FIBERED 3–MANIFOLDS AND VEECH GROUPS

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Abstract. We study Veech groups associated to the pseudo-Anosov monodromies of fibers and foliations of a fixed hyperbolic 3-manifold. Assuming Lehmer's Conjecture, we prove that the Veech groups associated to fibers generically contain no parabolic elements. For foliations, we prove that the Veech groups are always elementary.

1. Introduction

A pseudo-Anosov homeomorphism $f\colon S \to S$ on a surface determines a complex structure and holomorphic quadratic differential, (X, q), up to Teichmüller deformation, for which the vertical and horizontal foliations are the stable and unstable foliations of f. The pseudo-Anosov generates an infinite cyclic subgroup of the full group of orientation preserving affine homeomorphisms, $\mathrm{Aff}_+(X, q)$.

For a finite type surface S, we say that the pseudo-Anosov homeomorphism f is *lonely* if $\langle f\rangle < \mathrm{Aff}_+(S, q)$ has finite index. The motivation for this paper is the following; see e.g. Hubert-Masur-Schmidt-Zorich [HMSZ06] and Lanneau [Lan17]

Conjecture 1.1 (Lonely p-As). *There exist lonely pseudo-Anosov homeomorphisms. In fact, lonely pseudo-Anosov homeomorphisms are generic.*

There is not an agreed upon notion of "generic", and some care must be taken: work of Calta [Cal04] and McMullen [McM03a, McM03b] shows that *no* pseudo-Anosov homeomorphism on a surface of genus 2, with orientable stable/unstable foliation is lonely. In fact, in this case, not only are the pseudo-Anosov homeomorphisms not lonely, but their Veech groups always contain parabolic elements.

In this paper, we consider infinite families of pseudo-Anosov homeomorphism arising as follows; see §2.1. Suppose $f\colon S \to S$ is a pseudo-Anosov homeomorphism of a finite type surface S and M_f the mapping torus (which is hyperbolic by Thurston's Hyperbolization Theorem [Ota98]). The connected cross sections of the suspension flow are organized by their cohomology classes (up to isotopy), which are primitive integral classes in the cone on the open fibered face $F \subset H^1(M, \mathbb{R})$ of the Thurston norm ball containing the Poincaré-Lefschetz dual of the fiber S. Given such an integral class α, the first return map to the cross section S_α is a pseudo-Anosov homeomorphism $f_\alpha\colon S_\alpha \to S_\alpha$. When $b_1(M) > 1$, there are infinitely many such pseudo-Anosov homeomorphisms; in fact, $|\chi(S_\alpha)|$ is a linear function of α, and hence tends to infinity with α.

We let $\bar{\alpha} \in F$ denote the projection of the primitive integral class α in the cone over F, and let $F_{\mathbb{Q}}$ be the set of all such projections, which is precisely the (dense) set of rational points in F.

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Question 1.2. Given a fibered hyperbolic 3–manifold and fibered face F, are the pseudo-Anosov homeomorphism f_α for $\bar{\alpha} \in F_{\mathbb{Q}}$ generically lonely?

We will provide two pieces of evidence that the answer to this question is 'yes'. Write $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ for the orientation preserving affine group containing f_α; see §2.3 for more details.

Theorem 1.3. *Suppose F is the fibered face of a fibered hyperbolic 3–manifold. Assuming Lehmer's Conjecture, the set of $\bar{\alpha} \in F_{\mathbb{Q}}$ such that $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ contains a parabolic element is discrete in F.*

In certain examples, the set of classes whose associated Veech group contains parabolics is actually finite (again, assuming Lehmer's conjecture); see Theorem 4.2. In §3 we describe some explicit computations that illustrate this finite set.

Much of the defining structure survives for non-integral classes $\alpha \in F - F_{\mathbb{Q}}$; see §2.2 for details. Briefly, we first recall that every $\alpha \in F - F_{\mathbb{Q}}$ is represented by a closed 1–form ω_α which is positive on the vector field generating the suspension flow. The kernel of ω_α is tangent to a foliation $\mathcal{F}_\alpha$, and the flow can be reparameterized to send leaves of $\mathcal{F}_\alpha$ to other leaves. There is no longer a first return time, but rather a *higher rank abelian group* of return times, H_α, to any given leaf S_α of $\mathcal{F}_\alpha$. Work of McMullen [McM00] associates a *leaf-wise* complex structure and quadratic differential (X_α, q_α) to each $\alpha \in F - F_{\mathbb{Q}}$ so that the leaf-to-leaf maps of the flow are all Teichmüller maps. For every leaf S_α of $\mathcal{F}_\alpha$, the return maps to S_α thus determine an isomorphism from $H_\alpha < \mathbb{R}$ to a subgroup we denote $H_\alpha^{\mathrm{Aff}} < \mathrm{Aff}_+(X_\alpha, q_\alpha)$, an abelian group of pseudo-Anosov elements. Our second piece of evidence for a positive answer to Question 1.2 is the following.

Theorem 1.4. *If F is a fibered face of a closed, fibered, hyperbolic 3–manifold, then for all $\alpha \in F - F_{\mathbb{Q}}$, and any leaf S_α of $\mathcal{F}_\alpha$, the abelian group $H_\alpha^{\mathrm{Aff}} < \mathrm{Aff}_+(X_\alpha, q_\alpha)$ has finite index.*

For $\alpha \in F - F_{\mathbb{Q}}$, the leaves S_α are infinite type surfaces. In general, there is much more flexibility in constructing affine groups for infinite type surfaces, and exotic groups abound. Indeed, work of Przyticki-Schmithusen-Valdez [PSV11] and Ramírez-Valdez [RMV17] proves that *any* countable subgroup of $\mathrm{GL}_2(\mathbb{R})$ without contractions is the derivative-image of some affine group. (See also Bowman [Bow13] for a "naturally occurring" lonely pseudo-Anosov homeomorphism on an infinite type surface of finite area.) Theorem 1.4 says that for the leaves S_α of the foliations and their associated quadratic differentials, the situation is much more rigid.

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2. Definitions and background

2.1. **Fibered 3–manifolds.** Here we explain the set up and background for our work in more detail. For a pseudo-Anosov homeomorphism $f\colon S \to S$ of a finite type surface S, let $\lambda(f)$ denote its *stretch factor* (also called its *dilatation*); see [FLP79]. We write

$$M = M_f = S \times [0,1]/(x,1) \sim (f(x),0)$$

to denote the mapping torus of the pseudo-Anosov homeomorphism f. The suspension flow ψ_s of f is generated by the vector field $\xi = \frac{\partial}{\partial t}$, where t is the coordinate on the $[0,1]$ factor. Alternatively, we have the local flow of the same name $\psi_s(x,t) = (x, t+s)$ on $S \times [0,1]$, defined for $t, s+t \in [0,1]$, which descends to the suspension flow.

A *cross section* (or just *section*) of the flow is a surface $S_\alpha \subset M$ transverse to ξ, such that for all $x \in S_\alpha$, $\psi_s(x) \in S_\alpha$ for some $s > 0$. If $s(x) > 0$ is the smallest such number, then the *first return map* of ψ_s is the map $f_\alpha \colon S_\alpha \to S_\alpha$ defined by $f_\alpha(x) = \psi_{s(x)}(x)$ for $x \in S_\alpha$. Note that $S(= S \times \{0\}) \subset M$ is a section, and the first return map to S is precisely the map $f = \psi_1|_S$.

Cutting open along an arbitrary section S_α we get a product $S_\alpha \times [0,1]$ where the slices $\{x\} \times [0,1]$ are arcs of flow lines. Thus, M can also be expressed as the mapping torus of f_α, or alternatively, M fibers over the circle with *monodromy* f_α. Up to isotopy, the fiber S_α is determined by its Poincaré-Lefschetz dual cohomology class $\alpha = [S_\alpha] \in H^1(M;\mathbb{Z}) \subset H^1(M;\mathbb{R}) = H^1(M)$. To see how these are organized, we first recall the following theorem of Thurston [Thu86]

Theorem 2.1. *For $M = M_f$ as above, there is a finite union of open, convex, polyhedral cones $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset H^1(M)$ such that $\alpha \in H^1(M;\mathbb{Z})$ is dual to a fiber in a fibration over S^1 if and only if $\alpha \in \mathcal{C}_j$ for some j. Moreover, there is a norm $\|\cdot\|_T$ on $H^1(M)$ so that for each $\mathcal{C}_j$, $\|\cdot\|_T$ restricted to $\mathcal{C}_j$ is linear, and if $\alpha \in \mathcal{C}_j \cap H^1(M;\mathbb{Z})$ then $\|\alpha\|_T$ is the negative of the Euler characteristic of the fiber dual to α.*

The unit ball $\mathfrak{B}$ of $\|\cdot\|_T$ is a polyhedron, and each $\mathcal{C}_j$ is the cone over the interior of a top dimensional face F_j of $\mathfrak{B}$.

The cones in the theorem are called the *fibered cones* of M and the F_j the *fibered faces* of $\mathfrak{B}$. It follows from Thurston's proof of Theorem 2.1 that each of the sections S_α of (ψ_s) described above must lie in a single one of the fibered cones $\mathcal{C}$ over a fibered face F. The following theorem elaborates on this, combining results of Fried from [Fri83, Fri82].

Theorem 2.2. *For $M = M_f$ as above, there is a fibered cone $\mathcal{C} \subset H^1(M)$ such that $\alpha \in H^1(M;\mathbb{Z})$ is dual to a section of (ψ_s) if and only if $\alpha \in \mathcal{C}$. Moreover, there is a function $\mathfrak{h} \colon \mathcal{C} \to \mathbb{R}_+$ which is continuous, convex, and homogenous of degree -1, with the following properties.*

- *For any $\alpha \in \mathcal{C} \cap H^1(M;\mathbb{Z})$, f_α is pseudo-Anosov and $\mathfrak{h}(\alpha) = \log(\lambda(f_\alpha))$.*
- *For any $\{\alpha_n\} \subset \mathcal{C}$ with $\alpha_n \to \partial\mathcal{C}$, we have $\mathfrak{h}(\alpha_n) \to \infty$;*

We let $\mathcal{C}_\mathbb{Z} \subset \mathcal{C}$ denote the primitive integral classes in the fibered cone $\mathcal{C}$; that is, the integral points which are not nontrivial multiples of another element of $H^1(M;\mathbb{Z})$. These correspond precisely the the connected sections of (ψ_s).

McMullen [McM00] refined the analysis of $\mathfrak{h}$, proving for example that it is actually real-analytic. For this, he computed the stretch factors using his *Teichmüller polynomial* $\Theta_\mathcal{C}$. This polynomial

$$\Theta_\mathcal{C} = \sum_{g \in G} a_g g$$

is an element of the group ring $\mathbb{Z}[G]$ where $G = H_1(M;\mathbb{Z})/\text{torsion}$. For $\alpha \in \mathcal{C}_\mathbb{Z}$, the *specialization* of the Teichmüller polynomial is

$$\Theta^\alpha_\mathcal{C}(t) = \sum_{g \in G} a_g t^{\alpha(g)} \in \mathbb{Z}[t^{\pm 1}]$$

where we view $\alpha \in H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{G};\mathbb{Z})$. Further, $G \cong H \oplus \mathbb{Z}$ where $H = \mathrm{Hom}(H^1(S,\mathbb{Z})^f, \mathbb{Z}) \cong \mathbb{Z}^m$ and $H^1(S,\mathbb{Z})^f$ are the f–invariant cohomology classes. So we can regard $\Theta_{\mathcal{C}}$ as a Laurent polynomial on the generators $x_1, x_2, \ldots, x_m$ of H and the generator u of $\mathbb{Z}$. Then specialization to the dual of an element $(a_1, a_2, \ldots, a_m, b) \in \mathcal{C} \cap H^1(M;\mathbb{Z})$ amounts to setting $x_i = t^{a_i}$ for $1 \leqslant i \leqslant m$ and $u = t^b$. McMullen proves that the specializations and the pseudo-Anosov first return maps are related by the following.

Theorem 2.3. *For any $\alpha \in \mathcal{C}_{\mathbb{Z}}$, the stretch factor $\lambda(f_\alpha)$ is a root of $\Theta^\alpha_{\mathcal{C}}$ with the largest modulus.*

Combining the linearity of $\|\cdot\|_T$ on $\mathcal{C}$ together with the homogeneity of $\mathfrak{h}$, we have the following observation of McMullen; see [McM00].

Corollary 2.4. *The function $\alpha \mapsto \|\alpha\|_T \mathfrak{h}(\alpha)$ is continuous and constant on rays from 0. In particular, if $K \subset \mathcal{C}$ is any compact subset, then $\|\cdot\|_T \mathfrak{h}(\cdot)$ is bounded on $\mathbb{R}_+ K$.*

The key corollary for us is the following, also observed by McMullen from the same paper.

Corollary 2.5. *If $\{\alpha_n\}_n \subset \mathcal{C}_{\mathbb{Z}}$ is any infinite sequence of distinct elements, then $|\chi(S_{\alpha_n})| \to \infty$ and if the rays $\mathbb{R}_+\alpha_n$ do not accumulate on $\partial\mathcal{C}$, then*

$$\log(\lambda(f_{\alpha_n})) \asymp \frac{1}{|\chi(S_{\alpha_n})|}.$$

In particular, $\lambda(f_{\alpha_n}) \to 1$.

Remark 2.6. One can sometimes promote the final conclusion to *any* infinite sequence of distinct elements, without the assumption about non-accumulation to $\partial\mathcal{C}$; see the examples in §3. This is not always the case, and the accumulation set of stretch factors can be fairly complicated, as described by work of Landry-Minsky-Taylor [LMT21].

2.2. **Foliations in the fibered cone.** Fried's work described above [Fri83, Fri82] implies that any $\alpha \in \mathcal{C}$ may be represented by a closed 1–form ω_α for which $\omega_\alpha(\xi) > 0$ at every point of M. For integral classes, ω_α is the pull-back of the volume form from the fibration over the circle $\mathbb{R}/\mathbb{Z}$, and in general, ω_α is a convex combination of such 1–forms. The kernel of ω_α defines a foliation $\mathcal{F}_\alpha$ transverse to ξ whose leaves are injectively immersed surfaces $S_\alpha \subset M$. We consider the reparameterized flow $\{\psi^\alpha_s\}$ defined by scaling the generating vector field ξ by $\xi/\omega_\alpha(\xi)$. Then for every leaf $S_\alpha \subset M$ of $\mathcal{F}_\alpha$ and for every $s \in \mathbb{R}$, the image by the flow $\psi^\alpha_s(S_\alpha)$ is another leaf of $\mathcal{F}_\alpha$. The subgroup $H_\alpha < \mathbb{R}$ mentioned in the introduction is precisely the set of return times of ψ^α_s to S_α. As such, H_α acts on S_α so that $s \in H_\alpha$ acts by $s \cdot x = \psi^\alpha_s(x)$, for all $x \in S_\alpha$.

The group $H_\alpha \cong \mathbb{Z}^n$ for some $n = n_\alpha \leqslant b_1(M)$, and can alternatively be defined as the set of periods of α (i.e. the α–homomorphic image of $H_1(M;\mathbb{Z})$). A leaf S_α is a closed surface, and in fact a fiber as above if and only if $n_\alpha = 1$ in which case H_α is a discrete subgroup of $\mathbb{R}$ and $\bar{\alpha} \in F_{\mathbb{Q}}$. On the other hand, $n_\alpha \geqslant 2$ if and only if the group of return times H_α is indiscrete, and so S_α is *dense* in M.

2.3. **Teichmüller flows and Veech groups.** In [McM00], McMullen defines a conformal structure and quadratic differential, (X_α, q_α), on the leaves S_α of the foliation $\mathcal{F}_\alpha$, for all $\alpha \in \mathcal{C}$, with the following properties. For each $s \in \mathbb{R}$ and leaf S_α, the leaf-to-leaf map $\psi_s^\alpha \colon S_\alpha \to \psi_s^\alpha(S_\alpha)$ is a Teichmüller map with initial/terminal quadratic differentials given by q_α on the respective leaves. In fact, there exists some $K_\alpha > 1$ so that ψ_s^α is a $K_\alpha^{|s|}$–Teichmüller map, and hence $K_\alpha^{2|s|}$–quasi-conformal.

Remark 2.7. The notation (X_α, q_α) is somewhat ambiguous: this really denotes a family of structures, one on every leaf, though we abuse notation and also use this same notation to denote the restriction to any given leaf.

The vertical and horizontal foliations of q_α on the leaves S_α of $\mathcal{F}_\alpha$ are obtained by intersecting with a *fixed* singular foliation on the 3–manifold; namely, the suspension of the unstable/stable foliations for the original pseudo-Anosov homeomorphism f. In particular, the cone points (i.e. zeros) of q_α are precisely the intersections of S_α with the ψ_s–flowlines through the cone points on the original surface S. Consequently, the cone points are isolated, and the cone angles are bounded by those of the original surface, and are hence bounded independent of α.

For $s \in H_\alpha$, $\psi_s^\alpha \colon S_\alpha \to S_\alpha$ is (a remarking) of the Teichmüller map, and thus an affine pseudo-Anosov homeomorphism with respect to q_α. In this way, we obtain an isomorphism from H_α to a subgroup $H_\alpha^{\mathrm{Aff}} < \mathrm{Aff}_+(X_\alpha, q_\alpha)$, the group of orientation preserving affine homeomorphisms of the leaf S_α with respect to (X_α, q_α). The derivative with respect to the preferred coordinates defines a map

$$D_\alpha \colon \mathrm{Aff}_+(X_\alpha, q_\alpha) \to \mathrm{GL}_2^+(\mathbb{R})/\pm I,$$

which is called the *Veech group* of (X_α, q_α). A *parabolic* element of $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ is one whose image by D_α is parabolic.

Remark 2.8. The preferred coordinates for a quadratic differential are only defined up to translation and rotation through angle π, so the derivative is only defined up to sign. If all affine homeomorphisms are area preserving (e.g. if the surface has finite area) then the derivative maps to $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm \mathrm{I}$.

Since the vertical/horizontal foliations are the stable/unstable foliations, the image of H_α^{Aff}, which we denote $H_\alpha^D = D_\alpha(H_\alpha^{\mathrm{Aff}})$ is contained in the diagonal subgroup of $\mathrm{PSL}_2(\mathbb{R})$:

$$H_\alpha^D < \Delta = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; a > 0 \right\} / \pm I.$$

Define $\mathrm{SAff}(X_\alpha, q_\alpha) < \mathrm{Aff}_+(X_\alpha, q_\alpha)$ to be the area preserving subgroup of orientation preserving affine homeomorphisms; this is the preimage of $\mathrm{PSL}_2(\mathbb{R})$ under D_α. In particular, $H_\alpha^{\mathrm{Aff}} < \mathrm{SAff}(X_\alpha, q_\alpha)$.

2.4. **Trace fields.** A number field is *totally real* if the image of every embedding into $\mathbb{C}$ lies in $\mathbb{R}$. Hubert-Lanneau [HL06] proved the following.

Theorem 2.9. *If a nonelementary Veech group contains a parabolic element, then the trace field is totally real.*

A pseudo-Anosov f being lonely implies that there are no parabolic elements in the Veech group, but not conversely; see [HLM09].

McMullen [McM03b, Corollary 9.6] proved the following fact about the trace field of a Veech group; see also Kenyon-Smillie [KS00].

Theorem 2.10. *The trace field of a Veech group containing a pseudo-Anosov is generated by the trace of that pseudo-Anosov. That is, the trace field is given by* $\mathbb{Q}(\lambda(f) + \lambda(f)^{-1})$.

Thus, this trace field is totally real precisely when the trace of the pseudo-Anosov has only real Galois conjugates.

2.5. **Lehmer's Conjecture.** Theorem 1.3 is dependent on the validity of what is known as Lehmer's conjecture [Leh33] though Lehmer did not actually conjecture the statement we will use. See [Smy08]. To state this conjecture, we need the following.

Definition 2.11. Let $p(x) \in \mathbb{C}[x]$ with factorization over $\mathbb{C}$

$$p(x) = a_0 \prod_{i=1}^{m} (x - \gamma_i).$$

The **Mahler measure** of p is

$$\mathcal{M}(p) = |a_0| \prod_{i=1}^{m} (\max 1, |\gamma_i|).$$

With this definition, we state the conjecture we assume.

Conjecture 2.12 (Lehmer). *There is a constant* $\mu > 1$ *such that for every* $p(x) \in \mathbb{Z}[x]$ *with a root not equal to a root of unity* $\mathcal{M}(p) \geqslant \mu$.

3. Examples

Here we provide examples of fibered faces of fibered 3-manifolds and examine arithmetic features of the Veech groups of the corresponding pseudo-Anosov homeomorphisms.

3.1. **Example 1.** Let $\beta = \sigma_1 \sigma_2^{-1}$ be an element of the braid group B_3 on three strands (viewed as the mapping class group of a four-punctured sphere, S), where σ_1 and σ_2 denote the standard generators. Let M denote the mapping torus of β. McMullen computes the Teichmüller polynomial for this manifold in detail in [McM00]. See also Hironaka [Hir10].

Since β permutes the strands of the braid cyclically, $b_1(M) = 2$. Choosing appropriate bases, we obtain an isomorphism $H^1(M;\mathbb{Z}) \cong \mathbb{Z}^2$ so that the starting fiber surface S is dual to $(0, 1)$, the fibered cone is

$$\mathcal{C} = \{(a, b) \in \mathbb{R}^2 : b > 0, -b < a < b\}$$

and the Teichmüller polynomial for this cone is

$$\Theta_{\mathcal{C}}(x, u) = u^2 - u(x + 1 + x^{-1}) - 1.$$

Specialization to an integral class $(a, b) \in \mathcal{C}_{\mathbb{Z}}$ equates to setting $x = t^a$ and $u = t^b$ and yields

$$\Theta_{\mathcal{C}}^{(a,b)}(t) = \Theta_{\mathcal{C}}(t^a, t^b) = t^{2b} - t^{b+a} - t^b - t^{b-a} + 1.$$

We used the mathematics software system SageMath [S$^+$21] to factor $\Theta_{\mathcal{C}}^{(a,b)}(t)$ for all primitive integral pairs $(a, b) \in \mathcal{C}$ with $b < 50$, to determine the stretch factors $\lambda_{(a,b)}$ of the corresponding monodromies and their minimal polynomials. We then computed the conjugates of the corresponding traces, $\lambda_{(a,b)} + 1/\lambda_{(a,b)}$, to determine whether the trace field of each associated Veech group is totally real. The results are shown in Figure 1. Recall that by Theorem 2.9, when this trace field is not totally real, the Veech group has no parabolic elements.

These computations suggest that there are only finitely many pairs (a, b) where the trace field is not totally real. This is not a coincidence as we will see below. For this, we record the following improvement on Corollary 2.5 for the cone $\mathcal{C}$ for this example.

Lemma 3.1. *For any sequence* $\alpha_n = (a_n, b_n) \in \mathcal{C}_{\mathbb{Z}}$ *of distinct elements, we have* $\lambda(f_{\alpha_n}) \to 1$.

Proof. Since $\mathfrak{h}$ is convex, the maximum value of $\mathfrak{h}(a, b) = \log(\lambda(f_{(a,b)}))$, for points $(a, b) \in \mathcal{C}_{\mathbb{Z}}$ and a fixed b, occurs at either $(b-1, b)$ or $(1-b, b)$.

First we consider the points of the form $(b-1, b)$. The specialization of $\Theta_{\mathcal{C}}$ in this case takes the form

$$\Theta_{\mathcal{C}}^{(b-1,b)}(t) = t^{2b} - t^{2b-1} - t^b - t + 1.$$

Recall that $\lambda_b = \lambda(f_{(b-1,b)}) > 1$. As $b \to \infty$, we claim that $\lambda_b \to 1$. Suppose instead that the sequence is bounded below by $1 + \epsilon$, for $\epsilon > 0$ on some subsequence. Then in this subsequence we have

$$\begin{aligned}\Theta_{\mathcal{C}}^{(b-1,b)}(\lambda_b) &= \lambda_b^{2b}(1 - \lambda_b^{-1} - \lambda_b^{-b} - \lambda_b^{1-2b}) + 1 \\ &\geqslant (1+\epsilon)^{2b}\left(1 - (1+\epsilon)^{-1} - (1+\epsilon)^{-b} - (1+\epsilon)^{1-2b}\right)\end{aligned}$$

The first factor on the right hand side tends to infinity when b does, while the second factor tends toward $1 - (1+\epsilon)^{-1} = \epsilon/(1+\epsilon) > 0$. This implies that $\Theta_{\mathcal{C}}^{(b-1,b)}(\lambda_b)$ approaches infinity, whereas instead it is identically equal to 0. This contradiction proves the claim.

For points of the form $(1-b, b)$, the specialization takes the form

$$\Theta_{\mathcal{C}}^{(1-b,b)}(t) = t^{2b} - t - t^b - t^{2b-1} + 1 = \Theta_{\mathcal{C}}^{(b-1,b)}(t).$$

Therefore, $\lambda(f_{(1-b,b)}) = \lambda(f_{(b-1,b)}) = \lambda_b$ and as $b \to \infty$, these both tend to 1. $\square$

One of the difficulties in the proof of Theorem 1.3 is understanding the degrees of the trace field. This is complicated by the fact that the Teichmüller polynomial need not be irreducible in general. For example, when specialized to $(a, b) = (9, 14)$, the Teichmüller polynomial in this example splits into the cyclotomic polynomials $t^2 - t + 1$ and $t^4 - t^2 + 1$, plus the minimal polynomial of the corresponding stretch factor. However, in other cases, such as the specialization to $(a, b) = (5, 14)$, the Teichmüller polynomial remains irreducible. We refer the reader to [FG22] for more on the factorizations of the specialized polynomials in the example above. As we will see in the example below, the Teichmüller polynomial also sometimes admits additional non-cyclotomic factors aside from the minimal polynomial of the corresponding stretch factor.

3.2. **Example 2.** Let $\beta' = \beta^2$, for β from the preceding example. Let M' denote the mapping torus on β' and $\theta'_{\mathcal{C}'}$ the Teichmüller polynomial of the fibered cone $\mathcal{C}'$ containing the dual of β'. Here we will observe three different splitting behaviors of specializations of the Teichmüller polynomial. In particular, we see that certain specializations of $\theta'_{\mathcal{C}'}$ split into multiple non-cyclotomic factors, limiting what information can be derived about conjugates of the corresponding stretch factors and their traces by looking at the collection of all roots of $\theta'_{\mathcal{C}'}$.

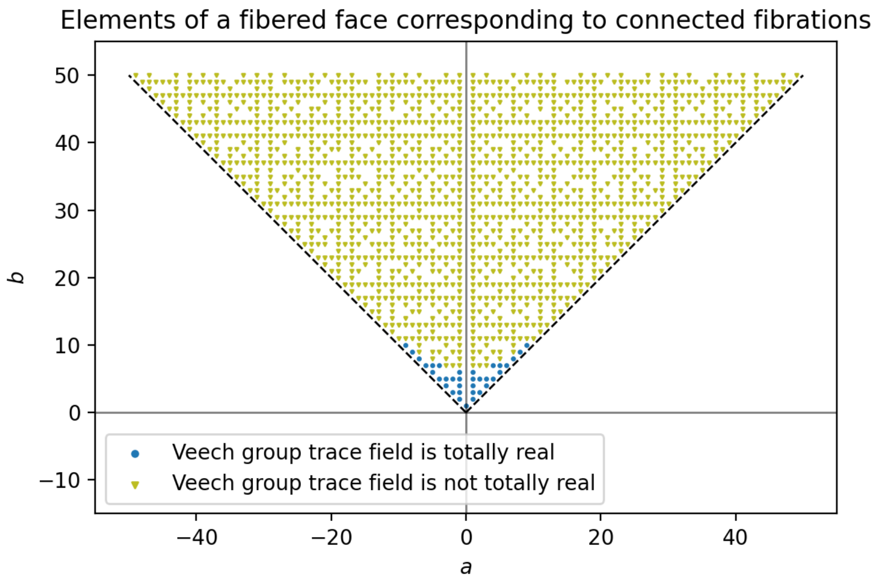

FIGURE 1. Primitive integral elements in a fibered cone for the mapping torus of the three-strand braid $\sigma_1\sigma_2^{-1}$. Elements marked with green triangles have corresponding Veech group with trace field that is not totally real.

The Teichmüller polynomial here is

$$\theta'_{\mathcal{C}'}(x,u) = u^2 - u(x^2 + 2x + 1 + 2x^{-1} + x^{-2}) + 1$$

over the cone

$$\mathcal{C} = \{(a,b) \in \mathbb{R}^2 : b > 0, -b/2 < a < b/2\}.$$

The specialization to $(a,b) = (6,17)$ is irreducible over $\mathbb{Z}$:

$$t^{34} - t^{29} - 2t^{23} - t^{17} - 2t^{11} - t^5 + 1,$$

while the specialization to $(a,b) = (7,17)$ splits as a cyclotomic and non-cyclotomic factor:

$$(t^4 + t^3 + t^2 + t + 1)(t^{30} - t^{29} - t^{27} + t^{26} + t^{25} - t^{24} - t^{22} + t^{21} - t^{20} + t^{19} - t^{17} + t^{16} - t^{15} + t^{14} - t^{13} + t^{11} - t^{10} + t^9 - t^8 - t^6 + t^5 + t^4 - t^3 - t + 1),$$

and the specialization to $(a,b) = (7,18)$ has multiple non-cyclotomic factors:

$$(t^2 - t + 1)(t^4 + t^3 + t^2 + t + 1)(t^{12} - t^9 - t^8 + t^7 + t^6 + t^5 - t^4 - t^3 + 1)(t^{18} - t^{16} - t^9 - t^2 + 1).$$

Figure 2 shows whether the Veech groups corresponding to elements of $\mathcal{C}'$ have totally real trace field. For all three specializations described in this example, the corresponding Veech group trace field is not totally real.

The analog to Lemma 3.1 holds in this example as well. M' is a 2-fold cover of M so the stretch factors in $\mathcal{C}'_{\mathbb{Z}}$ are at most squares of the stretch factors in $\mathcal{C}_{\mathbb{Z}}$.

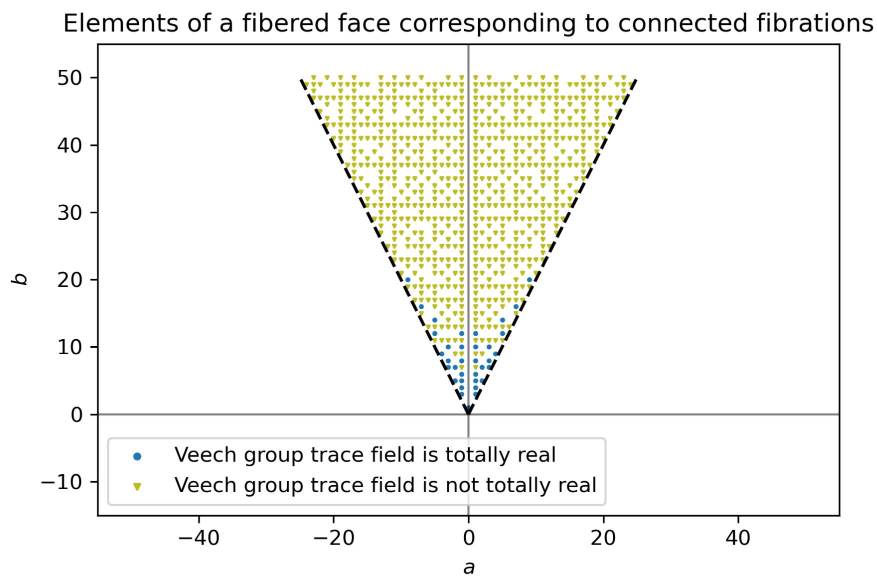

FIGURE 2. Primitive integral elements in a fibered cone for the mapping torus of the three-strand braid $(\sigma_1\sigma_2^{-1})^2$. Elements marked with green triangles have a not totally real corresponding Veech group.

4. MOST VEECH GROUPS HAVE NO PARABOLICS

We are now ready for the proof of the first theorem from the introduction.

Theorem 1.3. *Suppose F is the fibered face of a fibered hyperbolic 3–manifold. Assuming Lehmer's Conjecture, the set of $\bar{\alpha} \in F_{\mathbb{Q}}$ such that $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ contains a parabolic element is discrete in F.*

Proof. Consider any sequence of distinct elements α_n in $\mathcal{C}_{\mathbb{Z}}$ such that $\bar{\alpha}_n$ does not accumulate on ∂F. We need to show that $\mathrm{Aff}(X_\alpha, q_{\alpha_n})$ contains a parabolic for at most finitely many n. According to Theorem 2.9, it suffices to prove that the trace field is totally real for at most finitely many n. Setting $\lambda_n = \lambda(f_{\alpha_n})$, Theorem 2.10 implies that the trace field of $\mathrm{Aff}(X_{\alpha_n}, q_{\alpha_n})$ is $\mathbb{Q}(\lambda_n + \lambda_n^{-1})$.

Next, let N be the number of terms of the Teichmüller polynomial, $\Theta_{\mathcal{C}}$ for $\mathcal{C}$. The stretch factor λ_n is the largest modulus root of the specialization $\Theta_{\mathcal{C}}^{\alpha_n}(t)$ by Theorem 2.3. We observe that this polynomial has no more nonzero terms than $\Theta_{\mathcal{C}}$, and thus has at most N terms. Descartes's rule of signs implies that the number of real roots of $\Theta_{\mathcal{C}}^{\alpha_n}$ is at most $2N - 2$.

Suppose that $p_n(t)$ is the minimal polynomial of λ_n, which is thus a factor of $\Theta_{\mathcal{C}}^{\alpha_n}(t)$ (up to powers of t, which we will ignore). In particular, note that λ_n bounds the modulus of all other roots of $p_n(t)$. The stretch factors are always algebraic integers, and hence $p_n(t)$ is monic. The Mahler measure is therefore the product of the moduli of the roots outside the unit circle. There are at most $2N - 2$ real roots of $\Theta_{\mathcal{C}}^{\alpha_n}(t)$, and hence the same is true of $p_n(t)$. Write

$$\mathcal{M}(p_n) = A_n B_n$$

where A_n is the product of the moduli of the *real roots* and B_n is the product of the moduli of the non-real roots outside the unit circle (and 1 if there are none). Thus, we have

$$A_n \leqslant \lambda_n^{2N-2}. \tag{1}$$

Now, as $n \to \infty$, we have $|\chi(S_{\alpha_n})| = \|\alpha_n\|_T \to \infty$ as $n \to \infty$. Since $\bar{\alpha}_n$ does not accumulate on ∂F, Corollary 2.5 implies $\lambda_n = \lambda(f_{\alpha_n}) \to 1$. By (1), it follows that $A_n \to 1$ as $n \to \infty$. Since we are assuming Lehmer's Conjecture, it follows that $B_n > 1$ for all but finitely many n. That is, there is at least one non-real root ζ_n of $p_n(t)$ outside the unit circle. (In fact, the number of such roots tends to infinity linearly with $|\chi(S_{\alpha_n})|$ since λ_n has the maximum modulus of any root of $p_n(t)$).

Therefore, for all but finitely many n, the embedding of $\mathbb{Q}(\lambda_n + \lambda_n^{-1})$ to $\mathbb{C}$ sending $\lambda_n + \lambda_n^{-1}$ to $\zeta_n + \zeta_n^{-1}$ has non-real image, since ζ_n is non-real and lies off the unit circle. Therefore, $\mathbb{Q}(\lambda_n + \lambda_n^{-1})$ is totally real for at most finitely many n, as required. □

Remark 4.1. The proof of Theorem 1.3 follows a strategy of Craig Hodgson, [Hod], for understanding trace fields under hyperbolic Dehn filling.

The key ingredient is that for sequences $\{\alpha_n\}$ in $\mathcal{C}_{\mathbb{Z}}$ we $\lambda(f_{\alpha_n}) \to 1$.

Theorem 4.2. *Suppose F is the fibered face of a fibered hyperbolic 3–manifold and that 1 is the only accumulation point of the set*

$$\{\lambda(f_\alpha) \mid \bar{\alpha} \in F_{\mathbb{Q}}\}.$$

Assuming Lehmer's Conjecture, the set of $\bar{\alpha} \in F_{\mathbb{Q}}$ such $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ contains a parabolic element is finite.

Proof. This is exactly the same as the proof of Theorem 1.3, except that the assumption that 1 is the only accumulation point of $\{\lambda(f_\alpha) \mid \bar{\alpha} \in F_{\mathbb{Q}}\}$ replaces the references to Corollary 2.5, and does away with the requirement that $\bar{\alpha}_n$ does not accumulate on ∂F. □

Returning to the examples from Section 3, Lemmas 3.1 and the discussion in both implies that the hypotheses of Theorem 4.2 are satisfied. Thus only finitely many elements $\alpha \in \mathcal{C}_{\mathbb{Z}}$ are such that $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ can contain parabolics. We refer the reader to [LMT21] for more on accumulation set of $\{\lambda(f_\alpha) \mid \alpha \in \mathcal{C}_{\mathbb{Z}}\}$

5. Veech groups of leaves

We now turn our attention to the non-integral points in the cone and the second theorem from the introduction.

Theorem 1.4. *If F is a fibered face of a closed, fibered, hyperbolic 3–manifold, then for all $\alpha \in F - F_{\mathbb{Q}}$, and any leaf S_α of $\mathcal{F}_\alpha$, the abelian group $H_\alpha^{\mathrm{Aff}} < \mathrm{Aff}_+(X_\alpha, q_\alpha)$ has finite index.*

For the rest of the paper, we assume M is a closed, fibered, hyperbolic 3–manifold. The results of this section are only nontrivial if $b_1(M) > 1$, since otherwise $F - F_{\mathbb{Q}} = \varnothing$ for any fibered face F (since in that case $F = F_{\mathbb{Q}}$ is a point).

Given $\alpha \in F$, we recall that ψ_s^α is the reparameterized flow as in §2.2, that sends leaves of $\mathcal{F}_\alpha$ to leaves. Furthermore, (X_α, q_α) is the leaf-wise conformal structure and quadratic differential, and there is $K_\alpha > 1$ so that ψ_s^α is the $K_\alpha^{|s|}$–Teichmüller map, hence $K_\alpha^{2|s|}$–quasi-conformal and $K_\alpha^{|s|}$–bi-Lipschitz.

Lemma 5.1. *For any $\alpha \in F - F_{\mathbb{Q}}$ there exists a compact subsurface $Z \subset S_\alpha$ such that*

$$M = \bigcup_{s \in [0,1]} \psi_s^\alpha(Z).$$

Proof. Choose an exhaustion of S_α by a sequence of compact subsurfaces:

$$Z_1 \subsetneq Z_2 \subsetneq Z_3 \subsetneq \cdots S_\alpha, \text{ and } \bigcup_{n=1}^{\infty} Z_n = S_\alpha,$$

and observe that

$$\left\{ \bigcup_{s \in (0,1)} \psi_s^\alpha(\mathrm{int}(Z_n)) \right\}_{n=1}^{\infty}$$

is an open cover of M since every leaf is dense. Since M is compact, the open cover admits a finite subcover of M. As the compact surfaces Z_n are nested, there exists an index N such that for $Z = Z_N$ we have

$$M = \bigcup_{s \in [0,1]} \psi_s^\alpha(Z).$$

□

The isomorphism $H_\alpha \cong H_\alpha^{\mathrm{Aff}}$ is given by $s \mapsto \psi_s^\alpha|_{S_\alpha}$. We write

$$H_\alpha^{\mathrm{Aff}}[0,1] \subset H_\alpha^{\mathrm{Aff}}$$

for the image of $H_\alpha \cap [0,1]$ under this isomorphism. Note that every element of H_α^{Aff} is K_α^2–quasi-conformal and K_α–bi-Lipschitz since $s \leqslant 1$. As a consequence of Lemma 5.1, we have the following.

Corollary 5.2. *For $\alpha \in F - F_{\mathbb{Q}}$ and $Z \subset S_\alpha$ as in Lemma 5.1 we have*

$$S_\alpha = \bigcup_{h \in H_\alpha^{\mathrm{Aff}}[0,1]} h(Z).$$

Proof. Let $Z \subset S_\alpha$ be the compact subsurface from Lemma 5.1, so that for every $x \in S_\alpha \subseteq M$, we have $x \in \psi_s^\alpha(Z)$ for some $s \in [0,1]$. Since $x \in S_\alpha$, this implies that $s \in H_\alpha$. Therefore

$$S_\alpha = \bigcup_{s \in H_\alpha \cap [0,1]} \psi_s^\alpha(Z) = \bigcup_{h \in H_\alpha^{\mathrm{Aff}}[0,1]} h(Z).$$

□

Corollary 5.3. *For any $\alpha \in F - F_{\mathbb{Q}}$ there exists $C > 0$ so that for any leaf S_α of $\mathcal{F}_\alpha$, the geometry of q_α is bounded. Specifically, (1) there is a lower bound on the length of any saddle connection, in particular a lower bound on the distance between any two cone points, (2) all cone points have finite (uniformly bounded) cone angle, and (3) (X_α, q_α) is complete.*

Proof. Let S_α be any leaf, and consider the compact surface Z from Corollary 5.2. By making Z slightly larger, we can assume that no singular points of q_α lie on the boundary of Z. Denote the set of all singularities of q_α by A. Let $d_{\partial Z}(a)$ denote the distance of a singularity $a \in A$ to the boundary of Z, and let $d_Z(a,b)$ denote the minimal length of a saddle connection in Z between two (not necessarily distinct) singularities $a, b \in A \cap Z$. Since Z is compact, we have that

$$\epsilon = \min\left\{ \min_{a,b\in A\cap Z} d_Z(a,b), \min_{a\in A} d_{\partial Z}(a) \right\} > 0.$$

Pick a saddle connection ω connecting any singularity a to any singularity b. There exists an $h \in H_\alpha^{\mathrm{Aff}}[0,1]$ such that $h(Z)$ contains a. Since h is K_α–bi-Lipschitz, either ω is contained in $h(Z)$ and has length at least ϵK_α^{-1}, or it leaves $h(Z)$ and we again deduce that ω has length at least the distance from a to $\partial h(Z)$, which is at least ϵK_α^{-1}. In either case, we obtain a uniform lower bound ϵK_α^{-1} to the length of ω, proving (1).

As was noted in Section 2.3, we have that all cone points have finite cone angle which proves (2). Since Z is compact, there is an ϵ' so that the ϵ'–neighborhood of Z also has compact closure, which is thus complete. Any Cauchy sequence has a tail that is contained in the h-image of the closure of this neighborhood for some $h \in H_\alpha^{\mathrm{Aff}}[0,1]$. Since this h–image is also complete, the Cauchy sequence converges, and we have that (X_α, q_α) is complete which proves (3). □

Remark 5.4. Note that Corollary 5.3 implies that our surfaces are tame in the sense of Definition 2.1 of [PSV11].

An important observation is the following: for any element of $g \in \mathrm{Aff}_+(X_\alpha, q_\alpha)$, we can choose some element $h \in H_\alpha^{\mathrm{Aff}}[0,1]$ so that $h \circ g(Z) \cap Z \neq \varnothing$, and furthermore, if g is K–quasi-conformal, then $h \circ g$ is (KK_α^2)–quasi-conformal.

Proposition 5.5. *Suppose $\alpha \in F - F_{\mathbb{Q}}$, $K_0 > 1$, and $\{g_n\}_{n=1}^\infty \subset \mathrm{Aff}_+(X_\alpha, q_\alpha)$ is a sequence of elements with $K(g_n) \leqslant K_0$. Then there is a subsequence $\{g_{n_k}\}_{k=0}^\infty$ and $\{h_{n_k}\}_{k=0}^\infty \subset H_\alpha^{\mathrm{Aff}}[0,1]$ so that $h_{n_k} \circ g_{n_k} = h_{n_0} \circ g_{n_0}$ for all $k \geqslant 0$.*

Proof. From the observation before the statement, we can find $h_n \in H_\alpha^{\mathrm{Aff}}[0,1]$ so that $h_n \circ g_n(Z) \cap Z \neq \varnothing$. Next, observe that $h_n \circ g_n$ is $(K_0 K_\alpha^2)$–quasi-conformal, so by compactness of quasi-conformal maps, after passing to a subsequence, $h_{n_k} \circ g_{n_k}$ converges uniformly on compact sets to a map f. The maps $h_{n_k} \circ g_{n_k}$ are affine, so they must map cone points to cone points. Since the cone points are uniformly separated by Corollary 5.3, there are a pair of cone points a, b so that for k sufficiently large $h_{n_k} \circ g_{n_k}(a) = b$. Moreover, if we pick a pair of saddle connections in linearly independent directions emanating from a, then for n sufficiently large $h_{n_k} \circ g_{n_k}$ all agree on this pair, again by Corollary 5.3. But these conditions uniquely determines the affine homeomorphism, and hence $h_{n_k} \circ g_{n_k}$ is eventually constant, and passing to a tail-subsequence of this subsequence completes the proof. □

From this we can prove a special case of Theorem 1.4:

Proposition 5.6. *If $\alpha \in F - F_{\mathbb{Q}}$, then H_α^{Aff} has finite index in $\mathrm{SAff}(X_\alpha, q_\alpha)$.*

Proof. Suppose H_α^{Aff} is not finite index, consider the closure of the D_α–image in $\mathrm{PSL}_2(\mathbb{R})$:

$$G = \overline{D_\alpha(\mathrm{SAff}(X_\alpha, q_\alpha))}.$$

Since $\alpha \in F - F_{\mathbb{Q}}$, every leaf S_α of $\mathcal{F}_\alpha$ is dense in M. Therefore $H_\alpha^D < \Delta \cong \mathbb{R}$ is an abelian subgroup with rank at least 2, and hence is dense. Consequently, $\Delta < G$.

By the classification of Lie subalgebras of $\mathfrak{sl}_2(\mathbb{R})$ (or a direct calculations) we observe that, after replacing G with a finite index subgroup, we must be in one of the following situations:

(1) $G = \mathrm{PSL}_2(\mathbb{R})$,
(2) G is the subgroup of upper triangular matrices, or
(3) $G = \Delta$.

In any case, we claim that there is a sequence of elements $\{g_n\} \subset \mathrm{SAff}(X_\alpha, q_\alpha)$ such that $D_\alpha(g_n) \to I$ in $\mathrm{PSL}_2(\mathbb{R})$ and so that $H_\alpha^{\mathrm{Aff}} g_n$ are distinct cosets of H_α^{Aff}. Assuming the claim, we prove the proposition. For this, we simply apply Proposition 5.5, pass to a subsequence (of the same name) so that $h_n \circ g_n = h_0 \circ g_0$ for all $n \geqslant 0$. This contradicts the fact that $\{H_\alpha^{\mathrm{Aff}} g_n\}$ are all distinct cosets.

To prove the claim, notice that in the first two cases, a finite index subgroup of $D_\alpha(\mathrm{SAff}(X_\alpha, q_\alpha))$ is dense in the Lie subgroup $G \leqslant \mathrm{PSL}_2(\mathbb{R})$, and $\Delta < G$ is a 1–dimensional submanifold of G, which itself has dimenion 3 or 2 in cases (1) and (2), respectively. This implies that there exists a sequence $\{g_n\} \in \mathrm{SAff}(X_\alpha, q_\alpha)$ such that $D_\alpha(g_n) \to I$ as $n \to \infty$ but $D_\alpha(g_n) \notin \Delta$. By way of contradiction, suppose that there exists a subsequence $\{g_{n_i}\}$ such that g_{n_i} are in the same coset $H_\alpha^{\mathrm{Aff}} g$ where $D_\alpha(g) \notin \Delta$. This implies that $D_\alpha(g_{n_i}) \subset \Delta D_\alpha(g)$, which is a 1–manifold parallel to Δ and does not accumulate to I. This contradicts the fact that $D_\alpha(g_{n_i}) \to I$. Therefore, there exists a subsequence of $\{g_n\}$ such that $\{H_\alpha^{\mathrm{Aff}} g_n\}$ are all distinct cosets.

To prove the final case of the claim, we argue two distinct subcases. First, if H_α^D has infinite index in $D_\alpha(\mathrm{SAff}(X_\alpha, q_\alpha))$, then by definition there exists infinitely many distinct cosets $b_n^D H_\alpha^D$ of H_α^D in $D_\alpha(\mathrm{SAff}(X_\alpha, q_\alpha))$. Since H_α^D is dense in Δ, there are elements $a_n^D \in H_\alpha^D$ such that $b_n^D a_n^D \to I$ as $n \to \infty$. Choose a sequence $g_n \in \mathrm{SAff}(X_\alpha, q_\alpha)$ such that $D_\alpha(g_n) = b_n^D a_n^D$. Then $D_\alpha(g_n) \to I$ in $\mathrm{PSL}_2(\mathrm{R})$ and $H_\alpha^{\mathrm{Aff}} g_n$ are distinct cosets of H_α^{Aff}.

Secondly, suppose H_α^D has finite index in $D_\alpha(\mathrm{SAff}(X_\alpha, q_\alpha))$. Since we are assuming that H_α^{Aff} is infinite index in $\mathrm{SAff}(X_\alpha, q_\alpha)$, then we have infinitely many distinct cosets $b_n^{\mathrm{Aff}} H_\alpha^{\mathrm{Aff}}$ of H_α^{Aff} in $\mathrm{SAff}(X_\alpha, q_\alpha)$. Since H_α^D is dense in Δ, we can find a sequence $\{a_n^{\mathrm{Aff}}\} \in H_\alpha^{\mathrm{Aff}}$ such that $D(b_n^{\mathrm{Aff}}) D(a_n^{\mathrm{Aff}}) \to I$ as $n \to \infty$. Let $g_n = a_n^{\mathrm{Aff}} b_n^{\mathrm{Aff}}$. Then $D_\alpha(g_n) \to I$ in $\mathrm{PSL}_2(\mathrm{R})$ and $H_\alpha^{\mathrm{Aff}} g_n$ are distinct cosets of H_α^{Aff}. This completes the proof of the claim. Since we already proved the proposition assuming the claim, we are done. ☐

To complete the proof of Theorem 1.4, we need only prove the following.

Proposition 5.7. $\mathrm{Aff}_+(X_\alpha, q_\alpha) = \mathrm{SAff}(X_\alpha, q_\alpha)$.

Proof. First, observe that $\mathrm{SAff}_+(X_\alpha, q_\alpha)$ is a normal subgroup of $\mathrm{Aff}_+(X_\alpha, q_\alpha)$ since it is precisely the kernel of the homomorphism given by the determinant of the derivative. In fact, from this homomorphism, either $\mathrm{Aff}_+(X_\alpha, q_\alpha) = \mathrm{SAff}(X_\alpha, q_\alpha)$ or else the index is infinite; $[\mathrm{Aff}_+(X_\alpha, q_\alpha) : \mathrm{SAff}(X_\alpha, q_\alpha)] = \infty$.

After passing to a finite index subgroup, $\Gamma < \mathrm{Aff}_+(X_\alpha, q_\alpha)$, if necessary, the conjugation action of Γ on $\mathrm{SAff}_+(X_\alpha, q_\alpha)$ preserves the finite index subgroup H_α^{Aff} (and without loss of generality, $H_\alpha^{\mathrm{Aff}} < \Gamma$). It thus suffices to prove $\Gamma < \mathrm{SAff}_+(X_\alpha, q_\alpha)$, or equivalently, $D_\alpha(\Gamma) < \mathrm{PSL}_2(\mathbb{R})$.

Consider any element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in D_\alpha(\Gamma) \quad \text{and} \quad h = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in H_\alpha^D,$$

with $\lambda \neq \pm 1$. Then $ghg^{-1} \in H_\alpha^D$, and is given by

$$\begin{aligned} ghg^{-1} &= \frac{1}{\det(g)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{\det(g)} \begin{pmatrix} ad\lambda - bc\lambda^{-1} & ab(\lambda - \lambda^{-1}) \\ cd(\lambda - \lambda^{-1}) & ad\lambda^{-1} - bc\lambda \end{pmatrix}. \end{aligned}$$

In order for this element to be in H_α^D (hence diagonal), we must have that $ab = 0$ and $cd = 0$. Suppose that $a = 0$. If $c = 0$, then we have the zero matrix, so we must have that $c \neq 0$ and instead that $d = 0$. This gives us that g is a matrix of the form

$$g = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

We note that the square of a matrix of this form is a diagonal matrix. Similarly, if $b = 0$, we must have that $c = 0$ and we have that g is a matrix of the form

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Together, these two conclusions imply that either g or g^2 is diagonal.

Now we show that $D_\alpha(\Gamma) < \mathrm{PSL}_2(\mathbb{R})$. If not, then there exists $g \in D_\alpha(\Gamma)$ with $0 < \det(g) \neq 1$. After squaring and inverting if necessary, we may assume that g is diagonal,

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \sigma \end{pmatrix},$$

and $0 < \det(g) = \lambda\sigma < 1$. Without loss of generality, suppose $\lambda < 1$. Notice that there exists an element $h \in H_\alpha^D$ such that

$$h = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

and there exist $n, k \in \mathbb{Z}$ so that

$$m = g^n h^k = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

where $0 < r, s < 1$. Therefore, m^j is a contraction for all $j > 0$, which implies that it is contracting in both directions. Fixing a saddle connection ω of q_α, it follows that the length of $m^j(\omega)$ tends to 0 as $j \to \infty$. This contradicts Corollary 5.3, part (1), and thus proves that $g \in \mathrm{PSL}_2(\mathbb{R})$, as required. □

Remark 5.8. The final contradiction in the proof also follows from Theorem 1.1 of [PSV11], since $D_\alpha(\mathrm{Aff}_+(X_\alpha, q_\alpha))$ is necessarily of type (i) in that theorem.

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