Write $\gamma = \gamma_q$, $\ell = \ell(\gamma)$, and $T = R_j + t_{\mathrm{ret}}$. The initial points lie in fixed marked compact chart closures. An axis point is within $C_{\mathrm{clos}}\epsilon$ of $q$, so its base distance from $P$ is bounded by a uniform $\mathsf{D}_0$, proving (a). The period estimate gives

$$|\ell - T| \leq C_{\mathrm{clos}}\epsilon < 1, \qquad R_j - 1 \leq \ell \leq R_j + \rho_1 + 1,$$

which proves (b). The tracking estimate is

$$D_{\mathrm{AGY}}\big(q(t), \mathrm{axis}(\gamma)\big) \leq C_{\mathrm{clos}}\epsilon \qquad (0 \leq t \leq T). \tag{7.6}$$

Put $a_\gamma = [F_+(\gamma)]$ and choose $z \in \mathrm{axis}(\gamma)$ within $C_{\mathrm{clos}}\epsilon$ of $q$. Apply Theorem 4.11 with target and ball center $q$. The projection of $z$ is $Q_1(a_\gamma, F_\alpha)$; the theorem establishes that it is defined, principal, and at intrinsic distance at most $4C_{\mathrm{clos}}\epsilon < \mathsf{r}_{\mathrm{chart}}$ from $q$. Theorem 5.1(f) now gives $a_\gamma \in V_\alpha^{(1)}$, so this projection is $a_\gamma^\alpha(0)$. At each time $0 \leq t \leq T$, the same argument with target and center $q(t)$ gives

$$d_{\mathrm{AGY}}\big(q(t), a_\gamma^\alpha(t)\big) \leq 4C_{\mathrm{clos}}\epsilon. \tag{7.7}$$

All inputs lie inside the ambient 1/50–balls required by Theorem 4.11.

To follow the axis for all future time, observe that $D_{\mathrm{AGY}}(a_\gamma^\alpha(0), z) \leq 5C_{\mathrm{clos}}\epsilon < 1/200$. Apply the time-reversed projection theorem to project $z$ onto $\mathrm{Hor}_+(a_\gamma^\alpha(0))$. Its image $z'$ has the same two projective foliations as $z$, so it lies on the same axis, and

$$d_{\mathrm{AGY}}\big(a_\gamma^\alpha(0), z'\big) \leq 20C_{\mathrm{clos}}\epsilon.$$

Forward stable nonexpansion, the time-reversed form of (4.8), gives

$$D_{\mathrm{AGY}}\big(a_\gamma^\alpha(t), \mathrm{axis}(\gamma)\big) \leq 20C_{\mathrm{clos}}\epsilon < \frac{1}{100} \qquad (t \geq 0). \tag{7.8}$$

This proves (c). Since $g_\ell z' = \gamma z'$, it also gives

$$\begin{aligned} D_{\mathrm{AGY}}\big(\gamma^{-1}a_\gamma^\alpha(\ell), a_\gamma^\alpha(0)\big) &\leq D_{\mathrm{AGY}}\big(a_\gamma^\alpha(\ell), g_\ell z'\big) + D_{\mathrm{AGY}}\big(z', a_\gamma^\alpha(0)\big) \\ &\leq 40C_{\mathrm{clos}}\epsilon < 1/50. \end{aligned}$$

The closed ambient 1/50–neighborhood of $\bigcup_\alpha \overline{V_\alpha^{(1)}(0)}$ is compact in marked space. Its base distances from $P$ are bounded by a uniform $\mathsf{D}_1$, which proves (c') by Mod–invariance.

*Transferring the statistics.* Write $a(t) = a_\gamma^\alpha(t)$, and set

$$L = \overline{N_1\big(\{p(u) : p \in K,\ 0 \leq u \leq \rho_1\}\big)}.$$

The entire negative ray $a(t)$ lies in $K_0 \subset L$. On $[0, R_j]$, (7.7) puts $a(t)$ in $L$ whenever $q(t) \in N_{\delta_\theta}(K)$. Finally, since $q(R_j) \in N_{\delta_\theta}(K)$, (7.3) and (7.7) put the whole appended

segment $a([R_j, T])$ in $L$. These three observations transfer good statistics from $q(u)$ to $a(u)$ at every good time $u \in [0, R_j]$, and give good statistics also for $u \le 0$ and $R_j \le u \le T$. Indeed, every backward interval gains all its time before zero and after $R_j$, while the remaining part satisfies the original occupation bound. In particular, $a(T)$ has $(L, \eta)$–good statistics, and (7.5) gives, for every $v < T$,

$$\big|\{u \in [v, T] : a(u) \text{ fails } (L, \eta)\text{–good statistics}\}\big| \le \eta(T - v).$$

The period differs slightly from $T$. Put $h = \ell - T$, so $|h| \le C_{\mathrm{clos}}\epsilon < 1/300$. By (4.7), $D_{\mathrm{AGY}}(g_h r, r) \le |h|$; since $1 + |h| < 2$, our definition gives $g_h L \subset K_1$. Thus translating the entire past by $h$ sends every $(L, \eta)$–good time $u$ to a $(K_1, \eta)$–good time $u + h$. It follows that $a(\ell)$ has these statistics. Applying the preceding frequency estimate with $v = s - h$, also when this is negative, gives

$$\big|[s, \ell] \setminus G_\gamma\big| \le \eta(\ell - s) \qquad (0 \le s < \ell).$$

Since $\eta = \eta_1/2 < \eta_1$ and $a_\gamma \in V_\alpha^{(1)}$, this proves (d) and (e).

*Approximation and frequency.* For part (f), use (7.7) at time $T$, then (4.8) over the period correction:

$$\begin{aligned} d_{\mathrm{AGY}}\big(q(\ell), a_\gamma^\alpha(\ell)\big) &\le e^{2|\ell - T|} d_{\mathrm{AGY}}\big(q(T), a_\gamma^\alpha(T)\big) \\ &\le 4e^{2C_{\mathrm{clos}}\epsilon} C_{\mathrm{clos}}\epsilon < 8C_{\mathrm{clos}}\epsilon < \frac{1}{100\mathsf{k}}. \end{aligned}$$

Here the initial bounds $C_{\mathrm{clos}}\epsilon < 1/2000$ and $C_{\mathrm{clos}}\epsilon < 1/(800\mathsf{k})$ give the last two inequalities. Thus $\mathcal{G}_\alpha(R_j) \subset E_\alpha(R_j)$ for $j \ge j_0$. Put $g_n(q) = \#\{1 \le j \le n : q \in \mathcal{G}_\alpha(R_j)\}$. The omitted initial indices give $g_n(q) \le k_n(q) + j_0 - 1$, so, for $n \ge 2(j_0 - 1)/\eta_1$,

$$\{q : k_n(q) \le (1 - \eta_1)n\} \subset \{q : g_n(q) \le (1 - \eta_1/2)n\}.$$

Proposition A.30, applied with $\eta = \eta_1/2$, bounds the measure of the latter set by $\kappa(\eta_1/2)^n \mu(U_\alpha(0))$ for all sufficiently large $n$. Taking $\kappa_0 = \kappa(\eta_1/2)$ proves (f). Only the threshold for $n$, not this rate, depends on $\tau'$. □

Define the set of good geodesics by

$$\Gamma(R) = \bigcup_\alpha \Gamma_\alpha(R). \tag{7.9}$$

## 8. Convergence of rays with a common foliation

We prove a convergence estimate for two rays $q, q'$ with the same projective vertical foliation. If $q(T)$ has $(K, \eta)$–good statistics, then $q(T)$ and $q'(T + a)$ are close in the intrinsic AGY metric once $T$ exceeds the Teichmüller distance between their initial surfaces by an additive constant. The shift $a$ makes their measured vertical foliations agree. The additive constant depends on $K$, $\eta$, and the desired accuracy; the coefficient of the initial distance is one. We allow $q'$ to lie in any stratum, and prove that it must be principal.

We first use Hodge-norm contraction to prove convergence from a bounded initial distance once $q$ has spent enough time in $K$. The thin-triangle theorem of Dowdall–Duchin–Masur (DDM) then reduces the general case to this one.

All differentials in this section have area one, and $X_p$ denotes the base surface of $p$. Recall that a Mod–compact subset of $\mathcal{PQ}_1\mathcal{T}$ is $\mathrm{Mod}(S)$–invariant and has compact image in moduli space (Definition 2.1).

**Theorem 8.1.** *Fix a* Mod*–compact set $K \subset \mathcal{PQ}_1\mathcal{T}$, $0 < \eta < 1/2$, and $\epsilon > 0$. There exists $B = B(K, \eta, \epsilon) > 0$ with the following property. Let $q \in \mathcal{PQ}_1\mathcal{T}$ and $q' \in \mathcal{Q}_1\mathcal{T}$ have the same projective vertical foliation $[F]$, and put*

$$D = d_{\mathcal{T}}(X_q, X_{q'}), \qquad a = \frac{1}{2}\log\frac{\mathrm{Ext}_{X_{q'}}(F)}{\mathrm{Ext}_{X_q}(F)}.$$

*Suppose $T \geq D + B$, $q(T) \in K$, and*

$$\big|\{t \in [u, T] : q(t) \in K\}\big| \geq (1-\eta)(T-u) \qquad (0 \leq u < T). \tag{8.1}$$

*Then $q' \in \mathcal{PQ}_1\mathcal{T}$, $|a| \leq D$, $T + a > 0$, and*

$$d_{\mathrm{AGY}}\big(q(T), q'(T+a)\big) < \epsilon. \tag{8.2}$$

*If $F_+(q) = F_+(q')$, then $a = 0$.*

If $q(T)$ has $(K, \eta)$–good statistics (Definition 7.2), then $q(T) \in K$ and (8.1) holds.

The time adjustment is necessary because equality of projective foliations does not imply equality of their transverse measures. For a representative $F$ of the common projective foliation,

$$F_+(q) = \frac{F}{\sqrt{\mathrm{Ext}_{X_q}(F)}}, \qquad F_+(q') = \frac{F}{\sqrt{\mathrm{Ext}_{X_{q'}}(F)}}.$$

Since $F_+(g_t p) = e^t F_+(p)$, the value of $a$ in the theorem is exactly the adjustment for which $F_+(q'(a)) = F_+(q)$. Kerckhoff's extremal-length inequality gives $|a| \leq D$.

The following proposition treats a bounded initial distance. It uses the total time spent by $q$ in $K$ during $[0, t]$.

**Proposition 8.2.** *For every* Mod*–compact $K \subset \mathcal{PQ}_1\mathcal{T}$, $D_0 > 0$, and $\epsilon > 0$, there is $R = R(K, D_0, \epsilon) > 0$ such that the following holds. Suppose $q \in K$, $q' \in \mathcal{Q}_1\mathcal{T}$,*

$$F_+(q) = F_+(q'), \qquad d_{\mathcal{T}}(X_q, X_{q'}) \leq D_0.$$

*If*

$$\big|\{u \in [0, t] : q(u) \in K\}\big| \geq R,$$

*then $q' \in \mathcal{PQ}_1\mathcal{T}$ and*

$$d_{\mathrm{AGY}}\big(q(t), q'(t)\big) < \epsilon.$$

*The threshold is uniform over the common foliation, including foliations with saddle connections.*

**Convergence from a bounded initial distance.** We first work on a closed surface of genus at least two. On the principal part of a stable leaf $\mathrm{Hor}_+(q)$, let $d_{\mathrm{H}}$ be the path distance induced by the ordinary Hodge norm from (6.1). The stable tangent space is

$$E^s(q) = \left\{\alpha \in H^1_{\mathrm{odd}}(\widetilde{X}; \mathbb{R}) : \int_{\widetilde{X}} \mathrm{Re}\,(\omega_q) \wedge \alpha = 0\right\}.$$

Recall that a generic foliation has principal singularity type and no saddle connections. Every differential realizing it is principal, so we can use Hodge length on its entire stable leaf. We need an initial length bound that remains uniform as the foliation approaches one with saddle connections.

**Lemma 8.3.** *Suppose $S$ is closed of genus at least two. For every* Mod*–compact $K \subset \mathcal{PQ}_1\mathcal{T}$ and $D_0 > 0$ there is $M = M(K, D_0)$ such that*

$$d_{\mathrm{H}}(p, p') \le M$$

*whenever $p \in K$, $F_+(p) = F_+(p')$ is generic, and $d_{\mathcal{T}}(X_p, X_{p'}) \le D_0$.*

*Proof.* Put $F = F_+(p) = F_+(p')$, so that $\mathrm{Ext}_{X_p}(F) = \mathrm{Ext}_{X_{p'}}(F) = 1$. For the Teichmüller segment $Z_r$, $0 \le r \le 1$, joining the two bases, set

$$h_r = g_{\frac{1}{2}\log \mathrm{Ext}_{Z_r}(F)} Q_1([F], Z_r). \tag{8.3}$$

This is a continuous unit-area path from $p$ to $p'$ with measured vertical foliation $F$. Extremal-length comparison bounds its time adjustment by $D_0$, so its bases lie within $2D_0$ of $X_p$. After applying a common mapping class to $p, p'$, we may assume that $p$ lies in a fixed compact set of marked representatives for $K$. By properness of Teichmüller space, the pairs $(p, X_{p'})$ in the lemma then range over a compact parameter set after including nongeneric limits. Continuity of the Hubbard–Masur map and extremal length shows that $h_r$ depends continuously on this pair and on $r$. Thus all these paths, and their base surfaces, lie in fixed compact sets.

For points $b, c$ on the same leaf with bases in this compact set, the Euclidean and modified Hodge distances of [ABEM12, Section 3.5] satisfy

$$d_{\mathrm{H}}^{\mathrm{mod}}(b, c) \le C\, d_{\mathrm{E}}(b, c) |\log d_{\mathrm{E}}(b, c)|^{1/2} \qquad (0 < d_{\mathrm{E}}(b, c) < 1/2)$$

by [ABEM12, Theorem 3.10]; the stable version follows by $q \mapsto -q$. Uniform continuity of the path family gives one subdivision of $[0, 1]$ on which successive endpoints of every path have $d_{\mathrm{E}} < 1/2$. Summing over this subdivision bounds $d_{\mathrm{H}}^{\mathrm{mod}}(p, p')$ uniformly. The modified norm dominates the ordinary Hodge norm [ABEM12, Section 3.3.2], and a generic leaf is entirely principal. Thus the same bound applies to $d_{\mathrm{H}}(p, p')$, uniformly even near a nongeneric foliation. □

*Proof of Proposition 8.2 for closed surfaces.* First suppose the common foliation is generic. Choose a piecewise $C^1$ stable path $\gamma$ from $q$ to $q'$ of Hodge length at most $M(K, D_0) + 1$, and put $\mathcal{L}(u) = \mathrm{length}_{\mathrm{H}}(g_u\gamma)$. Hodge norms of stable vectors are non-increasing, and contract at a uniform positive rate on each Mod–compact subset of $\mathcal{PQ}_1\mathcal{T}$ [Fra22, Theorem 1.3]. Here the flow multiplies stable period vectors by $e^{-u}$. At any principal differential $p$, Frankel's estimate applies on the Hodge-orthogonal complement of $\mathrm{span}\{\mathrm{Re}\,\omega_p, \mathrm{Im}\,\omega_p\}$. The remaining stable direction is $\mathbb{R}\mathrm{Re}\,\omega_p$, which contracts by $e^{-2u}$. These two components remain Hodge orthogonal under the flow.

Fix a Mod–compact neighborhood $N \subset \mathcal{PQ}_1\mathcal{T}$ of $K$. Nested period charts around compact marked representatives of $K$, and uniform comparison of Hodge and Euclidean norms, give $\rho > 0$ such that every stable path starting in $K$ and leaving $N$ has Hodge length greater than $\rho$. Let $\lambda > 0$ be a strict contraction constant on $N$, and choose $C_N$ so that the AGY norm is at most $C_N$ times the Hodge norm there. At each visit $q(u) \in K$, the initial portion of $g_u\gamma$ of length $\min\{\mathcal{L}(u), \rho\}$ stays in $N$. Contraction on this portion and nonexpansion on the rest give

$$\mathcal{L}'(u) \le -\lambda\, \mathbf{1}_K(q(u)) \min\{\mathcal{L}(u), \rho\} \tag{8.4}$$

for almost every $u$. On each interval $[0, U]$, the transported paths stay in a compact subset of the principal stratum, so $\mathcal{L}$ is locally absolutely continuous.

Choose $0 < h < \min\{\rho/2, \epsilon/C_N\}$. Inequality (8.4) shows that time spent in $K$ decreases $\mathcal{L}$ at rate at least $\lambda\rho$ while $\mathcal{L} \geq \rho$, and gives exponential decay once $\mathcal{L} \leq \rho$. Thus after $R_0 = R_0(K, D_0, \epsilon)$ units of time in $K$, we have $\mathcal{L} \leq h$. Use $R_{\text{gen}} = R_0 + 1$ as the generic threshold. The additional unit of time in $K$ lets us choose $s \leq t$ with $q(s) \in K$ and $\mathcal{L}(s) \leq h$. Since $h < \rho$, the entire path $g_s\gamma$ lies in $N$, and hence

$$d_{\text{AGY}}(q(s), q'(s)) \leq C_N \mathcal{L}(s) < \epsilon.$$

Stable AGY nonexpansion gives the same bound at $t$.

We now pass to an arbitrary common foliation. Fix a Mod–compact neighborhood $K^+ \subset \mathcal{PQ}_1\mathcal{T}$ of $K$. Choose $0 < \delta < 1/100$ with $4\delta < \epsilon$ so that every ambient $2\delta$–ball centered at a point of $K^+$ lies in a period chart whose closure is compact and contained in $\mathcal{PQ}_1\mathcal{T}$. This choice is uniform by local norm comparison in finitely many nested charts around compact marked representatives of $K^+$. Set $R_* = R_{\text{gen}}(K^+, D_0 + 1, \delta)$ and use $R = R_* + 1$. The additional unit of time in $K$ again lets us choose $s \leq t$ with $q(s) \in K$ and at least $R_*$ units of time in $K$ during $[0, s]$.

Write $F = F_+(q) = F_+(q')$. Choose generic $F_n \to F$ with $\text{Ext}_{X_q}(F_n) = 1$, and put

$$q_n = Q_1([F_n], X_q), \qquad a_n = \frac{1}{2}\log \text{Ext}_{X_{q'}}(F_n), \qquad p_n = g_{a_n} Q_1([F_n], X_{q'}).$$

We have $F_+(q_n) = F_+(p_n) = F_n$, and $a_n \to 0$, $q_n \to q$, $p_n \to q'$ in the full bundle. For large $n$, the base surfaces of $q_n, p_n$ are at distance at most $D_0 + 1$. The rays $g_u q_n$ converge uniformly to $g_u q$ on $[0, s]$, so

$$q(u) \in K \quad \Longrightarrow \quad q_n(u) \in K^+ \qquad (0 \leq u \leq s) \tag{8.5}$$

for all sufficiently large $n$. The generic case therefore gives $d_{\text{AGY}}(q_n(s), p_n(s)) < \delta$ with the same threshold $R_*$.

For large $n$, the ambient triangle inequality gives $D_{\text{AGY}}(q(s), p_n(s)) < 2\delta$. Thus the points $p_n(s)$ lie in a fixed neighborhood whose closure is compact and contained in $\mathcal{PQ}_1\mathcal{T}$. Since $p_n(s) \to q'(s)$ in $\mathcal{Q}_1\mathcal{T}$, we have $q'(s) \in \mathcal{PQ}_1\mathcal{T}$; flow invariance then gives $q' \in \mathcal{PQ}_1\mathcal{T}$. We may now use continuity of the ambient AGY distance to obtain $D_{\text{AGY}}(q(s), q'(s)) \leq \delta$. Their measured vertical foliations agree, so the stable version of Lemma 4.12 gives $d_{\text{AGY}}(q(s), q'(s)) \leq 4\delta < \epsilon$. Stable nonexpansion finishes the proof at time $t$. □

**A delay linear in the initial distance.**

*Proof of Theorem 8.1 for closed surfaces.* We apply DDM far enough along the reference ray that the nearby point it gives cannot lie on $[X_q, X_{q'}]$. Passing to a limit gives a nearby point on the comparison ray. After matching measured foliations, we apply Proposition 8.2 over the remaining time.

Fix a Mod–compact neighborhood $K^+ \subset \mathcal{PQ}_1\mathcal{T}$ of $K$ and choose $\delta < 1/100$ with $4\delta < \epsilon$ as in the preceding proof. By [DDM14, Theorem A], there are $C > 0$ and $L \geq 1$ such that whenever an interval of length $L$ on one side of a finite Teichmüller triangle spends at least half its time over the projection of $K^+$, it comes within $C$ of one of the other two sides. Set

$$\begin{aligned} &D_* = 2(C + L), \qquad R_* = R(K^+, D_*, \delta), \\ &H = \frac{R_* + 1}{1 - \eta}, \qquad W = \frac{H + 2L + 1}{1 - 2\eta}, \qquad B = W + C + 1. \end{aligned} \tag{8.6}$$

Here $R_*$ comes from Proposition 8.2 with the fixed distance bound $D_*$. We reserve the last $H$ units of time for convergence and find an interval to which DDM applies in $[T-W, T-H]$. All these constants are independent of $D$.

With these constants fixed, suppose the common foliation is generic and uniquely ergodic. We claim that $T \geq D + B$ and (8.1) with $K^+$ in place of $K$ already imply $d_{\mathrm{AGY}}(q(T), q'(T+a)) < \delta$. Partition $[T-W, T-H]$ into intervals of length $L$ and a remainder of length $r < L$. If each full interval spent less than half its time in $K^+$, the $K^+$–time on $[T-W, T]$ would be less than

$$\frac{W-H-r}{2} + r + H < \frac{W+H+L}{2} < (1-\eta)W.$$

This contradicts (8.1) with $u = T - W$ and $K^+$ in place of $K$. Thus there is a full interval $I$ during which $q$ spends at least half its time in $K^+$. Choose $v \in I$ with $q(v) \in K^+$.

Apply DDM to the segment $X_{q(u)}$, $u \in I$, in the triangle with vertices $X_q, X_{q'}, X_{q(N)}$, for $N > T$. Every $u \in I$ satisfies $u \geq T - W \geq D + C + 1$, so this segment is more than $C$ from $[X_q, X_{q'}]$. DDM therefore gives a point on $[X_{q'}, X_{q(N)}]$ within $C$ of the reference segment. We show that these finite sides converge to the ray $q'$ as $N \to \infty$. Normalize $F$ by $\mathrm{Ext}_{X_q}(F) = 1$, and let $G_N$ be the vertical foliation of the unit differential directing the segment from $X_{q'}$ to $X_{q(N)}$, of length $t_N \geq N - D$. Then

$$\mathrm{i}(G_N, F)^2 \leq \mathrm{Ext}_{X_{q(N)}}(G_N)\, \mathrm{Ext}_{X_{q(N)}}(F) = e^{-2(t_N+N)}.$$

The $G_N$ lie on the compact $\mathrm{Ext}_{X_{q'}}$–unit sphere. Since $F$ is filling and uniquely ergodic, every limit is the representative of $[F]$ with extremal length one at $X_{q'}$. Hubbard–Masur continuity gives convergence of the segments to $q'$ on bounded time intervals. The points supplied by DDM stay within $C$ of the fixed reference segment, so their distances from $X_{q'}$ are bounded. Passing to a subsequence gives $b \geq 0$ and $u \in I$ with $d_{\mathcal{T}}(X_{q'(b)}, X_{q(u)}) \leq C$.

Since $|u - v| \leq L$, replacing $q(u)$ by $q(v)$ increases this base-distance bound by at most $L$. Extremal-length comparison then bounds the time needed to match the measured foliations:

$$d_{\mathcal{T}}(X_{q(v)}, X_{q'(b)}) \leq C + L, \qquad |b - v - a| \leq C + L.$$

Consequently,

$$d_{\mathcal{T}}(X_{q(v)}, X_{q'(v+a)}) \leq 2(C+L) = D_*, \qquad F_+(q(v)) = F_+(q'(v+a)).$$

Also $v + a \geq v - D \geq C + 1 > 0$. During $[v, T]$, the ray $q$ spends at least $(1-\eta)(T-v) \geq R_* + 1$ time in $K^+$, since $T - v \geq H$. Proposition 8.2, applied to $q(v)$ and $q'(v+a)$, gives the required bound $d_{\mathrm{AGY}}(q(T), q'(T+a)) < \delta$.

For the general case, choose generic uniquely ergodic $[F_n] \to [F]$. Such foliations are dense: unique ergodicity has full measure [Mas82], and nonprincipal types and saddle-connection walls have measure zero in foliation coordinates. Keep both starting surfaces fixed and put

$$q_n = Q_1([F_n], X_q), \qquad q_n' = Q_1([F_n], X_{q'}), \qquad a_n = \frac{1}{2} \log \frac{\mathrm{Ext}_{X_{q'}}(F_n)}{\mathrm{Ext}_{X_q}(F_n)}.$$

Then $q_n \to q$, $q_n' \to q'$, and $a_n \to a$. Their initial base distance is exactly $D$, and $|a_n| \leq D$. Uniform convergence on $[0, T]$ gives $q_n(u) \in K^+$ whenever $q(u) \in K$, for

all $u \in [0,T]$ and all sufficiently large $n$. Thus $q_n$ satisfies (8.1) with $K^+$ in place of $K$, and the preceding argument applies with the same constants, giving

$$d_{\mathrm{AGY}}(q_n(T), q_n'(T+a_n)) < \delta.$$

Since $q(T) \in K$ and $q_n(T) \to q(T)$, the points $q_n'(T+a_n)$ eventually lie in the ambient $2\delta$–ball about $q(T)$. This ball lies in a chart whose closure is compact and contained in $\mathcal{PQ}_1\mathcal{T}$. The limit $q'(T+a)$ is therefore principal. As in the proof of Proposition 8.2, continuity of the ambient AGY distance followed by Lemma 4.12 gives $d_{\mathrm{AGY}}(q(T), q'(T+a)) \le 4\delta < \epsilon$. Flow invariance gives $q' \in \mathcal{PQ}_1\mathcal{T}$, and $T+a \ge T - D \ge B > 0$. □

**Surfaces with punctures.** The Hodge-distance comparison used in Lemma 8.3 and the triangle theorem of [DDM14, Theorem A] are stated for closed surfaces. We transfer the preceding convergence estimates through a fixed cover branched only over the punctures. This preserves the principal quadratic stratum and the initial Teichmüller distance. The orienting cover also branches over simple zeros and would not preserve this stratum.

Suppose $S = S_{g,n}$, where $n > 0$, $3g - 3 + n > 0$, and $S \neq S_{0,4}$. The hyperbolic orbifold obtained by giving the punctures order two has a finite regular surface cover [Sco83, Section 2, Theorem 2.5]. Write this cover as $h\colon \widehat{S} \to \overline{S}$, where $\overline{S}$ is obtained by filling the punctures of $S$ and $\widehat{S}$ is closed of genus at least two. It has ramification index two over each puncture and is unbranched elsewhere. Fix this cover independently of the differential.

Write $d = \deg h$. Pullback gives an isometric embedding $\iota\colon \mathcal{T}(S) \to \mathcal{T}(\widehat{S})$ [RS09, Section 7]. Its lift to unit-area differentials is

$$\mathcal{L}(q) = \widehat{q} = d^{-1}h^*q.$$

The map $\mathcal{L}$ commutes with the flow and is a proper topological embedding: pullback is continuous and injective, $\iota$ is isometric, and the unit-area bundle over a compact subset of Teichmüller space is compact. If $q$ has order $k$ at a puncture, its pullback has order $2k+2$ at every point above it. A simple pole becomes a regular point; if $q$ is holomorphic at the puncture, its pullback has a multiple zero above that puncture. Thus $q$ is principal if and only if $\widehat{q}$ is principal. The normalization also gives

$$F_+(\widehat{q}) = d^{-1/2}h^*F_+(q), \qquad \mathrm{Ext}_{\iota(X)}(d^{-1/2}h^*F) = \mathrm{Ext}_X(F),$$

where the second identity follows by pulling back the Hubbard–Masur differential and comparing areas.

To transfer the estimates, we need closeness of the lifts to imply closeness downstairs, uniformly for $q$ in a Mod–compact set. The subgroup of $\mathrm{Mod}(S)$ that lifts through $h$ has finite index, since there are only finitely many covers of a fixed degree. Consequently, for every $\mathrm{Mod}(S)$–compact $K \subset \mathcal{PQ}_1\mathcal{T}(S)$, the set

$$\widehat{K} = \mathrm{Mod}(\widehat{S}) \cdot \mathcal{L}(K)$$

is $\mathrm{Mod}(\widehat{S})$–compact. Choose a compact set of representatives for $K$ under the liftable subgroup. Properness of $\mathcal{L}$ and the fact that the ambient AGY metric induces the principal-stratum topology show that, for each $\epsilon > 0$, there is $\delta > 0$ such that

$$D_{\mathrm{AGY}}(\widehat{q}, \widehat{p}) < \delta \quad \Longrightarrow \quad D_{\mathrm{AGY}}(q,p) < \min\{1/100, \epsilon/4\}$$

whenever $q \in K$, $p \in \mathcal{Q}_1\mathcal{T}(S)$, and $\widehat{p}$ is principal. The same $\delta$ works for every $q \in K$, since the mapping classes in this subgroup lift through $h$. If also $F_+(p) = F_+(q)$, Lemma 4.12, applied to $-q, -p$, gives $d_{\mathrm{AGY}}(q, p) < \epsilon$.

For Proposition 8.2, choose $\delta$ as above, let $\widehat{R}$ be the closed-surface threshold for $\widehat{K}, D_0, \delta$, and set $R = \widehat{R} + 1$. If $q$ spends at least $R$ time in $K$ during $[0, t]$, there is a return $s \le t$ to $K$ after at least $\widehat{R}$ time has been spent there: the cumulative occupation time is continuous, and one further unit of $K$–time remains after it reaches $\widehat{R}$. Every $K$–visit lifts to a $\widehat{K}$–visit. Apply the closed-surface proposition at $s$ to obtain $\widehat{q}'$ principal and $d_{\mathrm{AGY}}(\widehat{q}(s), \widehat{q}'(s)) < \delta$. The cover preserves principality, and the preceding comparison gives $d_{\mathrm{AGY}}(q(s), q'(s)) < \epsilon$; stable nonexpansion gives the same bound at $t$. The endpoint $q(t)$ need not lie in $K$.

For Theorem 8.1, the lifted ray satisfies (8.1) with $\widehat{K}$ in place of $K$ and the same $\eta$. Apply the closed-surface theorem with target distance $\delta$ and use the preceding comparison at $q(T)$ and $q'(T + a)$, where $q(T) \in K$. The extremal-length identity preserves the time shift $a$, and the isometry preserves the initial distance $D$. Thus the conclusion still holds for $T \ge D + B$ with coefficient one on $D$. Only the additive constant and the upstairs accuracy change.

For $S_{0,4}$, every nonzero quadratic differential is principal and its double cover is a flat torus. Two unit-area differentials $q, p$ with the same measured vertical foliation differ by a shear with parameter $c$, where $|c| = 2 \sinh d_{\mathcal{T}}(X_q, X_p)$. Under the flow this parameter becomes $ce^{-2t}$. Along the straight shear path a relative period $z = x + iy$ has $dz = ix\, dc$, so $|dz|/|z| \le |dc|$. This applies to lifts of saddle connections downstairs, and hence

$$d_{\mathrm{AGY}}(q(t), p(t)) \le 2 \sinh\big(d_{\mathcal{T}}(X_q, X_p)\big) e^{-2t}.$$

This proves Proposition 8.2, since elapsed time is at least compact occupation time. For the theorem, $p = q'(a)$ has the same measured vertical foliation as $q$, and $|a| \le D$ gives $d_{\mathcal{T}}(X_q, X_p) \le 2D$. Therefore

$$d_{\mathrm{AGY}}(q(T), q'(T + a)) \le 2 \sinh(2D) e^{-2T} \le e^{-2(T-D)}.$$

Choosing $B > 0$ sufficiently large proves the theorem with $T \ge D + B$ and $T + a > 0$, in this case without an occupation assumption.

**Choosing $K_2$.** We enlarge $K_1$ to accommodate the bounded time shifts and nearby differentials used in chart comparisons.

**Definition 8.4.** Let $K_2$ be a Mod–compact subset of $\mathcal{P}\mathcal{Q}_1\mathcal{T}$ of the form

$$K_2 = \{q \in \mathcal{P}\mathcal{Q}_1\mathcal{T} : \mathrm{sys}(q) \ge \mathsf{s}_2\},$$

where $\mathsf{s}_2 > 0$ is chosen small enough so that the following holds. If $q_1, q_2$ are points in $\mathcal{P}\mathcal{Q}_1\mathcal{T}$ and $u, v \in \mathbb{R}$ are such that

$$q_1 \in K_1, \quad \max(|u|, |v|) \le \max(\mathsf{D}_0, \mathsf{D}_{\mathrm{chart}}, \mathsf{D}_{\mathrm{temp}}) \qquad \text{and} \quad D_{\mathrm{AGY}}(q_1(u), q_2(v)) \le \tfrac{1}{100},$$

then $q_2 \in K_2$.

In later applications of Proposition 8.2, we use $K = K_2$ and $\epsilon = 1/500$ unless another accuracy is specified. The auxiliary Mod–compact neighborhoods used in the proofs do not change this choice of $K_2$.

## 9. Gromov distance on $\mathcal{PMF}$

We construct a metric on $\mathcal{PMF}$ using separation times in the charts of Section 5. We record the last time at which two rays are AGY-close and both have good statistics; later separation gives a smaller distance. We first construct chart metrics and then a global metric, whose balls we compare with horospherical AGY balls.

Fix $\eta_2 = 1/4$ for the remainder of the paper, and assume $0 < \eta_1 < \eta_2$. The smaller value of $\eta_1$ needed for the spike estimates will be specified in (11.1). Thus $\eta_2$ is fixed before the separation times and metric constants defined below.

For $x, y \in \mathcal{PMF}$ and $F \in \mathcal{MF}$, define the separation time

$$\tau_F(x,y) = \sup\Big\{ t \in \mathbb{R} : x_F(t) \text{ and } y_F(t) \text{ have } (K_2,\eta_2)\text{–good statistics} \\ \text{and} \quad d_{\mathrm{AGY}}\big(x_F(t), y_F(t)\big) \le \tfrac{1}{100} \Big\}. \tag{9.1}$$

The time $\tau_F(x,y)$ may be negative or infinite. We set it equal to $-\infty$ if the admissible set is empty or if either point lies outside $\mathcal{PMF}(F)$. Let

$$\beta = \beta_{K_2}, \tag{9.2}$$

$$C_{\mathrm{AGY}} = C_{K_2} \ge 1 \tag{9.3}$$

as in Corollary 4.9. For a chart $\alpha \in \mathcal{A}$, define

$$\tau_\alpha(x,y) = \begin{cases} \tau_{F_\alpha}(x,y), & x,y \in W_\alpha, \\ -1, & \text{otherwise.} \end{cases}$$

Define the chart pre-distance by

$$D^\alpha_G(x,y) = e^{-\beta \tau_\alpha(x,y)}.$$

Here $e^{-\beta(+\infty)} = 0$. The value $e^\beta$ outside the chart ensures that a chain starting in $W_\alpha$ with cost at most one stays inside it.

**Proposition 9.1.** *For any chart $\alpha \in \mathcal{A}$ and distinct $x, y \in W_\alpha$, the supremum in (9.1) is attained, and*

$$0 < D^\alpha_G(x,y) \le 1.$$

*If either $x$ or $y$ lies outside $W_\alpha$, then*

$$D^\alpha_G(x,y) = e^\beta > 1.$$

*Proof.* By Theorem 5.1(g), both negative rays lie in $K_0 \subset K_2$. Closedness of $K_2$ includes their endpoints at time zero, which therefore have $(K_2,\eta_2)$–good statistics. Equation (5.1) gives $d_{\mathrm{AGY}}(x^\alpha(0), y^\alpha(0)) \le 1/100$, so zero is admissible and $\tau_{F_\alpha}(x,y) \ge 0$.

Next we show that $\tau_{F_\alpha}(x,y)$ is finite. Otherwise there is a sequence $t_i \to +\infty$ such that both $x^\alpha(t_i)$ and $y^\alpha(t_i)$ have $(K_2,\eta_2)$–good statistics and

$$d_{\mathrm{AGY}}\big(x^\alpha(t_i), y^\alpha(t_i)\big) \le \frac{1}{100}.$$

For all sufficiently large $i$, Corollary 4.9 applies on the interval $[0, t_i]$ and gives

$$d_{\mathrm{AGY}}\big(x^\alpha(0), y^\alpha(0)\big) \le \frac{1}{100} C_{\mathrm{AGY}} e^{-\beta t_i}.$$

Letting $i \to \infty$ contradicts $x \neq y$.

Finally, the admissible set is closed. For each fixed backward interval length, occupation time varies continuously with the endpoint. Together with closedness of $K_2$, this makes the set of good-statistics times closed; the AGY distance condition is also closed. The finite supremum is therefore attained. The value outside $W_\alpha$ follows from the definition. □

To obtain the triangle inequality, take the infimum of the costs of finite chains:

$$d_G^\alpha(x,y) = \inf \sum_{i=0}^{k-1} D_G^\alpha(x_i, x_{i+1}),$$

where $x = x_0, \dots, x_k = y$ is a finite sequence in $\mathcal{PMF}$. We allow $k = 0$ when $x = y$, with cost zero. This defines the chart metric on $W_\alpha$ (Corollary 9.3), even though the pre-distance need not vanish on the diagonal.

**Lemma 9.2.** *For any chart $\alpha$ and $x, y \in W_\alpha$ (with $x \neq y$), let $s$ be a time such that either*

$$d_{\mathrm{AGY}}(x^\alpha(s), y^\alpha(s)) \leq \tfrac{1}{100} \qquad \text{or} \qquad d_G^\alpha(x,y) \leq e^{-\beta s}/2. \tag{9.4}$$

*Then*

$$100\, C_{\mathrm{AGY}}^{-1} d_{\mathrm{AGY}}(x^\alpha(s), y^\alpha(s)) e^{-\beta s} \leq d_G^\alpha(x,y). \tag{9.5}$$

*Also, for $t = \tau_{F_\alpha}(x,y)$, we have*

$$100\, C_{\mathrm{AGY}}^{-1} d_{\mathrm{AGY}}(x^\alpha(t), y^\alpha(t)) D_G^\alpha(x,y) \leq d_G^\alpha(x,y) \leq D_G^\alpha(x,y). \tag{9.6}$$

*Proof.* Take chains $x = x_0, \dots, x_k = y$ whose costs approach $d_G^\alpha(x,y)$, and delete consecutive repetitions. We may assume every vertex lies in $W_\alpha$: an edge leaving the chart costs $e^\beta > 1$, whereas the direct chain costs at most one. If $d_G^\alpha(x,y) \leq e^{-\beta s}/2$, we also take the chain costs less than $e^{-\beta s}$.

Put $t_i = \tau_{F_\alpha}(x_i, x_{i+1})$. These times are finite and attained by Proposition 9.1. If every $t_i \geq s$, the good statistics at $t_i$ and Corollary 4.9 give

$$d_{\mathrm{AGY}}\big(x_i^\alpha(s), x_{i+1}^\alpha(s)\big) \leq \frac{C_{\mathrm{AGY}}}{100} e^{-\beta(t_i - s)}.$$

Since $D_G^\alpha(x_i, x_{i+1}) = e^{-\beta t_i}$, rearranging and using the triangle inequality gives

$$\sum_{i=0}^{k-1} D_G^\alpha(x_i, x_{i+1}) \geq 100\, C_{\mathrm{AGY}}^{-1} d_{\mathrm{AGY}}\big(x^\alpha(s), y^\alpha(s)\big) e^{-\beta s}. \tag{9.7}$$

If some $t_i < s$, the chain has cost at least $e^{-\beta t_i} > e^{-\beta s}$. Our choice of chains excludes this case under the second alternative in (9.4). The first alternative must therefore hold:

$$d_{\mathrm{AGY}}\big(x^\alpha(s), y^\alpha(s)\big) \leq \frac{1}{100}. \tag{9.8}$$

Since $C_{\mathrm{AGY}} \geq 1$, the right-hand side of (9.7) is at most $e^{-\beta s}$, so the same bound holds in this case. Taking the infimum proves (9.5).

For (9.6), the direct chain gives the upper bound. The lower bound follows by applying (9.5) at the attained separation time $t = \tau_{F_\alpha}(x,y)$, when the first alternative in (9.4) holds and $e^{-\beta t} = D_G^\alpha(x,y)$. □

**Corollary 9.3.** *$d_G^\alpha$ is a metric on $W_\alpha$.*

*Proof.* The chain definition gives symmetry, the triangle inequality, and $d_G^\alpha(x,x)=0$. For $x\neq y$, Proposition 9.1 gives $D_G^\alpha(x,y)>0$, and (9.6) then gives $d_G^\alpha(x,y)>0$. □

We now combine the chart pre-distances using the partition of unity $\{\theta_\alpha\}$ from Theorem 5.1(e). For $x,y\in\mathcal{PMF}$, define

$$D_G(x,y)=\sum_{\alpha\in\mathcal{A}}\frac{\theta_\alpha(x)+\theta_\alpha(y)}{2}\cdot D_G^\alpha(x,y)$$

and

$$d_G(x,y)=\inf\sum_{i=0}^{k-1}D_G(x_i,x_{i+1}),$$

where the infimum is over finite chains $x=x_0,\dots,x_k=y$ in $\mathcal{PMF}$, again allowing the zero-edge chain when $x=y$.

Choose $\mathsf{r}_1$ small enough such that

$$4|\mathcal{A}|\cdot\mathsf{r}_1\le\frac{1}{2}\qquad\text{and}\qquad\frac{4C_{\text{AGY}}|\mathcal{A}|\cdot\mathsf{r}_1}{100}\le\mathsf{r}_{\text{chart}}.$$

This in particular means that

$$\Big(x\in V_\alpha^{(2)}\quad\text{and}\quad d_G^\alpha(x,y)\le 4|\mathcal{A}|\mathsf{r}_1\Big)\quad\Longrightarrow\quad y\in V_\alpha^{(3)}. \tag{9.9}$$

In fact, assuming the lefthand side of (9.9), we have $d_G^\alpha(x,y)\le 1/2$ which forces $y\in W_\alpha$, since an endpoint outside the chart would force an edge of cost $e^\beta$. Then, Lemma 9.2, applied with $s=0$, gives

$$d_{\text{AGY}}\big(x^\alpha(0),y^\alpha(0)\big)\le\frac{4C_{\text{AGY}}|\mathcal{A}|\mathsf{r}_1}{100}\le\mathsf{r}_{\text{chart}}.$$

Theorem 5.1(f) therefore gives $y\in V_\alpha^{(3)}$.

**Lemma 9.4.** *Let $x\in V_\alpha^{(2)}$ and $y\in\mathcal{PMF}$ be such that $d_G(x,y)\le\mathsf{r}_1$. Then*

$$D_G(x,y)\ge d_G(x,y)\ge\frac{1}{2|\mathcal{A}|}d_G^\alpha(x,y).$$

*Proof.* The first inequality is immediate from the definition of $d_G$. We prove the second inequality. Without loss of generality, assume $x\neq y$.

Take a chain $x=x_0,\dots,x_k=y$ with

$$\sum_{i=0}^{k-1}D_G(x_i,x_{i+1})\le 2\mathsf{r}_1.$$

This, in particular, means for every $j\le k$,

$$\sum_{i=0}^{j-1}D_G(x_i,x_{i+1})\le 2\mathsf{r}_1.$$

Assuming, by induction, that $\theta_\alpha(x_i)\ge\frac{1}{|\mathcal{A}|}$ for $i=0,\dots,(j-1)$, we have

$$\frac{1}{2|\mathcal{A}|}\sum_{i=0}^{j-1}D_G^\alpha(x_i,x_{i+1})\le 2\mathsf{r}_1,$$

which means $d_G^\alpha(x, x_j) \le 4|\mathcal{A}|\cdot \mathsf{r}_1$. Now (9.9) implies $x_j \in V_\alpha^{(3)}$ and hence $\theta_\alpha(x_j) \ge \frac{1}{|\mathcal{A}|}$. We have shown that, for $i = 0, \dots, k$, $\theta_\alpha(x_i) \ge \frac{1}{|\mathcal{A}|}$. Therefore,

$$\sum_{i=0}^{k-1} D_G(x_i, x_{i+1}) \ge \frac{1}{2|\mathcal{A}|} \sum_{i=0}^{k-1} D_G^\alpha(x_i, x_{i+1}) \ge \frac{1}{2|\mathcal{A}|} d_G^\alpha(x, y).$$

Taking the infimum over all such sequences $x_i$ we get

$$d_G(x, y) \ge \frac{1}{2|\mathcal{A}|} d_G^\alpha(x, y).$$

This finishes the proof. □

Let $R$ be the constant provided by Proposition 8.2 with $K = K_2$, accuracy $1/500$, and initial Teichmüller distance bound $D = \mathsf{D}_{\text{temp}} + \mathsf{D}_{\text{chart}}$. Choose

$$\mathsf{R}_1 > \max\left\{\mathsf{D}_{\text{chart}}, \frac{R}{\eta_2 - \eta_1}\right\}. \tag{9.10}$$

**Theorem 9.5.** *Let $x \in V_\alpha^{(3)}$, $y \in W_\alpha$, and $t > \mathsf{R}_1$. Assume that $x^\alpha(t)$ has $(K_1, \eta_1)$–good statistics, and set $r = e^{-\beta t}$. Then:*

(a)
$$d_{\text{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) \le \frac{1}{100\mathsf{k}} \qquad \Longrightarrow \qquad d_G(x, y) < r.$$

(b)
$$d_G^\alpha(x, y) \le \frac{r}{C_{\text{AGY}}\mathsf{k}} \qquad \Longrightarrow \qquad d_G(x, y) < r.$$

*Proof.* For part (a), we must transfer closeness and good statistics to every chart contributing to $D_G(x, y)$. The first $R$ units of time in $K_1$ let us apply Proposition 8.2 to bring representatives of a common foliation within AGY distance $1/500$. The margin $\eta_2 - \eta_1$ absorbs these $R$ units of compact time when we check good statistics in the new chart.

By (4.8),

$$d_{\text{AGY}}\big(x^\alpha(0), y^\alpha(0)\big) \le d_{\text{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) \le \frac{1}{100\mathsf{k}}.$$

Let $s_0 > 0$ be the first time at which

$$|\{u \in [0, s_0] \ : \ x^\alpha(u) \in K_1\}| = R.$$

The occupation function is continuous, and the good statistics at $t$ give at least $(1 - \eta_1)t > R$ units of time in $K_1$ by (9.10). Thus $s_0$ exists and $s_0 < t$.

Let $\mathcal{A}'$ be the set of charts $\alpha' \in \mathcal{A}$ such that

$$\theta_{\alpha'}(x) + \theta_{\alpha'}(y) > 0.$$

Fix $\alpha' \in \mathcal{A}'$. Since

$$\operatorname{supp}(\theta_{\alpha'}) \subset V_{\alpha'}^{(4)},$$

at least one of $x, y$ lies in $V_{\alpha'}^{(4)}$. Denote this point by $z$ and the other point by $w$. Thus

$$z, w \in W_\alpha, \qquad z \in V_{\alpha'}^{(4)} \subset W_{\alpha'}.$$

By part (b) of Lemma 5.2, there is a time $t_0$ with $|t_0| \le \mathsf{D}_{\text{chart}}$ such that

$$F_+\big(z^{\alpha'}(t_0)\big) = F_+\big(z^{\alpha}(0)\big).$$

Moreover, part (a) of that lemma gives

$$d_{\mathcal{T}}\big(z^{\alpha}(0), z^{\alpha'}(0)\big) \le \mathsf{D}_{\text{temp}},$$

and hence

$$d_{\mathcal{T}}\big(z^{\alpha}(0), z^{\alpha'}(t_0)\big) \le \mathsf{D}_{\text{temp}} + \mathsf{D}_{\text{chart}}.$$

The segment $x^\alpha([0, s_0])$ spends at least $R$ time in $K_1$. If $z = x$, the same is therefore true for $z^\alpha([0, s_0])$. If $z = y$, then for every $u \in [0, s_0]$,

$$d_{\text{AGY}}\big(x^\alpha(u), y^\alpha(u)\big) \le d_{\text{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) \le \frac{1}{100\mathsf{k}}.$$

Thus, whenever $x^\alpha(u) \in K_1$, the choice of $K_2$ implies $y^\alpha(u) \in K_2$. In either case,

$$|\{u \in [0, s_0] \ : \ z^\alpha(u) \in K_2\}| \ge R.$$

Proposition 8.2, applied to the two rays with common vertical foliation, gives

$$d_{\text{AGY}}\big(z^\alpha(s_0), z^{\alpha'}(s_0 + t_0)\big) \le \frac{1}{500}.$$

Stable nonexpansion (Theorem 4.5) gives

$$d_{\text{AGY}}\big(z^\alpha(u), z^{\alpha'}(u + t_0)\big) \le \frac{1}{500}, \qquad u \ge s_0.$$

We also have, using (4.8), for every $0 \le u \le t$,

$$d_{\text{AGY}}\big(z^\alpha(u), w^\alpha(u)\big) \le \frac{1}{100\mathsf{k}}.$$

Part (c) of Lemma 5.2 (applied with $x = z$, $y = w$, and $s_0 \le u \le t$) therefore gives

$$x, y \in W_{\alpha'}, \qquad d_{\text{AGY}}\big(x^{\alpha'}(t), y^{\alpha'}(t)\big) < 1/100,$$

and, for $s_0 \le u \le t$,

$$D_{\text{AGY}}\big(x^\alpha(u), p^{\alpha'}(u + t_0)\big) < 1/100 \qquad (p \in \{x, y\}).$$

This implies that, for $s_0 \le u \le t$, if

$$x^\alpha(u) \in K_1$$

then in view of the definition of $K_2$ (Definition 8.4), we have

$$y^\alpha(u) \in K_2, \qquad x^{\alpha'}(u) \in K_2 \qquad \text{and} \qquad y^{\alpha'}(u) \in K_2.$$

We now claim that $x^{\alpha'}(t)$ and $y^{\alpha'}(t)$ have $(K_2, \eta_2)$–good statistics. Let $z = x$ or $y$. For $s_0 \le s \le t$,

$$|\{u \in [s, t] \ : \ z^{\alpha'}(u) \in K_2\}| \ge |\{u \in [s, t] \ : \ x^\alpha(u) \in K_1\}| \ge (1 - \eta_1)(t - s),$$

and for $0 \le s \le s_0$,

$$\begin{aligned} |\{u \in [s, t] \ : \ z^{\alpha'}(u) \in K_2\}| &\ge |\{u \in [s_0, t] \ : \ z^{\alpha'}(u) \in K_2\}| \\ &\ge |\{u \in [s_0, t] \ : \ x^\alpha(u) \in K_1\}| \\ &= |\{u \in [0, t] \ : \ x^\alpha(u) \in K_1\}| - R \\ &\ge (1 - \eta_1)t - R \ge (1 - \eta_2)t \ge (1 - \eta_2)(t - s). \end{aligned}$$

For $s < 0$, Theorem 5.1(g) gives $z^{\alpha'}(s) \in K_0 \subset K_2$. Adding the interval $[s, 0]$ to the estimate for $[0, t]$ proves the required bound for every $s < 0$ as well.

We can extend these statistics a little past $t$. Indeed, hold $x^\alpha(t) \in K_1$ fixed. For $p \in \{x, y\}$ and all sufficiently small $h \geq 0$, continuity preserves the strict bound

$$D_{\mathrm{AGY}}\big(x^\alpha(t), p^{\alpha'}(t + h + t_0)\big) < 1/100.$$

Definition 8.4, with shifts $0, t_0$, puts $p^{\alpha'}(t+h)$ in $K_2$. This additional interval entirely in $K_2$ preserves good statistics, and continuity also preserves $d_{\mathrm{AGY}}(x^{\alpha'}(t + h), y^{\alpha'}(t + h)) < 1/100$. Hence

$$\tau_{F_{\alpha'}}(x, y) > t,$$

and therefore

$$D^{\alpha'}_G(x, y) < e^{-\beta t} = r.$$

It follows that

$$D_G(x, y) = \sum_{\alpha' \in \mathcal{A}'} \frac{\theta_{\alpha'}(x) + \theta_{\alpha'}(y)}{2} D^{\alpha'}_G(x, y) < r \sum_{\alpha' \in \mathcal{A}'} \frac{\theta_{\alpha'}(x) + \theta_{\alpha'}(y)}{2} = r.$$

Thus

$$d_G(x, y) \leq D_G(x, y) < r,$$

which proves part (a).

We now prove part (b). If $x = y$, the statement trivially holds. Suppose $x \neq y$ and

$$d^\alpha_G(x, y) \leq \frac{r}{C_{\mathrm{AGY}}\mathsf{k}}.$$

Since $\mathsf{k} \geq 2$ and $C_{\mathrm{AGY}} \geq 1$, the second alternative in (9.4) holds with $s = t$. Lemma 9.2 therefore gives

$$100\, C^{-1}_{\mathrm{AGY}} d_{\mathrm{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) e^{-\beta t} \leq d^\alpha_G(x, y) \leq \frac{r}{C_{\mathrm{AGY}}\mathsf{k}}.$$

Since $r = e^{-\beta t}$, we obtain

$$d_{\mathrm{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) \leq \frac{1}{100\mathsf{k}}.$$

Part (a) now gives

$$d_G(x, y) < r. \qquad \square$$

**Corollary 9.6.** *$d_G$ is a metric on $\mathcal{PMF}$. Furthermore, $(\mathcal{PMF}, d_G)$ has bounded diameter.*

*Proof.* The chain definition gives symmetry, the triangle inequality, and $d_G(x, x) = 0$. Lemma 9.4 and Corollary 9.3 give positivity when $x \neq y$. Since each chart predistance is at most $e^\beta$ and the partition-of-unity weights sum to one,

$$d_G(x, y) \leq D_G(x, y) \leq e^\beta. \qquad \square$$

The balls of $d^\alpha_G$ and $d_G$ are Borel in the usual topology on $\mathcal{PMF}$. With $F = F_\alpha$, the conditions in (9.1) are closed on $W^2_\alpha \times \mathbb{R}$, since the leafwise distance is continuous and the time spent in the closed set $K_2$ on each fixed backward interval is upper semicontinuous. For $r > 0$, exhaust the open subset where $e^{-\beta t} < r$ by compact sets and project the admissible triples to $(x, y)$. This writes $\{D^\alpha_G < r\}$ inside the chart as a countable union of compact sets, and the cost is constant off $W^2_\alpha$. Taking rational

upper bounds for the individual costs gives the same property for finite sums with continuous nonnegative weights, hence for the cost of a chain with a fixed number of edges. Projecting to the endpoints and taking the union over chain lengths gives the same property for $\{d_G^\alpha < r\}$ and $\{d_G < r\}$.

9.1. **Volumes of balls.** We use the horospherical volume forms of [AG13, Example 4.2]. We first specify their normalization and its relation to Thurston measure; the factor relating a hypersurface measure to the volume of its cone will be needed below. For $q \in \mathcal{PQ}_1\mathcal{T}$, recall that

$$\mathrm{Hor}(q) = \mathrm{Hor}_-(F_-(q)) = \big\{Q_1(x, F_-(q)) : x \in \mathcal{PMF}(F_-(q))\big\}.$$

Write the period of $q$ as $a + ib \in L(q) + iL(q)$, and let $\lambda_q$ be the real period volume form on $L(q)$ for which the integral period lattice has covolume one. Here the lattice is $L(q) \cap H^1(\widetilde{X}, \mathcal{Z}(q); \mathbb{Z})$, using the relative realization of $L(q)$. We use the same notation for its associated Lebesgue measure. Gauss–Manin transport preserves this measure. Let $A_b(u) = A(u, b)$ be the area functional with imaginary period $b$ fixed, so that $A_b(a) = 1$. In local period coordinates,

$$T_q \,\mathrm{Hor}(q) = \ker A_b.$$

Following Avila–Gouëzel, define the positive horospherical measure by the volume density

$$d\mu_q(v_1, \dots, v_{w-1}) = \big|\lambda_q(a, v_1, \dots, v_{w-1})\big|, \qquad v_j \in \ker A_b. \tag{9.11}$$

This density is constant along the affine horosphere: replacing $a$ by $a + u$, with $u \in \ker A_b$, does not change its restriction to $\ker A_b$. Thus $\mu_q$ depends only on the horosphere containing $q$.

We retain the Thurston measure $\mu_{\mathrm{Th}}$ fixed in Section 2. In real period coordinates it is $c_S\lambda_q$, where $c_S > 0$ is a fixed constant depending only on the surface and the two lattice normalizations. To see that this factor is independent of $q$, use the adapted train tracks of Proposition 3.5. Their transverse weights are signed real edge periods. The Thurston symplectic form in these coordinates is one half of the cup-product form on odd cohomology; this is the train-track calculation in [Dum15, proof of Theorem 5.8], applied to real periods. The same calculation includes punctured monogons at simple poles. Taking top exterior powers gives the stated constant factor. The local identities agree on overlaps and extend along positive rays by homogeneity.

For a measurable set $E$ in a small horospherical period chart, write $a_p + ib$ for the period of $p \in E$ and set

$$\mathcal{C}_q(E) = \{ta_p : p \in E,\ 0 < t \le 1\} \subset L(q).$$

This is a cone in the vector space; no extension of the period chart along the whole cone is required. Since $A_b(a_p) = 1$, the map $(t, p) \mapsto ta_p$ is injective and has radial Jacobian $t^{w-1}$ relative to (9.11). Therefore

$$\lambda_q(\mathcal{C}_q(E)) = \frac{1}{w}\mu_q(E), \qquad \mu_{\mathrm{Th}}\big(\{tF_+(p) : p \in E,\ 0 < t \le 1\}\big) = \frac{c_S}{w}\mu_q(E). \tag{9.12}$$

By subdividing into period charts, the second equality holds for every measurable $E$ in the principal-stratum part of $\mathrm{Hor}(q)$. Equivalently,

$$\mu_q(E) = \frac{w}{c_S}\,\mu_{\mathrm{Th}}\Big(\big\{F \in \mathcal{MF} : Q_1([F], F_-(q)) \in E,\ \mathrm{i}(F, F_-(q)) \le 1\big\}\Big). \tag{9.13}$$

For any $E \subset W_\alpha$ and $t \geq 0$, define

$$E_\alpha(t) = \{x^\alpha(t) : x \in E\}.$$

Then

$$E_\alpha(0) \subset \mathrm{Hor}(q_\alpha) \qquad \text{and} \qquad E_\alpha(t) \subset \mathrm{Hor}(g_t q_\alpha).$$

Since $F_-(g_t q_\alpha) = e^{-t} F_\alpha$, we have

$$\begin{aligned} \mu_{g_t q_\alpha}(E_\alpha(t)) &= \frac{w}{c_S}\, \mu_{\mathrm{Th}}\Big(\big\{F \in \mathcal{MF} : [F] \in E,\ \mathrm{i}(F, e^{-t}F_\alpha) \leq 1\big\}\Big) \\ &= \frac{w}{c_S}\, \mu_{\mathrm{Th}}\Big(\big\{F \in \mathcal{MF} : [F] \in E,\ \mathrm{i}(F, F_\alpha) \leq e^{t}\big\}\Big). \end{aligned}$$

By homogeneity of Thurston measure of degree $w = \dim(\mathcal{MF})$, this gives

$$\mu_{g_t q_\alpha}(E_\alpha(t)) = e^{wt} \mu_{q_\alpha}(E_\alpha(0)). \tag{9.14}$$

In what follows, when the base point is clear, we write $\mu$ for the corresponding horospherical measure $\mu_q$.

**Lemma 9.7.** *For any measurable set $E \subset W_\alpha$, we have*

$$\nu_\rho(E) \asymp \mu(E_\alpha(0)).$$

*Here $\mu$ is the horospherical measure on $\mathrm{Hor}(q_\alpha)$ defined above.*

*Proof.* For $x \in W_\alpha$, set $F_x = F_+(x^\alpha(0))$, so that $\mathrm{i}(F_x, F_\alpha) = 1$. The cone defining $\nu_\rho$ cuts the ray through $F_x$ at $\rho(F_x)^{-1} F_x$, whereas the cone in (9.13) cuts it at $F_x$. Homogeneity of degree $w$ and (9.12) therefore give the exact density

$$d\nu_\rho(x) = \frac{c_S}{w\, m_\rho}\, \rho(F_+(x^\alpha(0)))^{-w}\, d\mu_{q_\alpha}(x^\alpha(0)). \tag{9.15}$$

Here $m_\rho = \mu_{\mathrm{Th}}(\{\rho \leq 1\})$ is the normalizing mass from Section 2. The displayed density is positive and continuous on the compact chart closure, so it is bounded above and below there. □

We next bound the volume of a small intrinsic horospherical ball, uniformly even when its center leaves every compact set. Shuvaeva proves an ambient small-ball estimate for the holomorphic principal stratum [Shu26, Theorem 2.1.3]. For the horospherical measure we give a direct argument that also includes simple poles. The key is that the real AGY norm controls the Hodge norm, whose balls have fixed volume in odd cohomology.

**Lemma 9.8.** *There is a constant $\mathsf{V}_0 > 0$, depending only on $S$, such that for every $q \in \mathcal{PQ}_1\mathcal{T}$,*

$$\mu_q\big(B(q, \tfrac{1}{100})\big) \leq \mathsf{V}_0.$$

*Here $B(q, r)$ is the ball in $\mathrm{Hor}(q)$ for the intrinsic metric $d_{\mathrm{AGY}}$. Furthermore, for every* Mod*–compact set $K \subset \mathcal{PQ}_1\mathcal{T}$ there is a constant $\mathsf{V}_1 = \mathsf{V}_1(S, K) > 0$ such that, for every $q \in K$,*

$$\mu_q\big(B(q, \tfrac{1}{100})\big) \geq \mathsf{V}_1.$$

*Proof.* We first bound the volume of the real norm ball

$$\mathbb{B}_q = \{u \in L(q) : \|u\|_q \le 1\}.$$

We then place the cone over a small horospherical ball inside a fixed multiple of $\mathbb{B}_q$ and use (9.12).

Lemma 4.2 gives $\|u\|_{H,q} \le \sqrt{2}\,\|u\|_q$ for $u \in L(q)$, including at nongeneric centers and in the presence of simple poles.

The $\lambda_q$–volume of a Hodge unit ball in $L(q)$ depends only on $S$. To check this, let $\Omega$ be the cup-product form restricted to $L(q)$, and write $w = 2m$. The form $\Omega$ is nondegenerate, and the Hodge star $J_q$ preserves $L(q)$ because it commutes with the covering involution. The Hodge inner product is

$$G_q(u, v) = \Omega(u, J_q v).$$

Since $J_q^2 = -1$ and $\det J_q = 1$, the volume form of $G_q$ is $|\Omega^m|/m!$. Its ratio to $\lambda_q$ depends only on the topological orienting cover and its period lattice. In particular, (4.3) implies that there is a constant $C_{\mathrm{norm}} = C_{\mathrm{norm}}(S)$ such that

$$\lambda_q(\mathbb{B}_q) \le \lambda_q\big(\{u : \|u\|_{H,q} \le \sqrt{2}\}\big) \le C_{\mathrm{norm}} \qquad (q \in \mathcal{P}\mathcal{Q}_1\mathcal{T}). \tag{9.16}$$

The norm comparison used all saddle connections, without excluding any direction. The identification of odd relative and absolute cohomology and the volume of the Hodge ball are topological. Thus the bound is uniform also when the center has horizontal or vertical saddle connections.

Now put $r_0 = 1/200$ and $E = B(q, r_0)$. The uniform leaf chart of Lemma 4.3, valid also when the fixed foliation has saddle connections, identifies $E$ with a subset of $\ker A_b$. Its points have periods $a + u_p + ib$, and Proposition 4.4 gives

$$\|u_p\|_q \le 4r_0 = 1/50.$$

Also $\|a\|_q \le 1$, since the absolute real part of a saddle-connection period is at most its length. It follows that

$$\mathcal{C}_q(E) = \{t(a + u_p) : p \in E,\ 0 < t \le 1\} \subset \frac{51}{50}\mathbb{B}_q.$$

Consequently, (9.12) and (9.16) give

$$\mu_q(B(q, 1/200)) = w\lambda_q(\mathcal{C}_q(E)) \le w\left(\frac{51}{50}\right)^w C_{\mathrm{norm}}.$$

The doubling estimate [AG13, Corollary 5.4], recalled in Lemma A.2, gives

$$\mu_q(B(q, 1/100)) \le c_D^{-1}\mu_q(B(q, 1/200)).$$

Thus we may take $\mathsf{V}_0 = c_D^{-1} w (51/50)^w C_{\mathrm{norm}}$, which depends only on $S$.

For the lower bound, fix $q_0 \in K$ and choose a small period box around it. The spaces $\ker A_b$ admit a continuous local choice of basis. A fixed sufficiently small Euclidean box in these bases maps into $B(q, 1/100)$ for every center $q$ in a smaller neighborhood of $q_0$, by continuity of the AGY norm and the affine chart estimates. The density in (9.11) is positive and continuous, so these boxes have a common positive lower bound for their measures after shrinking the neighborhood. A finite cover of $K/\operatorname{Mod}$ by such neighborhoods gives $\mathsf{V}_1(S, K) > 0$. □

For $x \in \mathcal{PMF}$ and $r > 0$, write

$$B_G(x, r) = \{y \in \mathcal{PMF} : d_G(x, y) < r\}.$$

For $x \in W_\alpha$, write

$$B^\alpha_G(x, r) = \{y \in W_\alpha : d^\alpha_G(x, y) < r\}.$$

**Proposition 9.9.** *Let $W = w/\beta$. For every*

$$0 < r \le \frac{\mathsf{r}_1}{4|\mathcal{A}|C_{\text{AGY}}} \qquad \text{and} \qquad x \in \mathcal{PMF},$$

*we have*

$$\nu_\rho\big(B_G(x, r)\big) \prec r^W.$$

*Furthermore, suppose*

$$x \in V^{(1)}_\alpha, \qquad t > \mathsf{R}_1, \qquad r = e^{-\beta t},$$

*and $x^\alpha(t)$ has $(K_1, \eta_1)$–good statistics. Then*

$$\nu_\rho\big(B_G(x, r)\big) \succ r^W.$$

*Proof.* We first prove the upper bound. Choose a chart $\alpha$ such that $x \in U_\alpha \subset V^{(1)}_\alpha$. Set

$$\widehat{r} = 2|\mathcal{A}| \cdot r \qquad \text{which implies} \qquad \widehat{r} \le C^{-1}_{\text{AGY}}\mathsf{r}_1/2.$$

If $y \in B_G(x, r)$, then Lemma 9.4 gives

$$d^\alpha_G(x, y) \le 2|\mathcal{A}|\, d_G(x, y) < \widehat{r}.$$

Thus

$$B_G(x, r) \subset B^\alpha_G(x, \widehat{r}).$$

Note that $2C_{\text{AGY}}\widehat{r} \le \mathsf{r}_1 < 1$. Choose $s \ge 0$ so that

$$e^{-\beta s} = 2\, C_{\text{AGY}}\widehat{r}.$$

If $y \in B^\alpha_G(x, \widehat{r})$, then

$$d^\alpha_G(x, y) < \widehat{r} = \frac{1}{2} C^{-1}_{\text{AGY}} e^{-\beta s}.$$

The second alternative in (9.4) therefore holds, and Lemma 9.2 gives

$$100\, C^{-1}_{\text{AGY}} d_{\text{AGY}}\big(x^\alpha(s), y^\alpha(s)\big) e^{-\beta s} \le d^\alpha_G(x, y) < \widehat{r}.$$

Since $e^{-\beta s} = 2\, C_{\text{AGY}}\widehat{r}$, we obtain

$$d_{\text{AGY}}\big(x^\alpha(s), y^\alpha(s)\big) < \tfrac{1}{200}.$$

Let $E = B^\alpha_G(x, \widehat{r})$. Then

$$E_\alpha(s) \subset B\big(x^\alpha(s), \tfrac{1}{200}\big).$$

Lemmas 9.7 and 9.8, together with the volume growth formula (9.14), now give

$$\begin{aligned} \nu_\rho(B_G(x, r)) &\le \nu_\rho(E) \asymp \mu(E_\alpha(0)) = e^{-ws}\mu(E_\alpha(s)) \\ &\prec e^{-ws} = (2C_{\text{AGY}}\widehat{r})^W \asymp r^W. \end{aligned}$$

We now prove the lower bound. Define

$$\mathcal{E}_t = B\left(x^\alpha(t), \frac{1}{100\mathsf{k}}\right), \qquad E = \{y \in W_\alpha : y^\alpha(t) \in \mathcal{E}_t\}.$$

Every point of $\mathcal{E}_t$ has the form $y^\alpha(t)$ for some $y \in \mathcal{PMF}(F_\alpha)$. Moreover, by (4.8),

$$d_{\mathrm{AGY}}\big(x^\alpha(0), y^\alpha(0)\big) \le d_{\mathrm{AGY}}\big(x^\alpha(t), y^\alpha(t)\big) < \frac{1}{100\mathsf{k}} \le \mathsf{r}_{\mathrm{chart}}.$$

Since $x \in V_\alpha^{(1)}$, Theorem 5.1(f) puts $y$ in $V_\alpha^{(3)} \subset W_\alpha$. Thus $E_\alpha(t) = \mathcal{E}_t$. Theorem 9.5(a) gives

$$d_G(x, y) < e^{-\beta t} = r.$$

That is,

$$E \subset B_G(x, r).$$

Since $x^\alpha(t) \in K_1$, applying Lemma 9.8 and Lemma A.2 at the radius $1/(100\mathsf{k})$ gives

$$\mu\big(E_\alpha(t)\big) \succ 1.$$

Using (9.14) and Lemma 9.7, we obtain

$$\begin{aligned}\nu_\rho(B_G(x, r)) \ge \nu_\rho(E) &\asymp \mu(E_\alpha(0)) = e^{-wt}\mu(E_\alpha(t)) \\ &\succ e^{-wt} = r^W.\end{aligned}$$

□

**Corollary 9.10.** *For every $W' > W$, there is a constant $D_\rho > 0$, depending only on $\nu_\rho$, $W$, and $W'$, such that for every $x \in \mathcal{PMF}$ and $r > 0$,*

$$\int_{\mathcal{PMF} - B_G(x,r)} \frac{1}{d_G(y, x)^{W'}}\, d\nu_\rho(y) \le \frac{D_\rho}{r^{W'-W}}. \tag{9.17}$$

*Proof.* Since $\nu_\rho$ is a probability measure, enlarging the constant in Proposition 9.9 extends its upper bound to all radii. Thus there is $C_{\mathrm{vol}} > 0$ such that $\nu_\rho(B_G(x, s)) \le C_{\mathrm{vol}} s^W$ for all $x \in \mathcal{PMF}$ and $s > 0$. Decompose the domain of integration into the shells $2^j r \le d_G(x, y) < 2^{j+1} r$, $j \ge 0$. Then

$$\begin{aligned}\int_{\mathcal{PMF} - B_G(x,r)} \frac{d\nu_\rho(y)}{d_G(y, x)^{W'}} &\le C_{\mathrm{vol}} \sum_{j \ge 0} (2^j r)^{-W'} (2^{j+1} r)^W \\ &= \frac{2^W C_{\mathrm{vol}}}{1 - 2^{W-W'}}\, r^{W-W'}.\end{aligned}$$

This proves the claim with $D_\rho = 2^W C_{\mathrm{vol}}/(1 - 2^{W-W'})$. □

## 10. Good geodesics, revisited

We associate a Gromov ball to each mapping class in $\Gamma_\alpha(R)$. At each sufficiently large scale, these balls have uniformly bounded overlap. We also show that the set of points covered at no more than $(1 - \eta_1)n$ of the first $n$ scales has exponentially small measure. These are the covering estimates used in Section 14. For $\gamma \in \Gamma_\alpha(R)$, write

$$a_\gamma = [F_+(\gamma)],$$

and set

$$B_G(\gamma) = B_G\big(a_\gamma, e^{-\beta \ell(\gamma)}\big).$$

Let $\mathsf{R}_1$ be as in Theorem 9.5.

**Proposition 10.1.** *Let $\tau'$ be as in Proposition 7.4 with $\tau' > \mathsf{R}_1 + 1$ and let $R_j = j\tau'$. We have:*

*(a) The set $\Gamma(R_j)$ is finite. There is a constant $\mathsf{B} = \mathsf{B}(\eta_1)$ such that, for any sufficiently large $j$, a point in $\mathcal{PMF}$ belongs to at most $\mathsf{B}$ of the balls*

$$B_G(\gamma), \qquad \gamma \in \Gamma(R_j).$$

*(b) For each chart $\alpha$ and $x \in U_\alpha$, let*

$$k'_n(x) = \left| \left\{ j \in [1,n] \,\middle|\, x \in \bigcup_{\gamma \in \Gamma_\alpha(R_j)} B_G(\gamma) \right\} \right|.$$

*Then, for $n$ large enough, we have*

$$\nu_\rho\Big(\{x \in U_\alpha : k'_n(x) \le (1-\eta_1)n\}\Big) \le \kappa^n,$$

*where $0 < \kappa = \kappa(\eta_1) < 1$.*

*Proof.* By Proposition 7.4(a),(b), every $\gamma \in \Gamma(R_j)$ satisfies

$$d_{\mathcal{T}}(P, \gamma(P)) \le 2\mathsf{D}_0 + \ell(\gamma) \le 2\mathsf{D}_0 + R_j + \rho_1 + 1.$$

Since closed Teichmüller balls are compact and the mapping class group acts properly discontinuously, $\Gamma(R_j)$ is finite.

To bound the overlap, we work one chart at a time. Balls containing a common point will force the orbit points $\gamma_i(P)$ into a uniformly bounded Teichmüller neighborhood of one another. Proper discontinuity will then bound their number.

Fix $x_0 \in \mathcal{PMF}$ and $\alpha \in \mathcal{A}$, and list all elements with

$$x_0 \in B_G(\gamma_i), \qquad \gamma_i \in \Gamma_\alpha(R_j), \qquad i = 1, \dots, m_\alpha.$$

If the list is empty, its contribution to the overlap is zero. Otherwise, fix one element and call it $\gamma$. By Proposition 7.4(b), for all $i$ we have

$$|\ell(\gamma_i) - \ell(\gamma)| \le \rho_1 + 2.$$

Hence

$$d_G(a_{\gamma_i}, a_\gamma) \le d_G(a_{\gamma_i}, x_0) + d_G(x_0, a_\gamma) \le C_0 e^{-\beta \ell(\gamma)}$$

for a constant $C_0$ depending only on $\rho_1$.

By Proposition 7.4(d), both $a_\gamma$ and $a_{\gamma_i}$ lie in $V_\alpha^{(1)} \subset V_\alpha^{(2)}$. For $R_j$ large enough, the preceding inequality is smaller than $\mathsf{r}_1$. Therefore Lemma 9.4 gives

$$d_G^\alpha(a_{\gamma_i}, a_\gamma) \le C_1 e^{-\beta \ell(\gamma)}.$$

Choose a constant $L_0$ large enough, depending only on $C_1$ and $\beta$ so that

$$C_1 e^{-\beta \ell(\gamma)} \le \frac{1}{2} e^{-\beta(\ell(\gamma) - L_0)}.$$

If $a_{\gamma_i} \neq a_\gamma$, apply Lemma 9.2 with

$$s = \ell(\gamma) - L_0$$

to obtain a uniform bound

$$d_{\mathrm{AGY}}\big(a_{\gamma_i}^\alpha(s), a_\gamma^\alpha(s)\big) \le C_2.$$

If the centers coincide, the two chart representatives coincide, so the same bound holds with zero on the left. Using (4.8) over the bounded interval of length $L_0$, we obtain

$$d_{\mathrm{AGY}}\big(a_{\gamma_i}^\alpha(\ell(\gamma)), a_\gamma^\alpha(\ell(\gamma))\big) \le C_3.$$

By Proposition 7.4(d) and Definition 7.2, $a^\alpha_\gamma(\ell(\gamma)) \in K_1$. Apply a common mapping class so that this point lies in a fixed compact set of marked representatives of $K_1$. Since $D_{\text{AGY}} \le d_{\text{AGY}}$, the other differential lies in the closed ambient $C_3$–neighborhood of this compact set. This neighborhood is compact: the marked ambient AGY metric is a complete, locally compact length metric, as discussed in Section 4. Continuity of the Teichmüller distance therefore gives

$$d_{\mathcal{T}}\big(a^\alpha_{\gamma_i}(\ell(\gamma)), a^\alpha_\gamma(\ell(\gamma))\big) \le C_4.$$

Also, by Proposition 7.4(b),

$$d_{\mathcal{T}}\big(a^\alpha_{\gamma_i}(\ell(\gamma)), a^\alpha_{\gamma_i}(\ell(\gamma_i))\big) \le \rho_1 + 2.$$

By Proposition 7.4(c'), the points $a^\alpha_{\gamma_i}(\ell(\gamma_i))$ and $\gamma_i(P)$ are at uniformly bounded Teichmüller distance, and similarly $a^\alpha_\gamma(\ell(\gamma))$ is uniformly close to $\gamma(P)$. Therefore there is a constant $L_1 = L_1(\eta_1)$ such that

$$d_{\mathcal{T}}\big(\gamma_i(P), \gamma(P)\big) \le L_1$$

for every $i$. Equivalently,

$$d_{\mathcal{T}}\big(\gamma^{-1}\gamma_i(P), P\big) \le L_1.$$

Since the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ is properly discontinuous, there are only finitely many mapping classes $h$ with

$$d_{\mathcal{T}}(h(P), P) \le L_1.$$

Thus $m_\alpha$ is bounded by a constant depending only on $L_1$, and hence only on $\eta_1$. The total multiplicity $\mathsf{B}$ is bounded by $\sum_{\alpha\in\mathcal{A}} m_\alpha$. This proves part (a).

We now prove part (b) by showing that the approximations in Proposition 7.4(f) imply membership in the balls $B_G(\gamma)$. Let $E_\alpha(R_j)$ be the sets defined there, and suppose $q \in E_\alpha(R_j)$. Then there is $\gamma \in \Gamma_\alpha(R_j)$ such that

$$d_{\text{AGY}}\big(q(\ell(\gamma)), a^\alpha_\gamma(\ell(\gamma))\big) < \frac{1}{100\mathsf{k}}.$$

Let

$$x = [F_+(q)].$$

Since $q \in U_\alpha(0)$, we have $x \in U_\alpha \subset W_\alpha$ and

$$q = x^\alpha(0), \qquad q(t) = x^\alpha(t).$$

By Proposition 7.4(d), $a_\gamma \in V^{(1)}_\alpha \subset V^{(3)}_\alpha$ and $a^\alpha_\gamma(\ell(\gamma))$ has $(K_1, \eta_1)$–good statistics. Also, part (b) of that proposition and $\tau' > \mathsf{R}_1 + 1$ give

$$\ell(\gamma) \ge R_j - 1 = j\tau' - 1 > \mathsf{R}_1.$$

We may therefore apply Theorem 9.5(a) with base point $a_\gamma$, time $\ell(\gamma)$, and point $x$, to obtain

$$d_G(a_\gamma, x) < e^{-\beta\ell(\gamma)}.$$

Thus $x \in B_G(\gamma)$: every index counted by $k_n(x^\alpha(0))$ is counted by $k'_n(x)$. At the excluded initial indices, $\Gamma_\alpha(R_j) = E_\alpha(R_j) = \varnothing$, so

$$k'_n(x) \ge k_n(x^\alpha(0)) \qquad (x \in U_\alpha).$$

Under $x \mapsto x^\alpha(0)$, Lemma 9.7 compares $\nu_\rho$ with the horospherical measure $\mu$ on $\mathrm{Hor}(F_\alpha)$. Together with Proposition 7.4(f), the counting inequality therefore gives, for all sufficiently large $n$,

$$\begin{aligned}\nu_\rho\big(\{x \in U_\alpha : k_n'(x) \le (1-\eta_1)n\}\big) &\le C_\alpha\, \mu\big(\{q \in U_\alpha(0) : k_n(q) \le (1-\eta_1)n\}\big)\\ &\le C_\alpha \kappa_0^n \mu(U_\alpha(0)),\end{aligned}$$

where $C_\alpha > 0$ is independent of $n$ and $\kappa_0 = \kappa_0(\eta_1) < 1$. Set $\kappa = (1+\kappa_0)/2 < 1$. Since $\kappa > \kappa_0$, the last expression is at most $\kappa^n$ for all sufficiently large $n$. There are only finitely many charts, so the threshold can be chosen uniformly in $\alpha$. It may depend on $\eta_1$, $\tau'$, and $\rho$, whereas $\kappa$ depends only on $\eta_1$. This proves part (b). □

## 11. Lipschitz Estimates for spike functions

Recall that $\Gamma(R) = \bigcup_\alpha \Gamma_\alpha(R)$, with $\Gamma_\alpha(R)$ as in Proposition 7.4, and that $H(\gamma, x)$ is defined by (2.9). We estimate its size in terms of the distance from the attracting foliation of $\gamma$, and its variation between points at distance at most $e^{-\beta\ell(\gamma)}$. These estimates will verify the spike conditions in Section 12. As before, $w = \dim(\mathcal{MF}) = 6g - 6 + 2n$.

**Theorem 11.1.** *Choose*

$$0 < \eta_1 < \min\left\{\frac{1}{100}, \frac{1}{512w}\right\}, \qquad \eta = 256w\eta_1. \tag{11.1}$$

*There are constants $\mathsf{C} > 0$ and $\mathsf{R}_0 > 0$, depending on $\eta_1$, such that the following holds. Let $\gamma \in \Gamma(R)$ for $R \ge \mathsf{R}_0$ and let $a_\pm = [F_\pm(\gamma)]$ and $r = e^{-\beta\ell(\gamma)}$. Then*

(a) *For any point $x \in \mathcal{PMF}$,*

$$\frac{e^{w\ell(\gamma)}}{\mathsf{C}}\left(1 + \frac{d_G(a_+, x)}{r}\right)^{-\frac{2w+\eta}{\beta}} \le H(\gamma, x) \le \mathsf{C}\, e^{w\ell(\gamma)}\left(1 + \frac{d_G(a_+, x)}{r}\right)^{-\frac{2w-\eta}{\beta}}, \tag{11.2}$$

*In particular, if $x, y \in \mathcal{PMF}$ satisfy $d_G(x, y) \le r$, then*

$$\frac{1}{\mathsf{C}}\left(1 + \frac{d_G(a_+, x)}{r}\right)^{-\frac{2\eta}{\beta}} \le \frac{H(\gamma, y)}{H(\gamma, x)} \le \mathsf{C}\left(1 + \frac{d_G(a_+, x)}{r}\right)^{\frac{2\eta}{\beta}}, \tag{11.3}$$

(b) *For any $x, y \in \mathcal{PMF}$ with $d_G(x, y) \le r$,*

$$\Big|H(\gamma, x) - H(\gamma, y)\Big| \le \frac{\mathsf{C}}{r} H(\gamma, x)\left(1 + \frac{d_G(a_+, x)}{r}\right)^{\frac{\eta}{\beta}} d_G(x, y). \tag{11.4}$$

**Thin triangles.** Fix $\gamma \in \Gamma(R)$, and write

$$a_+ = [F_+(\gamma)], \qquad a_- = [F_-(\gamma)].$$

For $x \in \mathcal{PMF}$, we compare two rays toward $x$, starting near $P$ and $\gamma(P)$. We will choose times $\tau, \tau'$ whose sum is close to $\ell(\gamma)$ and whose corresponding points can be joined by a short geodesic segment. The time $\tau$ also controls $d_G(a_+, x)$. These comparisons will relate the change in the size of a foliation to its distance from $a_+$.

We use the following form of the thin-triangle theorem [DDM14, Theorem A]: a long interval that spends at least half its time in a fixed thick part comes within bounded distance of one of the other two sides.

**Theorem 11.2** (Dowdall–Duchin–Masur)**.** *For any* Mod*–compact set $\mathcal{K} \subset \mathcal{T}$ there are constants $D_{\mathrm{DDM}}$ and $L_{\mathrm{DDM}}$ such that the following holds. Let $[x, y]$ be a geodesic segment in $\mathcal{T}$ and let $I$ be a geodesic subinterval of $[x, y]$, with*

$$|I| \geq L_{\mathrm{DDM}}, \qquad \big|\{u \in I : u \in \mathcal{K}\}\big| \geq \frac{1}{2}|I|.$$

*Then for all $z \in \mathcal{T}$, we have*

$$I \cap N_{D_{\mathrm{DDM}}}([x, z] \cup [y, z]) \neq \emptyset.$$

Below we choose $D_{\mathrm{DDM}}$ and $L_{\mathrm{DDM}}$ for the projection of $K_2$ to $\mathcal{T}$.

For $x \in \mathcal{PMF}$, set

$$\bar{x} = \gamma^{-1}x.$$

If $\alpha_- \in \mathcal{A}$ is a chart with $\bar{x}, a_- \in W_{\alpha_-}$, we set

$$F' = \gamma(F_{\alpha_-}),$$

so that

$$x_{F'}(t) = \gamma\big(\bar{x}^{\alpha_-}(t)\big).$$

Proposition 11.3 gives the times shown schematically in Figure 3. We call the geodesic segment joining the selected points the *bridge*. Its length and the deficit $\ell(\gamma) - \tau - \tau'$ are bounded by fixed numerical multiples of $\eta_1\tau'$, up to additive errors depending on the compact sets.

These times explain the decay in (11.2). Ignoring the controlled errors and fixed multiplicative constants, the change of normalization gives $H(\gamma, x)^{1/w} \approx e^{\tau-\tau'}$, while $1 + d_G(a_+, x)/r \approx e^{\beta\tau'}$, where $r = e^{-\beta\ell(\gamma)}$. Together with $\tau + \tau' \approx \ell(\gamma)$, these relations give the decay exponent $-2w/\beta$. The estimates below quantify the errors in this picture.

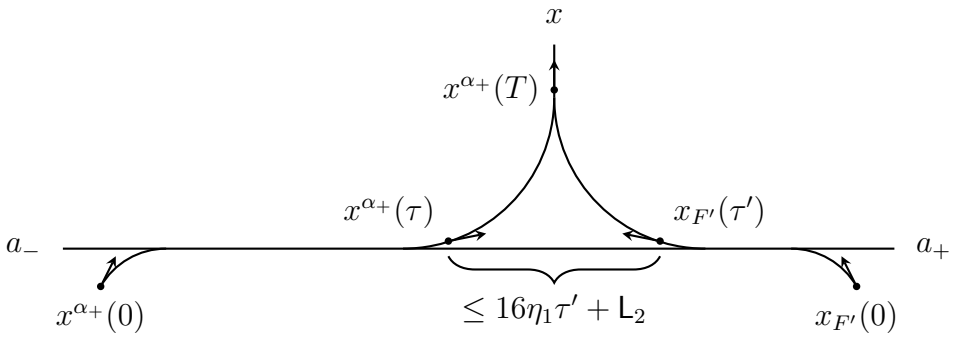


FIGURE 3. The two rays toward $x$ start near $P$ and $\gamma(P)$. The selected points at times $\tau$ and $\tau'$ are joined by a geodesic segment of length at most $16\eta_1\tau' + \mathsf{L}_2$. Part (d) of Proposition 11.3 compares rays from this segment with $x^{\alpha_+}(T)$.

**Proposition 11.3.** *Assume $0 < \eta_1 \leq 1/100$, and recall that $\eta_2 = 1/4$. There are constants*

$$\mathsf{C}_2 \geq 16, \qquad \mathsf{L}_2, \mathsf{T}_2, \mathsf{R}_2 > 0$$

*depending only on $K_1, K_2, \eta_1$ and the fixed chart system, such that if $\gamma \in \Gamma(R)$ and $\ell(\gamma) \geq \mathsf{R}_2$, then the following holds.*

*For every $x \in \mathcal{PMF}$, there exist charts $\alpha_+, \alpha_- \in \mathcal{A}$, with*

$$x, a_+ \in V^{(2)}_{\alpha_+}, \qquad \bar{x}, a_- \in V^{(2)}_{\alpha_-},$$

*and times*

$$0 \le \tau, \tau' \le \ell(\gamma),$$

*such that, setting $F' = \gamma(F_{\alpha_-})$, we have:*

*(a) $x^{\alpha_+}(\tau) \in K_2$ and*

$$d_G(a_+, x) \le \mathsf{C}_2 e^{-\beta\tau}. \tag{11.5}$$

*(b) Either $\tau' \le \mathsf{L}_2$, or*

$$\frac{1}{\mathsf{C}_2} e^{-\beta(\tau + 8\eta_1\tau')} \le d_G(a_+, x). \tag{11.6}$$

*(c) $x_{F'}(\tau') \in K_2$, and*

$$d_{\mathcal{T}}\big(x^{\alpha_+}(\tau), x_{F'}(\tau')\big) \le 16\eta_1\tau' + \mathsf{L}_2. \tag{11.7}$$

*Moreover,*

$$\ell(\gamma) - 8\eta_1\tau' - \mathsf{L}_2 \le \tau + \tau' \le \ell(\gamma). \tag{11.8}$$

*(d) Assume that there is a time*

$$T \ge \ell(\gamma) + \mathsf{T}_2$$

*such that $x^{\alpha_+}(T)$ has $(K_2, \eta_2)$–good statistics. Let $\zeta$ be a Teichmüller geodesic segment joining $x^{\alpha_+}(\tau)$ to $x_{F'}(\tau')$. For every point $Z$ in the image of $\zeta$, let*

$$q_Z = Q_1(x, Z).$$

*Then $q_Z \in \mathcal{PQ}_1\mathcal{T}$, and there exists a time $s_Z > 0$ such that*

$$d_{\mathrm{AGY}}\big(x^{\alpha_+}(T), q_Z(s_Z)\big) \le \tfrac{1}{100}. \tag{11.9}$$

The coefficients 8 and 16 are independent of the compact sets and of $\eta_1$. This will allow us to choose $\eta_1$ before the compact sets. We first fix the auxiliary constants and prove a comparison for ideal triangles; we then construct the two times in three cases.

Fix $\alpha$ such that $\gamma \in \Gamma_\alpha(R)$. Proposition 7.4(d) gives $a_+ \in V^{(1)}_\alpha$. The ideal endpoints $F_\alpha$, $F'$, and $a_\pm$ used below are generic and uniquely ergodic: they are the invariant foliations of pseudo-Anosov axes in the principal stratum. Write $L = \ell(\gamma)$ and $q(u) = a^\alpha_+(u)$, and put

$$G = \{u \in [0, L] : q(u) \text{ has } (K_1, \eta_1)\text{–good statistics}\}.$$

Proposition 7.4(e) says precisely that

$$|[s, L] \setminus G| \le \eta_1(L - s) \qquad (0 \le s < L). \tag{11.10}$$

All constants in the proof may depend on $K_1, K_2, \eta_1$ and the fixed chart system. They enter only additive errors or multiplicative bounds; the numerical coefficients displayed below do not depend on these data.

Choose $\delta > 0$ such that

$$8\delta < \mathsf{r}_{\mathrm{chart}}, \qquad 3\delta < \frac{1}{100}, \qquad \delta < \frac{1}{100\mathsf{k}}, \qquad 2\delta < \frac{1}{50}.$$

We may decrease $\delta$ further to work inside fixed local neighborhoods of $K_1$ and $K_2$. In particular, an AGY comparison of size at most $3\delta$ with a point of $K_1$ implies membership in $K_2$, by Definition 8.4. We use this also after reversing both differentials: $K_1$ and $K_2$ are invariant under $q \mapsto -q$.

Choose $D_{\text{aux}}$ to bound the following base distances: between any two chart origins $z^\alpha(0)$, for $z \in W_\alpha$; between a point within ambient AGY distance $\delta$ of $\pm q(L)$ and any $\gamma(z^{\alpha_-}(0))$, for $z \in W_{\alpha_-}$; and between $q(0)$ and an axis point within ambient AGY distance $1/80$ of it. These bounds are uniform. The chart slices have compact closures, and Proposition 7.4(c') puts $q(L)$ at bounded base distance from $\gamma(P)$. The fixed AGY neighborhoods also have compact closures modulo $\mathrm{Mod}(S)$. If two points have the same projective vertical foliation, extremal-length distortion bounds the time shift needed to match their measured foliations by their base distance. Thus $D_{\text{aux}}$ also bounds each such shift, and $2D_{\text{aux}}$ bounds the base distance after matching. We choose this constant before the occupation thresholds and buffer lengths.

Let $h_c > D_{\text{aux}}$ be large enough that the backward contraction from a separation time, over an interval of length $h_c$, reduces distance $1/100$ to $\delta$. This follows from Corollary 4.9, since good statistics with parameter $\eta_2 < 1/2$ supply the required half occupation on every backward interval.

We will sometimes replace a time by zero. Fix a threshold $R_*$ larger than $\mathsf{R}_1$, $D_{\text{aux}}$, and

$$D_{\text{aux}} + B(K_2, \eta_2, \delta),$$

where $B$ is supplied by Theorem 8.1. A change from a time at most $R_*$ to zero costs only a fixed additive amount.

The next lemma turns DDM's bounded base-distance comparison into AGY closeness at one end of an interval. The intervals on either side of the DDM interval provide the compact time needed for Proposition 8.2.

**Lemma 11.4.** *Let $A, B$ be distinct generic uniquely ergodic projective foliations, and let $C$ be a projective foliation. When $C$ differs from $A$ and $B$, suppose the pairs defining the three sides fill. Orient the reference side $\omega$ from $A$ to $B$. Set $l = \max(L_{\text{DDM}}, 1)$. There is an occupation threshold $R_c$ with the following property.*

*Suppose*

$$I_- = [u_0, u_1], \qquad I = [u_1, u_2], \qquad I_+ = [u_2, u_3]$$

*are consecutive intervals, $|I| = l$, at least half of $I$ lies in $K_2$, and each of $I_-, I_+$ contains at least $R_c$ time in $K_2$. Then one of the following holds:*

*(i) There is a differential $p_A$ on the side from $A$ to $C$, oriented toward $C$, such that*

$$F_-(p_A) = F_-(\omega(u_0)), \qquad d_{\text{AGY}}(\omega(u_0), p_A) \le \delta.$$

*(ii) There is a differential $p_B$ on the side from $C$ to $B$, oriented toward $B$, such that*

$$F_+(p_B) = F_+(\omega(u_3)), \qquad d_{\text{AGY}}(\omega(u_3), p_B) \le \delta.$$

*If $C = A$ or $C = B$, the corresponding comparison with the identical side holds directly.*

*Proof.* We first show that one of the other two ideal sides has a point whose base is within $D_{\text{DDM}}$ of $I$. We use the finite-triangle conclusion of Theorem 11.2. For punctured surfaces, this follows by lifting the triangle to the closed-surface cover described in Section 8, *Surfaces with punctures*; the lift is isometric and the lifted

set has compact image in moduli space. For the four-punctured sphere it is the thin-triangle property of its Teichmüller disk. The cases $C = A$ and $C = B$ use the identical side, so assume $C$ differs from both. Suppose first that $C$ is uniquely ergodic. Approximate $A$ and $B$ by $\omega(-N)$ and $\omega(N)$, and approximate $C$ by time $N$ on a ray toward $C$. The fixed interval $I$ lies on the first finite side. DDM gives a point on one of the other two finite sides within distance $D_{\mathrm{DDM}}$ of $I$. These points remain in a compact base set. Pass to a subsequence on which the side is fixed and the oriented unit-area differentials at these points converge.

For example, suppose the nearby point is on the side from $\omega(-N)$ to the vertex tending to $C$. Let $p_N$ be its differential, oriented toward $C$. Its distances to both vertices tend to infinity. The extremal lengths at the first vertex of a fixed representative of $A$ and of $F_-(p_N)$ both tend to zero. The intersection inequality therefore implies that the limiting horizontal foliation has zero intersection with $A$. Similarly, the limiting vertical foliation has zero intersection with $C$. Unique ergodicity identifies these projective foliations with $A$ and $C$. Thus the limiting differential lies on the actual side from $A$ to $C$. The other case gives a point on the side from $B$ to $C$.

For general $C$, approximate it by uniquely ergodic foliations $C_j$; the pairs with $A$ and $B$ still fill for all sufficiently large $j$. The nearby points again lie over a compact base set. After passing to a subsequence, their measured foliations are positive multiples of $A, C_j$, or of $B, C_j$. Unit-area normalization and continuity of extremal length bound these multiples above and below. The continuity of $Q(F, H)$ therefore identifies their limit with a point of the required side. The limiting differential is principal because its other foliation is generic. This proves the ideal-side comparison.

Choose a point $v \in I$ with $\omega(v) \in K_2$. Its base is within $D_{\mathrm{DDM}} + l$ of the point just found on the other side. Change the time on that side so that the foliation associated with the common ideal endpoint agrees as a measured foliation. The required shift has absolute value at most $D_{\mathrm{DDM}} + l$, by extremal-length distortion. The new base-distance bound is therefore at most $2(D_{\mathrm{DDM}} + l)$.

Choose $R_c$ using Proposition 8.2 for this bound, the compact set $K_2$, and tolerance $\delta$. For a point on the $A$-side, reverse both differentials and apply that proposition while moving back from $v$ to $u_0$. For a point on the $B$-side, orient its differential toward $B$ and apply the proposition from $v$ to $u_3$. The respective buffers supply the required compact occupation. These are exactly the two stated comparisons, with the indicated measured foliations and orientations. □

*Proof of Proposition 11.3.* We first construct the charts and times for (a)–(c), using the constants fixed above. Let $R_t$ be the occupation threshold in Proposition 8.2 for $K = K_2$ and initial base distance $2D_{\mathrm{aux}}$ and target $\delta$. Fix

$$h \geq \max(2R_c, 2R_t, 8l, 2D_{\mathrm{aux}}, 1), \qquad M = 8h + 4.$$

These constants are chosen before dividing the proof into cases.

Before treating the cases, we record how to obtain the chart inclusion in Proposition 11.3. A preliminary positive time $v > R_*$ below will satisfy

$$q(v) \text{ has } (K_1, \eta_1)\text{–good statistics}, \qquad d_{\mathrm{AGY}}(q(v), x^\alpha(v)) \leq \delta.$$

Backward nonexpansion transfers every negative-time suffix, so $x^\alpha(v)$ has $(K_2, \eta_2)$–good statistics. Choose $\alpha_+$ with $x, a_+ \in U_{\alpha_+}$. Applying Theorem 8.1 from the two

chart origins gives a shift $s_x$, with $|s_x| \le D_{\mathrm{aux}}$, for which

$$d_{\mathrm{AGY}}(x^\alpha(v), x^{\alpha_+}(v + s_x)) \le \delta. \tag{11.11}$$

Set $\tau = v + s_x$. Its endpoint belongs to $K_2$ by comparison with $q(v) \in K_1$. We reserve a buffer $D_{\mathrm{aux}}$ below so that $\tau + \tau' \le L$. If the preliminary choice is at most $R_*$, use $\tau = 0$ in the new chart instead; its endpoint is in $K_2$ by Proposition 9.1. The bounded chart change affects only additive errors and the multiplicative constants in (a) and (b).

For $x \in V_\alpha^{(3)}$, set $\sigma = \tau_\alpha(a_+, x)$. We distinguish whether this time is near $L$, earlier than $L - M$, or whether $x$ lies outside $V_\alpha^{(3)}$.

*Case 1: $x \in V_\alpha^{(3)}$ and $\sigma \ge L - M$.* Take $t_* = \min(\sigma, L)$, with $t_* = L$ if $\sigma = +\infty$. In the latter case $x = a_+$ by Proposition 9.1, so the two rays coincide. At an attained finite separation time the two rays are within $1/100$; backward nonexpansion gives the same bound at $t_*$ if $\sigma > L$. When $t_* = L$, this comparison and the good statistics of $q(L)$ also give good statistics for the companion ray at $L$. Thus backward contraction gives distance at most $\delta$ at $t_* - h_c$ and all earlier times.

By (11.10), an interval ending at $t_* - h_c$ and having length greater than $\eta_1(L - t_* + h_c)/(1 - \eta_1)$ cannot consist entirely of bad times. Since $L - t_* \le M$, choose a preliminary good time $v$ within a fixed distance before $t_* - h_c$. For sufficiently large $L$, we have $v > R_*$. Make the final chart change (11.11). The choice $h_c > D_{\mathrm{aux}}$ gives $0 \le \tau \le L$, and $L - \tau$ is bounded.

Choose $\alpha_-$ with $\bar{x}, a_- \in U_{\alpha_-}$ and put $\tau' = 0$. The point $x_{F'}(0)$ is in $K_2$, and its base point is a bounded distance from $\gamma(P)$. The same holds for $q(L)$. Consequently the distance between the two selected endpoints is bounded. The upper metric comparison at $v > \mathsf{R}_1$ gives (a), after absorbing the bounded chart shift into its prefactor. Part (b) holds by its first alternative. The bounded distance and deficit give (c), once $\mathsf{L}_2$ is chosen.

*Case 2: $x \in V_\alpha^{(3)}$ and $\sigma < L - M$.* Here $x \ne a_+$, since time $L$ is admissible when the two rays coincide. By Proposition 9.1, $\sigma \ge 0$ is finite and attained. Put $H = L - \sigma$ and $m = \eta_1 H$. We first find an interval after $\sigma$ to which Lemma 11.4 applies, leaving enough compact time afterward to compare with the ray from $\gamma(P)$.

The bad-time budget on $[\sigma, L]$ is at most $m$, so choose

$$t_0 \in G \cap (\sigma, \sigma + m + 1), \qquad \bar{t} \in G \cap (t_0 + 4h, t_0 + 4h + m + 1).$$

These intervals lie before $L$, and

$$\bar{t} - \sigma \le 2\eta_1 H + 4h + 2. \tag{11.12}$$

In particular,

$$L - \bar{t} \ge (1 - 2\eta_1)H - 4h - 2 \ge 3h.$$

Since $\bar{t}$ is good, at most $2\eta_1 h$ of $[\bar{t} - 2h, \bar{t} - h]$ lies outside $K_1$. Partition it into full $l$-blocks, discarding a remainder shorter than $l$. At least one block $I$ has half its time in $K_1$. The interval from $t_0$ to the beginning of $I$ contains $[\bar{t} - 3h, \bar{t} - 2h]$, with at least $(1 - 3\eta_1)h \ge R_c$ compact time. The interval from the end of $I$ to $\bar{t}$ contains $[\bar{t} - h, \bar{t}]$, with at least $(1 - \eta_1)h \ge R_c$ compact time.

Apply Lemma 11.4 to $F_\alpha, a_+, x$. Its first alternative gives the comparison with $x^\alpha(t_0)$ at exactly time $t_0$, because the measured horizontal foliations agree. Backward

nonexpansion would transfer every negative-time suffix of the good statistics at $t_0$ to $x^\alpha(t_0)$. This contradicts $t_0 > \sigma$. Hence the second alternative gives a differential $p$ oriented toward $a_+$, with

$$F_+(p) = F_+(q(\bar t)), \qquad d_{\mathrm{AGY}}(p, q(\bar t)) \le \delta.$$

Put $b = L - \bar t$ and define the $x$-ray $r(u) = -g_{b-u}p$. Forward stable nonexpansion gives

$$D_{\mathrm{AGY}}(r(u), -q(L-u)) \le \delta \qquad (0 \le u \le b).$$

Thus $r(0) \in K_2$, its base point is a bounded distance from $\gamma(P)$, and it spends at least $(1-\eta_1)b$ time in $K_2$. Choose $\alpha_-$ with $\bar x, a_- \in U_{\alpha_-}$. Match the measured vertical foliation of $r(0)$ to that of $x_{F'}(s_0)$, with $|s_0| \le D_{\mathrm{aux}}$. Since $b \ge 3h$, we have $b > D_{\mathrm{aux}}$ and $(1-\eta_1)b \ge R_t$. Proposition 8.2 therefore gives

$$\tau' = L - \bar t + s_0 > 0, \qquad D_{\mathrm{AGY}}(x_{F'}(\tau'), -q(\bar t)) \le 2\delta.$$

In particular $x_{F'}(\tau') \in K_2$. This terminal comparison has only a bounded time error.

We next choose a preliminary time $v \le \sigma$. Backward contraction from $\sigma$ gives distance at most $\delta$ by time $\sigma - h_c$. If the interval

$$J = \big(\sigma - h_c - [2\eta_1(H + h_c) + 1],\ \sigma - h_c\big)$$

lies in $[0, L]$, it contains a point of $G$. Indeed, otherwise its length $\Delta$ would satisfy $\Delta \le \eta_1(H + h_c + \Delta)$ by (11.10), which contradicts the displayed choice of $\Delta$. Choose such a point if it exceeds $R_*$; otherwise use $v = 0$. Also use $v = 0$ if the lower endpoint of $J$ is nonpositive. This includes the case that $J$ lies entirely before zero. In all cases,

$$\sigma - v \le 2\eta_1 H + B_2, \qquad \bar t - v \le 4\eta_1 H + B_3.$$

Make the final chart change described in (11.11) if $v > 0$, and use $\tau = 0$ in that chart otherwise. Since $\bar t - \sigma \ge 4h \ge 2D_{\mathrm{aux}}$, the two bounded time shifts leave $\tau + \tau' \le L$.

The endpoint comparisons and the segment of $q$ between $v$ and $\bar t$ give bridge length at most $4\eta_1 H + B_4$; the same bound holds for $L - \tau - \tau'$. Moreover

$$H = \tau' + \bar t - \sigma - s_0 \le \tau' + 2\eta_1 H + B_5.$$

Hence both quantities are bounded by

$$\frac{4\eta_1}{1 - 2\eta_1}\tau' + B_6 \le 5\eta_1\tau' + B_6.$$

Thus the distance and deficit satisfy (c), with the additive error to be included in $\mathsf{L}_2$.

It remains to relate these choices to $d_G(a_+, x)$. If $v > 0$, the upper metric comparison applies at $v > \mathsf{R}_1$; if $v = 0$, bounded diameter applies. The final chart shift is bounded, so we obtain the upper bound in (a) with a fixed prefactor. At $\bar t > \sigma$ the distance between $q(\bar t)$ and $x^\alpha(\bar t)$ exceeds $1/100$: otherwise its good statistics would transfer backwards and contradict the definition of $\sigma$. At $v$ the distance is smaller than $1/100$, including when $v = 0$ by the chart diameter. Thus there is $s \in [v, \bar t]$ where that distance equals $1/100$. Lemma 9.2 gives

$$d_G^\alpha(a_+, x) \ge C_{\mathrm{AGY}}^{-1} e^{-\beta s}.$$

Since $s \le \bar t \le \tau + 8\eta_1\tau' + B_7$, Lemma 9.4 gives, when $d_G(a_+, x) \le \mathsf{r}_1$,

$$d_G(a_+, x) \ge \frac{e^{-\beta(\tau + 8\eta_1\tau')}}{2|\mathcal{A}| C_{\mathrm{AGY}} e^{\beta B_7}}.$$

When $d_G(a_+, x) > \mathsf{r}_1$, use $\mathsf{r}_1$ as the lower bound in (b).

*Case 3:* $x \notin V_\alpha^{(3)}$. Since $a_+ \in V_\alpha^{(1)}$, Lemma 9.4 and (9.9) would give $x \in V_\alpha^{(3)}$ if $d_G(a_+, x) \le \mathsf{r}_1$. Thus $d_G(a_+, x) > \mathsf{r}_1$. Choose $\alpha_+, \alpha_-$ with $x, a_+ \in U_{\alpha_+}$ and $\bar{x}, a_- \in U_{\alpha_-}$. Set $\tau = 0$, so $x^{\alpha_+}(\tau) \in K_2$, and put

$$\omega(t) = \gamma(a_-^{\alpha_-}(t)), \qquad F' = \gamma F_{\alpha_-}.$$

Choose a point $z_0$ of the oriented axis with $D_{\mathrm{AGY}}(z_0, q(0)) < 1/80$. Such a point exists by Proposition 7.4(c); no attainment of the distance to the axis is needed. Choose $|h_0| \le D_{\mathrm{aux}}$ so that $a(u) = g_{u+h_0} z_0$ and $q(u)$ have the same measured vertical foliation. Put $m = \eta_1 L$ and $U = m + R_{\mathrm{ax}}$, where $R_{\mathrm{ax}} \ge \max(R_t, 2D_{\mathrm{aux}} + 2)$ is fixed. The interval $[0, U]$ has at least $R_{\mathrm{ax}}$ compact time, by (11.10) with $s = 0$. Proposition 8.2, followed by stable nonexpansion, gives

$$d_{\mathrm{AGY}}(q(u), a(u)) \le \delta \qquad (u \ge U).$$

Let $A(s) = -a(L - s)$. Match $\omega(c + s)$ to $A(s)$ with equal measured vertical foliations and $|c| \le D_{\mathrm{aux}}$. Their starting base distance is bounded by $2D_{\mathrm{aux}}$, since both starting points lie near $\gamma(P)$. Set $S_0 = 2R_t + 1$. For sufficiently large $L$, we have $L - S_0 \ge U$. The good statistics of $q(L)$ and the preceding comparison give at least $(1 - \eta_1)S_0 > R_t$ time in $K_2$ for $A$ on $[0, S_0]$. Also $A(0) \in K_2$. A second application of Proposition 8.2 therefore gives

$$d_{\mathrm{AGY}}(\omega(c + s), A(s)) \le \delta \qquad (s \ge S_0).$$

Consequently,

$$D_{\mathrm{AGY}}(\omega(L - u + c), -q(u)) \le 2\delta \quad (U \le u \le L - S_0). \tag{11.13}$$

In this range let $E = \{t : L - t + c \in G\}$; its complement has measure at most $m$. Every point of $E$ is a $K_2$ point of $\omega$. Set $A_0 = R_c + 1$, $\Delta_0 = 4m + 2A_0 + 2l$, and $b = L - U + c$. Choose

$$t_0 \in E \cap [b - \Delta_0 - 2m - 2, b - \Delta_0 - m - 1], \qquad \bar{t} \in E \cap [t_0 + \Delta_0, t_0 + \Delta_0 + m + 1].$$

Each choice interval has length $m + 1$. All these times lie in the range of (11.13) for sufficiently large $L$, since their corresponding $u$-parameters lie between $U$ and $U + 6m + 2A_0 + 2l + 2$, and $\eta_1 \le 1/100$.

The first and last intervals of length $m + A_0$ in $[t_0, \bar{t}]$ each have at least $R_c$ time in $E$. The middle interval has length at least $2m + 2l$; its full $l$-blocks have total length greater than $2m$, so one has half its time in $E$. Apply Lemma 11.4 to this block and the triangle $F', a_-, x$.

Its second alternative is impossible. Indeed, it gives a point $p$ oriented toward $a_-$, with the same measured vertical foliation as $\omega(\bar{t})$ and within distance $\delta$. Put $t_* = L + h_0 + c$. Then $A(t_* - c) = -z_0$, and $t_* \ge \bar{t}$ because $R_{\mathrm{ax}} > |h_0| + 1$. Flow $p$ and $\omega(\bar{t})$ forward to $t_*$, and put $v = -g_{t_* - \bar{t}} p$. Then $v$ has vertical class $x$ and $D_{\mathrm{AGY}}(v, z_0) \le 2\delta$. The projection $\pi^{\mathrm{cs}}_{q(0)}$ is defined on $B_{\mathrm{AGY}}(z_0, 1/50)$, sends $z_0$ to $q(0)$, and is 4–Lipschitz. Thus $q_x = \pi^{\mathrm{cs}}_{q(0)}(v)$ satisfies

$$[F_+(q_x)] = x, \qquad F_-(q_x) = F_\alpha, \qquad d_{\mathrm{AGY}}(q_x, q(0)) \le 8\delta < \mathsf{r}_{\mathrm{chart}}.$$

Thus $q_x = x^\alpha(0)$, and Theorem 5.1(f) would give $x \in V_\alpha^{(3)}$, a contradiction.

The first alternative consequently holds. Because the measured horizontal foliations agree, it compares $\omega(t_0)$ to $x_{F'}(t_0)$ at the exact same time. Set $\tau' = t_0$. This point is within $3\delta$ of $-q(u_0)$, where $u_0 = L - t_0 + c \in G$, so it belongs to $K_2$. Also,

$$u_0 \le 7\eta_1 L + B_9, \qquad d_{\mathcal{T}}(x^{\alpha_+}(0), x_{F'}(\tau')) \le u_0 + B_{10}, \qquad L - \tau' \le 7\eta_1 L + B_{11}.$$

Here the second estimate uses the bounded distance between the two chart origins and the segment of $q$ of length $u_0$. The choice $U > |c|$ ensures $\tau' \le L$; positivity holds when $L$ is sufficiently large. Eliminating $L$ by $\tau' \ge (1 - 7\eta_1)L - B_{11}$ gives

$$d_{\mathcal{T}}(x^{\alpha_+}(0), x_{F'}(\tau')),\ L - \tau' \le \frac{7\eta_1}{1 - 7\eta_1}\tau' + B_{12} \le 8\eta_1\tau' + B_{12}.$$

Finally, bounded diameter gives the upper bound in (a) with a fixed prefactor, and $d_G(a_+, x) > \mathsf{r}_1$ gives the required lower bound for (b).

We now choose the constants in (a)–(c). Take $\mathsf{L}_2 > 0$ to dominate the bounded distance and deficit in Case 1 and the additive errors $B_6, B_{12}$ in Cases 2 and 3. Choose $\mathsf{C}_2 \ge 16$ to dominate all upper metric prefactors and

$$2|\mathcal{A}|C_{\mathrm{AGY}}e^{\beta B_7}, \qquad \mathsf{r}_1^{-1}.$$

These choices give the lower metric bound in (b), with coefficient 8, and the distance and deficit bounds in (c), with coefficients 16 and 8. Choose $\mathsf{R}_2$ larger than all the length thresholds used in the three cases. Every choice depends only on the data in the proposition. The charts chosen above have both required points in $U_{\alpha_\pm} \subset V^{(2)}_{\alpha_\pm}$, completing (a)–(c).

For part (d), set $B_{\mathrm{br}} = B(K_2, \eta_2, 1/100)$ as in Theorem 8.1, and choose

$$\mathsf{T}_2 \ge \mathsf{L}_2 + B_{\mathrm{br}} + 1. \tag{11.14}$$

Let $T \ge \ell(\gamma) + \mathsf{T}_2$ be a time at which $x^{\alpha_+}(T)$ has $(K_2, \eta_2)$–good statistics, and put $S = T - \tau$. For every point $Z$ of the indicated geodesic bridge, (11.7) gives

$$D_Z := d_{\mathcal{T}}\big(Z, X_{x^{\alpha_+}(\tau)}\big) \le 16\eta_1\tau' + \mathsf{L}_2.$$

Since $\tau + \tau' \le \ell(\gamma)$, we have $S \ge \tau' + \mathsf{T}_2$. Consequently,

$$S - D_Z \ge (1 - 16\eta_1)\tau' + \mathsf{T}_2 - \mathsf{L}_2 \ge B_{\mathrm{br}} + 1,$$

where we used the initial restriction $\eta_1 \le 1/100$. Apply Theorem 8.1 with reference differential $p = x^{\alpha_+}(\tau)$, comparison differential $q_Z = Q_1(x, Z)$, and elapsed time $S$. Their projective vertical foliation is $x$. The reference endpoint is $x^{\alpha_+}(T) \in K_2$. Every terminal subinterval of $[0, S]$ corresponds to an interval ending at $T$, so the required occupation bound follows from its good statistics. The preceding inequality gives the required time bound.

Since $\mathrm{Ext}_{X_p}(F_+(p)) = 1$, the theorem gives

$$s_Z = S + \tfrac{1}{2}\log \mathrm{Ext}_Z(F_+(p)), \qquad |s_Z - S| \le D_Z.$$

In particular $s_Z \ge S - D_Z > 0$. It also proves that $q_Z$ is principal and gives the intrinsic AGY estimate (11.9). The theorem permits nongeneric $x$, and the same constant $B_{\mathrm{br}}$ applies to every $Z$ on the segment.

Thus the restriction on $\eta_1$ is imposed before choosing the statistical compact sets, whereas $\mathsf{T}_2$ is chosen last from (11.14). No smallness condition involving the compact-dependent prefactor $\mathsf{C}_2$ is used in (d). For the later applications to Theorem 11.1, take its length threshold $\mathsf{R}_0 \geq \mathsf{R}_2 + 1$, using Proposition 7.4(b). □

**Coarse estimate for the spike function.** We first compare the unit-area normalizations of $x$ in the two charts chosen by Proposition 11.3. Extremal-length distortion between the selected points then expresses $H(\gamma, x)$ in terms of $\tau, \tau'$. The metric bounds in that proposition will relate these times to $d_G(a_+, x)$.

*Proof of Theorem 11.1*(a). Fix $x \in \mathcal{PMF}$ and the charts and times from Proposition 11.3. Write $F' = \gamma(F_{\alpha_-})$ and put

$$p_0 = x^{\alpha_+}(0), \qquad p_1 = x^{\alpha_+}(\tau),$$
$$p_3 = x_{F'}(0), \qquad p_2 = x_{F'}(\tau').$$

The vertical foliations $F_0 = F_+(p_0)$ and $G_0 = F_+(p_3)$ represent the same projective class $x$, so $F_0 = cG_0$ for some $c > 0$. The points $p_0$ and $\gamma^{-1}p_3$ lie in the finitely many fixed chart slices. These slices have compact closures, so continuity and positivity of $\rho$ give uniform bounds

$$\rho(F_0) \asymp 1, \qquad \rho(\gamma^{-1}G_0) \asymp 1.$$

By (2.9) and homogeneity,

$$H(\gamma, x)^{1/w} = \frac{\rho(F_0)}{\rho(\gamma^{-1}F_0)} = \frac{\rho(F_0)}{c\,\rho(\gamma^{-1}G_0)} \asymp c^{-1}. \tag{11.15}$$

We estimate $c$ at the two ends of the connecting geodesic. The vertical foliations of $p_1, p_2$ are $e^{\tau}F_0, e^{\tau'}G_0$. Since both differentials have area one, the Hubbard–Masur area identity gives

$$\sqrt{\mathrm{Ext}_{X_{p_1}}(F_0)} = e^{-\tau}, \qquad \sqrt{\mathrm{Ext}_{X_{p_2}}(F_0)} = ce^{-\tau'}.$$

Set $D = d_{\mathcal{T}}(X_{p_1}, X_{p_2})$. Extremal-length distortion yields

$$e^{-D} \leq ce^{\tau-\tau'} \leq e^{D}, \qquad e^{\tau-\tau'-D} \leq c^{-1} \leq e^{\tau-\tau'+D}.$$

Using $D \leq 16\eta_1\tau' + \mathsf{L}_2$ from (11.7), we obtain

$$e^{\tau-\tau'-16\eta_1\tau'} \prec H(\gamma, x)^{1/w} \prec e^{\tau-\tau'+16\eta_1\tau'}.$$

The prefactors include $e^{\mathsf{L}_2}$ and the chart comparison constants. This argument applies directly to every $x$, including nongeneric foliations; it requires no lift of the connecting geodesic.

Write $L = \ell(\gamma)$ and $d = d_G(a_+, x)$, and abbreviate $C_+ = \mathsf{C}_2$, $C_- = \mathsf{C}_2^{-1}$, and $B = \mathsf{L}_2$. Combining the deficit estimate (11.8) with the scalar comparison gives

$$e^{L-(2+24\eta_1)\tau'-B} \prec H(\gamma, x)^{1/w} \prec e^{L-(2-16\eta_1)\tau'}. \tag{11.16}$$

The coefficient 24 is the sum of the deficit coefficient 8 and the bridge coefficient 16.

Put $r = e^{-\beta L}$ and $Z = 1 + d/r \geq 1$. If $\tau' \leq \mathsf{L}_2$, then

$$e^{\tau'} \leq e^{\mathsf{L}_2} \leq e^{\mathsf{L}_2} Z^{1/\beta}. \tag{11.17}$$

Otherwise the lower metric bound and $\tau + \tau' \leq L$ give

$$d/r \geq C_- e^{\beta(L-\tau-8\eta_1\tau')} \geq C_- e^{\beta(1-8\eta_1)\tau'}.$$

Combining the two cases, we obtain

$$e^{(1-8\eta_1)\tau'} \prec Z^{1/\beta}, \tag{11.18}$$

and hence

$$e^{\tau'} \prec Z^{1/(\beta(1-8\eta_1))}. \tag{11.19}$$

The denominator is positive because $\eta_1 \le 1/100$. For the reverse bound, $r \le e^{-\beta\tau}$ and (11.5) and (11.8) imply

$$Z \le (1 + C_+)e^{\beta(L-\tau)} \le (1 + C_+)e^{\beta B}e^{\beta(1+8\eta_1)\tau'}.$$

Therefore

$$e^{\tau'} \succ Z^{1/(\beta(1+8\eta_1))}. \tag{11.20}$$

Substituting these bounds in (11.16) and raising to the power $w$ gives

$$e^{wL}Z^{-w(2+24\eta_1)/(\beta(1-8\eta_1))} \prec H(\gamma, x) \prec e^{wL}Z^{-w(2-16\eta_1)/(\beta(1+8\eta_1))}.$$

The two errors in the numerator of the exponent are

$$\frac{40w\eta_1}{1-8\eta_1}, \qquad \frac{32w\eta_1}{1+8\eta_1}.$$

Both are at most $44w\eta_1$ for $\eta_1 \le 1/100$. Thus (11.2) holds with

$$\eta_{\mathrm{c}} = 44w\eta_1. \tag{11.21}$$

All dependence on $\beta$, the compact sets, and the additive errors is allowed in the multiplicative constant $\mathsf{C}$. The coefficient $44w$ depends only on the surface.

For the ratio estimate, set $Z_u = 1 + d_G(a_+, u)/r$. If $d_G(u, v) \le kr$ for a fixed $k \ge 1$, the triangle inequality gives

$$(k + 1)^{-1} \le Z_v/Z_u \le k + 1.$$

Taking the ratio of the two coarse bounds therefore yields

$$C_k^{-1}Z_u^{-2\eta_{\mathrm{c}}/\beta} \le \frac{H(\gamma, v)}{H(\gamma, u)} \le C_k Z_u^{2\eta_{\mathrm{c}}/\beta}. \tag{11.22}$$

The factor $(k + 1)^{(2w+\eta_{\mathrm{c}})/\beta}$ is part of $C_k$. Taking $k = 1$ proves (11.3).

The conclusions in part (a) remain valid with any $\eta \ge \eta_{\mathrm{c}}$. The choice $\eta = 256w\eta_1$ in (11.1) leaves room for the Lipschitz estimate and the passage from $D_G$ to $d_G$. Its restriction on $\eta_1$ depends only on the surface and is imposed before choosing $K_1, K_2$ and hence before determining $\beta$. In particular, $4\eta_{\mathrm{c}} \le \eta < 1/2$, so part (a) also holds with $\eta/4$ in place of $\eta$. We will check below that the same choice suffices for part (b), without any comparison between $\eta$ and $\beta$. □

**The Lipschitz estimate for the spike function.** To compare two nearby foliations, we transport their coordinate vectors from the source chart to the target chart along five paths. When $D_G(x, y)$ is sufficiently small compared with $r$, the two source rays are close at a time beyond $\ell(\gamma)$. We use this extra time to place both foliations in compatible coordinates along the paths. After bounding the transport and checking these domains, we estimate the relative coordinate errors in the quotient defining $H$.

Fix $x \in \mathcal{PMF}$, choose the charts and times from Proposition 11.3, and write $F' = \gamma(F_{\alpha_-})$. Put

$$p_0 = x^{\alpha_+}(0), \qquad p_1 = x^{\alpha_+}(\tau), \qquad p_2 = x_{F'}(\tau'),$$
$$p_3 = x_{F'}(0), \qquad p_4 = \gamma(q_{\alpha_-}), \qquad F_0 = F_+(p_0).$$

Thus $[F_0] = x$ and $p_1, p_2 \in K_2$. The four paths outside the bridge are always defined:

- $\xi_0$ joins $q_{\alpha_+}$ to $p_0$ in $\mathrm{Hor}(F_{\alpha_+})$;
- $\xi_1$ is $x^{\alpha_+}([0,\tau])$;
- $\xi_3$ is the reverse of $x_{F'}([0,\tau'])$;
- $\xi_4$ joins $p_3$ to $p_4$ in $\gamma\,\mathrm{Hor}(F_{\alpha_-})$.

Choose $\xi_0$ and $\xi_4$ to be piecewise $C^1$ paths in their fixed-horizontal, area-one slices, each of AGY length less than $1/300$. Such paths exist because the construction in Section 5 gives

$$d_{\mathrm{AGY}}\big(q_\alpha, z^\alpha(0)\big) < \frac{1}{400} \qquad (z \in W_\alpha). \tag{11.23}$$

Indeed, this bound was imposed on $\overline{W}'_\alpha$ and is preserved by the mapping class used to define $W_\alpha$. The definition of intrinsic distance then gives the required paths; no minimizing path is needed. Every point of $\xi_0$ has intrinsic distance less than $1/300$ from $q_{\alpha_+}$, and the analogous statement holds for $\gamma^{-1}\xi_4$ centered at $q_{\alpha_-}$. The uniform leaf coordinates of Lemma 4.3 therefore compute each short path's transport in one period chart.

Let $\zeta : [0,1] \to \mathcal{T}$ be the geodesic from $X_{p_1}$ to $X_{p_2}$. When its normalized lift is principal, define

$$\xi_2(s) = Q_1(x, \zeta(s)).$$

This path joins $p_1$ to $p_2$ and fixes the projective vertical foliation. It is principal whenever $F_0$ is generic, and also whenever the hypothesis of Proposition 11.3(d) holds. We only use its transport when this condition has been verified. In that case the concatenated path gives the diagram

$$L(q_{\alpha_+}) \xrightarrow{\mathrm{GM}_{\xi_0}} L(p_0) \xrightarrow{\mathrm{GM}_{\xi_1}} L(p_1) \xrightarrow{\mathrm{GM}_{\xi_2}} L(p_2),$$
$$L(p_2) \xrightarrow{\mathrm{GM}_{\xi_3}} L(p_3) \xrightarrow{\mathrm{GM}_{\xi_4}} L(p_4) \xrightarrow{(\gamma^{-1})^*} L(q_{\alpha_-}).$$

Whenever all the intervening paths exist, write

$$\mathrm{GM}_{ij} = \mathrm{GM}_{\xi_j} \circ \cdots \circ \mathrm{GM}_{\xi_i} \qquad (0 \le i \le j \le 4).$$

**Claim 11.5.** *There is a prefactor $M_{\mathrm{br}} \ge 1$, depending only on the fixed compact data and $\mathsf{L}_2$, for which the following estimates hold. All norms are AGY norms.*

*(0)* $\max\{\|\mathrm{GM}_{\xi_0}\|, \|\mathrm{GM}_{\xi_0}^{-1}\|\} \prec 1$, *and*

$$\|\Psi_{q_{\alpha_+}}(F_0)\|_{q_{\alpha_+}} \asymp \|\Psi_{p_0}(F_0)\|_{p_0}.$$

*(1)* $\max\{\|\mathrm{GM}_{\xi_1}\|, \|\mathrm{GM}_{\xi_1}^{-1}\|\} \asymp e^{\tau}$, *and*

$$\|\Psi_{p_1}(F_0)\|_{p_1} = \|\mathrm{GM}_{\xi_1}(\Psi_{p_0}(F_0))\|_{p_1} \asymp e^{-\tau}\|\Psi_{p_0}(F_0)\|_{p_0}.$$

*(2) If the lift $\xi_2$ is principal, then*

$$\max\{\|\mathrm{GM}_{\xi_2}\|, \|\mathrm{GM}_{\xi_2}^{-1}\|\} \le M_{\mathrm{br}} e^{16\eta_1\tau'}. \tag{11.24}$$

*The coefficient* 16 *is independent of the compact sets and statistical parameters; their dependence is confined to $M_{\mathrm{br}}$.*

*(3)* $\max\{\|\mathrm{GM}_{\xi_3}\|, \|\mathrm{GM}_{\xi_3}^{-1}\|\} \asymp e^{\tau'}$*, and*

$$\|\Psi_{p_3}(F_0)\|_{p_3} = \|\mathrm{GM}_{\xi_3}(\Psi_{p_2}(F_0))\|_{p_3} \asymp e^{\tau'}\|\Psi_{p_2}(F_0)\|_{p_2}.$$

*(4)* $\max\{\|\mathrm{GM}_{\xi_4}\|, \|\mathrm{GM}_{\xi_4}^{-1}\|\} \prec 1$*, and*

$$\|\Psi_{p_3}(F_0)\|_{p_3} \asymp \|\Psi_{p_4}(F_0)\|_{p_4}.$$

*Proof.* For $\xi_0$ and $\xi_4$, the length bound 1/300 and (4.5) bound both operator norms and their inverses by $e^{1/300}$. Along $\xi_0$, the point with projective vertical foliation $x$ and the same horizontal foliation is the fixed point $p_0$; it is at intrinsic distance less than 1/300 from every point of $\xi_0$. Along $\xi_4$, the corresponding point is $p_3$. Lemma 4.3 and its coordinate extension supply the domains and continuous leaf paths required by Lemma 3.7. Thus transport identifies the endpoint coordinates of $F_0$, giving the scalar comparisons as well.

For the flow segments $\xi_1, \xi_3$, Lemma 6.3 gives the foliation-vector identities and Lemma 4.6 gives the operator upper bounds and the scalar decay or growth. Applying the scalar decay to the inverse of $\xi_1$, and the scalar growth to $\xi_3$, gives the matching lower bounds for the maxima of the operator norms.

It remains to bound transport along $\xi_2$. Choose once a Mod–compact set $\widehat{K}_2$ in the principal stratum containing a neighborhood of $K_2$. This is auxiliary endpoint data; the standing set $K_2$ is unchanged. Put $D = d_{\mathcal{T}}(X_{p_1}, X_{p_2})$. If $x$ is generic, Proposition 6.1 and (6.11) give

$$\max\{\|\mathrm{GM}_{\xi_2}\|, \|\mathrm{GM}_{\xi_2}^{-1}\|\} \le A_{\widehat{K}_2}^2 e^{B_{\widehat{K}_2}} e^{D}.$$

Only the endpoints must lie in $\widehat{K}_2$.

Now suppose $x$ is nongeneric and $\xi_2$ is principal. Keep the base geodesic $\zeta$ fixed. Continuity of $(u, s) \mapsto Q_1(u, \zeta(s))$ and compactness of $[0, 1]$ give a contractible neighborhood $U$ of $x$ such that every lift over $U \times [0, 1]$ is principal and its endpoints lie in $\widehat{K}_2$. The pullback cohomology local system on this contractible product is trivial. In this trivialization all the lifts have the same transport matrix. Choose generic $x_n \to x$ in $U$ and apply the preceding estimate to $\xi_{2,n}(s) = Q_1(x_n, \zeta(s))$. Their base distance is exactly $D$. Continuity of the endpoint AGY norms passes both the operator and inverse-operator bounds to the limit.

Since $D \le 16\eta_1\tau' + \mathsf{L}_2$, we may therefore take

$$M_{\mathrm{br}} = A_{\widehat{K}_2}^2 e^{B_{\widehat{K}_2} + \mathsf{L}_2}$$

in (11.24). The neighborhood $U$ may depend on $\xi_2$, but its size does not enter the bound. The approximants require no good statistics or new choices of $\tau, \tau'$. The sets $K_2, \widehat{K}_2$ and the additive error $\mathsf{L}_2$ are fixed before $M_{\mathrm{br}}$; this prefactor enters no later smallness condition on $\eta_1$. The delay $\mathsf{T}_2$ in Proposition 11.3(d) remains the one chosen in (11.14). □

Whenever $\xi_2$ is principal, Lemma 6.3 also identifies the foliation vector across it. Composing with the four other identities just proved gives

$$\mathrm{GM}_{0i}\big(\Psi_{q_{\alpha_+}}(F_0)\big) = \Psi_{p_i}(F_0) \qquad (0 \le i \le 4). \tag{11.25}$$

The spike is a quotient of two powers of $\rho$. Nearby points give nearby vectors in the source chart, but transport may greatly increase their relative separation in the target chart. We therefore use a local estimate for the numerator and an estimate for reciprocal powers that applies to any two nonzero target vectors. The reciprocal-power estimate in Lemma 11.6(b) introduces the larger of the two spike values, which (11.22) controls.

We now express $\rho$ in the fixed charts. On $\Psi_{q_\alpha}(\mathbb{R}_+ W_\alpha)$, define

$$\rho_\alpha(v) = \rho\big(\Psi_{q_\alpha}^{-1}(v)\big). \tag{11.26}$$

This function is positive and homogeneous of degree one. We may extend it to $L(q_\alpha) \setminus \{0\}$ so that its restriction to the AGY unit sphere is positive and Lipschitz. Here is why the extension can be made with uniform bounds. In the construction of the charts in Section 5, the cone over $\overline{W}_\alpha$ is contained in $\mathbb{R}_+\mathcal{V}_+(q_\alpha)$. Its unit directions form a compact subset of the coordinate domain. On a neighborhood of this set, $\rho \circ \Psi_{q_\alpha}^{-1}$ is locally Lipschitz: the foliation coordinates are locally piecewise linear in period coordinates, as in Proposition 3.5, and $\rho$ is locally Lipschitz by our standing regularity convention. Its restriction to the compact set of unit directions is thus Lipschitz and has a positive minimum. Extend this restriction to the whole unit sphere by a Lipschitz extension, truncate its values to its original minimum and maximum, and then extend homogeneously. We continue to write $\rho_\alpha$ for this extension; all evaluations representing foliations will be in the original cone.

Since there are finitely many charts, there are constants $0 < c_\rho \le M_\rho$ and $L_\rho < \infty$, depending only on the fixed charts and $\rho$, such that

$$c_\rho \|v\|_{q_\alpha} \le \rho_\alpha(v) \le M_\rho \|v\|_{q_\alpha},$$

and every restriction to the unit sphere has Lipschitz constant at most $L_\rho$. In particular, these constants are chosen independently of the statistical compact sets.

**Lemma 11.6.** *Let $L$ be a finite-dimensional normed vector space, and let $\rho$ be the degree-one homogeneous extension of a positive Lipschitz function $\rho_0$ on its unit sphere. Fix $w \ge 1$. For nonzero $\mathbf{v}, \mathbf{w} \in L$, put $\delta = \|\mathbf{v} - \mathbf{w}\|_L / \|\mathbf{v}\|_L$. Then:*

(a) *If $\delta \le 1/2$, then*

$$|\rho(\mathbf{v})^w - \rho(\mathbf{w})^w| \prec_{w,\rho} \|\mathbf{v} - \mathbf{w}\|_L \, \rho(\mathbf{v})^{w-1}.$$

(b) *Without any restriction on $\delta$,*

$$|\rho(\mathbf{v})^{-w} - \rho(\mathbf{w})^{-w}| \prec_{w,\rho} \delta \max\{\rho(\mathbf{v})^{-w}, \rho(\mathbf{w})^{-w}\}. \tag{11.27}$$

*Proof.* Write $u = \mathbf{v}/\|\mathbf{v}\|_L$ and $u' = \mathbf{w}/\|\mathbf{w}\|_L$. The reverse triangle inequality gives $\|u - u'\|_L \le 2\delta$. Choose bounds $0 < c_\rho \le \rho_0 \le M_\rho$ on the unit sphere, and let $L_\rho$ be a Lipschitz constant for $\rho_0$. Expanding $\rho(\mathbf{v}) = \|\mathbf{v}\|_L \rho_0(u)$ and similarly for $\mathbf{w}$ gives

$$|\rho(\mathbf{v}) - \rho(\mathbf{w})| \le (M_\rho + 2L_\rho)\|\mathbf{v} - \mathbf{w}\|_L.$$

If $\delta \le 1/2$, the two norms are within a factor of two, so $\rho(\mathbf{w}) \asymp_\rho \rho(\mathbf{v})$. The mean value theorem therefore gives, for $b \in \{w, -w\}$,

$$|\rho(\mathbf{v})^b - \rho(\mathbf{w})^b| \prec_{b,\rho} \|\mathbf{v} - \mathbf{w}\|_L \, \rho(\mathbf{v})^{b-1}.$$

This proves (a). Taking $b = -w$ and using $\rho(\mathbf{v}) \ge c_\rho \|\mathbf{v}\|_L$ proves (b) when $\delta \le 1/2$. If $\delta > 1/2$, the difference of the two positive reciprocal powers is at most their maximum, which is at most $2\delta$ times that maximum. This proves (b) in all cases. □

We will estimate $H$ across a single small $D_G$-step. The next lemma passes this estimate to $d_G$ by summing over chains.

**Lemma 11.7** (One-step reduction)**.** *Assume that the coarse bounds in part* (a) *of Theorem 11.1 hold with $\eta/4$ in place of $\eta$. Suppose there are constants $\epsilon_{\mathrm{step}} > 0$ and $C_{\mathrm{step}} > 0$ such that, whenever*

$$D_G(u, v) \le \epsilon_{\mathrm{step}} r,$$

*we have*

$$\big|H(\gamma, u) - H(\gamma, v)\big| \le \frac{C_{\mathrm{step}}}{r} \, H(\gamma, u) \left(1 + \frac{d_G(a_+, u)}{r}\right)^{\eta/(2\beta)} D_G(u, v). \tag{11.28}$$

*Then part* (b) *of Theorem 11.1 holds, after increasing the constant* $\mathsf{C}$.

*Proof.* The coarse bounds with $\eta/4$ in place of $\eta$ give, whenever $d_G(u, v) \le 2r$,

$$H(\gamma, v) \prec H(\gamma, u) \left(1 + \frac{d_G(a_+, u)}{r}\right)^{\eta/(2\beta)}. \tag{11.29}$$

Indeed, take the ratio of the bounds in (11.2) and use the triangle inequality for $d_G$.

The conclusion is immediate if $x = y$. Otherwise, let $0 < d_G(x, y) \le r$, and for $0 < \varepsilon \le d_G(x, y)$ choose a chain $x = x_0, \dots, x_k = y$ with

$$\sum_{i=0}^{k-1} D_G(x_i, x_{i+1}) \le d_G(x, y) + \varepsilon \le 2r.$$

Every vertex satisfies $d_G(x, x_i) \le 2r$. Thus, with $Z = 1 + d_G(a_+, x)/r$, the triangle inequality and (11.29) give

$$1 + \frac{d_G(a_+, x_i)}{r} \le 3Z, \qquad H(\gamma, x_i) \prec H(\gamma, x) Z^{\eta/(2\beta)}.$$

If $D_G(x_i, x_{i+1}) \le \epsilon_{\mathrm{step}} r$, the assumed one-step estimate now implies

$$|H(\gamma, x_i) - H(\gamma, x_{i+1})| \prec \frac{1}{r} H(\gamma, x) Z^{\eta/\beta} D_G(x_i, x_{i+1}).$$

For a larger edge, bound the difference by $H(\gamma, x_i) + H(\gamma, x_{i+1})$. The preceding vertex bound, $Z \ge 1$, and $D_G(x_i, x_{i+1}) > \epsilon_{\mathrm{step}} r$ give the same estimate, with an additional factor $\epsilon_{\mathrm{step}}^{-1}$. Thus all edges are covered. Summing gives

$$|H(\gamma, x) - H(\gamma, y)| \le \frac{\mathsf{C}}{r} H(\gamma, x) Z^{\eta/\beta} \big(d_G(x, y) + \varepsilon\big).$$

Letting $\varepsilon \to 0$ proves part (b). □

We now prove the one-step estimate (11.28). Thus, until the end of the one-step argument, we assume

$$D_G(x,y) \le \epsilon_{\mathrm{step}} r, \qquad r = e^{-\beta \ell(\gamma)}.$$

The inequality is immediate when $y = x$, so assume $y \neq x$. We will check that $y \in W_{\alpha_+}$ and that the separation time $T = \tau_{\alpha_+}(x,y)$ is finite and attained. At time $T$, the source rays $x^{\alpha_+}$ and $y^{\alpha_+}$ are within $1/100$. To enter the target chart, we need the finer accuracy $\mathsf{r}_{\mathrm{chart}}$. We will move backward by a fixed time $h$ to bring the source points closer. We then project to horizontal horospheres at forward points on the rays from $\xi_2$, and flow the projected points back. Applying Theorem 8.1 with a smaller error and an additional delay ensures that these projections are defined after the backward move.

Fix the charts, $\rho$, the statistical compact sets, $\beta$, $C_{\mathrm{AGY}}$, and all constants of Proposition 11.3. Choose, in this order,

$$\delta_{\mathrm{dom}} = \min\left\{\frac{\mathsf{r}_{\mathrm{chart}}}{4}, \frac{1}{1000}\right\}, \qquad h \ge 1 \quad \text{with} \quad \frac{C_{\mathrm{AGY}}}{100} e^{-\beta h} < \delta_{\mathrm{dom}}. \tag{11.30}$$

Set

$$\epsilon_{\mathrm{br}} = \frac{e^{-2h}}{200}, \qquad B_{\mathrm{dom}} = B(K_2, \eta_2, \epsilon_{\mathrm{br}}),$$

where $B$ is from Theorem 8.1, and put

$$\mathsf{T}_{\mathrm{dom}} = \max\{\mathsf{T}_2, \mathsf{L}_2 + B_{\mathrm{dom}} + h + 1\}. \tag{11.31}$$

The error $\epsilon_{\mathrm{br}}$ compensates for the possible factor $e^{2h}$ of backward growth along a stable leaf. The added delay ensures both this finer approximation and a positive time after moving backward. These choices leave $K_1, K_2, \eta_1, \eta_2, \beta$, and $\mathsf{T}_2$ unchanged. We choose $\epsilon_{\mathrm{step}}$ only after these new constants are fixed.

Let $C_{\mathrm{src}} \ge 1$ bound the constant in (11.37) below. Its proof uses only the chart data and Lemma 9.2, once $2|\mathcal{A}|\epsilon_{\mathrm{step}} \le 1/2$; it is independent of $\epsilon_{\mathrm{step}}$. Let $C_{\mathrm{num}} \ge 1$ be the resulting constant in (11.40), obtained from $C_{\mathrm{src}}$, Lemma 11.6(a), and the fixed bounds for $\rho_\alpha$, under the preliminary restriction $C_{\mathrm{src}}\epsilon_{\mathrm{step}} \le 1/2$. Both constants are independent of $\epsilon_{\mathrm{step}}$. Choose

$$0 < \epsilon_{\mathrm{step}} \le \min\left\{1, \frac{1}{4|\mathcal{A}|}, \frac{e^{-\beta \mathsf{T}_{\mathrm{dom}}}}{2|\mathcal{A}|}, \frac{1}{4C_{\mathrm{src}}}, \frac{1}{2C_{\mathrm{num}}}\right\}. \tag{11.32}$$

The first three bounds give the required source chart and separation time; the last two make the source vectors and their numerator values comparable. There is no smallness requirement on the transported target vectors. Thus no transport prefactor enters a restriction on $\eta_1$ or $\eta$.

Since $x \in V^{(2)}_{\alpha_+}$, we have $\theta_{\alpha_+}(x) \ge 1/|\mathcal{A}|$ and hence

$$D_G^{\alpha_+}(x,y) \le 2|\mathcal{A}| D_G(x,y) \le 2|\mathcal{A}|\epsilon_{\mathrm{step}} r \le \tfrac{1}{2}.$$

An endpoint outside $W_{\alpha_+}$ would give $D_G^{\alpha_+}(x,y) = e^{\beta} > 1$, so $y \in W_{\alpha_+}$. Set $F = F_+(y^{\alpha_+}(0))$; as before, $F_0 = F_+(x^{\alpha_+}(0))$. Thus $[F] = y$ and $[F_0] = x$.

Before using transport, we verify that the middle lift is principal. By Proposition 9.1, the separation time $T = \tau_{\alpha_+}(x,y)$ is finite and attained. Since $e^{-\beta T} = D_G^{\alpha_+}(x,y)$, the preceding bound and (11.32) give

$$T \ge \ell(\gamma) + \mathsf{T}_{\mathrm{dom}}.$$

At this time $x^{\alpha+}(T)$ has $(K_2, \eta_2)$–good statistics and

$$d_{\mathrm{AGY}}\big(x^{\alpha+}(T), y^{\alpha+}(T)\big) \le \frac{1}{100}. \tag{11.33}$$

Since $\mathsf{T}_{\mathrm{dom}} \ge \mathsf{T}_2$, Proposition 11.3(d) shows that $\xi_2(s) = Q_1(x, \zeta(s))$ is principal, including when $x$ is nongeneric. Thus $\mathrm{GM}_{04}$ is defined and (11.24) applies. With $x$ and these paths fixed, define the linear map

$$\Upsilon_\gamma = (\gamma^{-1})^* \mathrm{GM}_{04} : L(q_{\alpha_+}) \longrightarrow L(q_{\alpha_-}).$$

We must still show that this map represents $\gamma^{-1}$ on nearby foliation coordinates. This requires checking the coordinate domains along all five paths.

**Claim 11.8.** *If $D_G(x, y) \le \epsilon_{\mathrm{step}} r$, then*

$$\bar{y} = \gamma^{-1} y \in V^{(3)}_{\alpha_-} \subset W_{\alpha_-}.$$

*For every nonzero $F \in \mathcal{MF}$ with $[F] = y$, both coordinates in the following identity are defined, and*

$$\Upsilon_\gamma\big(\Psi_{q_{\alpha_+}}(F)\big) = \Psi_{q_{\alpha_-}}(\gamma^{-1} F).$$

*The same identity holds for every positive multiple of $F_0$.*

*Proof.* For $F$ proportional to $F_0$, the identity follows from (11.25) and equivariance. For $[F] = y$, we construct, at each point $z$ of the concatenated path

$$\xi = \xi_4 * \xi_3 * \xi_2 * \xi_1 * \xi_0,$$

a unit-area differential with projective vertical foliation $y$ and horizontal foliation $F_-(z)$. Keeping this differential within intrinsic distance $1/100$ of $z$ will put $F$ in the leaf-coordinate domain (4.6). We first move backward from $T$ to make the source points closer. Projection will then give the required points along $\xi_2$ and the returning ray, close enough to enter the target chart.

Write

$$q(t) = x^{\alpha+}(t), \qquad p(t) = y^{\alpha+}(t).$$

The attained separation time $T$ satisfies (11.33), and $q(T)$ has $(K_2, \eta_2)$–good statistics. In particular,

$$\big|\{u \in [T-h, T] : q(u) \in K_2\}\big| \ge (1 - \eta_2)h > \tfrac{1}{2}h.$$

Corollary 4.9 and (11.30) therefore give

$$d_{\mathrm{AGY}}(q(T-h), p(T-h)) \le \frac{C_{\mathrm{AGY}}}{100} e^{-\beta h} < \delta_{\mathrm{dom}}. \tag{11.34}$$

Only good statistics at $T$ are used here; no good-statistics assertion at $T-h$ is needed. Also,

$$T - h \ge \ell(\gamma) + \mathsf{T}_{\mathrm{dom}} - h > \ell(\gamma) \ge \tau.$$

We check the first two paths. For $z \in \xi_0$, the length chosen for $\xi_0$ and (11.23) give

$$d_{\mathrm{AGY}}(z, p(0)) < \frac{1}{300} + \frac{1}{400} < \frac{1}{100}.$$

Here $z$ and $p(0)$ have the same horizontal foliation $F_{\alpha_+}$. The required point along $\xi_0$ is therefore $p(0)$, and (4.6) defines $\Psi_z(F)$. Along $\xi_1$, backward nonexpansion from (11.34) gives

$$d_{\mathrm{AGY}}(q(u), p(u)) < \delta_{\mathrm{dom}} < \frac{1}{100} \qquad (0 \le u \le \tau).$$

The points $p(u)$ give the same domain conclusion along $\xi_1$.

For $\xi_2$, we need a finer approximation than (11.9), so that it remains useful after moving backward by $h$. Fix $Z \in \zeta([0,1])$, and put

$$q_Z = Q_1(x, Z), \qquad S = T - \tau, \qquad D_Z = d_{\mathcal{T}}(Z, X_{q(\tau)}).$$

By (11.7) and $\tau + \tau' \le \ell(\gamma)$,

$$\begin{aligned} S - D_Z &\ge (1 - 16\eta_1)\tau' + \mathsf{T}_{\mathrm{dom}} - \mathsf{L}_2 \\ &\ge B_{\mathrm{dom}} + h + 1. \end{aligned}$$

The last inequality uses $\eta_1 \le 1/100$ and (11.31). Apply Theorem 8.1 to $q(\tau)$ and $q_Z$, with elapsed time $S$ and error $\epsilon_{\mathrm{br}}$. Its terminal-interval hypothesis follows from the good statistics of $q(T)$. The theorem applies to arbitrary $x$, including nongeneric foliations, with the same $B_{\mathrm{dom}}$ for every $Z$.

For $G = F_+(q(\tau))$, the resulting time is

$$s_Z = S + \tfrac{1}{2} \log \mathrm{Ext}_Z(G).$$

It satisfies

$$s_Z \ge S - D_Z > h, \qquad F_+(q_Z(s_Z)) = F_+(q(T)), \qquad d_{\mathrm{AGY}}(q(T), q_Z(s_Z)) < \epsilon_{\mathrm{br}}.$$

The equality of measured vertical foliations puts these two differentials in the same stable horosphere. The time-reversed form of (4.8) now gives

$$d_{\mathrm{AGY}}(q(T-h), q_Z(s_Z - h)) < e^{2h}\epsilon_{\mathrm{br}} = \frac{1}{200}. \tag{11.35}$$

We can now project. Set $q_Z^* = q_Z(s_Z - h)$ and define

$$p_Z^* = \pi^{\mathrm{cs}}_{q_Z^*}(p(T-h)).$$

To justify this definition, apply Theorem 4.11 with target $q_Z^*$ and ball centered at $q(T-h)$. Its assumption on the target follows from (11.35) and $D_{\mathrm{AGY}} \le d_{\mathrm{AGY}}$. Both inputs $q(T-h)$ and $p(T-h)$ lie strictly inside that open ball of radius $1/50$, by (11.34). The projection sends $q(T-h)$ to $q_Z^*$, since they have the same projective vertical foliation. Thus

$$d_{\mathrm{AGY}}(q_Z^*, p_Z^*) \le 4 D_{\mathrm{AGY}}(q(T-h), p(T-h)) < 4\delta_{\mathrm{dom}}.$$

In particular $p_Z^*$ is principal. Define $p_Z = g_{-(s_Z - h)} p_Z^*$. The projection preserves the projective vertical foliation and fixes the measured horizontal foliation, so uniqueness of the differential with prescribed filling foliations identifies

$$p_Z = Q_1(y, F_-^Z), \qquad F_-^Z = F_-(q_Z).$$

This construction establishes that the pair defining $p_Z$ fills; it does not assume its existence before applying the projection. Backward nonexpansion in the common horizontal horosphere gives

$$d_{\mathrm{AGY}}(q_Z, p_Z) < 4\delta_{\mathrm{dom}} \le \min\left\{ \mathsf{r}_{\mathrm{chart}}, \frac{1}{250} \right\}. \tag{11.36}$$

Consequently (4.6) defines $\Psi_{q_Z}(F)$. The formula for $p_Z$ and continuity of the filling-pair correspondence show that $p_Z$ varies continuously along $\xi_2$. The constants are independent of $Z$ and of the common foliation.

At the endpoint $q_Z = x_{F'}(\tau')$, the horizontal foliation is $e^{-\tau'}F'$, so $p_Z = y_{F'}(\tau')$. Flowing this pair backward proves

$$d_{\mathrm{AGY}}(x_{F'}(u), y_{F'}(u)) < 4\delta_{\mathrm{dom}} < \frac{1}{100} \qquad (0 \le u \le \tau').$$

Equation (4.6) therefore supplies the domain inclusion along $\xi_3$. At $u = 0$, apply $\gamma^{-1}$ to obtain

$$d_{\mathrm{AGY}}\big(\bar{x}_{F_{\alpha_-}}(0), \bar{y}_{F_{\alpha_-}}(0)\big) < 4\delta_{\mathrm{dom}} \le \mathsf{r}_{\mathrm{chart}}, \qquad \bar{y} = \gamma^{-1}y.$$

Both differentials have already been constructed in their common horizontal horosphere, so $\bar{x}, \bar{y} \in \mathcal{PMF}(F_{\alpha_-})$. Proposition 11.3 gives $\bar{x} \in V^{(2)}_{\alpha_-}$. Theorem 5.1(f) therefore yields

$$\bar{y} \in V^{(3)}_{\alpha_-} \subset W_{\alpha_-}.$$

Finally consider $\xi_4$. For $z \in \gamma^{-1}\xi_4$, the chosen length and (11.23) give

$$d_{\mathrm{AGY}}\big(z, \bar{y}^{\alpha_-}(0)\big) < \frac{1}{300} + \frac{1}{400} < \frac{1}{100}.$$

The required point along $\xi_4$ is $\gamma\bar{y}^{\alpha_-}(0)$. Equation (4.6) and equivariance give the domain.

The points constructed on consecutive pieces agree at the four joints: they are the unique unit-area differentials with projective vertical foliation $y$ and the horizontal foliation of the corresponding point of $\xi$. They therefore vary continuously along $\xi$, everywhere at intrinsic distance less than $1/100$. The joint leaf-coordinate construction after Lemma 4.3 supplies continuous affine leaf paths joining them to $\xi$. These paths remain principal, and all open-domain inequalities are strict. Lemma 3.7 now gives

$$\mathrm{GM}_{04}\big(\Psi_{q_{\alpha_+}}(F)\big) = \Psi_{\gamma(q_{\alpha_-})}(F).$$

Applying $(\gamma^{-1})^*$ and using equivariance proves

$$\Upsilon_\gamma\big(\Psi_{q_{\alpha_+}}(F)\big) = \Psi_{q_{\alpha_-}}(\gamma^{-1}F).$$

Every domain used is a positive cone, so the conclusion holds for every positive multiple of the chosen representative of $y$. □

The next claim bounds the relative coordinate errors at the source and target by fixed multiples of $D_G(x,y)$ and $r^{-1}e^{32\eta_1\tau'}D_G(x,y)$, respectively.

**Claim 11.9.** *We have*

$$\big\|\Psi_{q_{\alpha_+}}(F_0) - \Psi_{q_{\alpha_+}}(F)\big\|_{q_{\alpha_+}} \prec D_G(x,y)\ \Big\|\Psi_{q_{\alpha_+}}(F_0)\Big\|_{q_{\alpha_+}}, \tag{11.37}$$

*and*

$$\big\|\Upsilon_\gamma(\Psi_{q_{\alpha_+}}(F_0)) - \Upsilon_\gamma(\Psi_{q_{\alpha_+}}(F))\big\|_{q_{\alpha_-}} \prec e^{32\eta_1\tau'}\frac{D_G(x,y)}{r}\ \Big\|\Upsilon_\gamma(\Psi_{q_{\alpha_+}}(F_0))\Big\|_{q_{\alpha_-}}. \tag{11.38}$$

*Proof.* Write $D = D_G(x,y)$. For $s = 0$ and $s = \tau$, the alternative hypothesis of Lemma 9.2 follows from

$$d^{\alpha_+}_G(x,y) \le D^{\alpha_+}_G(x,y) \le 2|\mathcal{A}|\epsilon_{\mathrm{step}}e^{-\beta\ell(\gamma)} \le \tfrac{1}{2}e^{-\beta s},$$

since $\tau \le \ell(\gamma)$ and $2|\mathcal{A}|\epsilon_{\mathrm{step}} \le 1/2$. The lemma therefore gives

$$d_{\mathrm{AGY}}\big(x^{\alpha_+}(s), y^{\alpha_+}(s)\big) \prec D^{\alpha_+}_G(x,y)e^{\beta s} \le 2|\mathcal{A}|De^{\beta s}.$$

At $s = 0$, both points lie in the intrinsic $1/400$–ball about $q_{\alpha_+}$. Proposition 4.4 then bounds the difference of their coordinates by a fixed multiple of $D$. Since $\|\Psi_{q_{\alpha_+}}(F_0)\|_{q_{\alpha_+}} \asymp 1$ on the fixed chart slices, this proves (11.37).

At time $\tau$, the vertical foliations are $e^\tau F_0$ and $e^\tau F$. The construction in Claim 11.8 puts the two differentials at intrinsic distance less than $\delta_{\mathrm{dom}} \le 1/1000$. Applying Proposition 4.4 with center $p_1$, and then using homogeneity, gives

$$e^\tau \|\Psi_{p_1}(F) - \Psi_{p_1}(F_0)\|_{p_1} \prec D e^{\beta\tau}.$$

The real period has AGY norm one, as proved after Lemma 4.2, so

$$e^\tau \|\Psi_{p_1}(F_0)\|_{p_1} = \|\Psi_{p_1}(F_+(p_1))\|_{p_1} = 1.$$

Thus the factor $e^\tau$ cancels in the relative error.

To follow that relative error through the remaining paths, put

$$v_i = \mathrm{GM}_{0i}\big(\Psi_{q_{\alpha_+}}(F_0)\big), \qquad w_i = \mathrm{GM}_{0i}\big(\Psi_{q_{\alpha_+}}(F)\big) \quad (1 \le i \le 4).$$

The coordinate identities in (11.25) and the proof of Claim 11.8 identify these with $\Psi_{p_i}(F_0)$ and $\Psi_{p_i}(F)$, respectively. The preceding estimates give $\|w_1 - v_1\|_{p_1}/\|v_1\|_{p_1} \prec D e^{\beta\tau}$. Apply (11.24) to the difference and its inverse bound to the reference vector. Then

$$\begin{aligned} \frac{\|w_2 - v_2\|_{p_2}}{\|v_2\|_{p_2}} &\le M_{\mathrm{br}}^2 e^{32\eta_1\tau'} \frac{\|w_1 - v_1\|_{p_1}}{\|v_1\|_{p_1}} \\ &\prec D e^{\beta\tau + 32\eta_1\tau'}. \end{aligned} \tag{11.39}$$

The factor $M_{\mathrm{br}}^2$ enters the prefactor, leaving coefficient 32 unchanged.

Along the returning flow $\xi_3$, (4.9) multiplies the numerator by at most $e^{\tau'}$, while (4.10) multiplies the denominator by exactly $e^{\tau'}$. The relative error therefore does not increase. The final short path changes it by at most a fixed factor, by Claim 11.5(4). Finally, $(\gamma^{-1})^*$ is an AGY isometry and $e^{\beta\tau} \le e^{\beta\ell(\gamma)} = r^{-1}$. Using the definition of $\Upsilon_\gamma$ proves (11.38). □

We finish by comparing the numerator and the reciprocal denominator of the spike. Write

$$u_0 = \Psi_{q_{\alpha_+}}(F_0), \qquad u = \Psi_{q_{\alpha_+}}(F), \qquad v_0 = \Upsilon_\gamma u_0, \qquad v = \Upsilon_\gamma u,$$

$$D = D_G(x, y), \qquad Z = 1 + \frac{d_G(a_+, x)}{r}.$$

All four vectors are nonzero, and Claim 11.8 places them in the respective foliation-coordinate cones. At the source, (11.37) and (11.32) give relative error at most $C_{\mathrm{src}} D \le 1/4$, since $r < 1$. Applying Lemma 11.6(a) and the fixed comparison between $\rho_{\alpha_+}$ and the norm gives

$$|\rho_{\alpha_+}(u)^w - \rho_{\alpha_+}(u_0)^w| \le C_{\mathrm{num}} D\, \rho_{\alpha_+}(u_0)^w. \tag{11.40}$$

Since $C_{\mathrm{num}} D \le 1/2$, the two numerator values are within a factor of two.

At the target, the relative error need not be small. The unrestricted estimate in Lemma 11.6(b), together with (11.38), gives

$$|\rho_{\alpha_-}(v)^{-w} - \rho_{\alpha_-}(v_0)^{-w}| \prec \frac{D}{r} e^{32\eta_1\tau'} \max\{\rho_{\alpha_-}(v)^{-w}, \rho_{\alpha_-}(v_0)^{-w}\}. \tag{11.41}$$

By (2.9) and Claim 11.8,

$$H(\gamma,x)=\frac{\rho_{\alpha_+}(u_0)^w}{\rho_{\alpha_-}(v_0)^w},\qquad H(\gamma,y)=\frac{\rho_{\alpha_+}(u)^w}{\rho_{\alpha_-}(v)^w}.$$

Subtracting the two quotients gives

$$\begin{aligned}H(\gamma,x)-H(\gamma,y)=&\frac{\rho_{\alpha_+}(u_0)^w-\rho_{\alpha_+}(u)^w}{\rho_{\alpha_-}(v_0)^w}\\&+\rho_{\alpha_+}(u)^w\big(\rho_{\alpha_-}(v_0)^{-w}-\rho_{\alpha_-}(v)^{-w}\big).\end{aligned}$$

Applying (11.40) and (11.41), and using the comparability of the two numerators, gives

$$|H(\gamma,x)-H(\gamma,y)|\prec DH(\gamma,x)+\frac{D}{r}e^{32\eta_1\tau'}\max\{H(\gamma,x),H(\gamma,y)\}.$$

The global reciprocal-power estimate thus reduces the problem to controlling the larger spike value. Since $d_G(x,y)\le D\le\epsilon_{\mathrm{step}}r\le r$, the coarse comparison (11.22), with $k=1$, gives

$$\max\{H(\gamma,x),H(\gamma,y)\}\prec H(\gamma,x)Z^{2\eta_{\mathrm{c}}/\beta}.$$

Also, (11.19) gives

$$e^{32\eta_1\tau'}\prec Z^{32\eta_1/(\beta(1-8\eta_1))}.$$

The first term is absorbed since $r<1$ and $Z\ge1$. Consequently,

$$|H(\gamma,x)-H(\gamma,y)|\prec\frac{D}{r}H(\gamma,x)Z^{\left(32\eta_1/(1-8\eta_1)+2\eta_{\mathrm{c}}\right)/\beta}.\tag{11.42}$$

We can now see the entire error in the exponent. The bridge and its inverse contribute $32\eta_1/(1-8\eta_1)$, and the coarse comparison contributes $2\eta_{\mathrm{c}}$. For the initial choices (11.1),

$$\frac{32\eta_1}{1-8\eta_1}+2\eta_{\mathrm{c}}\le(35+88w)\eta_1\le128w\eta_1=\frac{\eta}{2},$$

where $w\ge1$. Thus

$$|H(\gamma,x)-H(\gamma,y)|\le\frac{C_{\mathrm{step}}}{r}H(\gamma,x)\left(1+\frac{d_G(a_+,x)}{r}\right)^{\eta/(2\beta)}D_G(x,y).$$

This is (11.28). All compact-dependent norm comparisons, the factor $M_{\mathrm{br}}^2$, and the constants in (11.22) and (11.19) enter the prefactor $C_{\mathrm{step}}$; the exponent remains $\eta/(2\beta)$. After fixing these constants and $\epsilon_{\mathrm{step}}$, the chain argument of Lemma 11.7 only enlarges $\mathsf{C}$, allowing dependence on $\epsilon_{\mathrm{step}}^{-1}$. Thus the order is $\eta_1,\eta$ first, then the compact sets and $\beta$, then the prefactors and the additional delay $\mathsf{T}_{\mathrm{dom}}$, and finally $\epsilon_{\mathrm{step}}$. The compact sets remain fixed throughout the domain argument in Claim 11.8. No comparison between $\eta$ and $\beta$ is required. Lemma 11.7 now proves part (b) of Theorem 11.1.

## 12. $Q$-Spikes

We now collect the properties of $H(\gamma, \cdot)$ that we will use in Section 13. These functions are large on a ball, decay away from it, and have controlled variation on the scale of its radius. The next section uses these properties to subtract positive sums of spikes while keeping the remainder positive and sufficiently regular to repeat the construction.

12.1. **Defining $Q$-spikes.** Let $(X, d_G, \nu_\rho)$ be a metric measure space, and let $B_G(a, r)$ denote the open ball of radius $r$ centered at $a$. Fix $W > 0$ and $C_{\text{vol}} > 0$ such that, for every $a \in X$ and $r > 0$,

$$\nu_\rho(B_G(a, r)) \le C_{\text{vol}} r^W. \tag{12.1}$$

The annular decomposition in the proof of Corollary 9.10 then gives, for every $W' > W$, $x \in X$, and $r > 0$,

$$\int_{X \setminus B_G(x,r)} \frac{d\nu_\rho(z)}{d_G(x, z)^{W'}} \le D_\rho \, r^{W - W'}, \tag{12.2}$$

where $D_\rho$ depends only on $C_{\text{vol}}$, $W$, and $W'$.

For a function $f : X \to \mathbb{R}$ and $s > 0$, define its *Lipschitz constant at $x$ on scale $s$* by

$$D_s f(x) = \sup_{\{y \,:\, 0 < d_G(x,y) \le s\}} \frac{|f(y) - f(x)|}{d_G(x, y)},$$

with value zero if the set is empty.

Fix $0 < \Xi < W$. We use a variant of Connell–Muchnik's $Q$-spikes [CM07a, Definitions 4.1–4.2], allowing the local comparison and Lipschitz bounds to deteriorate away from the center. A triple $(h, a, r)$, where $a \in X$, $r > 0$, and $h : X \to \mathbb{R}$ is nonnegative with $h(a) > 0$, is called a $Q$-spike if the following estimates hold with implicit constants independent of $x, y$.

First, $h$ decays away from $a$: for every $x \in X$,

$$h(x) \prec h(a) \left(1 + \frac{d_G(a, x)}{r}\right)^{-2W + \Xi/2}. \tag{12.3}$$

On the ball $B_G(a, r)$, we also require the lower bound

$$h(x) \succ h(a). \tag{12.4}$$

For any $x, y \in X$ with $d_G(x, y) \le r$, we require the comparison

$$h(y) \prec h(x) \left(1 + r^{-1} d_G(a, x)\right)^{\Xi}. \tag{12.5}$$

The selected ball must satisfy the volume lower bound

$$\nu_\rho(B_G(a, r)) \succ r^W. \tag{12.6}$$

Finally, the variation of $h$ on scale $r$ must satisfy, for every $x \in X$,

$$D_r h(x) \prec \frac{h(x)}{r} \left(1 + \frac{d_G(a, x)}{r}\right)^{\Xi/2}. \tag{12.7}$$

Combining (12.3) and (12.7) gives the following consequence, which controls variation far from the center:

$$D_r h(x) \prec \frac{h(a)}{r}\left(1+\frac{d_G(a,x)}{r}\right)^{-2W+\Xi}. \tag{12.8}$$

A family has *uniform spike estimates* if the same implicit constants work for all its members.

12.2. **Verification of the $Q$-spike conditions.** Take $X = \mathcal{PMF}$ with the Gromov metric from Section 9 and the measure $\nu_\rho$ from Section 2. Set

$$W = w/\beta \qquad \text{and} \qquad \Xi = 2\eta/\beta,$$

where $\eta$ is the constant from Theorem 11.1 and $\beta$ is as in Section 9. The initial choice (11.1) gives $2\eta = 512w\eta_1 < 1 \le w$, and hence $\Xi < W$. No further change to $\eta_1$ or the compact sets is needed. Proposition 9.9 gives (12.1) for sufficiently small $r$. Since $\nu_\rho$ is a probability measure, enlarging $C_{\mathrm{vol}}$ extends this bound to every $r > 0$.

**Theorem 12.1.** *Suppose $\gamma \in \Gamma(R)$ with $R > \max(\mathsf{R}_0, \mathsf{R}_1) + 1$. Let*

$$a = [F_+(\gamma)] \qquad \textit{and} \qquad r = e^{-\beta\ell(\gamma)}.$$

*Let $h(x) = H(\gamma, x)$. Then $(h, a, r)$ is a $Q$-spike, with constants independent of $\gamma$ and $R$ once the standing data are fixed.*

*Proof.* By (2.9), we have $h(a) = e^{w\ell(\gamma)}$. The two-sided estimate in Theorem 11.1(a) therefore gives (12.3) and (12.4). Its ratio estimate (11.3) gives (12.5), since $\Xi = 2\eta/\beta$.

For the volume lower bound, choose $\alpha \in \mathcal{A}$ such that $\gamma \in \Gamma_\alpha(R)$. Proposition 7.4(b) gives $\ell(\gamma) \ge R - 1 > \mathsf{R}_1$. By part (d), $a \in V_\alpha^{(1)}$ and $a^\alpha(\ell(\gamma))$ has $(K_1, \eta_1)$–good statistics. Proposition 9.9 therefore gives (12.6).

Finally, Theorem 11.1(b) gives (12.7), since $\eta/\beta = \Xi/2$. All constants are uniform in $\gamma$ and $R$. □

## 13. Approximation by positive sums of spikes

We now use the spikes from Section 12 to subtract a definite proportion of a positive function on the set covered by their balls. The remainder must stay positive and satisfy the same estimates at a smaller scale, so that we can repeat the construction. Proposition 13.1 gives this step. In Section 14, we start with $f \equiv 1$ and iterate it.

Let $(X, d_G, \nu_\rho)$ be as in §12.1. Let $(h_i, a_i, r_i)_{i=1}^N$ be a collection of $Q$-spikes. We assume that all members of this collection satisfy the $Q$-spike estimates with common bounds on their constants. We also assume there is a uniform constant $\mathsf{W}_0$ such that, for $i, j \in [1, N]$,

$$r_i \le \mathsf{W}_0 r_j.$$

Let $r_{\max} = \max_i r_i$, $r_{\min} = \min_i r_i$. Let $Y \subset X$ be a set covered by the balls $\{B_G(a_i, r_i)\}$. We further assume that there is a constant $\mathsf{B}$ such that each point in $X$ is covered by at most $\mathsf{B}$ balls. In this section, we call a constant *uniform* if it depends on the constants in the $Q$-spike estimates in §12.1, $C_{\mathrm{vol}}$, $W$, $\Xi$, $\mathsf{W}_0$, and $\mathsf{B}$. Set

$$\sigma = \frac{W - \Xi}{2}.$$

**Proposition 13.1.** *There exist uniform constants $\mathsf{m} \geq 2$ and $\epsilon_0 > 0$ such that the following holds: Suppose $f\colon X \to \mathbb{R}_+$ is a bounded positive function which is bounded away from 0, and suppose there is a $\delta > 0$ such that*

$$D_\delta f(x) \leq \frac{f(x)}{\delta}, \tag{13.1}$$

$$\frac{f(y)}{f(x)} \leq 4\left(1 + \frac{d_G(x,y)}{\delta}\right)^\sigma \text{ for all } x, y \in X. \tag{13.2}$$

*and*

$$r_{\max} \leq \delta/\mathsf{m}. \tag{13.3}$$

*Then we can find numbers $\mu_\Gamma^i \geq 0$ such that for*

$$g = \sum_{i=1}^N \mu_\Gamma^i h_i$$

*and $F = f - g$ we have*

$$\frac{1}{2} f(x) \leq F(x) \leq f(x) \qquad \text{for } x \in X,$$

$$F(x) \leq (1 - \epsilon_0) f(x), \qquad \text{for } x \in Y.$$

*Also, $F$ satisfies both* (13.1) *and* (13.2) *with $\delta' = \frac{r_{\min}}{4}$ in place of $\delta$.*

The local bound (13.1) controls variation within distance $\delta$. The global bound (13.2) controls contributions from distant centers, where $f(a_i)$ may be much larger than $f(x)$. Our choice of $\sigma$ gives $2W - \Xi - \sigma = W + \sigma > W$, so the integral estimate (12.2) still applies after allowing for this growth.

The next two lemmas express the decay bounds as integrals over the spike balls. Bounded overlap will then let us estimate sums of spikes.

**Lemma 13.2.** *Suppose $(h, a, r)$ is a Q-spike. For each $x \in B_G(a,r)^c$ we have*

$$0 \leq h(x) \prec h(a)\, r^{W-\Xi} \int_{B_G(a,r)} \frac{d\nu_\rho(y)}{d_G(x,y)^{2W-\Xi}}.$$

*Proof.* Since $d_G(x,a) \geq r$, for any $y \in B_G(a,r)$ we have

$$d_G(x,y) \leq d_G(x,a) + r \leq 2 d_G(x,a),$$

and hence

$$\frac{1}{d_G(x,a)} \prec \frac{1}{d_G(x,y)}.$$

Raising both sides to the power $2W - \Xi$ and integrating over $B_G(a,r)$, we obtain

$$\frac{\nu_\rho(B_G(a,r))}{d_G(x,a)^{2W-\Xi}} \prec \int_{B_G(a,r)} \frac{d\nu_\rho(y)}{d_G(x,y)^{2W-\Xi}}.$$

Multiplying both sides by $h(a) r^{W-\Xi}$ and using (12.6), we get

$$h(a)\left(\frac{r}{d_G(a,x)}\right)^{2W-\Xi} \prec h(a) r^{W-\Xi} \int_{B_G(a,r)} \frac{d\nu_\rho(y)}{d_G(x,y)^{2W-\Xi}}.$$

On the other hand, since $d_G(a,x) \geq r$, we have

$$\left(1+\frac{d_G(a,x)}{r}\right)^{-2W+\Xi} \asymp \left(\frac{d_G(a,x)}{r}\right)^{-2W+\Xi} = \left(\frac{r}{d_G(a,x)}\right)^{2W-\Xi}.$$

Since (12.3) implies the weaker bound with exponent $-2W+\Xi$, the lemma follows. □

**Lemma 13.3.** *Suppose $(h,a,r)$ is a Q-spike. For each $x \in B_G(a,r)^c$ we have*

$$D_r h(x) \prec \frac{h(a)}{r}\, r^{W-\Xi} \int_{B_G(a,r)} \frac{d\nu_\rho(y)}{d_G(x,y)^{2W-\Xi}}.$$

*Proof.* Apply the integral comparison in the proof of Lemma 13.2, with $h(a)/r$ in place of $h(a)$, to the decay bound (12.8). □

*Proof of Proposition 13.1.* We construct a sum greedily, filling the deficit at each center where the current sum is below $f$. We bound the sum above on $X$ and below on $Y$, then rescale it to control both the remainder and its variation.

Choose $\mathsf{m} \geq 2$ with $4\mathsf{m} \geq 2^{1+1/\sigma}$, large enough that the two tail bounds in the proof of Lemma 13.6 are at most $f(y)/4$ each. Lemmas 13.4 and 13.5 give these bounds with implicit constants independent of $\mathsf{m}$. Thus this choice uses only the uniform data; keep $\mathsf{m}$ fixed throughout.

Order the radii so that $r_i$ is nonincreasing. Let $g_0(x) = 0$ and

$$g_n(x) = \begin{cases} g_{n-1}(x) & \text{if} \quad g_{n-1}(a_n) \geq f(a_n) \\ g_{n-1}(x) + \frac{h_n(x)}{h_n(a_n)}(f(a_n) - g_{n-1}(a_n)) & \text{if} \quad g_{n-1}(a_n) < f(a_n) \end{cases}$$

This construction yields a sequence $\{\lambda_i\}_{i=1}^N$ with

$$g_n(x) = \sum_{i=1}^{n} \lambda_i h_i(x), \qquad \text{where } \lambda_i\, h_i(a_i) = \max(0, f(a_i) - g_{i-1}(a_i)).$$

In particular,

$$\lambda_i h_i(a_i) \leq f(a_i). \tag{13.4}$$

Fix $y \in X$. We first separate centers beyond distance $\delta$ and those between $\mathsf{m}r_i$ and $\delta$. For the remaining centers, we distinguish whether the ball $B_G(a_i,r_i)$ is farther than $r_n/2$ from $y$. This gives four index sets, whose dependence on $y$ we suppress in the notation:

$$I_0(n,N) = \big\{i \in \{n,...,N\} : d_G(y,a_i) > \delta\big\},$$
$$I_1(n,N) = \big\{i \in \{n,...,N\} : \mathsf{m}\, r_i < d_G(y,a_i) \leq \delta\big\},$$
$$I_2(n,N) = \big\{i \in \{n,...,N\} : d_G(y,a_i) \leq \mathsf{m}\, r_i \text{ and } d_G\big(y, B_G(a_i,r_i)\big) > r_n/2\big\},$$

and

$$I_3(n,N) = \big\{i \in \{n,...,N\} : d_G(y,a_i) \leq \mathsf{m}\, r_i \text{ and } d_G\big(y, B_G(a_i,r_i)\big) \leq r_n/2\big\}.$$

**Lemma 13.4.** *For any $y \in X$,*

$$\sum_{i \in I_0(1,N)} \lambda_i h_i(y) \prec \frac{f(y)}{\mathsf{m}^{W-\Xi}} \qquad \textit{and} \qquad \sum_{i \in I_0(1,N)} \lambda_i D_{r_i} h_i(y) \prec \frac{f(y)}{r_N\, \mathsf{m}^{W-\Xi}}.$$

*Proof.* From Lemma 13.2 and (13.4) we have

$$\sum_{i\in I_0(1,N)} \lambda_i h_i(y) \prec \sum_{i\in I_0(1,N)} f(a_i) r_i^{W-\Xi} \int_{B_G(a_i,r_i)} \frac{d\nu_\rho(x)}{d_G(y,x)^{2W-\Xi}} \tag{13.5}$$

Note that for all $i$, $r_i \le r_1 \le \delta/\mathsf{m} \le \delta/2$. For $i \in I_0(1,N)$ and $x \in B_G(a_i, r_i)$, we have $d_G(y,x) > \delta/2$ and

$$d_G(y,a_i) \le d_G(y,x) + r_i \le d_G(y,x) + \delta/2,$$

so

$$\left(1 + \frac{d_G(y,a_i)}{\delta}\right) \le \left(\frac{3}{2} + \frac{d_G(y,x)}{\delta}\right) \le \frac{4 d_G(y,x)}{\delta}.$$

and then, using (13.2),

$$f(a_i) \le 4 f(y) \left(1 + \frac{d_G(y,a_i)}{\delta}\right)^\sigma \le 4^{1+\sigma} f(y) \delta^{-\sigma} d_G(y,x)^\sigma.$$

Substituting into (13.5), we get

$$\sum_{i\in I_0(1,N)} \lambda_i h_i(y) \prec \sum_{i\in I_0(1,N)} f(y) r_i^{W-\Xi} \delta^{-\sigma} \int_{B_G(a_i,r_i)} \frac{d\nu_\rho(x)}{d_G(y,x)^{2W-\Xi-\sigma}}$$

Bounded overlap estimates the right-hand-side as

$$f(y) \mathsf{B} r_1^{W-\Xi} \delta^{-\sigma} \int_{X\setminus B_G(y,\delta/2)} \frac{d\nu_\rho(z)}{d_G(y,z)^{2W-\Xi-\sigma}}.$$

Using (12.2) applied with $W' = 2W - \Xi - \sigma = W + \sigma > W$, this is

$$\prec f(y) r_1^{W-\Xi} \delta^{-\sigma} \delta^{-(W-\Xi-\sigma)} = f(y) (r_1/\delta)^{W-\Xi} \le f(y)/\mathsf{m}^{W-\Xi}.$$

This proves the first estimate.

The proof of the second estimate is the same, using Lemma 13.3 instead of Lemma 13.2. The extra factor $1/r_i$ is bounded by $1/r_N$, since $r_i \ge r_N$. Thus

$$\sum_{i\in I_0(1,N)} \lambda_i D_{r_i} h_i(y) \prec \frac{f(y)}{r_N \mathsf{m}^{W-\Xi}}.$$

□

**Lemma 13.5.** *For any $y \in X$,*

$$\sum_{i\in I_1(1,N)} \lambda_i h_i(y) \prec \frac{f(y)}{(\mathsf{m}-1)^{W-\Xi}} \qquad \text{and} \qquad \sum_{i\in I_1(1,N)} \lambda_i D_{r_i} h_i(y) \prec \frac{f(y)}{r_N\,(\mathsf{m}-1)^{W-\Xi}}.$$

*Proof.* Since $\lambda_i h_i(a_i) \le f(a_i)$, Lemma 13.2 gives

$$\sum_{i\in I_1(1,N)} \lambda_i h_i(y) \prec \sum_{i\in I_1(1,N)} f(a_i) r_i^{W-\Xi} \int_{B_G(a_i,r_i)} \frac{d\nu_\rho(x)}{d_G(y,x)^{2W-\Xi}}.$$

For $i \in I_1(1,N)$, we have $d_G(a_i, y) \le \delta$, and hence, by (13.1),

$$f(a_i) \le 2 f(y).$$

Also $r_i \geq r_N$, and the balls $B_G(a_i, r_i)$ have multiplicity at most $\mathsf{B}$. Therefore

$$\sum_{i \in I_1(1,N)} \lambda_i h_i(y) \prec f(y) r_1^{W-\Xi} \int_{X - B_G(y,(\mathsf{m}-1)r_N)} \frac{d\nu_\rho(x)}{d_G(y,x)^{2W-\Xi}}.$$

By (12.2) applied with $W' = 2W - \Xi$, this is

$$\prec \frac{f(y) r_1^{W-\Xi}}{((\mathsf{m}-1)r_N)^{W-\Xi}} \leq \frac{\mathsf{W}_0^{W-\Xi} f(y)}{(\mathsf{m}-1)^{W-\Xi}}.$$

This proves the first estimate.

The proof of the second estimate is identical, using Lemma 13.3 and the extra factor $1/r_i \leq 1/r_N$. Hence

$$\sum_{i \in I_1(1,N)} \lambda_i D_{r_i} h_i(y) \prec \frac{\mathsf{W}_0^{W-\Xi} f(y)}{r_N (\mathsf{m}-1)^{W-\Xi}}. \qquad \square$$

**Lemma 13.6.** *There is a uniform constant $\mathsf{C}$ such that, for $y, y' \in X$ with $d_G(y, y') \leq r_n$, we have*

$$g_n(y) \leq \mathsf{C}\, g_n(y') + f(y)/2.$$

*Proof.* By our choice of $\mathsf{m}$, Lemmas 13.4 and 13.5, applied to the first $n$ spikes, give

$$\sum_{i \in I_0(1,n)} \lambda_i h_i(y) \leq \frac{f(y)}{4} \qquad \text{and} \qquad \sum_{i \in I_1(1,n)} \lambda_i h_i(y) \leq \frac{f(y)}{4}.$$

But, for $i \in I_2(1,n) \cup I_3(1,n)$, we have $d_G(a_i, y) \leq \mathsf{m} r_i$. Hence, $d_G(a_i, y') \leq (\mathsf{m}+1) r_i$. Applying (12.5) with $x = y'$ and $r = r_i$, we have

$$\sum_{i \in I_2(1,n) \cup I_3(1,n)} \lambda_i h_i(y) \prec \sum_{i \in I_2(1,n) \cup I_3(1,n)} \lambda_i h_i(y') \left(1 + \frac{(\mathsf{m}+1)\, r_i}{r_i}\right)^{\Xi} \prec g_n(y')\, (2+\mathsf{m})^{\Xi}.$$

Since $\mathsf{m}$ is now fixed, the last factor is uniform. Combining these estimates proves the lemma. $\square$

**Lemma 13.7.** *There is a uniform constant $\mathsf{M}_1$ such that*

$$g_N(y) \leq \mathsf{M}_1 f(y)$$

*for all $N \in \mathbb{N}$ and $y \in X$.*

*Proof.* Fix $y \in X$. If $g_N(y) < (3\mathsf{C} + 0.5) f(y)$ then the lemma holds for this $y$ as long $\mathsf{M}_1 \geq (3\mathsf{C} + 0.5)$. Otherwise, let $n$ be the smallest integer such that

$$g_n(y) \geq (3\mathsf{C} + 0.5) f(y).$$

Then

$$g_{n-1}(y) < (3\mathsf{C} + 0.5) f(y). \tag{13.6}$$

By Lemma 13.6, applied to $g_n$, if $d_G(y, y') \leq r_n$, then

$$g_n(y) \leq \mathsf{C} g_n(y') + \frac{1}{2} f(y).$$

Since $g_n(y) \geq (3\mathsf{C} + 0.5) f(y)$, this implies

$$g_n(y') \geq 3 f(y).$$

In particular, if $k > n$ and $d_G(y, a_k) \le r_n$, then

$$g_{k-1}(a_k) \ge g_n(a_k) \ge 3f(y) \ge f(a_k),$$

where the last inequality follows from (13.1), since $d_G(y, a_k) \le r_n \le \delta$. Hence, by construction we have $\lambda_k = 0$.

It remains to bound the contribution of indices $n \le i \le N$. We use the four sets $I_j(n, N)$ defined above. Lemmas 13.4 and 13.5 give

$$\sum_{i \in I_0(n,N) \cup I_1(n,N)} \lambda_i h_i(y) \prec f(y). \tag{13.7}$$

We now estimate the sums over $I_2(n, N)$ and $I_3(n, N)$. Since $\lambda_i h_i(a_i) \le f(a_i)$, Lemma 13.2 gives

$$\sum_{i \in I_2(n,N)} \lambda_i h_i(y) \prec \sum_{i \in I_2(n,N)} f(a_i) r_i^{W-\Xi} \int_{B_G(a_i, r_i)} \frac{d\nu_\rho(x)}{d_G(y, x)^{2W-\Xi}}.$$

For $i \in I_2(n, N)$, we have

$$d_G(a_i, y) \le \mathsf{m} r_i \le \mathsf{m} r_n \le \delta,$$

so (13.1) gives

$$f(a_i) \le 2f(y) \le 3f(y).$$

Also $r_i \le r_n$, and the balls have multiplicity at most $\mathsf{B}$. Hence

$$\sum_{i \in I_2(n,N)} \lambda_i h_i(y) \prec f(y) r_n^{W-\Xi} \int_{X - B_G(y, r_n/2)} \frac{d\nu_\rho(x)}{d_G(y, x)^{2W-\Xi}}.$$

By (12.2), applied with $W' = 2W - \Xi$,

$$\sum_{i \in I_2(n,N)} \lambda_i h_i(y) \prec f(y). \tag{13.8}$$

Now we consider the sum over $I_3(n, N)$. If $i \in I_3(n, N)$ and $r_i < \frac{r_n}{\mathsf{m}}$, then

$$d_G(y, a_i) \le \mathsf{m} r_i < r_n.$$

For $i > n$, this implies $\lambda_i = 0$ by the observation above. Thus the only indices in $I_3(n, N)$ which can contribute are contained in

$$I_4(n, N) = \left\{ n \le i \le N : r_i \ge \frac{r_n}{\mathsf{m}} \text{ and } d_G\big(y, B_G(a_i, r_i)\big) \le \frac{r_n}{2} \right\}.$$

For $i \in I_4(n, N)$, the ball $B_G(a_i, r_i)$ is contained in a fixed multiple of $B_G(y, r_n)$; say

$$B_G(a_i, r_i) \subset B_G(y, 3r_n).$$

Using the bounded-overlap assumption and (12.6) and (12.1) for each $B_G(a_i, r_i)$, we get

$$\#I_4(n, N) \left(\frac{r_n}{\mathsf{m}}\right)^W \prec \sum_{i \in I_4(n,N)} \nu_\rho(B_G(a_i, r_i)) \le \mathsf{B}\, \nu_\rho\big(B_G(y, 3r_n)\big) \prec r_n^W.$$

Hence

$$\#I_4(n, N) \prec 1,$$

with an implicit constant depending on $\mathsf{m}$.

For $i \in I_3(n,N)$ with $\lambda_i \neq 0$, we have $i \in I_4(n,N)$. Moreover, by (12.3) and $d_G(y,a_i) \leq \mathsf{m} r_i$,

$$h_i(y) \prec h_i(a_i).$$

Therefore

$$\sum_{i \in I_3(n,N)} \lambda_i h_i(y) \prec \sum_{i \in I_4(n,N)} \lambda_i h_i(a_i).$$

For $i \in I_4(n,N)$, the defining distance bound gives $d_G(y,a_i) \leq r_n/2 + r_i \leq 3r_n/2 \leq \delta$. Thus (13.4) and (13.1) give

$$\lambda_i h_i(a_i) \leq f(a_i) \leq 3f(y).$$

Thus

$$\sum_{i \in I_3(n,N)} \lambda_i h_i(y) \prec f(y). \tag{13.9}$$

This argument works for every $y$. Thus, there exist uniform constants $C_j$ such that

$$\sum_{i \in I_j(n,N)} \lambda_i h_i(y) \leq C_j f(y), \qquad 0 \leq j \leq 3.$$

Then,

$$g_N(y) = g_{n-1}(y) + \sum_{i=n}^{N} \lambda_i h_i(y) \leq (3\mathsf{C} + 1/2 + \sum_{j=0}^{3} C_j) f(y)$$

Thus the lemma holds for any $\mathsf{M}_1$ larger than $(3\mathsf{C} + 0.5) + \sum C_j$. □

**Lemma 13.8.** *For every $y \in Y$,*

$$g_N(y) \geq \frac{f(y)}{6\mathsf{C}}. \tag{13.10}$$

*Proof.* Assume that $n$ is the smallest integer such that $y \in B_G(a_n, r_n)$. By Lemma 13.6,

$$g_n(y) \geq \frac{1}{\mathsf{C}} \left( g_n(a_n) - \frac{1}{2} f(a_n) \right).$$

and, by construction, $g_n(a_n) \geq f(a_n)$. Also, as $d_G(y,a_n) \leq r_n \leq \delta$, we get, by (13.1), $f(y) \leq 2f(a_n)$. Therefore,

$$g_n(y) \geq \frac{g_n(a_n)}{2\mathsf{C}} \geq \frac{f(a_n)}{2\mathsf{C}} \geq \frac{f(y)}{6\mathsf{C}}.$$

But $g_N(y) \geq g_n(y)$ hence (13.10) holds. □

We first normalize the sum to ensure $g \leq f/2$. A further rescaling below will control its Lipschitz constant. Define

$$\mu_\Gamma^i = \frac{\lambda_i}{2\mathsf{M}_1} \qquad \text{and} \qquad g = \sum_{i=1}^{N} \mu_\Gamma^i \, h_i = \frac{1}{2\mathsf{M}_1} g_N.$$

For this normalization, set $\epsilon_0 = \frac{1}{12\mathsf{C}\mathsf{M}_1}$ and $F = f - g$. Then (13.10) gives, for $y \in Y$,

$$g(y) \geq \epsilon_0 f(y) \qquad \Longrightarrow \qquad F(y) \leq (1 - \epsilon_0) f(y)$$

and Lemma 13.7 implies that, for $x \in X$,

$$0 \le g(x) \le \frac{1}{2} f(x) \qquad \Longrightarrow \qquad \frac{f(x)}{2} \le F(x) \le f(x).$$

It remains to show that $F$ satisfies (13.1) and (13.2). To do this, we first estimate $D_{r_{\min}} g$. Fix $y \in X$, and let

$$g_{N,j}(x) = \sum_{i \in I_j(1,N)} \lambda_i h_i(x).$$

In the sum above, the sets $I_j(1,N)$ are defined using the point $y$ (and not $x$). For $j = 0$ or $j = 1$, the second estimates in Lemmas 13.4 and 13.5 give

$$D_{r_N} g_{N,j}(y) \prec \frac{f(y)}{r_N}.$$

Also, for $j = 2, 3$, we have $d_G(a_i, y) \le \mathsf{m} r_i$ for $i \in I_j(1,N)$. Using (12.7), we obtain

$$\begin{aligned} D_{r_N} g_{N,j}(y) &\le \sum_{i \in I_j(1,N)} \lambda_i D_{r_i} h_i(y) \prec \sum_{i \in I_j(1,N)} \lambda_i \frac{h_i(y)}{r_i} \left(1 + \frac{d_G(a_i, y)}{r_i}\right)^{\Xi} \\ &\prec \sum_{i \in I_j(1,N)} \lambda_i \frac{h_i(y)}{r_i} \le \frac{1}{r_N} \sum_{i \in I_j(1,N)} \lambda_i h_i(y) \\ &\le \frac{g_N(y)}{r_N} \prec \frac{f(y)}{r_N}, \end{aligned}$$

where the last inequality follows from Lemma 13.7. Thus,

$$D_{r_{\min}} g(y) = \frac{1}{2\mathsf{M}_1} D_{r_N} g_N(y) \le \frac{1}{2\mathsf{M}_1} \sum_{j=0}^{3} D_{r_N} g_{N,j}(y) \le \frac{\mathsf{L} f(y)}{r_{\min}}.$$

where $\mathsf{L} > 1$ is a uniform constant.

We now divide every coefficient $\mu_\Gamma^i$ and the lower-bound constant $\epsilon_0$ by the uniform constant $\mathsf{L}$ from the Lipschitz estimate. This preserves the inequalities $0 \le g \le f/2$ and $g \ge \epsilon_0 f$ on $Y$. We keep the notation $g, \epsilon_0$ for the rescaled quantities, and define $F = f - g$ using this final $g$. The final removal fraction is therefore $\epsilon_0 = (12\mathsf{C}\mathsf{M}_1\mathsf{L})^{-1}$. The constants were chosen in the order $\mathsf{m}, \mathsf{C}, \mathsf{M}_1, \mathsf{L}$, and none depends on $f$, $N$, or $\delta$.

Finally, we verify the two comparison estimates for $F = f - g$. Note that $g$ is now normalized so that

$$D_{r_{\min}} g(x) \le \frac{f(x)}{r_{\min}}.$$

Recall that $4\mathsf{m} \ge 2^{1+1/\sigma}$ and $\delta' \le \delta/(4\mathsf{m})$. Since $f/2 \le F \le f$, the function $F$ is positive, bounded, and bounded away from zero. The relative Lipschitz hypothesis and the normalized estimate for $g$ give

$$\begin{aligned} D_{\delta'} F(x) &\le D_{\delta'} f(x) + D_{\delta'} g(x) \\ &\le \left(\frac{1}{\delta} + \frac{1}{r_{\min}}\right) f(x) \\ &\le \frac{4F(x)}{r_{\min}} = \frac{F(x)}{\delta'}. \end{aligned}$$

The same estimate applies to pairs at distance exactly $\delta'$, since $\delta' < \min\{\delta, r_{\min}\}$.

For the polynomial comparison, write $d = d_G(x, y)$. If $d < \delta$, the relative Lipschitz hypothesis on $f$ gives

$$\frac{F(y)}{F(x)} \le 2\frac{f(y)}{f(x)} \le 2\left(1 + \frac{d}{\delta}\right) \le 4 \le 4\left(1 + \frac{d}{\delta'}\right)^{\sigma}.$$

If $d \ge \delta$, our choice of $\mathsf{m}$ gives

$$1 + \frac{d}{\delta'} \ge 4\mathsf{m}\frac{d}{\delta} \ge 2^{1/\sigma}\left(1 + \frac{d}{\delta}\right).$$

The polynomial comparison hypothesis on $f$ therefore implies

$$\frac{F(y)}{F(x)} \le 2\frac{f(y)}{f(x)} \le 8\left(1 + \frac{d}{\delta}\right)^{\sigma} \le 4\left(1 + \frac{d}{\delta'}\right)^{\sigma}.$$

Thus $F$ satisfies both hypotheses at scale $\delta'$.

□

The next lemma explains why a global comparison such as (13.2) is natural in this iteration. Starting from $f_0 \equiv 1$, if the scales decrease geometrically and each step decreases the function by at most half, the local bounds imply a global power bound. We do not need this observation in the proof of the main theorem.

**Lemma 13.9.** *Let $A > 1$, and suppose that positive scales $\delta_0, \dots, \delta_n$ satisfy*

$$\delta_j \le A^{-1}\delta_{j-1} \qquad (1 \le j \le n).$$

*Suppose $f_0 \equiv 1$, and that the positive functions $f_0, \dots, f_n$ satisfy*

$$\frac{1}{2}f_{j-1} \le f_j \le f_{j-1}, \tag{13.11}$$

*and*

$$D_{\delta_j} f_j(x) \le \frac{f_j(x)}{\delta_j} \quad \text{for all } x \in X \text{ and } 1 \le j \le n. \tag{13.12}$$

*Then, setting $\alpha = \log 2/\log A$, we have*

$$\frac{f_n(y)}{f_n(x)} \le 4\left(1 + \frac{d_G(x,y)}{\delta_n}\right)^{\alpha} \qquad \text{for all } x, y \in X. \tag{13.13}$$

*In particular, if $A \ge 2^{1/\sigma}$, then $f_n$ satisfies (13.2) with $\delta = \delta_n$.*

*Proof.* We record the following immediate consequence of (13.12):

$$f_j(y) \le 2f_j(x) \qquad \text{if } d_G(x,y) \le \delta_j. \tag{13.14}$$

Put $d = d_G(x, y)$.

For $d \le \delta_n$, (13.13) follows directly from (13.14). If

$$\delta_{n-k+1} < d \le \delta_{n-k} \qquad (1 \le k \le n),$$

the estimate (13.14) for $f_{n-k}$ and repeated use of (13.11) give

$$\frac{f_n(y)}{f_n(x)} \le 2^k \frac{f_{n-k}(y)}{f_{n-k}(x)} \le 2^{k+1}.$$

Since $d > \delta_{n-k+1} \geq A^{k-1}\delta_n$, this is bounded by $4(d/\delta_n)^\alpha$. Finally, if $d > \delta_0$, then by (13.11)

$$\frac{f_n(y)}{f_n(x)} \leq 2^n \leq \left(\frac{d}{\delta_n}\right)^\alpha.$$

□

Including (13.2) as one of the assumptions of Proposition 13.1 allows us to formulate Proposition 13.1 as a single inductive step, instead of keeping track of the entire history of earlier scales.

## 14. Construction of the measure $\mu_\Gamma$

We apply Proposition 13.1 successively to the spikes with translation lengths near $n\tau'$, starting with the constant function 1. Proposition 10.1 ensures that most points are covered at many scales, so the remainders tend to zero exponentially in $L^1$. The resulting coefficients satisfy the stationarity equation (2.10) and have an exponential Teichmüller moment.

Let $\beta$, $W = w/\beta$ and $\Xi = 2\eta/\beta$ be as in §12, with the initial choice (11.1), which already ensures $\Xi < W$. Let $\rho_1$, $\tau$, $\tau_0'$ and $\mathsf{B}$ be as in Proposition 7.4 and Proposition 10.1.

Set $\mathsf{W}_0 = e^{\beta(\rho_1+2)}$, and choose $\mathsf{m}$ and $\epsilon_0$ as in Proposition 13.1, decreasing $\epsilon_0$ to at most $1/2$ if necessary. The uniform spike estimates, $\mathsf{W}_0$, and $\mathsf{B}$ are independent of $\tau'$, so $\mathsf{m}$ and $\epsilon_0$ are now fixed. Choose $\tau' \in \tau\mathbb{N}$ with

$$\tau' > \max\left\{\tau_0',\ \rho_1 + 2 + \frac{\log(4\mathsf{m})}{\beta},\ \mathsf{R}_0 + 1,\ \mathsf{R}_1 + 1\right\}.$$

Here $\mathsf{R}_0$ and $\mathsf{R}_1$ are the thresholds in Theorem 12.1. The second bound will separate successive spike radii enough to apply Proposition 13.1.

For $j \geq 0$, let

$$R_j = j\tau'.$$

Choose $n_0 \geq 1$ so that Proposition 7.4 and the bounded-overlap conclusion of Proposition 10.1(a) apply for every $n \geq n_0$. Set $\Gamma_\alpha(R_n) = \varnothing$ for every $\alpha$ when $n < n_0$. For $n \geq 1$, let $\Gamma_n \subset \mathrm{Mod}$ be the finite set

$$\Gamma_n = \Gamma(R_n) = \bigcup_{\alpha\in\mathcal{A}} \Gamma_\alpha(R_n),$$

using the families from Proposition 7.4 when $n \geq n_0$. Their translation-length windows are disjoint by part (b) and $\tau' > \rho_1 + 2$, so the $\Gamma_n$ are pairwise disjoint. For $\alpha \in \mathcal{A}$, define

$$Y_n^\alpha = U_\alpha \cap \bigcup_{\gamma\in\Gamma_\alpha(R_n)} B_G(\gamma), \qquad \text{and set} \qquad Y_n = \bigcup_{\alpha\in\mathcal{A}} Y_n^\alpha.$$

By Proposition 10.1(a), the balls $\{B_G(\gamma) : \gamma \in \Gamma_n\}$ cover each point at most $\mathsf{B}$ times.

Let $\mathcal{Z}_n$ denote the collection of spikes

$$\mathcal{Z}_n = \{(H(\gamma,\cdot), [F_+(\gamma)], e^{-\beta\ell(\gamma)}) \ : \ \gamma \in \Gamma(R_n)\}.$$

Theorem 12.1 and the choice of $\tau'$ give uniform spike estimates for these families. Set

$$\delta_n = \tfrac{1}{4}e^{-\beta(R_n+\rho_1+1)} \qquad (n \geq 0), \qquad \sigma = (W - \Xi)/2.$$

We construct positive functions $f_n : \mathcal{PMF} \to \mathbb{R}$ inductively, with $2^{-n} \le f_n \le 1$ and satisfying (13.1) and (13.2) at scale $\delta_n$. These bounds supply the boundedness and positive lower bound required by Proposition 13.1; its constants do not depend on this lower bound. Start with

$$f_0(x) = 1,$$

which satisfies all these conditions at scale $\delta_0$.

Suppose $f_{n-1}$ has been constructed. If $\Gamma_n = \varnothing$, set $g_n = 0$ and $f_n = f_{n-1}$. Otherwise, the radii in $\mathcal{Z}_n$ are $e^{-\beta\ell(\gamma)}$, so Proposition 7.4(b) gives

$$4\delta_n = e^{-\beta(R_n+\rho_1+1)} \le r_{\min,n} \le r_{\max,n} \le e^{-\beta(R_n-1)}. \tag{14.1}$$

In particular, $r_{\max,n}/r_{\min,n} \le \mathsf{W}_0$. By the choice of $\tau'$,

$$r_{\max,n} \le e^{-\beta(R_n-1)} = 4\delta_{n-1}e^{-\beta(\tau'-(\rho_1+2))} \le \frac{\delta_{n-1}}{\mathsf{m}}.$$

We can therefore apply Proposition 13.1 with $Y = Y_n$, $f = f_{n-1}$, $\delta = \delta_{n-1}$ and the collection $\mathcal{Z}_n$. Denote its nonnegative coefficients by $\mu_\Gamma(\gamma)$, $\gamma \in \Gamma_n$, and put

$$g_n(x) = \sum_{\gamma\in\Gamma_n} \mu_\Gamma(\gamma) H_\rho(\gamma, x).$$

Let $f_n = f_{n-1} - g_n$ be the remainder supplied by the proposition. Since the $\Gamma_n$ are disjoint, the coefficients chosen at different steps do not conflict.

The two comparison estimates hold at scale $\delta_n' = r_{\min,n}/4 \ge \delta_n$, by (14.1). They persist at the smaller scale $\delta_n$:

$$D_{\delta_n} f_n(x) \le D_{\delta_n'} f_n(x) \le \frac{f_n(x)}{\delta_n'} \le \frac{f_n(x)}{\delta_n},$$

and, for $x, y \in \mathcal{PMF}$,

$$\frac{f_n(y)}{f_n(x)} \le 4\left(1 + \frac{d_G(x,y)}{\delta_n'}\right)^\sigma \le 4\left(1 + \frac{d_G(x,y)}{\delta_n}\right)^\sigma.$$

If $\Gamma_n$ is empty, the same comparisons with $\delta_{n-1}$ in place of $\delta_n'$ show that $f_n = f_{n-1}$ satisfies both estimates. In either case,

$$\frac{f_{n-1}(x)}{2} \le f_n(x) \le f_{n-1}(x),$$

which completes the induction and gives, for all $n \ge 0$,

$$2^{-n} \le f_n(x) \le 1. \tag{14.2}$$

Also, by Proposition 13.1, for $x \in Y_n$,

$$f_n(x) \le (1-\epsilon_0) f_{n-1}(x). \tag{14.3}$$

Each time $x$ belongs to $Y_j$, the remainder is multiplied by a factor at most $1-\epsilon_0$. Thus we count these visits by setting

$$k_n(x) = \#\{j \in [1,n] : x \in Y_j\}.$$

It follows from (14.3) that

$$f_n(x) \le (1-\epsilon_0)^{k_n(x)}. \tag{14.4}$$

At points with $k_n(x) > n/2$, the remainder is exponentially small. We bound the measure of the complement using Proposition 10.1(b). For a fixed chart $\alpha \in \mathcal{A}$, let

$$k_n^\alpha(x) = \# \left\{ j \in [1,n] : x \in \bigcup_{\gamma \in \Gamma_\alpha(R_j)} B_G(\gamma) \right\}.$$

The initial cutoff discards at most $n_0$ coverage times. Since $\eta_1 < 1/2$, we have $n/2 + n_0 \le (1-\eta_1)n$ for all sufficiently large $n$. Proposition 10.1(b), with its threshold $(1-\eta_1)n$, therefore gives

$$\nu_\rho\big(\{x \in U_\alpha : k_n^\alpha(x) \le n/2\}\big) \le \kappa^n,$$

where $\kappa = \kappa(\eta_1) < 1$ is the constant in that proposition. Since the sets $U_\alpha$ cover $\mathcal{PMF}$, if $k_n(x) \le n/2$, then for every chart $\alpha$ with $x \in U_\alpha$ we have $k_n^\alpha(x) \le n/2$. Hence

$$\{x \in \mathcal{PMF} : k_n(x) \le n/2\} \subset \bigcup_{\alpha \in \mathcal{A}} \{x \in U_\alpha : k_n^\alpha(x) \le n/2\}.$$

Therefore

$$\nu_\rho\big(\{x \in \mathcal{PMF} : k_n(x) \le n/2\}\big) \le |\mathcal{A}|\kappa^n. \tag{14.5}$$

Split the integral according to whether $k_n(x) > n/2$. Using (14.4) on this set and $f_n \le 1$ on its complement, (14.5) gives, for sufficiently large $n$,

$$\|f_n\|_{L^1(\nu_\rho)} \le (1-\epsilon_0)^{n/2} + |\mathcal{A}|\kappa^n.$$

Set $\kappa_1 = \max\{\sqrt{1-\epsilon_0}, \kappa\} < 1$. Choosing $C_1 \ge 1 + |\mathcal{A}|$ large enough to cover the finitely many initial indices gives, for all $n \ge 0$,

$$\|f_n\|_{L^1(\nu_\rho)} \le C_1 \kappa_1^n. \tag{14.6}$$

*Proof of Theorem 1.1.* By construction, $f_n = f_0 - \sum_{j=1}^n g_j$. Since $f_0 \equiv 1$ and $f_n \to 0$ in $L^1(\nu_\rho)$ by (14.6), we have

$$\sum_{n=1}^\infty g_n(x) = f_0(x) = 1$$

in $L^1(\nu_\rho)$. Setting $\mu_\Gamma(\gamma) = 0$ for $\gamma \notin \bigcup_{n=1}^\infty \Gamma_n$, we obtain

$$\sum_{\gamma \in \mathrm{Mod}(S)} \mu_\Gamma(\gamma) H_\rho(\gamma, x) = 1$$

in $L^1(\nu_\rho)$. The coefficients are nonnegative, so Tonelli's theorem and the normalization

$$\int_{\mathcal{PMF}} H_\rho(\gamma, x)\, d\nu_\rho(x) = 1$$

give

$$\sum_{\gamma \in \mathrm{Mod}(S)} \mu_\Gamma(\gamma) = 1.$$

Thus $\mu_\Gamma$ is a probability measure. Since

$$d(\gamma_* \nu_\rho)(x) = H_\rho(\gamma, x)\, d\nu_\rho(x),$$

the preceding identity is exactly the stationarity equation

$$\nu_\rho = \mu_\Gamma * \nu_\rho.$$

We now prove the exponential moment estimate. Since $0 \le g_n \le f_{n-1}$, the normalization of $H_\rho$ and (14.6) give

$$\mu_\Gamma(\Gamma_n) = \|g_n\|_{L^1(\nu_\rho)} \le \|f_{n-1}\|_{L^1(\nu_\rho)} \le C_1 \kappa_1^{n-1}.$$

By Proposition 7.4(a),(b),

$$d_{\mathcal{T}}(P, \gamma P) \le n\tau' + C_0 \quad (\gamma \in \Gamma_n), \qquad C_0 = 2\mathsf{D}_0 + \rho_1 + 1.$$

Thus, for every $0 < a < -\log \kappa_1/\tau'$,

$$\begin{aligned}\sum_{\gamma \in \mathrm{Mod}(S)} \mu_\Gamma(\gamma) e^{a d_{\mathcal{T}}(P,\gamma P)} &\le C_1 e^{aC_0} \sum_{n \ge 1} e^{an\tau'} \kappa_1^{n-1} \\ &= \frac{C_1 e^{a(C_0+\tau')}}{1 - \kappa_1 e^{a\tau'}} < \infty.\end{aligned}$$

The same exponent works at every base point $P'$, since $d_{\mathcal{T}}(P', \gamma P') \le d_{\mathcal{T}}(P, \gamma P) + 2d_{\mathcal{T}}(P, P')$.

Finally, the semigroup generated by $\mathrm{supp}(\mu_\Gamma)$ is non-elementary. It is enough to show that $\mathrm{supp}(\mu_\Gamma)$ has two independent pseudo-Anosov elements. We have proved equality with 1 almost everywhere. The pointwise identities $\sum_{n=1}^{N} g_n(x) = 1 - f_N(x) \le 1$ also imply $\sum_\gamma \mu_\Gamma(\gamma) H_\rho(\gamma, x) \le 1$ at every $x$. If all positive-weight pseudo-Anosovs had the same fixed pair $a_+, a_-$, then (2.9) would give, for each such $\gamma$,

$$H_\rho(\gamma, a_+) + H_\rho(\gamma, a_-) = e^{w\ell(\gamma)} + e^{-w\ell(\gamma)} > 2.$$

Summing with weights $\mu_\Gamma(\gamma)$ contradicts the preceding bounds at $a_+$ and $a_-$ and $\sum_\gamma \mu_\Gamma(\gamma) = 1$. The fixed pairs of two pseudo-Anosovs are either equal or disjoint, so failure of a common pair supplies two independent elements already in the support.

Kaimanovich–Masur's theorem [KM94, Theorem 2.2.4] now gives uniqueness of the non-atomic stationary measure. The extension to punctured surfaces is recalled in [Gad14, Remark 3.3]. This completes the proof of Theorem 1.1. □

## Appendix A. Large deviation for Horospheres

For $q \in \mathcal{PQ}_1\mathcal{T}$, define

$$V(q) = \max\left(1, \frac{1}{\mathrm{sys}(q)}\right).$$

Along a Teichmüller geodesic, $V(q)$ changes at most exponentially fast. That is, for $t > 0$, we always have

$$V(q)\ e^{-t} \le V(q(t)) \le V(q)\ e^{t}.$$

Let $\mu_q$ be the Avila–Gouëzel horospherical measure fixed in Section 9.1, with density (9.11), and let $B(q, r) \subset \mathrm{Hor}(q)$ be the ball of radius $r$ in the intrinsic $d_{\mathrm{AGY}}$-metric around $q$.

We prove that most initial points in a chart slice have many sampled times at which the geodesic has good backward statistics and visits a prescribed neighborhood of large measure. Horospherical estimates for $V$ control excursions into the thin

part. Tracking transfers these estimates from random walks to geodesics in Proposition A.18. Mixing supplies the returns in Corollary A.28, and the two estimates give Proposition A.30.

We first recall a few statements from [AG13].

**Proposition A.1** (Proposition 6.1 in [AG13])**.** *For $\alpha \in (0, 1/4)$, there exists $C_\alpha > 0$ such that, for any $q \in \mathcal{PQ}_1\mathcal{T}$ and any $t \geq 0$,*

$$\frac{1}{\mu_q(B(q, \frac{1}{100}))} \int_{B(q,\frac{1}{100})} V(y(t))\, d\mu_q(y) \leq \tfrac{1}{2} C_\alpha e^{-(1-2\alpha)t} \cdot V(q) + \tfrac{1}{2} C_\alpha.$$

*Moreover, the function $\log V$ is 1-Lipschitz for the $d_{\mathrm{AGY}}$–metric.*

For $\tau > 0$ choose $\alpha = \alpha(\tau) \in (0, 1/4)$ with $\alpha(\tau) \to 0$ and $\log C_{\alpha(\tau)}/\tau \to 0$ as $\tau \to \infty$.

Let $V_0 = e^{(1-2\alpha)\tau}$. Then, Proposition A.1 implies

$$\frac{1}{\mu_q(B(q, \frac{1}{100}))} \int_{B(q,\frac{1}{100})} V(y(\tau))\, d\mu_q(y) \leq \begin{cases} C_\alpha e^{-(1-2\alpha)\tau} \cdot V(q) & \text{if} \quad V(q) \geq V_0 \\ C_\alpha & \text{if} \quad V(q) \leq V_0 \end{cases} \tag{A.1}$$

We recall the following Lemma (slightly restated) from [AG13].

**Lemma A.2** ([AG13], Corollary 5.4)**.** *There is a constant $0 < c_D < 1$ such that, for every $q \in \mathcal{PQ}_1\mathcal{T}$ and for $0 \leq 2r \leq \frac{1}{100}$, if $p \in \mathrm{Hor}(q)$ and $d_{\mathrm{AGY}}(p, q) \leq r$ then*

$$\mu_q(B(p, r)) \geq c_D \cdot \mu_q(B(q, 2r)).$$

A.1. **A random walk model.** We now define a random walk as follows. Let $\tau > 0$ be a large number and $p_0 \in \mathcal{PQ}_1\mathcal{T}$. For $n \geq 1$, define

$$\Omega_{n,p_0} = \Big\{(q_0, q_1, \ldots, q_n) : q_0 \in B(p_0, \tfrac{1}{100}),\ q_{j+1} \in g_\tau B(q_j, \tfrac{1}{100}),\ 0 \leq j \leq n-1\Big\}.$$

We equip $\Omega_{n,p_0}$ with a Markov measure $\mathbb{P}_{n,p_0}$ as follows. For $0 \leq j \leq n$, let $\mu_j$ be shorthand for $\mu_q$, where $q$ is any point of $\mathrm{Hor}(p_0(j\tau))$. This is well-defined because the conditional measure $\mu_q$ depends only on the horosphere containing $q$. In particular, $\mu_0 = \mu_{p_0}$.

By (9.14), these measures satisfy, for every measurable $E \subset \mathrm{Hor}(p_0(j\tau))$,

$$\mu_{j+1}(g_\tau E) = e^{w\tau} \mu_j(E).$$

Then, define $\mathbb{P}_{n,p_0}$ to be the probability measure on $\Omega_{n,p_0}$ as follows. The law of $q_0$ is the normalized restriction $\mu_0|_{B(p_0,\frac{1}{100})}/\mu_0(B(p_0, \frac{1}{100}))$. For $0 \leq j \leq n-1$, conditionally on $q_j$, the law of $q_{j+1}$ is

$$(g_\tau)_* \left( \frac{\mu_j|_{B(q_j,\frac{1}{100})}}{\mu_j(B(q_j, \frac{1}{100}))} \right).$$

Equivalently, one may temporarily choose $q_j' \in B(q_j, \frac{1}{100})$ according to the normalized restriction of $\mu_j$ and then set $q_{j+1} = q_j'(\tau)$.

Define

$$f_0 \colon \mathcal{PQ}_1\mathcal{T} \to \mathbb{R}_+ \qquad \text{by} \qquad f_0(q) = \begin{cases} 1 & V(q) > 3V_0 e^\tau. \\ 0 & V(q) \leq 3V_0 e^\tau. \end{cases}$$

and

$$f_1\colon \mathcal{PQ}_1\mathcal{T} \to \mathbb{R}_+ \qquad \text{by} \qquad f_1(q) = \begin{cases} 1 & V(q) > 9V_0e^{3\tau}. \\ 0 & V(q) \le 9V_0e^{3\tau}. \end{cases}$$

For $\omega \in \Omega_{n,p_0}$ and $0 \le j \le n$, define

$$\widetilde{V}(\omega, j) = V(q_j) \cdot \exp\left(\sum_{i=0}^{j-1} \int_0^\tau f_0(q_i(t))\, dt\right),$$

where the sum is empty when $j = 0$.

**Proposition A.3.** *For all sufficiently large $\tau > 0$ there exists $\alpha_1 = \alpha_1(\tau) > 0$ with $\alpha_1 \to 0$ as $\tau \to \infty$ such that, for $n \ge 1$ and $p_0 \in \mathcal{PQ}_1\mathcal{T}$, we have*

$$\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, n)) \le e^{\alpha_1 \tau\, n} \cdot V(p_0).$$

*Proof.* Let $\mathcal{F}_j$ be the $\sigma$-algebra generated by $q_0, \dots, q_j$. We claim, for $0 \le j < n$,

$$\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, j+1)|\mathcal{F}_j) \le C_\alpha \cdot e^{2\alpha\tau} \cdot \widetilde{V}(\omega, j). \tag{A.2}$$

Note that

$$\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, j+1) \mid \mathcal{F}_j) = \frac{\exp\left(\sum_{i=0}^{j} \int_0^\tau f_0(q_i(t))\, dt\right)}{\mu_j(B(q_j, \frac{1}{100}))} \int_{B(q_j, \frac{1}{100})} V(q'(\tau))\, d\mu_j(q').$$

Dividing both sides by $\widetilde{V}(\omega, j)$ we get

$$\frac{\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, j+1) \mid \mathcal{F}_j)}{\widetilde{V}(\omega, j)} = \frac{\exp\left(\int_0^\tau f_0(q_j(t))\, dt\right)}{V(q_j) \cdot \mu_j(B(q_j, \frac{1}{100}))} \int_{B(q_j, \frac{1}{100})} V(q'(\tau))\, d\mu_j(q'). \tag{A.3}$$

Since $f_0 \le 1$,

$$\exp\left(\int_0^\tau f_0(q_j(t))\, dt\right) \le e^\tau. \tag{A.4}$$

We consider two cases. First assume $V(q_j) \ge V_0$. Then (A.1), (A.3) and (A.4) imply

$$\frac{\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, j+1)|\mathcal{F}_j)}{\widetilde{V}(\omega, j)} \le e^\tau \cdot C_\alpha e^{-(1-2\alpha)\tau} = C_\alpha e^{2\alpha\tau}.$$

If $V(q_j) \le V_0$, then for $0 \le t \le \tau$ we have $V(q_j(t)) \le e^\tau V_0 < 3e^\tau V_0$, so $f_0(q_j(t)) = 0$. Thus

$$\exp\left(\int_0^\tau f_0(q_j(t))\, dt\right) = 1.$$

Then using (A.1), (A.3) and the fact that $V(q_j) \ge 1$, we have

$$\frac{\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, j+1)|\mathcal{F}_j)}{\widetilde{V}(\omega, j)} \le C_\alpha.$$

This finishes the proof of (A.2). Since $q_0 \in B(p_0, \frac{1}{100})$ and $\log V$ is 1–Lipschitz, we also have

$$\mathbb{E}_{n,p_0}(\widetilde{V}(\omega, 0)) = \mathbb{E}_{n,p_0}(V(q_0)) \le e^{\frac{1}{100}} V(p_0).$$

Iterating (A.2) and using the tower property gives

$$\mathbb{E}_{n,p_0}(\widetilde{V}(\omega,n)) \le C_\alpha^n e^{2\alpha n\tau}\mathbb{E}_{n,p_0}(V(q_0)) \le e^{\frac{1}{100}} C_\alpha^n e^{2\alpha n\tau} V(p_0).$$

We now choose $\alpha_1$ such that

$$e^{\frac{1}{100}} C_\alpha^n e^{2\alpha n\tau} \le e^{\alpha_1 n\tau}$$

and $\alpha_1 \to 0$ as $\tau \to \infty$. (After enlarging $C_\alpha$ so that $C_\alpha \ge 1$, one may take $\alpha_1(\tau) = 2\alpha(\tau) + \big(\log C_{\alpha(\tau)} + \frac{1}{100}\big)/\tau$.) This finishes the proof. □

For every quadratic differential $p \in B(p_0, \frac{1}{200})$, define

$$\widetilde{N}_{\frac{1}{200},p_0}(p) = \Big\{\omega \in \Omega_{n,p_0} : d_{\mathrm{AGY}}(p(j\tau), q_j) < \tfrac{1}{200}, \quad 0 \le j \le n\Big\}.$$

That is, $\widetilde{N}_{\frac{1}{200},p_0}(p)$ is the set of random walks that start near $p_0$ and fellow travel the Teichmüller geodesic defined by $p$.

Recall that $c_D$ is defined in Lemma A.2 (and is independent of $\tau$ or $p_0$).

**Lemma A.4.** *For any $p_0 \in \mathcal{PQ}_1\mathcal{T}$ and $p \in B(p_0, \frac{1}{200})$, we have*

$$\mathbb{P}_{n,p_0}(\widetilde{N}_{\frac{1}{200},p_0}(p)) \cdot \mu_0(B(p_0, \tfrac{1}{100})) \ge c_D^n e^{-wn\tau} \cdot \mu_n(B(p(n\tau), \tfrac{1}{200})).$$

*Proof.* Define

$$A_j = \mu_j(B(p(j\tau), \tfrac{1}{200})).$$

Let $T_j(p)$ be the event that $q_i \in B(p(i\tau), \frac{1}{200})$ for $0 \le i \le j$. On $T_j(p)$, backward nonexpansion gives

$$g_{-\tau}B(p((j+1)\tau), \tfrac{1}{200}) \subset B(q_j, \tfrac{1}{100}).$$

Thus on $T_j(p)$ we have

$$\begin{aligned}\mathbb{P}_{n,p_0}\Big(q_{j+1} \in B(p((j+1)\tau), \tfrac{1}{200}) \mid \mathcal{F}_j\Big) &= \frac{\mu_j(g_{-\tau}B(p((j+1)\tau), \frac{1}{200}))}{\mu_j(B(q_j, \frac{1}{100}))} \\ &= \frac{e^{-w\tau}A_{j+1}}{\mu_j(B(q_j, \frac{1}{100}))} \ge c_D \cdot e^{-w\tau} \cdot \frac{A_{j+1}}{A_j}.\end{aligned}$$

where $c_D$ is as in Lemma A.2. Note that $T_n(p) = \widetilde{N}_{\frac{1}{200},p_0}(p)$. Since

$$\mathbb{P}_{n,p_0}(T_{j+1}(p)) = \mathbb{E}_{n,p_0}[\mathbf{1}_{T_j(p)}\mathbb{P}_{n,p_0}(q_{j+1} \in B(p((j+1)\tau), \tfrac{1}{200}) \mid \mathcal{F}_j)],$$

we have

$$\mathbb{P}(T_{j+1}(p)) \ge c_D \cdot e^{-w\tau} \cdot \frac{A_{j+1}}{A_j}\mathbb{P}(T_j(p)).$$

Thus,

$$\begin{aligned}\mathbb{P}_{n,p_0}(\widetilde{N}_{\frac{1}{200},p_0}(p)) &\ge \mathbb{P}_{n,p_0}\left(T_0(p)\right)\prod_{j=0}^{n-1} c_D \cdot e^{-w\tau}\frac{A_{j+1}}{A_j} \\ &\ge \frac{A_0}{\mu_0(B(p_0, \frac{1}{100}))}\prod_{j=0}^{n-1} c_D \cdot e^{-w\tau}\frac{A_{j+1}}{A_j} = c_D^n \cdot e^{-wn\tau}\frac{A_n}{\mu_0(B(p_0, \frac{1}{100}))}.\end{aligned}$$

This finishes the proof. □

For $p \in \mathrm{Hor}(p_0)$, let

$$\widetilde{V}_1(p,n) = V(p(n\tau)) \exp\left(\int_0^{n\tau} f_1(p(t))\,dt\right). \tag{A.5}$$

**Lemma A.5.** *For any $p \in B(p_0, \frac{1}{200})$, let $E_{p,n} = g_{-n\tau}(B(p(n\tau), \frac{1}{100}))$. Then,*

$$\frac{1}{\mu_0(E_{p,n})} \int_{E_{p,n}} \widetilde{V}_1(q,n)\,d\mu_0(q) \le 3\,\frac{1}{\mathbb{P}_{n,p_0}(\widetilde{N}_{\frac{1}{200},p_0}(p))} \int_{\widetilde{N}_{\frac{1}{200},p_0}(p)} \widetilde{V}(\omega,n)\,d\mathbb{P}_{n,p_0}(\omega).$$

*Proof.* Fix $q \in E_{p,n}$ and $\omega \in \widetilde{N}_{\frac{1}{200},p_0}(p)$. Since $d_{\mathrm{AGY}}(q(n\tau), p(n\tau)) \le \frac{1}{100}$, the monotonicity (4.8) of the AGY metric along a horosphere gives, for $0 \le j \le n$,

$$d_{\mathrm{AGY}}(q(j\tau), p(j\tau)) \le \tfrac{1}{100}.$$

Consequently,

$$d_{\mathrm{AGY}}(q(j\tau), q_j) \le \tfrac{1}{100} + \tfrac{1}{200} = \frac{3}{200}.$$

Let $t \in [j\tau, (j+1)\tau]$. If $f_1(q(t)) = 1$, then

$$V(q(t)) \ge 9V_0 e^{3\tau}.$$

Using the exponential change of $V$ along the flow, the 1–Lipschitz property of $\log V$, and $e^{-3/200} > 1/3$, we obtain

$$\begin{aligned} V(q(j\tau)) &\ge e^{-\tau} V(q(t)) \ge 9V_0 e^{2\tau}, \\ V(q_j) &\ge e^{-3/200} V(q(j\tau)) \ge 3V_0 e^{2\tau}, \\ V(q_j(t-j\tau)) &\ge e^{-\tau} V(q_j) \ge 3V_0 e^{\tau}. \end{aligned}$$

Thus $f_0(q_j(t-j\tau)) = 1$, and hence

$$\int_0^{n\tau} f_1(q(t))\,dt \le \sum_{j=0}^{n-1} \int_0^{\tau} f_0(q_j(s))\,ds.$$

At time $n\tau$, the triangle inequality and the 1–Lipschitz property of $\log V$ also give

$$V(q(n\tau)) \le e^{3/200} V(q_n) < 3V(q_n).$$

It follows that $\widetilde{V}_1(q,n) \le 3\widetilde{V}(\omega,n)$. Since this holds for every pair $(q,\omega) \in E_{p,n} \times \widetilde{N}_{\frac{1}{200},p_0}(p)$, averaging over the two factors proves the claim. □

**Lemma A.6.** *There exists $\alpha_2 = \alpha_2(\tau) > 0$ with $\alpha_2 \to 0$ as $\tau \to \infty$ such that*

$$\frac{1}{\mu_0(B(p_0, \frac{1}{200}))} \int_{B(p_0, \frac{1}{200})} \widetilde{V}_1(q,n)\,d\mu_0(q) \le e^{\alpha_2 n\tau} \cdot V(p_0).$$

*Proof.* Consider the set $Z = g_{n\tau} B(p_0, \frac{1}{200}) \subset \mathrm{Hor}(p_0(n\tau))$. Let $X'$ be a maximal $\frac{1}{100}$-separated subset of $Z$ (in the AGY metric), and let $X = g_{-n\tau}X'$. Then,

- if $q \in B(p_0, \frac{1}{200})$ then $q \in E_{p,n}$ for some $p \in X$ where $E_{p,n}$ is as in Lemma A.5.
- for distinct $p, p' \in X$, $B(p(n\tau), \frac{1}{200}) \cap B(p'(n\tau), \frac{1}{200}) = \emptyset$.

Then,

$$\int_{B(p_0,\frac{1}{200})} \widetilde{V}_1(q,n)\,d\mu_0(q) \le \sum_{p\in X} \int_{E_{p,n}} \widetilde{V}_1(q,n)\,d\mu_0(q), \tag{A.6}$$

By Lemma A.5, for every $p \in X$,

$$\int_{E_{p,n}} \widetilde{V}_1(q,n)\,d\mu_0(q) \le 3\,\frac{\mu_0(E_{p,n})}{\mathbb{P}_{n,p_0}(\widetilde{N}_{\frac{1}{200},p_0}(p))} \int_{\widetilde{N}_{\frac{1}{200},p_0}(p)} \widetilde{V}(\omega,n)\,d\mathbb{P}_{n,p_0}(\omega).$$

Note that by Lemma A.2,

$$\mu_0(E_{p,n}) = e^{-wn\tau}\mu_n(B(p(n\tau),\tfrac{1}{100})) \le c_D^{-1}e^{-wn\tau}\mu_n(B(p(n\tau),\tfrac{1}{200})).$$

Then, by Lemma A.4 we get

$$\frac{\mu_0(E_{p,n})}{\mathbb{P}_{n,p_0}(\widetilde{N}_{\frac{1}{200},p_0}(p))} \le c_D^{-(n+2)}\mu_0(B(p_0,\tfrac{1}{200})).$$

Thus, for every $p \in X$,

$$\frac{1}{\mu_0(B(p_0,\frac{1}{200}))}\int_{E_{p,n}} \widetilde{V}_1(q,n)\,d\mu_0(q) \le 3\,c_D^{-(n+2)} \int_{\widetilde{N}_{\frac{1}{200},p_0}(p)} \widetilde{V}(\omega,n)\,d\mathbb{P}_{n,p_0}(\omega).$$

A walk tracking $p$ ends in $B(p(n\tau),\frac{1}{200})$, so the tracking events for distinct $p \in X$ are disjoint. Substituting into (A.6), we get

$$\frac{1}{\mu_0(B(p_0,\frac{1}{200}))}\int_{B(p_0,\frac{1}{200})} \widetilde{V}_1(q,n)\,d\mu_0(q) \le 3\,c_D^{-(n+2)}\mathbb{E}_{n,p_0}(\widetilde{V}(\omega,n)).$$

Now, the statement follows from Proposition A.3. (For example, one may take $\alpha_2 = \alpha_1 + (\log 3 + 3\log(c_D^{-1}))/\tau$). □

We record whether some interval $[j\tau, n\tau]$, with $0 \le j < n$ an integer, spends more than a proportion $\rho$ of its time in $\{f_1 = 1\}$. For $0 < \rho < 1$, $q \in \mathcal{PQ}_1\mathcal{T}$, and $n \in \mathbb{N}$, let

$$H_\rho(q,n) = \max_{0\le j\le n}\left\{\int_{j\tau}^{n\tau} f_1(q(t))\,dt - \rho(n-j)\tau\right\}.$$

Let

$$Y_\rho(q,n) = \begin{cases} 1 & \text{if } H_\rho(q,n) > 0 \\ 0 & \text{otherwise.}\end{cases}$$

**Proposition A.7.** *For every $0 < \rho < 1$ there exists $\tau_0 = \tau_0(\rho)$ and $\epsilon_\rho = \epsilon_\rho(\tau) \to 0$ as $\tau \to \infty$ such that the following holds. For every $\tau > \tau_0$, every $p_0 \in \mathcal{PQ}_1\mathcal{T}$ and every $n \ge 1$,*

$$\frac{1}{\mu_{p_0}(B(p_0,\frac{1}{200}))}\int_{B(p_0,\frac{1}{200})} V(q(n\tau))Y_\rho(q,n)\,d\mu_{p_0}(q) \le e^{-(\rho-\epsilon_\rho(\tau))\tau}\cdot V(p_0).$$

*Proof.* Put $r = \frac{1}{200}$ and $\gamma = 1 - 2\alpha$, and write $\nu_t$ for the conditional measure on the horosphere containing $p_0(t)$. We first claim that there is a uniform constant $C_* > 0$ such that, for $1 \le \ell \le n$,

$$\frac{1}{\nu_0(B(p_0,r))} \int_{B(p_0,r)} V(q(n\tau)) \exp\left( \int_{(n-\ell)\tau}^{n\tau} f_1(q(t))\, dt \right) d\nu_0(q)$$
$$\le C_* C_\alpha e^{\alpha_2 \ell \tau} \left( e^{-\gamma(n-\ell)\tau} V(p_0) + 1 \right). \tag{A.7}$$

Set $s = (n-\ell)\tau$ and $E = g_s B(p_0, r)$. After changing variables by $g_s$, the left hand side of (A.7) is

$$\frac{1}{\nu_s(E)} \int_E F_\ell(z)\, d\nu_s(z), \qquad F_\ell(z) = V(z(\ell\tau)) \exp\left( \int_0^{\ell\tau} f_1(z(t))\, dt \right).$$

Choose a maximal $r$–separated set $\{x_i\} \subset E$. The balls $B(x_i, r)$ cover $E$, while the balls $B(x_i, r/2)$ are pairwise disjoint. Lemma A.6, applied with base point $x_i$, gives

$$\int_{B(x_i,r)} F_\ell(z)\, d\nu_s(z) \le e^{\alpha_2 \ell \tau} V(x_i) \nu_s(B(x_i, r)).$$

By Lemma A.2 and the 1–Lipschitz property of $\log V$,

$$V(x_i)\nu_s(B(x_i, r)) \le C_* \int_{B(x_i, r/2)} V(z)\, d\nu_s(z).$$

Consequently,

$$\int_E F_\ell(z)\, d\nu_s(z) \le C_* e^{\alpha_2 \ell \tau} \int_{E^+} V(z)\, d\nu_s(z), \qquad E^+ = \bigcup_i B(x_i, r/2).$$

If $z \in E^+$, then the monotonicity of the AGY metric along a horosphere gives

$$z(-s) \in B(p_0, 3r/2) \subset B(p_0, 2r) = B(p_0, \tfrac{1}{100}).$$

It follows from the horospherical volume-scaling formula and Proposition A.1 that

$$\int_{E^+} V(z)\, d\nu_s(z) \le e^{ws} \int_{B(p_0, \frac{1}{100})} V(q(s))\, d\nu_0(q)$$
$$\le \tfrac{1}{2} C_\alpha e^{ws} \nu_0(B(p_0, \tfrac{1}{100})) \left( e^{-\gamma s} V(p_0) + 1 \right).$$

Since

$$\nu_s(E) = e^{ws} \nu_0(B(p_0, r))$$

and $\nu_0(B(p_0, \frac{1}{100}))/\nu_0(B(p_0, r))$ is uniformly bounded by Lemma A.2, this proves (A.7).

If $Y_\rho(q, n) = 1$, then for some $1 \le \ell \le n$,

$$\int_{(n-\ell)\tau}^{n\tau} f_1(q(t))\, dt > \rho \ell \tau.$$

Therefore,

$$Y_\rho(q, n) \le \sum_{\ell=1}^{n} e^{-\rho\ell\tau} \exp\left( \int_{(n-\ell)\tau}^{n\tau} f_1(q(t))\, dt \right).$$

Multiplying by $V(q(n\tau))$, integrating, and applying (A.7), we obtain

$$\begin{aligned}\frac{1}{\nu_0(B(p_0,r))}\int_{B(p_0,r)} V(q(n\tau))Y_\rho(q,n)\,d\nu_0(q)\\ &\le C_*C_\alpha\sum_{\ell=1}^{n} e^{-(\rho-\alpha_2)\ell\tau}\left(e^{-\gamma(n-\ell)\tau}V(p_0)+1\right)\\ &\le \frac{2C_*C_\alpha}{1-e^{-(\rho-\alpha_2)\tau}}e^{-(\rho-\alpha_2)\tau}V(p_0),\end{aligned}$$

where in the last inequality we used $V(p_0)\ge 1$ and took $\tau$ large enough that $\alpha_2(\tau)<\rho$.

Define

$$\epsilon_\rho(\tau)=\alpha_2(\tau)+\frac{1}{\tau}\log\left(\frac{2C_*C_{\alpha(\tau)}}{1-e^{-(\rho-\alpha_2(\tau))\tau}}\right).$$

Then the preceding upper bound is

$$e^{-(\rho-\epsilon_\rho(\tau))\tau}V(p_0).$$

Finally, $\alpha_2(\tau)\to 0$, $\log C_{\alpha(\tau)}/\tau\to 0$, and $e^{-(\rho-\alpha_2(\tau))\tau}\to 0$. Hence $\epsilon_\rho(\tau)\to 0$, as required. □

A.2. **An alternative random walk model.** The next walk has prescribed distributions on the horospheres, so probability bounds will give horospherical measure bounds. Its transition densities are Lipschitz, allowing us to apply mixing.

In this section, we let $\tau'=m\tau$, where $\tau$ is as in §A.1 and $m\in\mathbb{N}$ is a large parameter. Let $T=T(K_0)$ be the constant in Corollary 4.10, and fix once and for all a constant

$$L>800T(w+1).$$

Let $U(0)$ denote $U_\alpha(0)$ where $\alpha\in\mathcal{A}$. Let $\mathrm{Hor}_0=\mathrm{Hor}_\alpha$ be the horosphere containing $U(0)$, and let $\mathrm{Hor}_n=\mathrm{Hor}_0(n\tau')$ be the horosphere containing $U(n\tau')$. For $q\in\mathrm{Hor}_0$, define

$$\lambda_0(q)=e^{-Ld_{\mathrm{AGY}}(q,U(0))},$$

and for $q\in\mathrm{Hor}_n$, let

$$\lambda_n(q)=\lambda_0(q(-n\tau')).$$

Then, by Theorem 4.5, $\log\lambda_n$ is $L$-Lipschitz, i.e. for any $p$, $q$ in $\mathrm{Hor}_n$,

$$e^{-Ld_{\mathrm{AGY}}(p,q)}\le\frac{\lambda_n(p)}{\lambda_n(q)}\le e^{Ld_{\mathrm{AGY}}(p,q)}. \tag{A.8}$$

We use the abbreviation $\bar\mu_n$ to denote the conditional measure $\mu_{p_n}$ where $p_n$ is any point of $\mathrm{Hor}_n$. (In particular, if $p_0\in\mathrm{Hor}_0$, $\bar\mu_0=\mu_0$). We then define

$$\mu_n'=\lambda_n\bar\mu_n=\lambda_n\mu_{p_n}.$$

Then, for $E\subset\mathrm{Hor}_0$,

$$\mu_n'(g_{n\tau'}(E))=e^{wn\tau'}\mu_0'(E). \tag{A.9}$$

The weight $\lambda_0$ equals 1 on $U(0)$ and decays away from it. The next lemma proves that $\mu_0'$ is finite; then (A.9) gives finiteness of every $\mu_n'$. These measures are not normalized.

**Lemma A.8.** *We have*

$$\int_{\mathrm{Hor}_0} V(q)\, d\mu_0'(q) < \infty.$$

*Proof.* Recall that $q_\alpha \in U(0)$ and that $q_\alpha(t) \in K_0$ for every $t \in \mathbb{R}$. Choose $h > 0$ sufficiently small that $e^{2h} \le 2$, and put

$$r_- := \tfrac{1}{200}, \qquad r_+ := e^{2h} r_- \le \tfrac{1}{100}.$$

For $k \ge 0$, define a shell in the horosphere through $q_\alpha(-kh)$ by

$$S_k = \{z \in \mathrm{Hor}(q_\alpha(-kh)) : r_- \le d_{\mathrm{AGY}}(z, q_\alpha(-kh)) < r_+\}, \qquad \mathcal{E}_k = g_{kh} S_k \subset \mathrm{Hor}_0 .$$

We first claim that

$$\mathrm{Hor}_0 \subset B(q_\alpha, r_-) \cup \bigcup_{k \ge 0} \mathcal{E}_k.$$

Indeed, for $q \in \mathrm{Hor}_0$, let

$$D_q(t) = d_{\mathrm{AGY}}(q(-t), q_\alpha(-t)), \qquad t \ge 0.$$

By (4.8), $D_q$ is nonincreasing. We use the proof of Corollary 4.10 with initial path segments of length $r_- = \frac{1}{200}$ in place of $\frac{1}{100}$. More explicitly, if $D_q(t) \ge r_-$, then $D_q(s) \ge r_-$ for $0 \le s \le t$. On each of the $\lfloor t/T \rfloor$ full blocks of length $T$, Corollary 4.8 halves the distance between the endpoints of the initial segment of length $r_-$ of a close-to-minimizing path, while (4.8) does not increase the length of the rest. Here the base orbit stays in $K_0$ throughout every block. Letting the path-length error tend to zero, each block therefore decreases the distance by at least $r_-/2 = 1/400$. By monotonicity on the remaining time interval, for $t \ge 2T$ we obtain

$$D_q(t) \le D_q(0) - \frac{1}{400} \left\lfloor \frac{t}{T} \right\rfloor \le D_q(0) - \frac{t}{800T} \qquad \text{if } D_q(t) \ge r_-.$$

Consequently $D_q(t)$ must eventually be smaller than $r_-$; otherwise the displayed upper bound would become negative. If $D_q(0) \ge r_-$, let $k \ge 1$ be the first integer such that $D_q(kh) < r_-$. Then $D_q((k-1)h) \ge r_-$, while (4.8) gives

$$D_q((k-1)h) \le e^{2h} D_q(kh) < r_+.$$

Thus $q \in \mathcal{E}_{k-1}$, proving the claim.

Lemma 9.8 and the choice $r_+ \le \frac{1}{100}$ give

$$\mu_{q_\alpha(-kh)}(S_k) \le \mathsf{V}_0.$$

The horospherical volume-scaling formula therefore implies

$$\bar{\mu}_0(\mathcal{E}_k) = e^{wkh} \mu_{q_\alpha(-kh)}(S_k) \le \mathsf{V}_0 e^{wkh}.$$

Since $q_\alpha(-kh) \in K_0$, the function $V$ is uniformly bounded along the orbit of $q_\alpha$. If $q = z(kh) \in \mathcal{E}_k$, the Lipschitz property of $\log V$ and the exponential change of $V$ along the flow give

$$V(q) \le e^{kh} V(z) \le C e^{kh}.$$

For $q \in \mathcal{E}_k$ we have $D_q(kh) \ge r_-$. Thus, when $kh \ge 2T$, the contraction estimate just proved yields

$$d_{\mathrm{AGY}}(q, q_\alpha) \ge d_{\mathrm{AGY}}(q(-kh), q_\alpha(-kh)) + \frac{kh}{800T} \ge \frac{kh}{800T}.$$

Since $U(0) \subset W_\alpha(0)$ and (5.1) gives $\sup_{u\in U(0)} d_{\mathrm{AGY}}(u, q_\alpha) \le \frac{1}{200}$, it follows that

$$\lambda_0(q) \le e^{L\frac{1}{200}} \exp\left(-\frac{Lkh}{800T}\right) \qquad (q \in \mathcal{E}_k,\ kh \ge 2T).$$

Combining the preceding three estimates, we obtain

$$\int_{\mathcal{E}_k} V(q)\, d\mu_0'(q) = \int_{\mathcal{E}_k} V(q)\lambda_0(q)\, d\bar{\mu}_0(q) \le C \exp\left(\left(w + 1 - \frac{L}{800T}\right) kh\right).$$

The finitely many indices for which $kh < 2T$ contribute a finite amount by the volume and $V$ estimates above. The remaining series over $k$ converges by the choice of $L$. Finally, the integral over $B(q_\alpha, r_-)$ is finite by Lemma 9.8, the 1–Lipschitz property of $\log V$, and the inequality $0 < \lambda_0 \le 1$. The covering claim now proves the lemma. □

Let $0 < \delta \le 1/400$. We now construct a partition of unity on each $\mathrm{Hor}_n$.

For each non-negative integer $n$, we choose a maximal $\delta/8$-separated subset $\Delta_n$ of $\mathrm{Hor}_n$. We denote this set as $\{x_\sigma\}_{\sigma\in\Delta_n}$. We now claim that

$$\bigcup_{\sigma\in\Delta_n} B(x_\sigma, \delta/8) = \mathrm{Hor}_n\,.$$

Indeed, if $y \in \mathrm{Hor}_n$, then there exists $x_\sigma \in \Delta_n$ with $d_{\mathrm{AGY}}(x_\sigma, y) < \delta/8$ (or else we would add $y$ to $\Delta_n$). Thus, $\{B(x_\sigma, \delta/4)\}_{\sigma\in\Delta_n}$ is an open cover of $\mathrm{Hor}_n$. Let $M$ be the multiplicity of this cover, which is bounded by a universal constant in view of Lemma A.2. (Indeed for all centers whose $\delta/4$–balls contain a fixed $p$, the $\delta/16$–balls are disjoint and contained in $B(p, \delta/2)$. Three applications of Lemma A.2 give $\bar{\mu}_n(B(x_\sigma, \delta/16)) \ge c_D^3 \bar{\mu}_n(B(p, \delta/2))$, so there are at most $c_D^{-3}$ such centers.) We now choose a partition of unity $\phi_\sigma$ subordinate to this cover as follows. For each $\sigma \in \Delta_n$, choose a smooth $\psi_\sigma : \mathrm{Hor}_n \to [0, 1]$ with

$$\psi_\sigma = 1 \text{ on } B(x_\sigma, \delta/8), \qquad \psi_\sigma = 0 \text{ outside } B(x_\sigma, \delta/4), \qquad \mathcal{C}^1(\psi_\sigma) = O(1/\delta).$$

The bound on $\mathcal{C}^1(\psi_\sigma)$ follows from the proof of [AG13, Proposition 5.8]. Now set

$$\phi_\sigma(q) = \frac{\psi_\sigma(q)}{\sum_{\rho\in\Delta_n} \psi_\rho(q)}.$$

On $B(x_\sigma, \delta/16)$, the numerator is 1 and at most $M$ terms, each at most 1, contribute to the denominator. Hence

$$\phi_\sigma(x) \ge \frac{1}{M} \qquad \text{for } x \in B(x_\sigma, \delta/16).$$

Note that $\phi_\sigma = 0$ outside $B(x_\sigma, \delta/4)$. From the construction, $\phi_\sigma$ is smooth with $\mathcal{C}^1(\phi_\sigma) = O(1/\delta)$. Here $\mathcal{C}^1(\phi)$ denotes the $C^1$-norm of the function $\phi$, as defined in [AG13].

We now define a random walk as follows. Define

$$\Omega_n' = \Big\{(q_0, q_1, \dots, q_n) : q_j \in \mathrm{Hor}_j,\ 0 \le j \le n, \qquad d_{\mathrm{AGY}}(q_j, q_{j+1}(-\tau')) \le \delta/2,\ 0 \le j \le n-1\Big\}.$$

We equip $\Omega_n'$ with a measure $\mathbb{P}_n'$ as follows. We define the law of $q_0 \in \mathrm{Hor}_0$ to be $\frac{1}{\mu_0'(\mathrm{Hor}_0)}\, d\mu_0'(q_0)$.

For $p, q \in \mathrm{Hor}_j$ define

$$\bar{P}_p(q) = \sum_{\sigma \in \Delta_j} \frac{\phi_\sigma(p)}{\int \phi_\sigma \, d\mu'_j} \phi_\sigma(q), \qquad P_p(q) = \lambda_j(q) \bar{P}_p(q).$$

Note that for any $p$, $q$, the sum has at most $M$ terms.

For $0 \le j \le n-1$, given $q_j \in \mathrm{Hor}_j$, we define the law of $q' \in \mathrm{Hor}_j$ to be

$$\bar{P}_{q_j}(q') \, d\mu'_j(q') = P_{q_j}(q') \, d\bar{\mu}_j(q'),$$

and then define $q_{j+1} = q'(\tau') \in \mathrm{Hor}_{j+1}$.

For a bounded Lipschitz function $f$ on a horosphere, write

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + \mathrm{Lip}_{d_{\mathrm{AGY}}}(f).$$

Since $d_{\mathrm{AGY}}$ is the Finsler length metric on the horosphere, $\|f\|_{\mathrm{Lip}} \le \mathcal{C}^1(f)$ for $C^1$ functions $f$.

The immediate properties of this random walk are summarized in the following:

**Lemma A.9.** *Under the measure $\mathbb{P}'_n$, for any $0 \le j \le n$, the following hold:*

(i) *For $1 \le j \le n$, we have $q_j(-\tau') \in B(q_{j-1}, \delta/2)$.*

(ii) *The law of $q_j$ is $\frac{1}{\mu'_j(\mathrm{Hor}_j)} \, d\mu'_j(q_j)$.*

(iii) *For any $p \in \mathrm{Hor}_j$ and any $\sigma \in \Delta_j$ with $\phi_\sigma(p) > 0$,*

$$\int \phi_\sigma \, d\mu'_j \ge c_\delta \lambda_j(p) \bar{\mu}_j(B(p, \tfrac{1}{200})).$$

(iv) *For any $p \in \mathrm{Hor}_j$,*

$$\sup_q P_p(q) \le C_\delta \bar{\mu}_j(B(p, \tfrac{1}{200}))^{-1}.$$

(v) *For any $p \in \mathrm{Hor}_j$, the function $P_p$ is Lipschitz, and*

$$\|P_p\|_{\mathrm{Lip}} \le C_\delta \bar{\mu}_j(B(p, \tfrac{1}{200}))^{-1}.$$

*Proof.* The statement (i) follows immediately from the definition of the random walk law, and the statement (ii) follows immediately by induction on $j$. To see (iii) note that $\phi_\sigma \ge 1/M$ on a ball of radius $\delta/16$, and use Lemma A.2 and (A.8). Then (iv) follows from (iii), (A.8) and the definition of $P_p(q)$. For (v), put $A_p = \bar{\mu}_j(B(p, \frac{1}{200}))^{-1}$. By (iii), the bounded multiplicity of the partition, and its derivative bounds,

$$\mathcal{C}^1(\bar{P}_p) \le C_\delta \lambda_j(p)^{-1} A_p.$$

Every term in $\bar{P}_p$ is supported where $d_{\mathrm{AGY}}(p, q) \le \delta/2$, so (A.8) gives $\lambda_j(q) \le e^{L\delta/2} \lambda_j(p)$ there. Also, $\|d\lambda_j\|_{\mathrm{AGY}} \le L\lambda_j$ almost everywhere. The product rule for locally Lipschitz functions and (iv) therefore give

$$\|dP_p\|_{\mathrm{AGY}} \le LP_p + \lambda_j \|d\bar{P}_p\|_{\mathrm{AGY}} \le C_\delta A_p \quad \text{almost everywhere.}$$

The resulting local Lipschitz bound gives the same global bound for the Finsler length metric. Together with (iv), this proves (v). Here the fixed constant $L$ is absorbed into $C_\delta$.

□

We also record some easy estimates for later use. For $\tau' > 0$, let

$$V_0' = e^{3\tau'/4}. \tag{A.10}$$

Then, Proposition A.1, applied with the fixed choice $\alpha = 1/8$, implies

$$\frac{1}{\mu_q(B(q, \frac{1}{100}))} \int_{B(q,\frac{1}{100})} V(y(\tau'))\, d\mu_q(y) \le \begin{cases} C_{1/8} e^{-3\tau'/4} \cdot V(q) & \text{if} \quad V(q) \ge V_0' \\ C_{1/8} & \text{if} \quad V(q) \le V_0'. \end{cases} \tag{A.11}$$

**Lemma A.10.** *There exists a constant $C_\delta$ depending only on $\delta$ such that for any $p \in \mathrm{Hor}_n$,*

$$\int P_p(q) V(q(\tau'))\, d\bar{\mu}_n(q) \le \begin{cases} C_\delta e^{-3\tau'/4} \cdot V(p) & \text{if} \quad V(p) \ge V_0' \\ C_\delta & \text{if} \quad V(p) \le V_0'. \end{cases} \tag{A.12}$$

*Proof.* Note that $P_p$ is supported in $B(p, \frac{1}{100})$. Then the lemma follows from Lemma A.9(iv), (A.11) and Lemma A.2. □

A.3. **Tracking and good statistics.** For $\omega \in \Omega_n'$ and $0 \le j \le n$, define

$$\widetilde{V}_1(\omega, j) = V(q_j) \cdot \exp\left( \sum_{i=0}^{j-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt \right).$$

We note that the definition of $f_1$ involves $\tau$ (and not $\tau'$).

Let $\mathcal{F}_j'$ be the smallest $\sigma$-algebra for which $q_i = q_i(\omega)$ is measurable for all $0 \le i \le j$.

**Proposition A.11.** *For every $\tau' \in \tau\mathbb{N}$ there exists a constant $\alpha_3 = \alpha_3(\tau, \delta, \tau') > 0$ with $\alpha_3 \to \alpha_2(\tau)$ as $\tau' \to \infty$ such that for any $n \ge 1$, we have*

$$\mathbb{E}_n'(\widetilde{V}_1(\omega, n) \mid \mathcal{F}_0') \le e^{\alpha_3 n \tau'} \cdot V(q_0).$$

*Here $\mathbb{E}_n'$ denotes expectation with respect to the measure $\mathbb{P}_n'$.*

*Proof.* We claim that there exists $\alpha_3 = \alpha_3(\tau, \delta, \tau')$ with $\alpha_3 \to \alpha_2(\tau)$ as $\tau' \to \infty$ such that for $0 \le j < n$,

$$\mathbb{E}_n'(\widetilde{V}_1(\omega, j+1) | \mathcal{F}_j') \le e^{\alpha_3 \tau'} \cdot \widetilde{V}_1(\omega, j). \tag{A.13}$$

Note that

$$\frac{\mathbb{E}_n'(\widetilde{V}_1(\omega, j+1) \mid \mathcal{F}_j')}{\widetilde{V}_1(\omega, j)} = \frac{1}{V(q_j)} \int P_{q_j}(q) V(q(\tau')) \exp\left( \int_0^{\tau'} f_1(q(t))\, dt \right) d\bar{\mu}_j(q).$$

Recall the definition $\widetilde{V}_1(q, n)$ in (A.5), and write $\tau' = m\tau$. Then, the right hand side of the above equation can be rewritten as

$$\frac{1}{V(q_j)} \int P_{q_j}(q) \widetilde{V}_1(q, m)\, d\bar{\mu}_j(q).$$

Note that $P_{q_j}(q) = 0$ if $q \notin B(q_j, \frac{1}{200})$. Therefore, in view of Lemma A.9 (iv),

$$\frac{\mathbb{E}_n'(\widetilde{V}_1(\omega, j+1) \mid \mathcal{F}_j')}{\widetilde{V}_1(\omega, j)} \le \frac{1}{V(q_j)} \frac{C_\delta}{\bar{\mu}_j(B(q_j, \frac{1}{200}))} \int_{B(q_j, \frac{1}{200})} \widetilde{V}_1(q, m)\, d\bar{\mu}_j(q) \le C_\delta e^{\alpha_2 \tau'},$$

where $\alpha_2 = \alpha_2(\tau) < 1$ be as in Lemma A.6, and we have used Lemma A.6 with the base point $p_0 = q_j$ for the last estimate.

For fixed $\tau$ and $\delta$, we now choose $\alpha_3 = \alpha_3(\tau')$ to be the smallest so that $C_\delta e^{\alpha_2 \tau'} \leq e^{\alpha_3 \tau'}$. Then, $\alpha_3 \to \alpha_2(\tau)$ as $\tau' \to \infty$, and (A.13) holds. The rest of the proof is the same as the proof of Proposition A.3. □

**Lemma A.12.** *For every $\eta > 0$ there exists $\tau_0 = \tau_0(\eta) > 0$ such that, for every $0 < \delta \leq 1/400$ and every $\tau \geq \tau_0$, there exists $\tau_0' = \tau_0'(\eta, \delta, \tau) > 0$ such that if $\tau' > \tau_0'$ (and $\tau' \in \tau\mathbb{N}$), the following holds:*

*Let $n \geq 0$ be an integer, and let $G_+(\omega, \eta)$ denote the set of $k \in [0, n] \cap \mathbb{N}$ such that for all $k < j \leq n$,*

$$\sum_{i=k}^{j-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt \leq \eta(j-k)\tau'.$$

*Then*

$$\mathbb{P}_n'\Big(|G_+(\omega, \eta)| < (1 - 8\eta)n\Big) < e^{-\frac{\eta^4}{4} n\tau'}.$$

*Proof.* If $\eta \geq 1/8$, the event in the statement is empty, and if $n = 0$, the claim is immediate. Thus, we may assume $0 < \eta < 1/8$ and $n \geq 1$. Let $\alpha_2 = \alpha_2(\tau)$ be as in Lemma A.6. Since $\alpha_2(\tau) \to 0$, choose $\tau_0 = \tau_0(\eta)$ so that $\alpha_2(\tau) < \eta^4/16$ for every $\tau \geq \tau_0$. Suppose $0 < \delta \leq 1/400$ and $\tau \geq \tau_0$.

In view of Proposition A.11 and Lemma A.8, for $\tau' \in \tau\mathbb{N}$ large enough (depending on $\eta$, $\delta$, and $\tau$), we have

$$\mathbb{E}_n'(\widetilde{V}_1(\omega, n)) \leq \mu_0'(\mathrm{Hor}_0)^{-1} \left( \int_{\mathrm{Hor}_0} V \, d\mu_0' \right) e^{(\eta^4/16)n\tau'} \leq e^{(\eta^4/8)n\tau'},$$

where the last inequality is obtained by taking $\tau'$ sufficiently large. Then, by the Markov inequality,

$$\mathbb{P}_n' \left( \sum_{i=0}^{n-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt \geq \frac{\eta^4}{2} n\tau' \right) \leq e^{-(\eta^4/4)n\tau'}.$$

In the rest of the proof, we assume that $\omega$ is such that

$$\sum_{i=0}^{n-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt < \frac{\eta^4}{2} n\tau'. \tag{A.14}$$

Let $H_+(\omega, \eta) = ([0, n] \cap \mathbb{N}) \setminus G_+(\omega, \eta)$. If $k \in H_+(\omega, \eta)$ then there exists $k < j \leq n$ such that

$$\sum_{i=k}^{j-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt > \eta(j-k)\tau'.$$

We can rewrite this as

$$(j-k) < \frac{1}{\eta\tau'} \sum_{i=k}^{j-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt.$$

The sets $[k, j-1] \cap \mathbb{N}$ cover $H_+(\omega, \eta)$ and we can choose a subcover such that each point is covered at most twice. Then,

$$|H_+(\omega, \eta)| \leq \frac{2}{\eta\tau'} \sum_{i=0}^{n-1} \int_0^{\tau'} f_1(q_{i+1}(-\tau' + t))\, dt \leq \eta^3 n,$$

where the last estimate follows from (A.14). Therefore,

$$|G_+(\omega,\eta)| \geq (1-\eta^3)n \geq (1-8\eta)n.$$

□

**Lemma A.13.** *For every $0<\eta<1/4$, every $0<\delta\leq 1/400$ and every $\tau>0$ there exists $\tau_0'=\tau_0'(\eta,\delta,\tau)$ such that if $\tau'>\tau_0'$ (and $\tau'\in\tau\mathbb{N}$) the following holds:*

*Suppose $\omega=(q_0,\dots,q_n)\in\Omega_n'$ and $k\in G_+(\omega,\eta)$. Then,*

$$d_{\mathrm{AGY}}(q_k, q_n(-(n-k)\tau')) < \delta.$$

Note that $q_k$ and $q_n(-(n-k)\tau')$ are both in the same horosphere.

*Proof.* Let

$$\hat{q}(t) = q_{j+1}(-(j+1)\tau'+t) \text{ for } j\tau'\leq t<(j+1)\tau'.$$

Let

$$F_j = \{t\in[k\tau',(k+j)\tau'] \ : \ f_1(\hat{q}(t))=0\}.$$

Since $k\in G_+(\omega,\eta)$, we have

$$|F_j|\geq(1-\eta)j\tau'. \tag{A.15}$$

Consider the function

$$h_j(t) = d_{\mathrm{AGY}}(q_{k+j}(t-(k+j)\tau'), \hat{q}(t)).$$

Note that $h_j$ may not be continuous at the points $i\tau'$, $i\in\mathbb{N}$. However, the jump $|h_j(i\tau'+)-h_j(i\tau'-)|$ is bounded by $\delta/2$. Also, $h_j$ is increasing on each interval $[i\tau',(i+1)\tau']$, and $h_j((k+j)\tau'-)=0$.

Let

$$E_j = \{t\in[k\tau',(k+j)\tau'] \ : \ h_j(t)>\tfrac{1}{100}\}.$$

We claim that

$$|E_j\cap F_j|\leq\eta j\tau'. \tag{A.16}$$

Let $T=T(\operatorname{supp}1-f_1)$, where $T(\cdot)$ is as in Corollary 4.10. Let

$$\mathcal{J} = \{i\in[k,k+j-1] \ : \ |E_j\cap F_j\cap[i\tau',(i+1)\tau']|\geq 2T\},$$

and for $i\in\mathcal{J}$ let $t_i'$ denote the smallest $t\in[i\tau',(i+1)\tau']$ such that $h_j(t_i')\geq\frac{1}{100}$.

By Corollary 4.10 applied to the interval $[t_i',(i+1)\tau']$, we get

$$|E_j\cap F_j\cap[i\tau',(i+1)\tau']\leq 400T\left(h_j((i+1)\tau'-)-h_j(i\tau'+)\right).$$

For $i\notin\mathcal{J}$, we have by definition,

$$|E_j\cap F_j\cap[i\tau',(i+1)\tau']\leq 2T.$$

Sum these bounds over all $j$ blocks. The increases of $h_j$ between jumps sum to at most $j\delta$, since $h_j\geq 0$, $h_j((k+j)\tau'-)=0$, and each jump has size at most $\delta/2$. Thus

$$|E_j\cap F_j|\leq 400Tj\delta+2Tj.$$

We choose $\tau_0'$ so that for $\tau'>\tau_0'$, $400T\delta+2T\leq\eta\tau'$. This proves (A.16). Then, in view of (A.15), and since $\eta<1/4$,

$$|F_j\setminus E_j|\geq(1-2\eta)j\tau'\geq\frac{1}{2}j\tau'. \tag{A.17}$$

Let $K$ be the Mod-compact set

$$K = \{q \quad : \quad \text{there exists } p \in \mathrm{Hor}(q) \text{ with } f_1(p) = 0 \text{ and } d_{\mathrm{AGY}}(p, q) \leq \tfrac{1}{100}\},$$

and let $C_K$, $\beta_K$ be as in Corollary 4.9. Choose $\tau' = \tau'(\eta, \delta, \tau)$ so that $C_K e^{-\beta_K \tau'} < 0.01$. We now claim that for $0 \leq j \leq n - k$,

$$d_{\mathrm{AGY}}(q_k, q_{k+j}(-j\tau')) \leq \frac{\delta}{2}\left(1 + \sum_{i=1}^{j-1} C_K e^{-\beta_K i \tau'}\right) \leq \delta. \tag{A.18}$$

This will be done by induction on $j$. For $j = 0$ there is nothing to prove, and for $j = 1$ it follows from $d_{\mathrm{AGY}}(q_k, q_{k+1}(-\tau')) \leq \delta/2$. Now suppose $j \geq 1$ and the statement holds for $j$.

By the triangle inequality, it is enough to show that

$$d_{\mathrm{AGY}}(q_{k+j}(-j\tau'), q_{k+j+1}(-(j+1)\tau')) \leq \frac{\delta}{2} C_K e^{-\beta_K j \tau'}.$$

Note that for $t \in F_j \setminus E_j$,

$$d_{\mathrm{AGY}}(q_{k+j}(-(k+j)\tau' + t), \hat{q}(t)) \leq \tfrac{1}{100},$$

which together with the definitions of $F_j$ and $K$ implies that $q_{k+j}(-(k+j)\tau' + t) \in K$. Also, note that

$$d_{\mathrm{AGY}}(q_{k+j+1}(-\tau'), q_{k+j}) \leq \frac{\delta}{2}.$$

Then, by Corollary 4.9 and (A.17),

$$d_{\mathrm{AGY}}(q_{k+j+1}(-(j+1)\tau'), q_{k+j}(-j\tau')) \leq \frac{\delta}{2} C_K e^{-\beta_K j \tau'},$$

as required.

The final estimate is just the case $j = n - k$ of (A.18). □

A.4. **Good Statistics.** In this subsection, we take $\delta = 1/400$. As in the previous subsection, let $m$ be a large integer and put $\tau' = m\tau$.

For $\omega \in \Omega'_n$ and $0 \leq j \leq n$, define

$$V^*(\omega, j) = V(q_j) \exp\left(\sum_{i=0}^{j-1} Y_\rho(q_{i+1}(-\tau'), m)\right).$$

We note that the definition of $Y_\rho$ involves $\tau$ (and not $\tau'$).

Let $\mathcal{F}'_j$ be the smallest $\sigma$-algebra for which $q_i = q_i(\omega)$ is measurable for all $0 \leq i \leq j$.

**Proposition A.14.** *For every $0 < \rho < 1$ there exist $\tau_0 = \tau_0(\rho) > 0$ a constant $C \geq 1$ and a function $\beta_\rho(\tau)$ with $\beta_\rho(\tau) \to 0$ as $\tau \to \infty$ such that the following holds. For every $\tau > \tau_0$, every $m, n \in \mathbb{N}$ with $\tau' = m\tau$, we have, almost surely*

$$\mathbb{E}'_n(V^*(\omega, n) \mid \mathcal{F}'_0) \leq C e^{\beta_\rho(\tau) n} \cdot V(q_0),$$

*Here $\mathbb{E}'_n$ denotes expectation with respect to the measure $\mathbb{P}'_n$.*

*Proof.* Set

$$\kappa_\rho(\tau) = e^{-(\rho - \epsilon_\rho(\tau))\tau}.$$

After increasing $\tau_0(\rho)$, we may assume that $\epsilon_\rho(\tau) < \rho$ for $\tau > \tau_0$.

For $p \in \mathrm{Hor}_j$, define

$$T_j f(p) = \int P_p(q) f(q(\tau'))\, d\bar{\mu}_j(q),$$
$$R_j f(p) = \int P_p(q) Y_\rho(q,m) f(q(\tau'))\, d\bar{\mu}_j(q).$$

Using Proposition A.1 with the fixed choice $\alpha = 1/8$, there are absolute constants $b \geq 1$ and $C_0 \geq 1$, and a number $a_\tau = C_0 e^{-3\tau/4}$, such that

$$T_j V \leq a_\tau V + b. \tag{A.19}$$

Here we used Lemma A.9(iv), the fact that $P_p$ is supported in $B(p, \delta/2) \subset B(p, \frac{1}{100})$, and the inequality $\tau' = m\tau \geq \tau$.

Increase $\tau_0$ once more so that $a_\tau \leq 1/2$. It follows from (A.19) and $T_j 1 = 1$ that every string of $\ell$ consecutive ordinary transitions satisfies

$$\begin{aligned} T_j T_{j+1} \cdots T_{j+\ell-1} V &\leq a_\tau^\ell V + b \sum_{s=0}^{\ell-1} a_\tau^s \\ &\leq V + 2b \leq C_1 V, \end{aligned} \tag{A.20}$$

where $C_1 = 1 + 2b$ and we used $V \geq 1$.

On the other hand, Proposition A.7 and Lemma A.9(iv) give

$$R_j V \leq C_0 \kappa_\rho(\tau) V. \tag{A.21}$$

Indeed, since $P_p$ is supported in $B(p, \delta/2) \subset B(p, \frac{1}{200})$,

$$\begin{aligned} R_j V(p) &\leq \frac{C_\delta}{\bar{\mu}_j(B(p, \frac{1}{100}))} \int_{B(p, \frac{1}{200})} V(q(m\tau)) Y_\rho(q,m)\, d\bar{\mu}_j(q) \\ &\leq C_0 \kappa_\rho(\tau) V(p). \end{aligned}$$

Since $Y_\rho$ takes values in $\{0,1\}$, we have $e^{Y_\rho} = 1 + (e-1)Y_\rho$. Thus, if

$$K_j f(p) = \int P_p(q) e^{Y_\rho(q,m)} f(q(\tau'))\, d\bar{\mu}_j(q),$$

then

$$K_j = T_j + (e-1) R_j.$$

By the Markov property and the definition of $V^*$,

$$\mathbb{E}'_n\big(V^*(\omega, n) \mid \mathcal{F}'_0\big) = (K_0 K_1 \cdots K_{n-1} V)(q_0).$$

Expand the product $K_0 K_1 \cdots K_{n-1}$. A term containing exactly $k$ occurrences of the operators $R_j$ consists of those $k$ bad transitions separated by at most $k+1$ strings of ordinary transitions. By (A.20) and (A.21), each such term is bounded by

$$C_1^{k+1} \big(C_0 \kappa_\rho(\tau)\big)^k V(q_0).$$

There are $\binom{n}{k}$ such terms. Consequently,

$$\begin{aligned}\mathbb{E}'_n\big(V^*(\omega,n)\mid\mathcal{F}'_0\big) &\le C_1\sum_{k=0}^{n}\binom{n}{k}\big((e-1)C_0C_1\kappa_\rho(\tau)\big)^kV(q_0)\\ &= C_1\big(1+(e-1)C_0C_1\kappa_\rho(\tau)\big)^nV(q_0)\\ &\le C_1\exp\big((e-1)C_0C_1\kappa_\rho(\tau)n\big)V(q_0).\end{aligned}$$

The proposition follows by taking $C=C_1$ and

$$\beta_\rho(\tau)=(e-1)C_0C_1e^{-(\rho-\epsilon_\rho(\tau))\tau}.$$

Clearly $\beta_\rho(\tau)\to 0$ as $\tau\to\infty$. □

We need two backward counting conditions. The set $G_-$ controls the frequency of blocks detected by $Y_\rho$ in every interval ending at the chosen index. The set $G_*$ controls the frequency of indices outside $G_+$, for which Lemma A.13 gives no tracking estimate.

**Lemma A.15.** *Suppose* $0<\eta<1/16$, *and put*

$$\rho=\frac{\eta^2}{100}.$$

*There exist* $\tau_0=\tau_0(\eta)>0$ *and* $0<\kappa=\kappa(\eta)<1$ *such that, for every* $\tau\ge\tau_0$, *there exists* $\tau'_0=\tau'_0(\eta,\tau)>0$ *such that if* $\tau'>\tau'_0$ *(and* $\tau'\in\tau\mathbb{N}$*), the following holds:*

*Let* $n\ge 0$ *be an integer, put* $m=\tau'/\tau$, *and let* $G_-(\omega,\eta)$ *denote the set of* $k\in[1,n]\cap\mathbb{N}$ *such that, for every* $0\le j<k$,

$$\sum_{i=j}^{k-1}Y_\rho(q_{i+1}(-\tau'),m)\le\eta^2(k-j).$$

*Let* $G_*(\omega,\eta)$ *be the set of* $k\in[1,n]\cap\mathbb{N}$ *such that, for every* $0\le j<k$,

$$\big|\{\ell\in[j+1,k]\cap\mathbb{N}:\ell\notin G_+(\omega,\eta)\}\big|\le\eta^2(k-j).$$

*Then, for all sufficiently large* $n$ *(depending on* $\eta$, $\tau$, *and* $\tau'$*),*

$$\mathbb{P}'_n\Big(\big|G_+(\omega,\eta)\cap G_-(\omega,\eta)\cap G_*(\omega,\eta)\big|<(1-8\eta)n\Big)\le\kappa^n.$$

*Proof.* Choose $\tau_0=\tau_0(\eta)$ as in Lemma A.12. By Proposition A.14, after increasing $\tau_0$ we may assume that

$$\beta_\rho(\tau)<\frac{\eta^4}{4},\qquad\rho=\frac{\eta^2}{100}.$$

for every $\tau\ge\tau_0$. Suppose $\tau\ge\tau_0$, and choose $\tau'_0=\tau'_0(\eta,\tau)\ge 1$ so that Lemma A.12 applies with $\delta=1/400$.

By Proposition A.14 and Lemma A.8, there is a constant $C_\eta$ such that

$$\mathbb{E}'_n\big(V^*(\omega,n)\big)\le C_\eta e^{\beta_\rho(\tau)n}.$$

Since $V \geq 1$, Markov's inequality gives

$$\mathbb{P}'_n \left( \sum_{i=0}^{n-1} Y_\rho(q_{i+1}(-\tau'), m) \geq \frac{\eta^4}{2} n \right) \leq C_\eta \exp\left( - \left( \frac{\eta^4}{2} - \beta_\rho(\tau) \right) n \right) \leq C_\eta e^{-\eta^4 n/4}.$$

Consequently, for some $\kappa_- = \kappa_-(\eta) < 1$ and every sufficiently large $n$, the probability above is at most $\kappa_-^n$.

In the remainder of the proof, suppose that

$$\sum_{i=0}^{n-1} Y_\rho(q_{i+1}(-\tau'), m) < \frac{\eta^4}{2} n. \tag{A.22}$$

Let

$$H_-(\omega, \eta) = ([1, n] \cap \mathbb{N}) \setminus G_-(\omega, \eta).$$

For every $k \in H_-(\omega, \eta)$, there is $0 \leq j < k$ such that

$$\sum_{i=j}^{k-1} Y_\rho(q_{i+1}(-\tau'), m) > \eta^2 (k - j).$$

The intervals $[j+1, k] \cap \mathbb{N}$ cover $H_-(\omega, \eta)$, and we may choose a subcover of multiplicity at most two. Therefore,

$$|H_-(\omega, \eta)| \leq \frac{2}{\eta^2} \sum_{i=0}^{n-1} Y_\rho(q_{i+1}(-\tau'), m) \leq \eta^2 n.$$

The proof of Lemma A.12 shows that, outside an event of probability at most

$$e^{-\eta^4 n \tau'/4},$$

we have

$$|H_+(\omega, \eta)| \leq \eta^3 n, \qquad H_+(\omega, \eta) = ([1, n] \cap \mathbb{N}) \setminus G_+(\omega, \eta).$$

Let

$$H_*(\omega, \eta) = ([1, n] \cap \mathbb{N}) \setminus G_*(\omega, \eta).$$

The same interval-covering argument gives

$$|H_*(\omega, \eta)| \leq \frac{2}{\eta^2} |H_+(\omega, \eta)| \leq 2\eta n.$$

Combining these bounds, we obtain

$$|H_+ \cup H_- \cup H_*| \leq (\eta^3 + \eta^2 + 2\eta) n < 8\eta n.$$

Since $\tau' > 1$, we have $e^{-\eta^4 n \tau'/4} \leq e^{-\eta^4 n/4}$. Choose

$$\max\{\kappa_-, e^{-\eta^4/4}\} < \kappa(\eta) < 1.$$

Then the probability in the statement is at most $\kappa_-^n + e^{-\eta^4 n/4} \leq \kappa(\eta)^n$ for all sufficiently large $n$. The lemma follows.

□

**Lemma A.16.** *For every $\tau > 0$ there exists a* Mod*–compact set $K = K(\tau)$, containing $K_0$, with the following property. For every $0 < \eta < 1/16$, put $\rho = \eta^2/100$. There exists $\tau_0' = \tau_0'(\eta, \tau) > 0$ such that, for every $m, n \in \mathbb{N}$ with $\tau' = m\tau \geq \tau_0'$, every $\omega = (q_0, \dots, q_n) \in \Omega_n'$, and every $k \in [1, n] \cap \mathbb{N}$ satisfying*

$$k \in G_-(\omega, \eta) \cap G_*(\omega, \eta),$$

*the following hold:*

(i) *The point $q_n(-(n-k)\tau')$ belongs to $K$, and for every $0 \leq a < k\tau'$,*

$$\big|\{v \in [a, k\tau'] : \ q_n(-n\tau' + v) \in K\}\big| \geq (1 - 8\eta)(k\tau' - a).$$

(ii) *For every $0 \leq b < k\tau'$, we have*

$$\big|\{u \in [b, k\tau'] : \ q_n(-n\tau' + u) \in K \text{ and, for every } 0 \leq a < u,$$
$$\big|\{v \in [a, u] : q_n(-n\tau' + v) \in K\}\big| \geq (1 - 8\eta)(u - a)\}\big| \geq (1 - 8\eta)(k\tau' - b).$$

*Proof.* Put $T = k\tau'$, write $p(u) = q_n(-n\tau' + u)$, and set

$$K = K_0 \cup \{q \in \mathcal{PQ}_1\mathcal{T} : V(q) \leq 9eV_0e^{4\tau}\}.$$

This is a Mod–compact set depending only on $\tau$. Choose $\tau_0'$ sufficiently large that Lemma A.13 applies with $\delta = 1/400$.

For $0 \leq i < k$ and $0 \leq r < m$, call

$$I_{i,r} = [i\tau' + r\tau, \ i\tau' + (r+1)\tau]$$

good if $f_1(q_{i+1}(-\tau' + t)) = 0$ for some $t \in [r\tau, (r+1)\tau]$, and bad otherwise. If $i + 1 \in G_+(\omega, \eta)$, tracking and backward monotonicity of $d_{\text{AGY}}$ give

$$d_{\text{AGY}}\big(q_{i+1}(-\tau' + t), p(i\tau' + t)\big) < \delta \qquad (0 \leq t \leq \tau').$$

Since $f_1(q) = 0$ implies $V(q) \leq 9V_0e^{3\tau}$, the exponential change of $V$ along the flow and the 1–Lipschitz property of $\log V$ imply that

$$p(u) \in K \qquad \text{for all } u \in I_{i,r} \tag{A.23}$$

whenever $i + 1 \in G_+(\omega, \eta)$ and $I_{i,r}$ is good.

Call $i$ admissible if $Y_\rho(q_{i+1}(-\tau'), m) = 0$ and $i + 1 \in G_+(\omega, \eta)$. For an admissible $i$, the definition of $Y_\rho$ and the fact that $f_1 = 1$ throughout every bad interval give

$$\#\{r \in [a, m-1] \cap \mathbb{Z} : I_{i,r} \text{ is bad}\} \leq \rho(m - a) \qquad (a \in [0, m] \cap \mathbb{Z}). \tag{A.24}$$

Since $k \in G_-(\omega, \eta) \cap G_*(\omega, \eta)$, we also have

$$\#\{i \in [j, k-1] \cap \mathbb{Z} : i \text{ is not admissible}\} \leq 2\eta^2(k - j) \qquad (j \in [0, k-1] \cap \mathbb{Z}). \tag{A.25}$$

In particular, $k - 1$ is admissible since $2\eta^2 < 1$.

Let $B$ be the union of the nonadmissible $\tau'$–pieces and the bad $\tau$–intervals in admissible pieces. By (A.23), $p(u) \in K$ outside $B$. The function $a \mapsto |B \cap [a, T]|$ is affine between consecutive grid points. Thus (A.24) within each admissible piece and (A.25) for the nonadmissible pieces, followed by linear interpolation and addition, give

$$|B \cap [a, T]| \leq (2\eta^2 + \rho)(T - a) \qquad (0 \leq a \leq T). \tag{A.26}$$

The last small interval is good by (A.24), since $k - 1$ is admissible and $\rho < 1$. Hence $p(T) \in K$, and (A.26) proves (i).

For (ii), define

$$H = \{u \in [0,T] : |B \cap [a,u]| > 8\eta(u-a) \text{ for some } 0 \le a < u\}.$$

We claim that

$$|H \cap [b,T]| \le \frac{2(2\eta^2+\rho)}{8\eta}(T-b) = \frac{201\eta}{400}(T-b) < 8\eta(T-b) \tag{A.27}$$

for every $0 \le b < T$. To see this, take a nonempty compact set $A \subset H \cap [b,T]$. Slightly extending witnessing intervals at their right endpoints, choose a finite cover of $A$ by intervals $J \subset [0,T]$ of multiplicity at most two, each with right endpoint at least $b$ and $|B \cap J| > 8\eta|J|$. This is possible since (A.26) implies $T \notin H$. Let $v$ be the minimum of $b$ and their left endpoints. Their union contains $A \cup (v,b)$, so

$$8\eta(|A| + b - v) \le 8\eta \sum_J |J| \le \sum_J |B \cap J| \le 2(2\eta^2+\rho)(T-v).$$

Since $2(2\eta^2+\rho)/(8\eta) < 1$, rearranging and taking the supremum over $A$ proves (A.27). Every interior point of $B$ belongs to $H$ since $8\eta < 1$. Thus, apart from the finitely many subdivision endpoints, every $u \notin H$ has $p(u) \in K$ and satisfies the condition in (ii). Equation (A.27) therefore proves (ii). □

We introduce a variant of the definition of good statistics. For a Mod-compact set $K$, a constant $0 < \eta < 1/2$ and $t > 0$ we say a quadratic differential $q \in g_t(\mathrm{Hor}_0)$ has $(K,\eta)$–good statistics relative to $\mathrm{Hor}_0$ if $q \in K$ and for all $s \in [-t,0]$,

$$|\{u \in [s,0] : q(u) \in K\}| \ge (1-\eta)\cdot|s|. \tag{A.28}$$

**Definition A.17.** Let $K \subset \mathcal{PQ}_1\mathcal{T}$ be Mod–compact and let $0 < \eta < 1/2$. For $R > 0$, we say $q \in \mathrm{Hor}_0$ is $(K,\eta)$-generic at $R$ if the following hold:

(i) $q(R)$ has $(K,\eta)$–good statistics relative to $\mathrm{Hor}_0$, and
(ii) For $0 \le s \le R$,

$$|\{u \in (s,R) : q(u) \text{ has } (K,\eta)\text{–good statistics relative to } \mathrm{Hor}_0\}| \ge (1-\eta)(R-s).$$

**Proposition A.18.** *For every $0 < \eta < 1/2$ there exist $0 < \kappa = \kappa(\eta) < 1$ and $\tau_0 = \tau_0(\eta) > 0$ such that, for every $\tau \ge \tau_0$, there exists $\tau_0' = \tau_0'(\eta,\tau) > 0$ such that if $\tau' > \tau_0'$ (and $\tau' \in \tau\mathbb{N}$), the following holds:*

*Let $K = K(\tau) \supset K_0$ be the* Mod*–compact set supplied by Lemma A.16, and set $R_j = j\tau'$.*

*Then, for*

$$\bar{k}_n(x) = \#\Big\{j \in [1,n] : x \text{ is } (K,\eta)\text{-generic at } R_j\Big\},$$

*we have, for all sufficiently large $n$ (depending on $\eta$, $\tau$, and $\tau'$), that for any $\alpha \in \mathcal{A}$,*

$$\mu_0(\{x \in U_\alpha(0) : \bar{k}_n(x) \le (1-\eta)n\}) \le \kappa^n \mu_0(U_\alpha(0)).$$

*Proof.* Since $\mathcal{A}$ is finite, it is enough to prove the proposition for a fixed $\alpha$. Let

$$H_n = G_+(\omega,\epsilon) \cap G_-(\omega,\epsilon) \cap G_*(\omega,\epsilon), \qquad \epsilon = \eta/16,$$

and let $E_n$ be the event

$$|H_n| \ge (1-8\epsilon)n \ge (1-\eta/2)n.$$

Choose $\tau_0 = \tau_0(\eta)$ and $0 < \kappa_0 = \kappa_0(\eta) < 1$ as in Lemma A.15, with $\eta/16$ in place of $\eta$. Suppose $\tau \geq \tau_0$, and let $\tau_0' = \tau_0'(\eta, \tau)$ be as in that lemma. Then, for $\tau' > \tau_0'$ (and $\tau' \in \tau\mathbb{N}$), $\mathbb{P}'_n(E_n) \geq 1 - \kappa_0^n$ for all sufficiently large $n$.

Now suppose $E_n$ holds. We can enlarge $\tau_0'$ so that Lemma A.16 applies. It follows from Lemma A.16 that for each $k \in H_n$, $q = q_n(-n\tau')$ is $(K, \eta)$–generic at $R_k = k\tau'$. Therefore,

$$\bar{k}_n(q_n(-n\tau')) \geq (1 - \eta/2)n > (1 - \eta)n.$$

Let $F_n$ be the event $q_n \in U_\alpha(n\tau')$. Then, by Lemma A.9(ii) and (A.9), $\mathbb{P}'_n(F_n) \geq c$ for some constant $c$ independent of $n$. Then, $\mathbb{P}'_n(E_n \mid F_n) \geq 1 - \kappa(\eta)^n$ for some $\kappa(\eta) < 1$. Now the proposition follows since $\lambda_0 = 1$ on $U_\alpha(0)$ and the law of $q_n(-n\tau')$, conditioned on $F_n$, is the normalized restriction of $\mu_0$ to $U_\alpha(0)$.

□

A.5. **Mixing.** In this subsection, let $0 < \delta \leq 1/400$, and use the random walk constructed above with this value of $\delta$.

**Lemma A.19.** *There exist $c_1 > 0$ and $N_1 > 0$ such that for all $x \in \mathcal{PQ}_1\mathcal{T}$,*

$$\mu_x(B(x, \tfrac{1}{100})) \geq c_1 V(x)^{-N_1}.$$

*Proof.* Compare the return to a compact set in [EMM22, Theorem 2.5 and the proof of Lemma 2.6]. We record the adaptation to the present height and horospherical measure. Fix $\alpha \in (0, 1/4)$, put $b = 1 - 2\alpha$, and take $r = \frac{1}{100}$ and $t = b^{-1} \log V(x)$. Proposition A.1 and Lemma A.2 give

$$\frac{1}{\mu_x(B(x, r/2))} \int_{B(x,r/2)} V(y(t))\, d\mu_x(y) \leq \frac{C_\alpha}{2c_D}\big(e^{-bt}V(x) + 1\big) = \frac{C_\alpha}{c_D}.$$

Thus some $y \in B(x, r/2)$ has $z = y(t)$ in the fixed Mod–compact set $K = \{V \leq 2\max(1, C_\alpha/c_D)\}$. The compact-set lower bound in Lemma 9.8 and doubling give $\mu_z(B(z, r/2)) \geq c_1 > 0$ uniformly on $K$. This bound follows from local period-coordinate comparison and positivity of the horospherical volume form. By backward monotonicity in (4.8),

$$g_{-t}B(z, r/2) \subset B(y, r/2) \subset B(x, r).$$

The conditional measures at $x$ and $y$ agree, so (9.14) yields

$$\mu_x(B(x, r)) \geq \mu_x\big(g_{-t}B(z, r/2)\big) = e^{-wt}\mu_z(B(z, r/2)) \geq c_1 V(x)^{-w/b}.$$

Take $N_1 = w/b$.

□

Suppose $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ is a Mod–compact set and let $N_\delta(\Omega)$ be the $\delta$-neighborhood of $\Omega$ (in the $D_{\text{AGY}}$ metric).

We recall a few lemmas from [EMM22]. As in [EMM22, Section 2.1], we work with embedded period boxes in a fixed finite cover without orbifold points. More precisely, fix a torsion-free finite-index subgroup $\Gamma_0 < \text{Mod}$, put $X_0 = \mathcal{PQ}_1\mathcal{T}/\Gamma_0$, and write $\pi_0 : \mathcal{PQ}_1\mathcal{T} \to X_0$ for the projection. Let $\nu_0$ be the Masur–Veech probability measure on $X_0$. The inverse image in $X_0$ of a measurable subset of $\mathcal{PQ}_1\mathcal{M}$ has the same normalized measure as that subset. Thus the estimates below for Mod–invariant target sets descend to $\mathcal{PQ}_1\mathcal{M}$. The cover is fixed throughout, so constants may depend on it.

**Lemma A.20** ([EMM22, Lemma 2.6])**.** *There exist $c_3 > 0$ and $N_3 \geq 1$ such that, on setting*

$$r_3(x) = c_3 V(x)^{-N_3},$$

*the restriction of $\pi_0$ to $B(x, r)$ is injective for every $x \in \mathcal{PQ}_1\mathcal{T}$ and $0 < r \leq r_3(x)$. We may also require that the period box of radius $10r_3(x)$ centered at $x$ embeds in $X_0$.*

*Proof.* We spell out the argument of the cited lemma with the present height $V$. Fix the parameters in [AG13, Lemmas 6.2 and 6.4]. The height comparison in Lemma 6.2 and the horocycle estimate in Lemma 6.4 give constants $A \geq 1$ and $b > 0$ such that

$$\int_0^1 V(g_t u_s x)\, ds \leq Ae^{-bt}V(x) + A \qquad (t \geq 0),$$

where $u_s = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$. Taking $t = b^{-1}\log(2AV(x))$, we find $s \in [0, 1]$ for which $V(g_t u_s x) \leq A + 1$. Consequently $\pi_0(g_t u_s x)$ belongs to a fixed compact set $K \subset X_0$, and $t \leq C_0 + b^{-1}\log V(x)$.

Write $\mathcal{B}_a(x)$ for the period box of radius $a$ defined in [EMM22, Section 2.9]. Since $X_0$ has no orbifold points, compactness of $K$ gives $a_K > 0$ such that the period box $\mathcal{B}_{a_K}(z)$ embeds in $X_0$ whenever $\pi_0(z) \in K$. The linear action in period coordinates and [EMM22, Lemma 2.4] imply, with $h = g_t u_s$,

$$h\mathcal{B}_a(x) \subset \mathcal{B}_{C_1 V(x)^{2/b} a}(hx)$$

whenever the radius on the right is sufficiently small. Choose $N_3 \geq \max(1, 2/b)$ and then choose $c_3 > 0$ so small that $10C_1c_3 < a_K$ and all these period boxes are defined. It follows that $\mathcal{B}_{10r_3(x)}(x)$ embeds: any identification in that box would, after applying $h$, give an identification in $\mathcal{B}_{a_K}(hx)$, since the $SL(2, \mathbb{R})$ action commutes with $\Gamma_0$.

Finally, [AG13, Proposition 5.3] gives $B(x, r) \subset \mathcal{B}_{2r}(x)$ for all sufficiently small $r$. Decreasing $c_3$ once more, we may assume this for $r \leq r_3(x)$ and also that $r_3(x) \leq \frac{1}{100}$. This proves the lemma. The fixed factor $c_3$ is retained because $V$ can equal 1. □

Let $\mathcal{C}^1(\varphi)$ denote the $C^1$ norm of $\varphi$ as defined in [AG13]. The following is a rephrased version of [EMM22, Lemma 2.10].

**Lemma A.21.** *There exists $N_5 > 0$ so that the following holds. Let $x \in \mathcal{PQ}_1\mathcal{T}$, and let $\epsilon = 0.01e^{-N_3\frac{1}{100}}r_3(x)$. There exists a finite collection $\{\psi_i\}$ of $C^\infty$ functions on* $\mathrm{Hor}(x)$ *with the following properties:*

*(1) $0 \leq \psi_i \leq 1$ for all $i$.*
*(2) $\mathcal{C}^1(\psi_i) \leq N_5 V(x)^{N_5}$.*
*(3) For every $i$, $\psi_i$ is supported on $B(y_i, \epsilon)$ for some $y_i \in B(x, \frac{1}{100})$.*
*(4) The covering $\{B(y_i, \epsilon)\}$ of $B(x, \frac{1}{100})$ has multiplicity at most $N_5$.*
*(5) $\sum \psi_i \leq 1$, and the equality holds on a neighborhood of $B(x, \frac{1}{100})$.*

The extra fixed factor in $\epsilon$ ensures compatibility with Lemma A.20: since $\log V$ is 1–Lipschitz on a horosphere and $y_i \in B(x, \frac{1}{100})$, we have

$$r_3(y_i) \geq e^{-N_3\frac{1}{100}}r_3(x), \qquad \epsilon \leq 0.01 r_3(y_i).$$

All fixed factors are absorbed in $N_5$ and the constants below.

Let $N(x)$ denote the number of functions $\psi_i$ in Lemma A.21. Then, in view of (iv),

$$\sum_{i=1}^{N(x)} \mu_x(B(y_i,\epsilon)) \le N_5 \mu_x(B(x,\tfrac{1}{100}+\epsilon))$$

Thus, in view of Lemma A.2, $N(x) \le N_5 \epsilon^{-d}$, where $d$ depends only on the dimension. WLOG, $N_5 > d$, thus

$$N(x) \le N_5 \epsilon^{-N_5}. \tag{A.29}$$

**Lemma A.22** (Boundary layers of metric neighborhoods)**.** *Let $(X,d)$ be a proper geodesic metric space with a locally finite Borel measure $\nu$. Suppose that, for some $D \ge 1$ and $r_0 > 0$,*

$$0 < \nu(B_d(z,2r)) \le D\nu(B_d(z,r)) \qquad (z \in X,\ 0 < 2r \le r_0).$$

*For a nonempty set $F \subset X$, write $F_a = \{z : d(z,F) < a\}$. Let $\delta > 0$ and $r_* = \min(\delta, r_0)$, and assume $\nu(F_{\delta+r_*}) < \infty$. Then, for $0 < \epsilon < r_*$,*

$$\nu(F_{\delta+\epsilon} \setminus F_{\delta-\epsilon}) \le C \left(\frac{\epsilon}{r_*}\right)^{\beta} \nu(F_{\delta+r_*}), \tag{A.30}$$

*where*

$$C = 1 + D^{-5}, \qquad \beta = \frac{\log(1+D^{-5})}{\log 32} > 0.$$

*In particular, if $\nu(X) = 1$, then for every $0 < \epsilon < \delta$,*

$$\nu(F_{\delta+\epsilon} \setminus F_{\delta-\epsilon}) \le C_\delta \epsilon^\beta, \qquad C_\delta = C \min(\delta, r_0)^{-\beta},$$

*uniformly in $F$.*

*Proof.* Replace $F$ by its closure, and put

$$\rho(z) = d(z,F), \qquad A_s = \{z : \delta - s \le \rho(z) < \delta + s\}.$$

We claim that

$$\nu(A_s) \le \lambda \nu(A_{32s}), \qquad 0 < s \le r_*/32, \qquad \lambda = (1+D^{-5})^{-1} < 1. \tag{A.31}$$

Choose a maximal pairwise disjoint family $B_i = B_d(z_i, 8s)$ with $z_i \in A_s$. Properness implies separability, so this family is countable. Maximality gives

$$A_s \subset \bigcup_i B_d(z_i, 16s).$$

For each $i$, choose a nearest point of $F$ to $z_i$, and follow a minimizing geodesic toward it for distance $4s$, reaching $w_i$. This is possible since $\rho(z_i) \ge \delta - s \ge 31s$. Then $\rho(w_i) = \rho(z_i) - 4s$. The 1–Lipschitz property of $\rho$ shows that the ball $H_i = B_d(w_i, s)$ satisfies

$$H_i \subset \{\delta - 6s < \rho < \delta - 2s\} \subset A_{32s} \setminus A_s.$$

Also $H_i \subset B_i$, so the $H_i$ are pairwise disjoint. Since

$$B_d(z_i, 16s) \subset B_d(w_i, 20s) \subset B_d(w_i, 32s),$$

five applications of doubling give $\nu(B_d(z_i, 16s)) \le D^5 \nu(H_i)$; all radii are at most $r_0$. Consequently,

$$\begin{aligned}\nu(A_{32s}) &\ge \nu(A_s) + \sum_i \nu(H_i) \\ &\ge \nu(A_s) + D^{-5} \sum_i \nu(B_d(z_i, 16s)) \\ &\ge (1 + D^{-5})\nu(A_s).\end{aligned}$$

This proves (A.31).

Set $n = \lfloor \log_{32}(r_*/\epsilon) \rfloor$. Iterating gives

$$\nu(A_\epsilon) \le \lambda^n \nu(A_{32^n \epsilon}) \le \lambda^n \nu(F_{\delta + r_*}).$$

Since $32^{-\beta} = \lambda$, we have $\lambda^n \le \lambda^{-1}(\epsilon/r_*)^\beta$, proving (A.30). If $\nu(X) = 1$, this proves the last assertion for $\epsilon < r_*$; for $\epsilon \ge r_*$ it follows from the trivial bound by 1. □

We need a mixing estimate for the indicator of $N_\delta(\Omega)$, with constants independent of $\Omega$. The following variant of [EMM22, Corollary 3.4] follows from exponential mixing [AGY06] by approximating this indicator with smooth functions.

Let $\chi$ be the characteristic function of the complement of $N_\delta(\Omega)$.

**Proposition A.23.** *For every $\delta > 0$ there exist $\kappa > 0$ and $C_2 > 0$ (both depending on $\delta$) such that for every $q \in \mathcal{PQ}_1\mathcal{T}$ and every Lipschitz function $\varphi : \mathrm{Hor}(q) \to \mathbb{R}$ with compact support in $B(q, 0.1 r_3(q))$, for every* Mod*-compact set $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ and every $t > 0$, we have*

$$\left| \int \varphi(x)(1 - \chi)(x(t))\, d\mu_q(x) - \mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \cdot \int \varphi \, d\mu_q \right| \le \|\varphi\|_{\mathrm{Lip}} C_2 \cdot e^{-\kappa t}$$

*Proof.* We first bound the measure near the boundary of $N_\delta(\Omega)$, uniformly in $\Omega$, using Lemma A.22. We then bracket its indicator by smooth functions and apply mixing.

To apply the boundary estimate, we verify its local doubling hypothesis. On the marked unit-area stratum, the slow variation of the AGY norm [AG13, Proposition 5.5], together with the period-box and local-product coordinates of [EMM22, Sections 2.9–2.10], gives uniform local comparisons with a normed vector space and Lebesgue measure. In particular, writing $\widetilde{\nu}_0$ for the lift of $\nu_0$ and $m = \dim_{\mathbb{R}} \mathcal{PQ}_1\mathcal{T}$, there are a fixed radius $r_1 > 0$ and numbers $J(z) > 0$ such that

$$\widetilde{\nu}_0(B_{\mathrm{AGY}}(z, a)) \asymp J(z) a^m \qquad (0 < a \le r_1),$$

with comparison constants independent of $z$ and $a$. Indeed, the AGY norm is uniformly comparable to its value at the center in a fixed-size period chart; in the unit-area local-product coordinates the volume density is uniformly comparable to its value at the center. Comparing the corresponding norm balls gives the displayed estimate. Thus marked balls are uniformly locally doubling and have uniform packing bounds at these scales.

These bounds also hold on $X_0$, even for balls larger than its local injectivity radius. Here is the multiplicity argument. For a lift $z$ of $\bar{z} \in X_0$, let

$$M_z(a) = \#\{\gamma \in \Gamma_0 : D_{\mathrm{AGY}}(z, \gamma z) < a\}.$$

For all sufficiently small $r > 0$, the packing bound covers $B_{\mathrm{AGY}}(z, 2r)$ by at most $L$ balls of radius $r/2$, with $L$ uniform. Each such ball contains at most $M_z(r)$ points of

the orbit $\Gamma_0 z$: translate one orbit point in the ball back to $z$. Hence $M_z(2r) \le LM_z(r)$. Counting lifts over quotient balls gives

$$\frac{\widetilde{\nu}_0(B_{\mathrm{AGY}}(z,r))}{M_z(2r)} \le \nu_0(B_{X_0}(\bar{z},r)),$$
$$\nu_0(B_{X_0}(\bar{z},2r)) \le \frac{\widetilde{\nu}_0(B_{\mathrm{AGY}}(z,3r))}{M_z(r)}.$$

For the first inequality, a point lifted to $B_{\mathrm{AGY}}(z,r)$ has at most $M_z(2r)$ lifts there. For the second, a point lifted to $B_{\mathrm{AGY}}(z,2r)$ has at least $M_z(r)$ lifts in $B_{\mathrm{AGY}}(z,3r)$. Combining these inequalities with marked doubling proves that there exist uniform $D$ and $r_0 > 0$ such that

$$\nu_0(B_{X_0}(\bar{z},2r)) \le D\nu_0(B_{X_0}(\bar{z},r)) \qquad (0 < 2r \le r_0). \tag{A.32}$$

The AGY length metric is complete and locally compact, so $X_0$ is a proper geodesic space.

Let $F \subset X_0$ be the inverse image of $\Omega/\operatorname{Mod}$, and set $E = N_\delta(F)$. We may replace $F$ by its closure. It is compact, and $E$ is the inverse image of $N_\delta(\Omega)/\operatorname{Mod}$; in particular,

$$\nu_0(E) = \mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}).$$

By (A.32) and Lemma A.22, there are $\beta > 0$ and $C_\delta$ independent of $F$ such that

$$\nu_0(N_{\delta+\epsilon}(F) \setminus N_{\delta-\epsilon}(F)) \le C_\delta \epsilon^\beta \qquad (0 < \epsilon < \delta). \tag{A.33}$$

For $0 < \epsilon < \min(\delta/10, r_0/10, 1/10)$, choose smooth compactly supported functions $f_\epsilon^-$ and $f_\epsilon^+$ on $X_0$ such that

$$0 \le f_\epsilon^- \le 1_E \le f_\epsilon^+ \le 1, \qquad \mathcal{C}^1(f_\epsilon^\pm) \le C\epsilon^{-1}, \tag{A.34}$$
$$f_\epsilon^- = 1 \text{ on } N_{\delta-3\epsilon}(F), \qquad \operatorname{supp} f_\epsilon^+ \subset N_{\delta+3\epsilon}(F).$$

For completeness, start with Lipschitz cutoffs of $d(\cdot,F)$: the lower cutoff is 1 for $d \le \delta - 2\epsilon$ and 0 for $d \ge \delta - \epsilon$, and the upper cutoff is 1 for $d \le \delta + \epsilon$ and 0 for $d \ge \delta + 2\epsilon$. Their Lipschitz constants are at most $\epsilon^{-1}$. Smooth in period charts of sufficiently small diameter and combine with a locally finite smooth partition of unity $\{\zeta_j\}$. If $h_j$ is a local smoothing of a cutoff $h$, choose its value error $a_j$ so small that

$$\sum_j a_j \|d\zeta_j\|_{\mathrm{AGY}} \le \epsilon^{-1}, \qquad a_j \le 1/10.$$

Local norm comparison bounds $\|dh_j\|_{\mathrm{AGY}}$ by $C\epsilon^{-1}$. The identity

$$d\Big(\sum_j \zeta_j h_j\Big) = \sum_j \zeta_j\, dh_j + \sum_j (h_j - h)\, d\zeta_j$$

therefore gives the same uniform derivative bound after gluing. Compose with a fixed smooth function equal to 0 on $(-\infty, 1/5]$ and to 1 on $[4/5, \infty)$, taking values in $[0,1]$. Choosing the chart diameters and convolution radii small enough preserves the indicated inner and outer margins. This proves (A.34); the chart sizes may depend on $F$, but the first-derivative bound does not. Properness gives compact support. It follows from (A.33) that

$$0 \le \nu_0(f_\epsilon^+) - \nu_0(f_\epsilon^-) \le C_\delta \epsilon^\beta. \tag{A.35}$$

By [EMM22, Proposition 3.2], applied on the fixed cover $X_0$, there exist $C > 0$ and $\kappa_0 > 0$ such that, for smooth compactly supported $f$ and smooth weights $\varphi$ with the stated support,

$$\left| \int_{\mathrm{Hor}(q)} \varphi(x) f(x(t))\, d\mu_q(x) - \nu_0(f) \int_{\mathrm{Hor}(q)} \varphi\, d\mu_q \right| \le C\mathcal{C}^1(\varphi)\mathcal{C}^1(f) e^{-\kappa_0 t}. \tag{A.36}$$

The support condition is admissible by Lemma A.20. The fixed finite cover changes only the mixing constants; compare also [EMM22, Theorem 8.2 and its proof].

The same estimate holds for compactly supported Lipschitz weights, with $\|\varphi\|_{\mathrm{Lip}}$ in place of $\mathcal{C}^1(\varphi)$. Indeed, identify the horospherical period chart with a subset of a real vector space and freeze the AGY norm at $q$. By [AG13, Proposition 5.3], this norm and the horospherical metric are uniformly comparable on the chart. Extend $\varphi$ by zero in these coordinates and convolve with a nonnegative smooth approximate identity. The resulting smooth functions $\varphi_a$ satisfy

$$\varphi_a \longrightarrow \varphi \quad \text{uniformly}, \qquad \mathcal{C}^1(\varphi_a) \le C\|\varphi\|_{\mathrm{Lip}},$$

where $C$ is independent of $q$ and $a$. Since the original support is a compact subset of $B(q, 0.1 r_3(q))$, sufficiently small convolution radii keep the supports in this ball. Convolution also preserves nonnegativity. For each fixed $t$, uniform convergence on this common support, which has finite $\mu_q$-measure, lets us pass to the limit in (A.36). The first-derivative bound is uniform as $a \to 0$, so only a fixed factor is lost and the decay exponent is unchanged. We use this Lipschitz version below.

First suppose $\varphi \ge 0$. Apply (A.36) with $f = f_\epsilon^-$ and $f = f_\epsilon^+$ and use the pointwise brackets. Since $\int \varphi\, d\mu_q \le C\|\varphi\|_{\mathrm{Lip}}$ by Lemma 9.8, equations (A.34)–(A.35) give

$$\left| \int \varphi(x) 1_E(x(t))\, d\mu_q(x) - \nu_0(E) \int \varphi\, d\mu_q \right| \le C_\delta \|\varphi\|_{\mathrm{Lip}} \big(\epsilon^{-1} e^{-\kappa_0 t} + \epsilon^\beta\big). \tag{A.37}$$

This also holds for signed Lipschitz $\varphi$. Apply the nonnegative case to $\varphi_+ = \max(\varphi, 0)$ and $\varphi_- = \max(-\varphi, 0)$ and subtract. Each has Lipschitz norm at most $\|\varphi\|_{\mathrm{Lip}}$, so only a fixed factor is lost.

Fix $0 < a_\delta < \min(\delta/10, r_0/10, 1/10)$ and put

$$\epsilon = a_\delta e^{-\kappa_0 t/(1+\beta)}, \qquad \kappa = \frac{\beta\kappa_0}{1+\beta} > 0.$$

Both terms in (A.37) are bounded by a constant depending only on $\delta$ times $e^{-\kappa t}$. Finally, $1_E(x(t)) = (1-\chi)(x(t))$ and $\nu_0(E) = \mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod})$, so the result follows. □

**Proposition A.24.** *Suppose $\theta > 0$, $0 < \delta \le 1/400$ and suppose $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ is a Mod-compact set satisfying $\mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \ge (1-\theta)$. Then there exists $t_\theta = t_\theta(\theta, \delta) > 0$ and $N_* = N_*(\delta)$ such that for every $n \ge 0$, for every $q \in \mathrm{Hor}_n$, if $t \ge t_\theta + N_* \log(V(q))$, then*

$$\int P_q(x)\chi(x(t))\, d\bar{\mu}_n(x) \le 2\theta. \tag{A.38}$$

*Proof.* Let $\{\psi_i\}_{1 \le i \le N(q)}$ be the functions of Lemma A.21. Let $\varphi_i(x) = P_q(x)\psi_i(x)$. By Lemma A.9(v), Lemma A.19 and Lemma A.21(2),

$$\|\varphi_i\|_{\mathrm{Lip}} \le C_\delta V(q)^{N_1 + N_3 N_5}.$$

Here we use the product inequality $\|P_q\psi_i\|_{\mathrm{Lip}} \le \|P_q\|_{\mathrm{Lip}}\|\psi_i\|_{\mathrm{Lip}}$ and $\|\psi_i\|_{\mathrm{Lip}} \le \mathcal{C}^1(\psi_i)$.

For each $i$ we apply Proposition A.23 with $\varphi$ replaced by $\varphi_i$. Summing over $i$ and noting that $\int P_q(x)\, d\bar{\mu}_n(x) = 1$, we get

$$\Big| \int P_q(x)(1-\chi)(x(t))\, d\bar{\mu}_n(x) - \mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \Big| \le$$
$$\le N(q)(C_\delta V(q)^{N_1+N_3N_5})C_2 \cdot e^{-\kappa t} \le C'_\delta V(q)^{N_1+2N_3N_5} C_2 \cdot e^{-\kappa t}.$$

Let $N_* = 2(N_1 + N_3N_5)/\kappa$. Choose $t_\theta = t_\theta(\theta,\delta) > 0$ so that $C'_\delta C_2 e^{-\kappa t_\theta} \le \theta$. If $t \ge t_\theta + N_* \log V(q)$, then the right hand side of the preceding estimate is at most $C'_\delta C_2 e^{-\kappa t_\theta}$, since $\kappa N_* \ge N_1 + 2N_3N_5$ and $V(q) \ge 1$. The proposition follows.

□

Let $M$ be a large parameter. Let

$$V_M(q) = \max(V(q), M).$$

It follows from Lemma A.10 that for any $n$ and any $p \in \mathrm{Hor}_n$,

$$\frac{1}{V_M(p)} \int P_p(q) V_M(q(\tau'))\, d\bar{\mu}_n(q) \le \begin{cases} C_\delta e^{-3\tau'/4} + \frac{M}{V(p)} & \text{if} \quad V(p) \ge V'_0 \\ \frac{C_\delta + M}{V_M(p)} & \text{if} \quad V(p) \le V'_0. \end{cases} \tag{A.39}$$

For $\omega \in \Omega'_n$ and $0 \le j \le n$, define

$$\widehat{V}(\omega, j) = V_M(q_j) \cdot \exp\left( \sum_{i=1}^{j} \chi(q_i) \right).$$

**Lemma A.25.** *Suppose $\theta > 0$, $0 < \delta \le 1/400$ and suppose $\Omega$ is a* Mod*-compact subset of $\mathcal{PQ}_1\mathcal{T}$ with $\mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \ge 1 - \theta$. Then there exist $M = M(\theta,\delta) \ge 1$ and $\tau'_0 = \tau'_0(\theta,\delta) > 0$ such that for all $\tau' > \tau'_0$ with $\tau' \in \tau\mathbb{N}$ and all $n \ge 1$ we have*

$$\mathbb{E}'_n(\widehat{V}(\omega, n) \mid \mathcal{F}'_0) \le e^{8\theta n} V_M(q_0).$$

*Proof.* Choose $M = M(\theta,\delta) \ge 1$ so that

$$1 + \frac{eC_\delta}{M} + 2\theta(e-1) < e^{8\theta}, \tag{A.40}$$

where $C_\delta$ is as in (A.39). Choose $\tau'_0 = \tau'_0(\theta,\delta) > 0$ large enough that, for $\tau' > \tau'_0$, we have $V'_0 > M$ and $\tau' > t_\theta + N_* \log M$.

We claim that

$$\mathbb{E}'_n(\widehat{V}(\omega, j+1) | \mathcal{F}'_j) \le e^{8\theta} \cdot \widehat{V}(\omega, j). \tag{A.41}$$

Note that

$$\mathbb{E}'_n(\widehat{V}(\omega, j+1) \mid \mathcal{F}'_j) = \exp\left( \sum_{i=1}^{j} \chi(q_i) \right) \int P_{q_j}(q) e^{\chi(q(\tau'))} V_M(q(\tau'))\, d\bar{\mu}_j(q).$$

Dividing both sides by $\widehat{V}(\omega, j)$ we get

$$\frac{\mathbb{E}'_n(\widehat{V}(\omega, j+1) \mid \mathcal{F}'_j)}{\widehat{V}(\omega, j)} = \frac{1}{V_M(q_j)} \int P_{q_j}(q) e^{\chi(q(\tau'))} V_M(q(\tau'))\, d\bar{\mu}_j(q).$$

We consider three cases. First assume $V(q_j) \geq V_0'$. (We choose $\tau'$ large enough so that $V_0' > M$). Then, using (A.39) and the trivial bound $e^{\chi(q(\tau'))} \leq e$, we get

$$\frac{1}{V_M(q_j)} \int P_{q_j}(q) e^{\chi(q(\tau'))} V_M(q(\tau'))\, d\bar{\mu}_j(q) \leq e \cdot C_\delta e^{-3\tau'/4} + \frac{e \cdot M}{V_0'} \leq e \cdot (C_\delta + M) e^{-3\tau'/4}.$$

Now assume $V(q_j) < V_0'$ but $t_\theta + N_* \log V(q_j) \geq \tau'$. (In particular, with $\tau'$ chosen large enough, $V(q_j) \geq M$). Then, again using (A.39) and the trivial bound $e^{\chi(q(\tau'))} \leq e$, we get,

$$\frac{1}{V_M(q_j)} \int P_{q_j}(q) e^{\chi(q(\tau'))} V_M(q(\tau'))\, d\bar{\mu}_j(q) \leq \frac{e \cdot (C_\delta + M)}{V(q_j)} \leq e \cdot (C_\delta + M) e^{-(\tau' - t_\theta)/N_*}$$

Now suppose $V(q_j) < V_0'$ but $t_\theta + N_* \log V(q_j) < \tau'$. (This is the remaining case). We decompose

$$\int P_{q_j}(q) e^{\chi(q(\tau'))} V_M(q(\tau'))\, d\bar{\mu}_j(q) = I_1 + I_2,$$

where $I_1$ is the integral over $\{q \ : \ V(q(\tau')) > M\}$ and $I_2$ is the integral over $\{q \ : \ V(q(\tau')) \leq M\}$. We have, by Lemma A.10,

$$\frac{I_1}{V_M(q_j)} \leq \frac{e}{M} \int_{V(q(\tau')) \geq M} P_{q_j}(q) V(q(\tau'))\, d\bar{\mu}_j(q) \leq \frac{e \cdot C_\delta}{M}.$$

Also,

$$\begin{aligned} \frac{I_2}{V_M(q_j)} &\leq \int P_{q_j}(q) e^{\chi(q(\tau'))}\, d\bar{\mu}_j(q) \\ &= \int P_{q_j}(q)(1 + (e-1)\chi(q(\tau')))\, d\bar{\mu}_j(q) \\ &\leq 1 + 2\theta(e-1), \end{aligned}$$

where we have used Proposition A.24 for the last estimate.

Thus, combining the estimates in the three cases we get

$$\frac{\mathbb{E}_n'(\widehat{V}(\omega, j+1) | \mathcal{F}_j')}{\widehat{V}(\omega, j)} \leq \max\Big( e \cdot (C_\delta + M) e^{-3\tau'/4}, e \cdot (C_\delta + M) e^{-(\tau' - t_\theta)/N_*}, \frac{e \cdot C_\delta}{M} + 1 + 2\theta(e-1) \Big).$$

Thus, in view of (A.40), after increasing $\tau_0'$ if necessary, (A.41) holds for every $\tau' > \tau_0'$. The rest of the proof is the same as the proof of Proposition A.3. □

Let

$$S(\omega) = \{k \in [0, n] \ : \ q_k \in N_\delta(\Omega).\}.$$

**Lemma A.26.** *Let $\eta > 0$ and $0 < \delta \leq 1/400$. Then, for $\theta > 0$ sufficiently small depending on $\eta$, any* Mod*-compact $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ with $\mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \geq 1 - \theta$, any $\tau' \in \tau\mathbb{N}$ sufficiently large (depending on $\theta$, $\delta$) and any $n$ large enough,*

$$\mathbb{P}_n'\Big( |S(\omega)| < (1 - \eta) n \Big) < e^{-\frac{\eta}{2} n}.$$

*Proof.* Choose $0 < \theta < \eta/32$. Let $M = M(\theta, \delta)$ and $\tau_0' = \tau_0'(\theta, \delta)$ be as in Lemma A.25, and suppose $\tau' \in \tau\mathbb{N}$ with $\tau' > \tau_0'$. By Lemma A.8 and the finiteness of $\mathcal{A}$, choose $A \geq 1$ so that $\mathbb{E}_n' V(q_0) \leq A$ for every chart. The initial law is independent of $\tau$, $\tau'$, $\theta$, $\delta$, and $\Omega$, so $A$ can be chosen independently of these parameters. Increasing $\tau_0'$ if necessary, we may assume that $M < V_0' = e^{3\tau'/4}$. Hence

$$\mathbb{E}_n' V_M(q_0) \leq A + M \leq A + e^{\tau'}.$$

Then, by Lemma A.25,

$$\mathbb{E}_n'(\widehat{V}(\omega, n)) \leq (A + e^{\tau'})e^{8\theta n} \leq (A + e^{\tau'})e^{\frac{\eta}{4}n}.$$

Note that if $|S(\omega)| < (1-\eta)n$, we have $\widehat{V}(\omega, n) \geq e^{\eta n}$. Then, by the Markov inequality,

$$\mathbb{P}_n'\Big(|S(\omega)| < (1-\eta)n\Big) \leq (A + e^{\tau'})e^{-\frac{3\eta}{4}n} < e^{-\eta n/2}$$

provided $n > \frac{4}{\eta}\log(A + e^{\tau'})$. Thus the lower bound on $n$ can be chosen to depend only on $\eta$ and $\tau'$. The lemma follows.

□

**Lemma A.27.** *Suppose $0 < \eta < 1/2$ and $0 < \delta \leq 1/400$ and $\tau > 0$ are arbitrary, and suppose $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ is a* Mod*-compact set. Then there exists $\tau_0' = \tau_0'(\eta, \delta, \tau) > 0$ such that if $\tau' \in \tau\mathbb{N}$ and $\tau' > \tau_0'$, and if $k \in S(\omega) \cap G_+(\omega, \eta/16)$, then*

$$q_n(-n\tau' + k\tau') \in N_{2\delta}(\Omega).$$

*Proof.* Let $q = q_n(-n\tau')$. Since $k \in G_+(\omega, \eta/16)$, by Lemma A.13, $D_{\text{AGY}}(q_k, q(k\tau')) \leq d_{\text{AGY}}(q_k, q(k\tau')) \leq \delta$. Since $k \in S(\omega)$, we have $q_k \in N_\delta(\Omega)$. Therefore, $q(k\tau') \in N_{2\delta}(\Omega)$. □

**Corollary A.28** (Geodesic returns)**.** *For every $0 < \eta < 1/2$ there exist $\tau_0 = \tau_0(\eta) > 0$, $0 < \kappa_{\text{v}} = \kappa_{\text{v}}(\eta) < 1$, and $\theta_0 = \theta_0(\eta) > 0$ such that, for every $\tau \geq \tau_0$, every $0 < \theta < \theta_0$, every $0 < \delta \leq 1/400$, and every* Mod*–compact $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ satisfying*

$$\mu_{\text{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \geq 1 - \theta,$$

*there exists $\tau_0' = \tau_0'(\eta, \tau, \theta, \Omega, \delta) > 0$ such that if $\tau' > \tau_0'$ (and $\tau' \in \tau\mathbb{N}$), the following holds:*

*Let $n \geq 0$ be an integer, and set*

$$v_n(x) = \#\{1 \leq j \leq n : x(j\tau') \in N_{2\delta}(\Omega)\}.$$

*Then, for all sufficiently large $n$ (depending on $\eta$, $\tau$, $\theta$, $\Omega$, $\delta$, and $\tau'$) and every $\alpha \in \mathcal{A}$,*

$$\mu_0\big(\{x \in U_\alpha(0) : v_n(x) \leq (1 - \eta/2)n\}\big) \leq \kappa_{\text{v}}^n \mu_0(U_\alpha(0)).$$

*Proof.* Let $\epsilon = \eta/64$, and choose $\tau_0 = \tau_0(\eta)$ as in Lemma A.12, with $\epsilon$ in place of $\eta$. Choose $\theta_0 = \theta_0(\eta) > 0$ so that Lemma A.26 applies with $\eta/4$ in place of $\eta$. Suppose $\tau \geq \tau_0$ and $0 < \theta < \theta_0$.

Let

$$E_n^+ = \{|G_+(\omega, \epsilon)| \geq (1 - \eta/8)n\}, \qquad E_n^\Omega = \{|S(\omega)| \geq (1 - \eta/4)n\}.$$

Then, by Lemmas A.12 and A.26, for $\tau' \in \tau\mathbb{N}$ sufficiently large,

$$\mathbb{P}_n'((E_n^+)^c) \leq e^{-\epsilon^4 n\tau'/4}, \qquad \mathbb{P}_n'((E_n^\Omega)^c) \leq e^{-\eta n/8}$$

for all sufficiently large $n$. We also assume that $\tau' > 1$ and that $\tau'$ is large enough for Lemma A.27 to apply with $\eta/4$ in place of $\eta$.

Let $q = q_n(-n\tau')$. By Lemma A.27, $q(j\tau') \in N_{2\delta}(\Omega)$ for every $j \in G_+(\omega,\epsilon) \cap S(\omega)$. Hence, on $E_n^+ \cap E_n^\Omega$,

$$v_n(q) \geq \big|G_+(\omega,\epsilon) \cap S(\omega) \cap [1,n]\big| \geq (1 - 3\eta/8)n - 2 > (1-\eta/2)n$$

for sufficiently large $n$. Here we have omitted the index zero from the two sets.

By Lemma A.9(ii) and (A.9), the law of $q$ is

$$\frac{\mu_0'}{\mu_0'(\mathrm{Hor}_0)}.$$

Fix $\alpha \in \mathcal{A}$, and let $F_n = \{q \in U_\alpha(0)\}$. Since $\lambda_0 = 1$ on $U_\alpha(0)$, we have

$$\mathbb{P}_n'(F_n) = c_\alpha := \frac{\mu_0(U_\alpha(0))}{\mu_0'(\mathrm{Hor}_0)} > 0, \qquad \mathbb{P}_n'(q \in A \mid F_n) = \frac{\mu_0(A)}{\mu_0(U_\alpha(0))} \quad (A \subset U_\alpha(0)).$$

Since $\mu_0'(\mathrm{Hor}_0) < \infty$ by Lemma A.8, it follows that

$$\frac{\mu_0(\{x \in U_\alpha(0) : v_n(x) \leq (1-\eta/2)n\})}{\mu_0(U_\alpha(0))} \leq c_\alpha^{-1}\big(e^{-\epsilon^4 n\tau'/4} + e^{-\eta n/8}\big).$$

Choose $\kappa_{\mathrm{v}} = \kappa_{\mathrm{v}}(\eta)$ so that

$$\max\{e^{-\epsilon^4/4}, e^{-\eta/8}\} < \kappa_{\mathrm{v}} < 1.$$

Since $\tau' > 1$, the right hand side of the preceding estimate is at most $\kappa_{\mathrm{v}}^n$ for all sufficiently large $n$. Since $\mathcal{A}$ is finite, this holds for all $\alpha \in \mathcal{A}$ when $n$ is sufficiently large.

□

In the next definition, good statistics has the full backward-time meaning of Definition 7.2.

**Definition A.29.** For $R > 0$, we say $q \in \mathrm{Hor}_0$ is $(K,\eta,\delta,\Omega)$-generic at $R$ if the following hold:

(i) $q(R) \in N_{2\delta}(\Omega)$ and has $(K,\eta)$–good statistics, and
(ii) For all $0 \leq s < R$,

$$|\{u \in (s,R) : q(u) \text{ has } (K,\eta)\text{–good statistics}\}| \geq (1-\eta)(R-s).$$

**Proposition A.30.** *For any $1/2 > \eta > 0$, there exists $\kappa = \kappa(\eta) < 1$ and $\tau = \tau(\eta) > 0$ a* Mod*–compact set $K = K(\tau) \supset K_0$ and $\theta_0 = \theta_0(\eta) > 0$ such that the following holds: For any $0 < \theta < \theta_0$ and any $0 < \delta \leq 1/400$, for any* Mod*–compact $\Omega \subset \mathcal{PQ}_1\mathcal{T}$ with*

$$\mu_{\mathrm{MV}}(N_\delta(\Omega)/\operatorname{Mod}) \geq (1-\theta),$$

*there exists $\tau_0' = \tau_0'(\eta,\theta,\Omega,\delta)$ such that if $\tau' \in \tau\mathbb{N}$ and $\tau' > \tau_0'$ the following holds: Let $R_j = j\tau'$. Then, for*

$$k_n''(x) = \#\Big\{j \in [1,n] : x \text{ is } (N_\delta(K),\eta,\delta,\Omega)\text{-generic at } R_j\Big\},$$

*we have, for all sufficiently large $n$ (depending on $\tau'$, $\eta$, $\Omega$, $\delta$), and for all $\alpha \in \mathcal{A}$, that*

$$\mu_0(\{x \in U_\alpha(0) : k_n''(x) \leq (1-\eta)n\}) \leq \kappa^n \mu_0(U_\alpha(0)).$$

*Proof.* Choose $\tau = \tau(\eta)$ sufficiently large for Proposition A.18, with $\eta/4$ in place of $\eta$, and Corollary A.28. Let $K = K(\tau) \supset K_0$ and $0 < \kappa_{\mathrm{g}} = \kappa_{\mathrm{g}}(\eta) < 1$ be as in the proposition, and let $\theta_0 = \theta_0(\eta)$ and $\kappa_{\mathrm{v}} = \kappa_{\mathrm{v}}(\eta)$ be as in the corollary. Choose $\kappa = \kappa(\eta)$ so that

$$\max\{\kappa_{\mathrm{g}}, \kappa_{\mathrm{v}}\} < \kappa < 1.$$

Suppose $0 < \theta < \theta_0$, $0 < \delta \leq 1/400$, and $\Omega$ satisfies the measure hypothesis. Choose $\tau_0'$ large enough for both Proposition A.18 and Corollary A.28, and suppose $\tau' > \tau_0'$ (and $\tau' \in \tau\mathbb{N}$). Put $R_j = j\tau'$. Fix $\alpha \in \mathcal{A}$. For $x \in U_\alpha(0)$, let

$$\mathcal{I}_n(x) = \{1 \leq j \leq n : x \text{ is } (K, \eta/4)\text{–generic at } R_j\},$$
$$\mathcal{J}_n(x) = \{1 \leq j \leq n : x(R_j) \in N_{2\delta}(\Omega)\}.$$

Then, by Proposition A.18 and Corollary A.28,

$$\mu_0(\{x \in U_\alpha(0) : |\mathcal{I}_n(x)| \leq (1 - \eta/4)n\}) \leq \kappa_{\mathrm{g}}^n \mu_0(U_\alpha(0)),$$
$$\mu_0(\{x \in U_\alpha(0) : |\mathcal{J}_n(x)| \leq (1 - \eta/2)n\}) \leq \kappa_{\mathrm{v}}^n \mu_0(U_\alpha(0))$$

for all sufficiently large $n$.

Outside the union of these two exceptional sets,

$$|\mathcal{I}_n(x) \cap \mathcal{J}_n(x)| > (1 - 3\eta/4)n > (1 - \eta)n.$$

For every $j \in \mathcal{I}_n(x) \cap \mathcal{J}_n(x)$, we have $x(R_j) \in N_{2\delta}(\Omega)$ and $x$ is $(K, \eta/4)$–generic at $R_j$ in the sense of Definition A.17. Since $x \in U_\alpha(0)$ and $K \supset K_0$, by Lemma 7.3, the relative good statistics in that definition imply good statistics on the full backward geodesic, also at the intermediate times. Since $\eta/4 < \eta$ and $K \subset N_\delta(K)$, it follows that $j$ is counted by $k_n''(x)$.

It follows that

$$\mu_0(\{x \in U_\alpha(0) : k_n''(x) \leq (1 - \eta)n\})$$
$$\leq (\kappa_{\mathrm{g}}^n + \kappa_{\mathrm{v}}^n)\mu_0(U_\alpha(0)) \leq \kappa^n \mu_0(U_\alpha(0))$$

for all sufficiently large $n$. Since $\mathcal{A}$ is finite, this holds for all $\alpha \in \mathcal{A}$ when $n$ is sufficiently large. The proposition follows.

□

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