The shadow of a Thurston geodesic to the curve graph

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Abstract

We study the geometry of the Thurston metric on Teichmüller space by examining its geodesics and comparing them to Teichmüller geodesics. We show that, similar to a Teichmüller geodesic, the shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic. However, we show that the set of short curves along the two geodesics are not identical.

1. Introduction

In [22], Thurston introduced a metric on Teichmüller space in terms of the least possible value of the global Lipschitz constant between two hyperbolic surfaces of finite volume. Even though this is an asymmetric metric, Thurston constructed geodesics connecting any pair of points in Teichmüller space that are concatenations of stretch paths. However, there is no unique geodesic connecting two points in Teichmüller space $\mathcal{T}(S)$. We construct some examples to highlight the extent of non-uniqueness of geodesics:

**Theorem 1.1.** For every $D > 0$, there are points $X, Y, Z \in \mathcal{T}(S)$ and Thurston geodesic segments $\mathcal{G}_1$ and $\mathcal{G}_2$ starting from $X$ and ending in $Y$ with the following properties.

1. Geodesics $\mathcal{G}_1$ and $\mathcal{G}_2$ do not follow travel each other; the point $Z$ lies in path $\mathcal{G}_1$ but is at least $D$ away from any point in $\mathcal{G}_2$.
2. The geodesic $\mathcal{G}_1$ parameterized in any way in the reverse direction is not a geodesic. In fact, the point $Z$ is at least $D$ away from any point in any geodesic connecting $Y$ to $X$.

In view of these examples, one may ask whether geodesics connecting $X$ to $Y$ have any common features. There is a mantra that all notions of a straight line in Teichmüller space behave the same way at the level of the curve graph. That is, the shadow of any such line to the curve graph is a reparameterized quasi-geodesic. This has already been shown for Teichmüller geodesics [15], lines of minima [8], grafting rays [7], certain geodesics in the Weil–Petersson metric [6], and Kleinian surface groups [18]. (See also [2] of an analogous result in Outer Space.)

In this paper, we show the following theorem.

**Theorem 1.2.** The shadow of a Thurston geodesic to the curve graph is a reparameterized quasi-geodesic.

Since the curve graph is Gromov hyperbolic [15], quasi-geodesics with common endpoints fellow travel. Hence:

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Corollary 1.3. The shadow to the curve graph of different Thurston geodesics connecting $X$ to $Y$ fellow travel each other.

This builds on the analogy established in [14] between Teichmüller geodesics and Thurston geodesics. We showed that if the Teichmüller geodesic connecting $X$ and $Y$ stays in the thick part of Teichmüller space, so does any Thurston geodesic connecting $X$ to $Y$ and in fact all these paths fellow travel each other. However, this analogy does not extend much further; we show that the converse of the above statement is not true:

Theorem 1.4. There is an $\epsilon_0 > 0$ such that, for every $\epsilon > 0$, there are points $X, Y \in T(S)$ and a Thurston geodesic connecting $X$ to $Y$ that stays in the $\epsilon_0$-thick part of Teichmüller space, whereas the associated Teichmüller geodesic connecting $X$ to $Y$ does not stay in the $\epsilon$-thick part of Teichmüller space.

In particular, this means that the set of short curves along a Teichmüller geodesic and a Thurston geodesic are not the same.

1.1. Outline of the proof

To prove Theorem 1.2, one needs a suitable definition for when a curve is sufficiently horizontal along a Thurston geodesic. This is in analogy with both the study of Teichmüller geodesics and geodesics in Outer space after [2, 15]. In the Teichmüller metric, geodesics are described by a quadratic differential, which in turn defines a singular flat structure on a Riemann surface. The flat metric is then deformed by stretching the horizontal foliation and contracting the vertical foliation of the flat surface. If a curve is not completely vertical, then its horizontal length grows exponentially fast along the Teichmüller geodesic. Similarly, for Outer space, the geodesics are described as folding paths associated to train-tracks. If an immersed curve has a sufficiently long legal segment at a point along a folding path, then the length of the horizontal segment grows exponentially fast along the folding path. These two facts, respectively, play important roles in the proofs of the hyperbolicity of curve complexes and free factor complexes.

The notion of horizontal foliation in the setting of Teichmüller geodesics is replaced by the maximally stretched lamination in the setting of Thurston geodesics (see Subsection 2.9). However, it is possible for a curve $\alpha$ on the surface to fellow travel the maximally stretched lamination $\lambda$ for a long time only to have its length go down later along the Thurston geodesic. That is, the property of fellow-traveling $\lambda$ geometrically does not persist (see Example 4.4).

In Section 4, we define the notion of a curve $\alpha$ being sufficiently horizontal along a Thurston geodesic to mean that $\alpha$ fellow travels $\lambda$ for sufficiently long time both topologically and geometrically. We show that if a curve is sufficiently horizontal at a point along a Thurston geodesic, then it remains sufficiently horizontal throughout the geodesic, with exponential growth of the length of its horizontal segment (Theorem 4.2). In Section 5, we define a projection map from the curve complex to the shadow of a Thurston geodesic sending a curve $\alpha$ first to the earliest time in the Thurston geodesic where $\alpha$ is sufficiently horizontal and then to a curve of bounded length at that point. We show in Theorem 5.6 that this map is a coarse Lipschitz retraction. Theorem 1.2 follows from this fact using a standard argument. In Section 6, we construct the examples of Thurston geodesics that illustrate the deviant behaviors of Thurston geodesics from Teichmüller geodesics, as indicated by Theorems 1.1 and 1.4.

The proof is somewhat technical, because all we know about a Thurston geodesic is that the length of the maximally stretched lamination (which may not be a filling lamination) is growing exponentially. Using this and some delicate hyperbolic geometry arguments, we are able to control the geometry of the surface. For the ease of exposition, we have collected several technical lemmas in Section 3. These statements should be intuitively clear to a reader familiar
with hyperbolic geometry and the proofs can be skipped in the first reading of the paper. Section 6 is also less technical and can be read independently from the rest of the paper.

2. Background

We briefly review some background material needed for this paper. We refer the reader to [13, 19, 22] and the references therein for background on hyperbolic surfaces and the Thurston metric on Teichmüller space.

2.1. Notation

We adopt the following notation to simplify some calculations. Call a constant $C$ universal if it depends only on the topological type of a surface, and not on a hyperbolic metric on the surface. Then, given a universal constant $C$ and two quantities $a$ and $b$, we write

(i) $a \preceq b$ if $a \leq Cb$;
(ii) $a \succeq b$ if $a \preceq b$ and $b \preceq a$;
(iii) $a \bowtie b$ if $a \preceq b + C$;
(iv) $a \bowreverses b$ if $a \bowtie b$ and $b \bowtie a$;
(v) $a \bowlesssim b$ if $a \bowtie Cb$;
(vi) $a \bowlesssim b$ if $a \bowlesssim b$ and $b \bowlesssim a$.

We will also write $a = O(1)$ to mean $a \bowlesssim 1$.

2.2. Coarse maps

Given two metric spaces $X$ and $Y$, a multivalued map $f : X \to Y$ is called a coarse map if the image of every point has uniformly bounded diameter. The map $f$ is (coarsely) Lipschitz if

$$d_Y(f(x), f(y)) \bowlesssim d_X(x, y)$$

for all $x, y \in X$, where

$$d_Y(f(x), f(y)) = \text{diam}_Y(f(x) \cup f(y)).$$

Given a subset $A \subset X$, a coarse Lipschitz map $f : X \to A$ is a coarse retraction if $d_X(a, f(a)) = O(1)$ for all $a \in A$.

2.3. Curve graph

Let $S$ be a connected oriented surface of genus $g$ with $p$ punctures with $3g + p - 4 \geq 0$. By a curve on $S$ we will mean an essential simple closed curve up to free homotopy. Essential means the curve is not homotopic to a point or a puncture of $S$. For two curves $\alpha$ and $\beta$, let $i(\alpha, \beta)$ be the minimal intersection number between the representatives of $\alpha$ and $\beta$. Two distinct curves are disjoint if their intersection number is 0. A multicurve on $S$ is a collection of pairwise disjoint curves. A pair of pants is homeomorphic to a thrice-punctured sphere. A pants decomposition on $S$ is a multicurve whose complement in $S$ is a disjoint union of pairs of pants.

We define the curve graph $\mathcal{C}(S)$ of $S$ as introduced by Harvey [11]. The vertices of $\mathcal{C}(S)$ are curves on $S$, and two curves span an edge if they intersect minimally on $S$. For a surface with $3g + p - 4 > 0$, the minimal intersection number is 0; for the once-punctured torus, the minimal intersection number is 1; and for the four-times punctured sphere, the minimal intersection number is 2. The curve graph of a pair of pants is empty since there are no essential curves. By an element or a subset of $\mathcal{C}(S)$, we will always mean a vertex or a subset of the vertices of $\mathcal{C}(S)$.

Assigning each edge of $\mathcal{C}(S)$ to have length 1 endows $\mathcal{C}(S)$ with a metric structure. Let $d_{\mathcal{C}(S)}(\bullet, \bullet)$ be the induced path metric on $\mathcal{C}(S)$. The following fact will be useful for bounding
curve graph distances \([21]\): for any \(\alpha, \beta \in \mathcal{C}(S)\),
\[
d_{\mathcal{C}(S)}(\alpha, \beta) \leq \log_2 i(\alpha, \beta) + 1.
\] (1)

By \([15]\), for any surface \(S\), the graph \(\mathcal{C}(S)\) is hyperbolic in the sense of Gromov. More recently, it was shown contemporaneously and independently by \([1, 5, 9, 12]\) that there is a uniform \(\delta\) such that \(\mathcal{C}(S)\) is \(\delta\)-hyperbolic for all \(S\).

2.4. Teichmüller space

A marked hyperbolic surface is a complete finite-area hyperbolic surface equipped with a fixed homeomorphism from \(S\). Two marked hyperbolic surfaces \(X\) and \(Y\) are considered equivalent if there is an isometry from \(X\) to \(Y\) in the correct homotopy class. The collection of equivalence classes of all marked hyperbolic surfaces is called the Teichmüller space \(T(S)\) of \(S\). This space \(T(S)\) equipped with its natural topology is homeomorphic to \(\mathbb{R}^{6g-6+2p}\).

2.5. Short curves and collars

Given \(X \in T(S)\) and a simple geodesic \(\omega\) on \(X\), let \(\ell_X(\omega)\) be the arc length of \(\omega\). Since \(X\) is marked by a homeomorphism to \(S\), its set of curves is identified with the set of curves on \(S\). For a curve \(\alpha\) on \(S\), let \(\ell_X(\alpha) = \ell_X(\alpha^*)\), where \(\alpha^*\) is the geodesic representative of \(\alpha\) on \(X\). A curve is called a systole of \(X\) if its hyperbolic length is minimal among all curves. Given a constant \(C\), a multicurve on \(X\) is called \(C\)-short if the length of every curve in the set is bounded above by \(C\). The Bers constant \(\epsilon_B = \epsilon_B(S)\) is the smallest constant such that every hyperbolic surface \(X\) admits an \(\epsilon_B\)-short pants decomposition. In most situations, we will assume a curve or multicurve is realized by geodesics on \(X\).

We state the well-known Collar Lemma with some additional properties (see \([13, \text{Subsection 3.8}]\)).

**Lemma 2.1 (Collar Lemma).** Let \(X\) be a hyperbolic surface. For any simple closed geodesic \(\alpha\) on \(X\), the regular neighborhood about \(\alpha\)
\[
U(\alpha) = \left\{ p \in X \mid d_X(p, \alpha) \leq \sinh^{-1} \frac{1}{\sinh(0.5\ell_X(\alpha)))} \right\}
\]
is an embedded annulus. If two simple closed geodesics \(\alpha\) and \(\beta\) are disjoint, then \(U(\alpha)\) and \(U(\beta)\) are disjoint. Moreover, given a simple closed geodesic \(\alpha\) and a simple geodesic \(\omega\) (not necessarily closed, but complete), if \(\omega\) does not intersect \(\alpha\) and does not spiral toward \(\alpha\), then it is disjoint from \(U(\alpha)\).

We will refer to \(U(\alpha)\) as the standard collar of \(\alpha\). There is a universal upper and lower bound on the arc length of the boundary of \(U(\alpha)\) provided that \(\alpha\) is \(\epsilon_B\)-short.

Using the convention
\[
\log(x) = \begin{cases} 
\ln(x) & \text{if } x \geq e, \\
1 & \text{if } x \leq e,
\end{cases}
\] (2)
we note that, for \(0 \leq x \leq \epsilon_B\),
\[
\sinh^{-1}(1/\sinh(0.5x)) \leq \log(1/x).
\]

A consequence of the Collar Lemma is the existence of a universal constant \(\delta_B\) such that if a curve \(\beta\) intersects an \(\epsilon_B\)-short curve \(\alpha\), then
\[
\ell_X(\beta) \geq i(\alpha, \beta)\delta_B.
\]
Also, for any geodesic segment $\omega$,
\[ \ell_X(\omega) \geq (i(\alpha, \omega) - 1)\delta_B. \]
We will refer to $\delta_B$ as the dual constant to the Bers constant $\epsilon_B$.

2.6. Various notions of twisting

In this section, we will define several notions of relative twisting of two objects or structures about a simple closed curve $\gamma$. The notation will always be twist$_\gamma(\cdot, \cdot)$.

First suppose $A$ is a compact annulus and $\gamma$ is the core curve of $A$. Given two simple arcs $\eta$ and $\omega$ with endpoints on the boundary of $A$, we define
\[ \text{twist}_\gamma(\eta, \omega) = i(\eta, \omega), \]
where $i(\eta, \omega)$ is the minimal number of interior intersections between isotopy classes of $\eta$ and $\omega$ fixing the endpoints pointwise.

Now suppose that $\gamma$ is a curve in $S$. The annular cover $\hat{A}$ of $S$ corresponding to $\langle \gamma \rangle < \pi_1(S)$ can be compactified in an intrinsic way. Let $\hat{\gamma}$ be the core curve of $\hat{A}$. Given two simple geodesics or curves $\eta$ and $\omega$ in $S$, let $\hat{\eta}$ and $\hat{\omega}$ be any lifts to $\hat{A}$ that join the boundary of $\hat{A}$ (such lifts exist when $\eta$ and $\omega$ intersect $\gamma$). The relative twisting of $\eta$ and $\omega$ about $\gamma$ is
\[ \text{twist}_\gamma(\eta, \omega) = \text{twist}_\gamma(\hat{\eta}, \hat{\omega}). \]
This definition is well defined up to an additive error of 1 with different choices of $\hat{\eta}$ and $\hat{\omega}$.

Now suppose $X \in T(S)$ and let $\omega$ be a geodesic arc or curve in $X$. We want to measure the number of times $\omega$ twists about $\gamma$ in $X$. To do this, represent $\gamma$ by a geodesic and lift the hyperbolic metric of $X$ to the annular cover $\hat{A}$. Let $\hat{\tau}$ be any geodesic perpendicular to $\hat{\gamma}$ joining the boundary of $\hat{A}$. We define the twist of $\omega$ about $\gamma$ on $X$ to be
\[ \text{twist}_\gamma(\omega, X) = i(\hat{\omega}, \hat{\tau}), \]
where $\hat{\omega}$ is any lift of $\omega$ joining the boundary of $\hat{A}$. Since there may be other choices of $\hat{\tau}$, this notion is well defined up to an additive error of at most one. Note that if twist$_\gamma(\omega, X) = 0$ and twist$_\gamma(\eta, \omega) = n$, then twist$_\gamma(\eta, X) \geq n$.

When $\gamma$ is $\epsilon_B$-short, fix a perpendicular arc $\tau$ to the standard collar $U(\gamma)$, then the quantity $i(\omega, \tau)$ differs from twist$_\gamma(\omega, X)$ by at most one [17, Lemma 3.1].

Given $X, Y \in T(S)$, the relative twisting of $X$ and $Y$ about $\gamma$ is
\[ \text{twist}_\gamma(X, Y) = i(\hat{\tau}_X, \hat{\tau}_Y), \]
where $\hat{\tau}_X$ is an arc perpendicular to $\hat{\gamma}$ in the metric $X$, and $\hat{\tau}_Y$ is an arc perpendicular to $\hat{\gamma}$ in the metric $Y$. Again, choosing different perpendicular arcs changes this quantity by at most one.

2.7. Subsurface projection and bounded combinatorics

Let $\Sigma \subset S$ be a compact and connected subsurface such that each boundary component of $\Sigma$ is an essential simple closed curve. We assume that $\Sigma$ is not a pair of pants or an annulus. From [16], we recall the definition of subsurface projection $\pi_\Sigma: \mathcal{C}(S) \to \mathcal{P}(\mathcal{C}(\Sigma))$ from the curve graph of $S$ to the space of subsets of the curve graph of $\Sigma$.

Equip $S$ with a hyperbolic metric and represent $\Sigma$ as a convex set with geodesic boundary. (The projection map does not depend on the choice of the hyperbolic metric.) Let $\hat{\Sigma}$ be the Gromov compactification of the cover of $S$ corresponding to $\pi_1(\Sigma) < \pi_1(S)$. There is a natural homeomorphism from $\hat{\Sigma}$ to $\Sigma$, which allows us to identify $\mathcal{C}(\Sigma)$ with $\mathcal{C}(\hat{\Sigma})$. For any curve $\alpha$ on $S$, let $\hat{\alpha}$ be the closure of the lift of $\alpha$ in $\hat{\Sigma}$. For each component $\beta$ of $\hat{\alpha}$, let $N_\beta$ be a regular neighborhood of $\beta \cup \partial \hat{\Sigma}$. The isotopy class of each component $\beta'$ of $\partial N_\beta$, with isotopy relative
to $\partial \hat{\Sigma}$, can be regarded as an element of $\mathcal{P}(\mathcal{C}(\Sigma))$; $\beta'$ is the empty set if $\beta'$ is isotopic into $\partial \hat{\Sigma}$. We define

$$\pi_{\Sigma}(\alpha) = \bigcup_{\beta \subset \alpha, \beta' \subset \partial N_{\beta}} \{\beta'\}.$$  

The projection distance between two elements $\alpha, \beta \in \mathcal{C}(S)$ in $\Sigma$ is

$$d_{\mathcal{C}(\Sigma)}(\alpha, \beta) = \text{diam}_{\mathcal{C}(\Sigma)}(\pi_{\Sigma}(\alpha) \cup \pi_{\Sigma}(\beta)).$$

Given a subset $K \subset \mathcal{C}(S)$, we also define $\pi_{\Sigma}(K) = \bigcup_{\alpha \in K} \pi_{\Sigma}(\alpha)$, and the projection distance between two subsets of $\mathcal{C}(S)$ in $\Sigma$ is likewise defined. For any $\Sigma \subset S$, the projection map $\pi_{\Sigma}$ is a coarse Lipschitz map [16].

For any $X \in \mathcal{T}(S)$, a pants decomposition $\mathcal{P}$ on $X$ is called short if $\sum_{\alpha \in \mathcal{P}} \ell_X(\alpha)$ is minimized. Note that a short pants decomposition is always $\epsilon_B$-short, and two short pants decompositions have bounded diameter in $\mathcal{C}(S)$.

Let $X_1, X_2 \in \mathcal{T}(S)$. For $i = 1, 2$, let $\mathcal{P}_i$ be a short pants decomposition on $X_i$. We will say $X_1$ and $X_2$ have $K$-bounded combinatorics if there exists a constant $K$ such that the following two properties hold.

(i) For $\Sigma = S$, or $\Sigma$ a subsurface of $S$,

$$d_{\mathcal{C}(\Sigma)}(\mathcal{P}_1, \mathcal{P}_2) \le K.$$

(ii) For every curve $\gamma$ in $S$

$$\text{twist}_{\gamma}(X_1, X_2) \le K.$$

2.8. Geodesic lamination

Let $X$ be a hyperbolic metric on $S$. A geodesic lamination $\mu$ is a closed subset of $S$ which is a union of disjoint simple complete geodesics in the metric of $X$. These geodesics are called leaves of $\mu$, and we will call their union the support of $\mu$. A basic example of a geodesic lamination is a multicurve (realized by its geodesic representative).

Given another hyperbolic metric on $S$, there is a canonical one-to-one correspondence between the two spaces of geodesic laminations. We therefore will denote the space of geodesic laminations on $S$ by $\mathcal{GL}(S)$ without referencing to a hyperbolic metric. The set $\mathcal{GL}(S)$ endowed with the Hausdorff distance is compact. A geodesic lamination is said to be chain-recurrent if it is in the closure of the set of all multicurves.

A transverse measure on a geodesic lamination $\mu$ is a Radon measure on arcs transverse to the leaves of the lamination. The measure is required to be invariant under projections along the leaves of $\mu$. When $\mu$ is a simple closed geodesic, the transverse measure is just the counting measure times a positive real number. It is easy to see that an infinite isolated leaf spiraling toward a closed leaf cannot be in the support of a transverse measure.

The stump of a geodesic lamination $\mu$ is a maximal (with respect to inclusion) compactly supported sublamination of $\mu$ which admits a transverse measure of full support.

2.9. Thurston metric

In this section, we will give a brief overview of the Thurston metric, sometimes referred to as the Lipschitz metric or Thurston’s ‘asymmetric’ metric in the literature. All facts in this section are due to Thurston and contained in [22]. We also refer the reader to [19] for additional reference.

Given $X, Y \in \mathcal{T}(S)$, the distance from $X$ to $Y$ in the Thurston metric is defined to be

$$d_{\text{Th}}(X, Y) = \log L(X, Y),$$
where \( L(X, Y) \) is the infimum of Lipschitz constants over all homeomorphisms from \( X \) to \( Y \) in the correct homotopy class. Since the inverse of a Lipschitz map is not necessarily Lipschitz, there is no reason for the metric to be symmetric. In fact, \( L(X, Y) \) is in general not equal to \( L(Y, X) \), as shown in the example on page 5 of [22].

Thurston showed that the quantity \( L(X, Y) \) can be computed using ratios of lengths of curves on \( S \).

**Theorem 2.2** [22]. For any \( X, Y \in T(S) \),

\[
L(X, Y) = \sup_{\alpha} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)},
\]

where \( \alpha \) ranges over all curves on \( S \).

The length function extends continuously to measured laminations, and the space of projectivized measured laminations is compact. Hence there is a measured lamination that realizes the supremum above. It might not be unique, but one can assign to an ordered pair \( (X, Y) \) a geodesic lamination \( \mu(X, Y) \) admitting a transverse measure that contains the supports of all the measured laminations realizing the supremum.

For any sequence \( \{\alpha_i\} \) of curves on \( S \) with \( \lim_{i \to \infty} \ell_Y(\alpha_i)/\ell_X(\alpha_i) \to L(X, Y) \), let \( \alpha_\infty \) be a limiting geodesic lamination of \( \{\alpha_i\} \) in the Hausdorff topology. Set

\[
\lambda(X, Y) = \bigcup \{\alpha_\infty\},
\]

where the union on the right-hand side ranges over the limits of all such sequences. Thurston showed that \( \lambda(X, Y) \) is a geodesic lamination, called the *maximally stretched lamination* from \( X \) to \( Y \), which contains \( \mu(X, Y) \) as its stump. Moreover, there is a \( L(X, Y) \)-Lipschitz homeomorphism from \( X \) to \( Y \) in the correct homotopy class that stretches \( \lambda(X, Y) \) by \( L(X, Y) \) and whose local Lipschitz constant outside \( \lambda(X, Y) \) is strictly less than \( L(X, Y) \). In particular, the infimum is realized in the definition of \( L(X, Y) \) and \( d_{Th}(X, Y) \). The existence of such a map follows from the fact that one can connect \( X \) to \( Y \) by a concatenation of finitely many stretch paths (see Subsection 2.10), all of which contain \( \lambda(X, Y) \) in its stretch locus. We will call a homeomorphism \( f : X \to Y \) *optimal* if \( f \) is a \( L(X, Y) \)-Lipschitz map. Note that our sense of ‘optimal’ is more in the sense of \( L^\infty \) metric than \( L^1 \), since we only require the global Lipschitz constant to be minimized.

Thurston showed that \( T(S) \) equipped with the Thurston metric is a (asymmetric) geodesic metric space. That is, for any \( X, Y \in T(S) \), there exists a geodesic from \( X \) to \( Y \), that is, a parameterized path \( \Gamma : [0, d] \to T(S) \) such that \( d = d_{Th}(X, Y) \), \( \Gamma(0) = X \), \( \Gamma(d) = Y \), and for any \( 0 \leq s < t \leq d \), \( d_{Th}(\Gamma(s), \Gamma(t)) = t - s \). Any geodesic from \( X \) to \( Y \) is characterized by the property that the maximally stretched lamination \( \lambda(X, Y) \) is stretched maximally at all times. Thus, there is only one such geodesic only when \( \lambda(X, Y) \) is a maximal lamination (the complement of \( \lambda(X, Y) \) are ideal triangles). In general, the set of geodesics from \( X \) to \( Y \) can have uncountable cardinality: the idea is that one is free to deform any part of the surface that is not forced to be maximally stretched. We refer to the proof of Theorem 1.1 in Section 6 for an example of such deformation.

Given a geodesic segment \( \Gamma : [a, b] \to T(S) \), we will often denote by \( \lambda_\Gamma \) the maximally stretched lamination from \( \Gamma(a) \) to \( \Gamma(b) \). The maximally stretched lamination is well defined for geodesic rays or bi-infinite geodesics. Suppose \( \Gamma : \mathbb{R} \to T(S) \) is a bi-infinite geodesic. Consider two sequences \( \{t_n\}_n, \{s_m\}_m \subset \mathbb{R} \) with

\[
\lim_{n \to \infty} t_n \to \infty \quad \text{and} \quad \lim_{m \to \infty} s_m \to -\infty.
\]

Set \( X_m = \Gamma(s_m) \) and \( Y_n = \Gamma(t_n) \). The sequence \( \lambda(X_m, Y_n) \) is increasing by inclusion as \( m, n \to \infty \), hence \( \lambda = \bigcup_{m,n} \lambda(X_m, Y_n) \) is defined. The lamination \( \lambda \) is independent of the sequences \( X_m \)}
and $Y_n$, hence $\lambda = \lambda_G$ is the maximally stretched lamination for $G$. Similarly, the maximally stretched lamination $\Lambda_G$ is defined for a geodesic ray $G: [a, \infty) \to T(S)$.

Throughout this paper, we will always assume that Thurston geodesics are parameterized by arc length.

2.10. Stretch paths

To prove $T(S)$ is a geodesic metric space, Thurston introduced a special family of geodesics called stretch paths. Namely, let $\lambda$ be a maximal geodesic lamination (all complementary components are ideal triangles). Then $\lambda$, together with a choice of basis for relative homology, defines shearing coordinates on Teichmüller space (see [3]). In fact, in this situation, the choice of basis does not matter. For any hyperbolic surface $X$, and time $t$, define $\text{stretch}(X, \lambda, t)$ to be the hyperbolic surface where the shearing coordinates are $e^t$ times the shearing coordinates at $X$. That is, the path $t \mapsto \text{stretch}(X, \lambda, t)$ is a straight line in the shearing coordinate system associated to $\lambda$. Thurston showed [22] that this path is a geodesic in $T(S)$.

In the case where $\lambda$ is a finite union of geodesics, the shearing coordinates are easy to understand. An ideal triangle has an inscribed circle tangent to each edge at a point which we refer to as an anchor point. Then the shearing coordinate associated to two adjacent ideal triangles is the distance between the anchor points coming from the two triangles (see Figure 1). To obtain the surface $\text{stretch}(X, \lambda, t)$, one has to slide every pair of adjacent triangles against each other such that the distance between the associated pairs of anchor points is increased by a factor of $e^t$.

2.11. An example

We illustrate some possible behaviors along a stretch path in the following basic example.

Fix a small $0 < \epsilon \ll 1$. Let $A_0$ be an annulus which is glued out of two ideal triangles as follows. One pair of the sides is glued with a shift of 2, and another pair is glued with a shift of $2 + 2\epsilon$, as in Figure 1. That is, if $p, p'$ are the anchor points associated to one triangle and $q, q'$ are anchor points associated to the other triangle, and the sides containing $p$ and $q$ are glued, and same for $p'$ and $q'$, then the segments $[p, q]$ and $[p', q']$ have lengths 2 and $2 + 2\epsilon$, respectively. Note that by adding enough ideal triangles to this construction and gluing them appropriately, one can obtain a hyperbolic surface of arbitrarily large complexity. For example,
adding an ideal triangle to the top and to the bottom of $A_0$ with zero shift and then identifying the top edges with zero shift and the bottom edges with zero shift gives rise to a hyperbolic surface $X_0$ which is topologically a sphere with four points removed. Also, the sides of the four ideal triangles define a maximal geodesic lamination $\lambda$ on $X_0$. When two triangles are glued with zero shift, the associated shearing coordinate remains unchanged under a stretch map. Hence, we concentrate on how the geometry of $A_0$ changes only.

Let $r$ and $r'$ be the midpoints of $[p, q]$ and $[p', q']$. There is an isometry of $A_0$ that switches the two triangles and fixes $r$ and $r'$. Hence, if $\gamma$ is the core curve of the annulus $A_0$, then the geodesic representative of $\gamma$, which is unique and is fixed by this isometry, passes through points $r$ and $r'$.

Define $X_t = \text{stretch}(X_0, \lambda, t)$. We give an estimate for the hyperbolic length of $\gamma$ at $X_t$ for $t \in \mathbb{R}_+$.

**Claim.** For $t \in \mathbb{R}_+$, we have $\ell_{X_t}(\gamma) \preceq e^{\epsilon t} + e^{-\epsilon t}$.

**Proof.** Let $A_t$ be the annulus obtained by gluing two triangles when the shifts are $2\epsilon t$ and $(2 + 2\epsilon)\epsilon t$ and let $p_t, q_t, r_t$ and $r'_t$ be points that are defined similar to $p, q, r$, and $r'$ in $A_0$. The geodesic representative of the core curve of $A_t$, which we still denote by $\gamma$, passes through the points $r_t$ and $r'_t$. Denote the length of the segment $[r_t, r'_t]$ by $d(r_t, r'_t)$. Then

$$\ell_{X_t}(\gamma) = 2d(r_t, r'_t).$$

To estimate $d(r_t, r'_t)$, we work in one of the ideal triangles. Let $s_t \in [p_t, q_t]$ be the point that is on the same horocycle as $r'_t$. Consider the triangle $[r_t, r'_t] \cup [r'_t, s_t] \cup [s_t, r_t]$. By the triangle inequality, we have

$$d(r_t, r'_t) \leq d(r_t, s_t) + d(s_t, r'_t) \leq 2 \max\{d(r_t, s_t), d(s_t, r'_t)\}.$$ 

On the other hand, the angle between the segments $[r'_t, s_t]$ and $[s_t, r_t]$ is at least $\frac{\pi}{2}$, which implies that the side $[r_t, r'_t]$ is the largest of the triangle. Hence, we also have

$$d(r_t, r'_t) \geq \max\{d(r_t, s_t), d(s_t, r'_t)\}.$$ 

That is, up to a multiplicative error of at most 4, the length $\ell_t(\gamma)$ is $d(r_t, s_t) + d(s_t, r'_t)$. The distance $d(s_t, r'_t)$ is asymptotically (as $t \to +\infty$) equal to the length of the horocycle between $s_t$ and $r'_t$. It is straightforward to see that since $d(p_t, s_t) = e^t(1 + \epsilon)$, the length of the horocycle is $e^{-e^t(1+\epsilon)}$. Also $d(r_t, s_t) = e^{\epsilon t}$, and we have

$$\ell_{X_t}(\gamma) \preceq e^{\epsilon t} + e^{-e^t(1+\epsilon)}.$$ 

The second term in the sum can be replaced with $e^{-e^t}$ without increasing the multiplicative error by much. This is true because the first term $e^{\epsilon t}$ in the sum is bigger than the second term when $e^{\epsilon t}$ is bigger than 1. This proves the claim.

We can now approximate the minimum of $\ell_{X_t}(\gamma)$ for $t \in \mathbb{R}_+$. At $t = 0$ and $t = \log \frac{1}{\epsilon}$ the length of $\gamma$ is basically 1. If $\epsilon$ is small enough, there is $t_0 > 0$ such that $e^{\epsilon t_0} = e^{-e^t}$. Then we have

$$\ell_{X_t}(\gamma) \preceq \begin{cases} e^{-e^t}, & t < t_0, \\ e^{\epsilon t}, & t > t_0. \end{cases}$$

This means in particular that the length of $\gamma$ decreases super-exponentially fast, reaches its minimum, and grows back up exponentially fast. We will not compute the exact value of $t_0$, but if we take log twice we see that $t_0 \approx \log \log(1/\epsilon)$, with additive error at most log 2. Then $\ell_{t_0}(\gamma) \preceq \epsilon \log(1/\epsilon)$ and this is, up to a multiplicative error, the minimum of $\ell_{X_t}(\gamma)$. 
The reason the curve $\gamma$ gets short and then long again is because it is more efficient to twist around $\gamma$ when $\gamma$ is short. To see this, we estimate the relative twisting of $X_0$ and $X_t$ around $\gamma$ at $t = \log(1/\epsilon)$, that is when the length of $\gamma$ grew back to approximately 1. The lamination $\lambda$ is nearly perpendicular to $\gamma$ at $X_0$ and does not twist around it. Hence, we need to compute how many times it twists around $\gamma$ in $X_t$.

For a fixed $t > 0$, choose lifts $\hat{\gamma}$ and $\hat{\lambda}$ to $\mathbb{H}$ of the geodesic representative of $\gamma$ and of the leaf of $\lambda$ containing $[p_t, q_t]$, such that $\hat{\gamma}$ and $\hat{\lambda}$ intersect. Let $\ell$ be the length of the orthogonal projection of $\lambda$ to $\hat{\gamma}$. Then (see [17, Section 3])

$$\text{twist}_{\gamma}(\lambda, X_t) \lesssim \frac{\ell}{D_{X_t}(\gamma)}.$$ 

To find $\ell$, we note that $\cosh \ell/2 = 1/\sin \alpha$, where $\alpha$ is the angle between $\hat{\gamma}$ and $\hat{\lambda}$. In the triangle $[r_t, r'_t] \cup [r'_t, s_t] \cup [s_t, r_t]$, $\alpha$ is the angle between segments $[s_t, r_t]$ and $[r_t, r'_t]$. Let $\beta$ be the angle between $[r'_t, s_t]$ and $[s_t, r_t]$. Since $\beta$ is asymptotically $\pi/2$, by the hyperbolic sine rule, we have

$$\cosh \ell/2 \asymp \frac{\sinh(d(r'_t, r_t))}{\sinh(d(r'_t, s_t))}.$$ 

Assuming $t = \log(1/\epsilon)$, we have $\sinh(d(r'_t, r_t)) \asymp 1$ and $\sinh(d(r'_t, s_t)) \asymp e^{-1/\epsilon}$ which implies $\ell \asymp 2/\epsilon$. Hence

$$\text{twist}_{\gamma}(\lambda, X_{\log(1/\epsilon)}) \lesssim \frac{1}{\epsilon}.$$ 

To summarize, the surface $X_{\log(1/\epsilon)}$ is close to $D^n_{\gamma}(X_0)$, where $D_\gamma$ is a Dehn twist around $\gamma$ and $n \asymp 1/\epsilon$. The stretch path stretch($X_0, \lambda, t$) from $X_0$ to $X_{\log(1/\epsilon)}$ changes only an annular neighborhood of $\gamma$, first decreasing the length of $\gamma$ super-exponentially fast to $\epsilon \log(1/\epsilon)$ and then increasing it exponentially fast. In fact, further analysis shows that essentially all the twisting is done near the time $t_0$ when the length of $\gamma$ is minimum.

2.12. **Shadow map**

For any $X \in T(S)$, the set of systoles on $X$ has uniformly bounded diameter in $C(S)$. We will call the coarse map $\pi: T(S) \to C(S)$ sending $X$ to the set of systoles on $X$ the **shadow map**. The following lemma shows that the shadow map is Lipschitz.

**Lemma 2.3.** The shadow map $\pi: T(S) \to C(S)$ satisfies, for all $X, Y \in T(S)$,

$$d_{C(S)}(\pi(X), \pi(Y)) \ll d_{T\mathbb{H}}(X, Y).$$

**Proof.** Let $X, Y \in T(S)$ and let $K = L(X, Y)$. Let $\alpha$ be a systole on $X$ and let $\beta$ be a systole on $Y$. Recall that $\epsilon_B$ is the Bers constant and $\delta_B$ its dual constant defined in Subsection 2.5. We have $\ell_X(\alpha) \leq \epsilon_B$ and $\ell_Y(\beta) \leq \epsilon_B$. Now

$$i(\alpha, \beta) \leq \frac{\ell_Y(\alpha)}{\delta_B} \leq \frac{K \ell_X(\alpha)}{\delta_B} \leq K \frac{\epsilon_B}{\delta_B}.$$ 

Therefore, equation (1) implies $d_{C(S)}(\alpha, \beta) \ll \log K = d_{T\mathbb{H}}(X, Y)$. \hfill $\square$

For simplicity, we will often write $d_{C(S)}(X, Y) := d_{C(S)}(\pi(X), \pi(Y))$.

3. **Hyperbolic geometry**

In this section, we establish some basic properties of the hyperbolic plane $\mathbb{H}$ and hyperbolic surfaces. Many of these results are known in spirit, but to our knowledge the exact statements do not directly follow from what is written in the literature.
Recall that $\mathbb{H}$ is Gromov hyperbolic, that is, there is a constant $\delta_\mathbb{H}$ such that all triangles in $\mathbb{H}$ are $\delta_\mathbb{H}$-slim: every edge of a triangle is contained in a $\delta_\mathbb{H}$-neighborhood of the union of the other two edges.

3.1. Geodesic arcs on hyperbolic surfaces

Let $\alpha$ be a simple closed geodesic on a hyperbolic surface $X$ and let $U(\alpha)$ be the standard collar of $\alpha$. When $\omega$ is a geodesic segment contained in $U(\alpha)$ with endpoints $p$ and $p'$, we denote the distance between $p$ and $p'$ in $U(\alpha)$ by $d_{U(\alpha)}(\omega)$. The following lemma can be read as saying that all the twisting around a curve $\alpha$ takes place in $U(\alpha)$.

**Lemma 3.1.** Let $P$ be a pair of pants in a hyperbolic surface $X$ with geodesic boundary lengths less than $\epsilon_B$. For each connected component $\alpha \subset \partial P$, there is an arc $\tau_\alpha$ in $U(\alpha)$ perpendicular to $\alpha$ such that the following holds. Any finite subarc $\omega$ of a simple complete geodesic $\lambda$ that is contained in $P$ can be subdivided into three pieces

$$\omega = \omega_\alpha \cup \omega_0 \cup \omega_\beta$$

such that

(a) the interior of $\omega_0$ is disjoint from every $U(\gamma)$, for $\gamma \subset \partial P$, and $\ell_X(\omega_0) = O(1)$;

(b) the segment $\omega_\alpha$, $\alpha \subset \partial P$, is contained in $U(\alpha)$ and intersects any curve in $U(\alpha)$ that is equidistant to $\alpha$ at most once. That is, as one travels along $\omega_\alpha$, the distance to $\alpha$ changes monotonically. Furthermore,

$$\ell_X(\omega_\alpha) \preceq i(\tau_\alpha, \omega)\ell(\alpha) + d_{U(\alpha)}(\omega_\alpha),$$

(c) the same holds for $\omega_\beta$ ($\alpha$ and $\beta$ may be the same curve).

**Proof.** Note that if $\omega$ intersects $U(\alpha)$, then by Lemma 2.1, $\lambda$ either intersects $\alpha$ or spirals toward $\alpha$. This implies that $\omega$ intersects at most two standard collars, say $U(\alpha)$ and $U(\beta)$. We allow the possibility that $\alpha = \beta$. Let $\omega_\alpha = \omega \cap U(\alpha)$, $\omega_\beta = \omega \cap U(\beta)$, and $\omega_0$ be the remaining middle segment. In $P$, there is a unique geodesic segment $\eta$ perpendicular to $\alpha$ and $\beta$. Set $\tau_\alpha = \eta \cap U(\alpha)$ and $\tau_\beta = \eta \cap U(\beta)$. If $\alpha = \beta$, then $\eta$ is the unique simple segment intersecting $U(\alpha)$ twice and perpendicular to $\alpha$, and $\tau_\alpha$ and $\tau_\beta$ are the two components of $\eta$ in $U(\alpha)$.

In the universal cover, a lift $\tilde{\omega}$ of $\omega$ is in a $2\delta_\mathbb{H}$-neighborhood of the union of a lift $\tilde{\alpha}$ of $\alpha$, a lift $\tilde{\beta}$ of $\beta$ and a lift $\tilde{\eta}$ of $\eta$.

In fact, a point in $\tilde{\omega}$ is either in a $\delta_B$-neighborhood of $\tilde{\alpha} \cup \tilde{\beta}$, where $\delta_B$ is the dual constant to $\epsilon_B$, or is uniformly close to $\tilde{\eta}$. This fact has two consequences. First, because $U(\alpha)$ and $U(\beta)$ have thicknesses bigger than $\delta_B$, the lift $\tilde{\omega}'$ of $\omega'$ is contained in a uniform bounded neighborhood of $\tilde{\eta}$ and hence, the length of $\omega'$ is comparable to the length of $\eta$ outside of $U(\alpha)$ and $U(\beta)$, which is uniformly bounded.

Secondly, $\tilde{\omega}_\alpha$ is in a $\delta_\mathbb{H}$-neighborhood of $\tilde{\tau}_\alpha$ and $\tilde{\alpha}$. The portion that is in the neighborhood of $\tilde{\tau}_\alpha$ has a length that is, up to an additive error, equal to $d_{U(\alpha)}(\omega_\alpha)$. The portion that is in the neighborhood of $\tilde{\alpha}$ has a length that is (up to an additive error) equal to $i(\tau_\alpha, \omega_\alpha)\ell(\alpha)$.

The formula in part (b) follows from adding these two estimates.

The only remaining point is that, in the above, the choice of $\tau_\alpha$ depends on $\beta$. However, we observe that the choice of $\tau_\alpha$ is not important and for any other segment $\tau^\prime_\alpha$ perpendicular to $\alpha$ that spans the width of $U(\alpha)$, we have

$$i(\tau_\alpha, \omega_\alpha) \preceq i(\tau^\prime_\alpha, \omega_\alpha).$$

This completes the proof. ■
Let $X \in T(S)$ and $P$ be a pair of pants in $X$ with boundary lengths at most $\epsilon_B$. Roughly speaking, the following technical lemma states that if a subsegment $\omega$ of $\lambda$ intersects a closed (non-geodesic) curve $\gamma$ enough times and the consecutive intersection points are far enough apart, then $\omega$ cannot be contained in $P$.

**Lemma 3.2.** There exist $D_0 > 0$ and $K_0 > 0$ with the following property. Let $\gamma$ be a simple closed curve ($\gamma$ may not be a geodesic) that intersects $\partial P$ and let $\omega$ be a simple geodesic in $P$ where the consecutive intersections of $\omega$ with $\gamma$ are at least $D_0$ apart in $\omega$. Let $\overline{\omega}$ be a subarc of $\omega$ with endpoints on $\gamma$, and let $\omega_1$ and $\omega_2$ be the connected components of $\omega \setminus \overline{\omega}$. Then at least one of $\omega$, $\omega_1$, or $\omega_2$ has length bounded by $K_0 (\ell_X(\gamma) + 1)$.

**Proof.** For $i = 1, 2$, let $p_i$ be the endpoint of $\overline{\omega}$ that is also an endpoint of $\omega_i$. Let $\alpha_i$ be the boundary curve of $P$ with $p_i \in U(\alpha_i)$, where $U(\alpha_i)$ is the standard collar of $\alpha_i$. If $p_i$ does not belong in any of the collar neighborhoods of a curve in $\partial P$, then we choose $\alpha_i$ arbitrarily. For simplicity, denote $U(\alpha_i)$ by $U_i$ and let $\overline{\omega}_i$ be the component of $\overline{\omega} \cap U_i$ with endpoint $p_i$ ($\overline{\omega}_i$ may be empty). It is enough to show that, for $i = 1, 2$, we have either

$$\ell_X(\overline{\omega}_i) < \ell_X(\gamma) \quad \text{or} \quad \ell_X(\omega_i) < \ell_X(\gamma).$$

(3)

This is because if $\omega_1$ and $\omega_2$ are both very long, then equation (3) implies that $\overline{\omega}_1$ and $\overline{\omega}_2$ both have length bounded by $\ell_X(\gamma)$ up to a small error. Since the middle part of $\overline{\omega}$ has bounded length (Lemma 3.1), this implies the desired upper bound for the length of $\overline{\omega}$.

We now prove equation (3). The point $p_i$ subdivides $U_i$ into two sets $V_i$ and $W_i$, where $V_i$ and $W_i$ are regular annuli (their boundaries are equidistance curves to $\alpha$) with disjoint interiors and $p_i$ is on the common boundary of $V_i$ and $W_i$. By part (c) of Lemma 3.1, one of these annuli contains $\omega_i$ and the other contains $\overline{\omega}_i$. Also, since $\gamma$ passes through $p_i$, it intersects both boundaries of either $V_i$ or $W_i$.

Let $V$ be either $V_i$ or $W_i$ such that $\gamma$ intersects both boundaries of $V$ and let $\eta$ be either $\omega_i$ or $\omega_i$ that is contained in $V$. We want to show

$$\ell_X(\eta) < \ell_X(\gamma),$$

which is equivalent to equation (3).

From Lemma 3.1, we have

$$\ell_X(\eta) \lesssim i(\eta, \tau_\alpha) \ell_X(\alpha) + d_V(\gamma).$$

(4)

Let $\sigma$ be the geodesic representative of the subarc of $\gamma$ connecting the boundaries of $V$. Then

$$d(\eta, V) \lesssim \ell_X(\sigma) \leq \ell_X(\gamma).$$

(5)

The intersection numbers between arcs in an annulus satisfy the triangle inequality up to a small additive error. Hence,

$$i(\eta, \tau_\alpha) \lesssim i(\eta, \sigma) + i(\sigma, \tau_\alpha) \lesssim i(\eta, \gamma) + i(\sigma, \tau_\alpha).$$

(6)

Let $D_0 > \epsilon_B$ be any constant. By assumption, consecutive intersections of $\omega$ with $\gamma$ are at least $D_0$ apart in $\omega$. We have $i(\eta, \gamma) \leq (1/D_0) \ell_X(\eta)$. Also, $i(\sigma, \tau_\alpha) \leq \ell_X(\sigma)/\ell_X(\alpha) + 1$. These facts and equation (6) imply

$$i(\eta, \tau_\alpha) \lesssim \frac{1}{D_0} \ell_X(\eta) + \frac{\ell_X(\sigma)}{\ell_X(\alpha)}.$$
Combining this with equations (4) and (5), we have
\[
\ell_X(\eta) < \left( 1 - \frac{\epsilon B}{D_0} \right) \ell_X(\eta) + 2\ell_X(\gamma).
\]
That is,
\[
\left( 1 - \frac{\epsilon B}{D_0} \right) \ell_X(\eta) < 2\ell_X(\gamma).
\]
The constant \(1 - \epsilon B/D_0\) is positive, since \(D_0 > \epsilon B\). So taking \(K_0\) sufficiently larger than \(2(1 - \epsilon B/D_0)^{-1}\) completes the proof.

**Lemma 3.3.** Let \(X \in \mathcal{T}(S)\) and let \(\alpha\) be a simple closed geodesic in the hyperbolic metric of \(X\). Let \(\omega\) be a simple geodesic arc in \(X\). If
\[
i(\omega, \alpha) \geq 3,
\]
then any curve \(\gamma\) in the homotopy class of \(\alpha\) that is disjoint from \(\alpha\) intersects \(\omega\) at least once.

**Remark 3.4.** Note that the statement is sharp in the sense that if \(\gamma\) is not disjoint from \(\alpha\) or if \(\omega\) intersects \(\alpha\) less than three times, \(\gamma\) can be disjoint from \(\omega\).

**Proof.** Curves \(\alpha\) and \(\gamma\) as above bound an annulus \(A\) in \(X\). Suppose that \(\omega\) intersects \(\alpha\) at points \(p_1\), \(p_2\) and \(p_3\), the points being ordered by their appearances along \(\omega\). For \(i = 1, 2\), let \([p_i, p_{i+1}]\) be the segment of \(\omega\) between \(p_i\) and \(p_{i+1}\). Since \(\alpha\) and \(\omega\) are geodesics, their segments cannot form bigons. Therefore, one of the segments \([p_1, p_2]\) and \([p_2, p_3]\) intersects the interior, and therefore both boundary components of \(A\).

### 3.2. We now prove several useful facts about geodesics in the hyperbolic plane.

**Proposition 3.5.** Let \(\phi : \mathbb{H} \to \mathbb{H}\) be a hyperbolic isometry with axis \(\gamma\) and translation length \(\ell(\phi) \leq \epsilon\) for some \(\epsilon > 0\). Suppose that \(\alpha\) and \(\beta\) are two geodesic lines such that the following two conditions hold:

1. \(\alpha\) and \(\beta\) intersect \(\gamma\) at the points \(a\) and \(b\), respectively, with \(d_H(a, b) \leq \epsilon\);
2. let \(c\) be the point on \(\beta\) which is closest to \(\alpha\) (the intersection point between \(\alpha\) and \(\beta\) if they intersect). Assume \(d_H(c, b) \geq 2\epsilon\).

Fix an endpoint \(\beta_+\) of \(\beta\) and let \(\alpha_+\) be the endpoint of \(\alpha\) that is on the same side of \(\gamma\) as \(\beta_+\). Then, there exists \(k \in \mathbb{Z}\) such that \(\alpha_+\) is between \(\beta_+\) and \(\phi^k(\beta_+)\) and \(|k|\ell(\phi) \leq 4\epsilon + 3\).

**Proof.** We assume that \(\beta_+\) is the endpoint of the ray \(\overrightarrow{cb}\) (see Figure 2). The case when \(\beta_+\) is the endpoint of \(bc\) is similar.

By exchanging \(\phi\) with \(\phi^{-1}\) if necessary, then we may assume that \(\alpha_+\) is between \(\beta_+\) and the attracting fixed point \(\gamma_+\) of \(\phi\). Let \(k\) be a positive integer with
\[
3\epsilon + 3 \leq k\ell(\phi) \leq 4\epsilon + 3.
\]
Such a \(k\) exists since \(\ell(\phi) \leq \epsilon\). Let \(\beta' = \phi^k(\beta),\ b' = \phi^k(b)\) and \(\beta_+ = \phi^k(\beta_+)\).

Since \(k\ell(\phi) \geq \epsilon\), we know \(a\) is between \(b\) and \(b'\). To show \(\alpha_+\) is between \(\beta_+\) and \(\beta'_+\), we need to show that the rays \(\overrightarrow{aa_+}\) and \(\overrightarrow{b'b'_+}\) do not intersect. Note that
\[
d_H(b', a) = d_H(b', b) - d_H(a, b) \geq (3\epsilon + 3) - \epsilon = 2\epsilon + 3 > d_H(a, b).
\]
Let $A = \angle \alpha - a \gamma = \angle \alpha - a \gamma$ and $B = \angle \beta - b \gamma = \angle \beta - b \gamma$. If $\alpha$ and $\beta$ are disjoint, then $\overrightarrow{a \alpha}$ and $\overrightarrow{b \beta}$ are also disjoint and we are done. Hence, we can assume that $\alpha$ and $\beta$ intersect at $c$.

Using the law of cosines, $\overrightarrow{a \alpha}$ and $\overrightarrow{b \beta}$ do not intersect if

$$\sin A \sin B \cosh d_H(a, b) - \cos A \cos B \geq \cos 0 = 1.$$  \hspace{1cm} (7)

Since $d_H(a, b') \geq 2\epsilon + 3$,  

$$\cosh d_H(a, b') \geq \cosh(2\epsilon + 3) > \frac{e^{2\epsilon+3}}{2}. $$

Hence, by equation (7), it suffices to show that

$$\frac{e^{2\epsilon+3}}{2} \geq \frac{1 + \cos A \cos B}{\sin A \sin B}. $$  \hspace{1cm} (8)

Before starting the calculations, we make an elementary observation. For any $y > 0$, the function

$$f(x) = \frac{\sinh(x + y)}{\sinh x}$$

is decreasing and $f(x) \leq 2e^y$ for all $x \geq y$. This is because $f'(x) = -\sinh(y)/\sinh^2(x) < 0$. Hence, for all $x \geq y$,

$$f(x) \leq f(y) = \frac{\sinh 2y}{\sinh y} = 2 \cosh y \leq 2e^y.$$

We argue in three cases. Suppose $A > \pi/2$. Since $d_H(c, a) \geq d_H(c, b) - d_H(a, b) \geq \epsilon$, we have

$$\frac{1 + \cos A \cos B}{\sin A \sin B} = \frac{1 - \cos(\pi - A) \cos B}{\sin A \sin B} \leq \frac{\sin^2(\max\{A, B\})}{\sin^2(\min\{A, B\})} \leq \frac{\sin^2(\max\{A, B\})}{\sin^2(\min\{A, B\})} = \frac{\sin^2 A}{\sin^2 B} \leq \frac{\sin^2 d_H(c, b)}{\sin^2 d_H(c, a)} \leq \frac{\sin^2 (d_H(c, a) + \epsilon)}{\sin^2 d_H(c, a)} \leq 4e^{2\epsilon}. $$
There is a lower bound on the distance between any point in $Q_p$ and any point in $Q_q$.

Similarly, if $B \geq \pi/2$ instead, then
\[
\frac{1 + \cos A \cos B}{\sin A \sin B} \leq \frac{\sinh^2(d_{\mathbb{H}}(c, b) + \ell(\phi))}{\sinh^2(d_{\mathbb{H}}(c, b))} \leq 4e^{2\epsilon}.
\]

In the case where both $A$ and $B$ are at most $\pi/2$, let $w$ be the point on the segment $ab$ which is the foot of the perpendicular from $c$ to $ab$. We have
\[
\frac{1 + \cos A \cos B}{\sin A \sin B} \leq 2 \frac{\sinh d_{\mathbb{H}}(c, a) \sinh d_{\mathbb{H}}(c, b)}{\sinh d_{\mathbb{H}}(c, w) \sinh d_{\mathbb{H}}(c, w)} \leq 2 \frac{\sinh^2(d_{\mathbb{H}}(c, w) + \ell(\phi))}{\sinh^2 d_{\mathbb{H}}(c, w)} \leq 8e^{2\epsilon}.
\]

But $8e^{2\epsilon} < e^{2\epsilon+3}/2$ and we are done.

We need some definitions for the next two lemmas. Let $\gamma$ be a geodesic in $\mathbb{H}$ with endpoints $\gamma_+$ and $\gamma_-$. Fix a $\delta$-neighborhood $U$ of $\gamma$ and let $p$ be any point on the boundary of $U$. The geodesic through $p$ with endpoint $\gamma_+$ and the geodesic through $p$ with endpoint $\gamma_-$ together subdivide $\mathbb{H}$ into four quadrants. The quadrant disjoint from the interior of $U$ will be called the upper quadrant at $p$, and the quadrant diametrically opposite will be called the lower quadrant at $p$.

**Lemma 3.6.** For every $\delta_0 > 0$ and $M > 0$, there is $d_0 > 0$ such that the following holds (see the left-hand side of Figure 3). Fix a geodesic $\gamma$ in $\mathbb{H}$ and let $U$ be the $\delta$-neighborhood of $\gamma$ with $\delta \geq \delta_0$. Let $p$ and $q$ be points on the same boundary component of $U$. Suppose $d_{\mathbb{H}}(p, q) \geq M$. Then any point in the upper quadrant $Q_p$ at $p$ is at least $d_0$ away from any point in the lower quadrant $Q_q$ at $q$.

**Proof.** It suffices to find a lower bound for $d_{\mathbb{H}}(p, \xi)$, where $\xi$ is the closest boundary component of $Q_q$ to $p$. Assume that $\xi$ has the endpoint $\gamma_-$. Note that $d_{\mathbb{H}}(p, \xi)$ increases with $d_{\mathbb{H}}(p, q)$, since any geodesic from $\gamma_-$ that intersects the boundary of $U$ between $p$ and $q$ separates $p$ from $\xi$. 

![Figure 3](image-url)
Now fix $d_{\tilde{H}}(p, q) = M$ and see the right-hand side of Figure 3 for the rest of the proof. Fix a perpendicular geodesic $\nu$ to $\gamma$ and let $\partial$ be a boundary component of the $M$-neighborhood of $\nu$. Fix an endpoint $\nu_-$ of $\nu$ and let $\partial_s$ be a boundary curve of the $s$-neighborhood of $\gamma$ contained in the complement $\mathbb{H} \setminus \gamma$ determined by $\nu_-$. Let $q_s$ be the intersection of $\nu$ and $\partial_s$ and let $p_s$ be the intersection of $\partial$ and $\partial_s$. Let $\xi_s$ be the geodesic through $q_s$ with endpoint $\gamma_-$ and let $\beta$ be the geodesic connecting $\gamma_-$ and $\nu_-$. Since $\beta$ is asymptotic to $\nu$, $d_{\tilde{H}}(p_s, \beta)$ is increasing as a function of $s$, and for $s$ big enough, $\beta$ separates $p_s$ from $\xi_s$. Therefore, there exist $d_1 > 0$ and $s_1 \geq d_0$ such that $d_{\tilde{H}}(p_s, \xi_s) > d_{\tilde{H}}(p_s, \beta) > d_1 > 0$ for all $s \geq s_1$. Finally, since $d_{\tilde{H}}(p_s, \xi_s)$ is continuous as a function of $s$ and is only zero when $s = 0$, it is bounded away from zero on the segment $[d_0, s_1]$. This completes the proof.

**Lemma 3.7.** Given $d_0 > 0$ and $M > 0$, the constant $d_0$ of Lemma 3.6 also satisfies the following property. Let $\gamma$ be a geodesic in $\mathbb{H}$. Let $U$ be the $\delta$-neighborhood of $\gamma$ with $\delta \geq d_0$. Let $\omega_1$ be a geodesic intersecting $\gamma$ and let $\omega_2$ be the image of $\omega_1$ under an isometry fixing $\gamma$. Let $p_i$ be the intersection of $\omega_i$ with a fixed boundary component of $U$. Let $\omega_1^\perp$ be the component of $\omega_1$ contained in the upper quadrant at $p_1$. Suppose $d_{\tilde{H}}(p_1, p_2) \geq M$. Then $d_{\tilde{H}}(\omega_1^\perp, \omega_2) \geq d_0$.

**Proof.** Let $\omega_2^\perp$ be the component of $\omega_2$ contained in the upper quadrant at $p_2$. By Lemma 3.6, it is enough to show $d_{\tilde{H}}(\omega_1^\perp, \omega_2^\perp) \geq d_0$. Let $r_1$ be the point on $\omega_1$ such that $d_{\tilde{H}}(r_1, r_2)$ realizes the distance between $\omega_1$ and $\omega_2$. Because of the symmetry of $\omega_1$ and $\omega_2$ relative to a rotation fixing $\gamma$, either both $r_1$ and $r_2$ are contained in $U$, and hence each $r_i$ is contained in the lower quadrant at $p_i$, or one is in the lower quadrant and the other is in the upper quadrant.

Identify the space of pairs of $(x, y)$, where $x \in \omega_1$ and $y \in \omega_2$, with $\mathbb{R}^2$. Then the function $\mathbb{R}^2 \to \mathbb{R}$ sending $(x, y)$ to $d_{\tilde{H}}(x, y)$ is a convex function realizing its minimum at $(r_1, r_2) \in \mathbb{R}^2$.

In the first case, since $d_{\tilde{H}}(p_i, \omega_{i-1}^\perp) \geq d_0$ by Lemma 3.6 and the distance between $\omega_1^\perp$ and $\omega_2^\perp$ increases from $p_1$ and $p_2$ on, and we can conclude $d_{\tilde{H}}(\omega_1^\perp, \omega_2^\perp) \geq d_0$. In the second case, invoking Lemma 3.6 again implies $d_{\tilde{H}}(r_1, r_2) \geq d_0$, hence $d_{\tilde{H}}(\omega_1^\perp, \omega_2^\perp) \geq d_0$ by minimality of $d_{\tilde{H}}(r_1, r_2)$.

4. A notion of being sufficiently horizontal

Let $I$ be a closed connected subset of $\mathbb{R}$, let $G \colon I \to T(S)$ be a Thurston geodesic and let $\lambda_G$ be its maximally stretched lamination. The main purpose of this section is to develop a notion of a closed curve $\alpha$ being sufficiently horizontal along $G$, such that if $\alpha$ is horizontal, then it remains horizontal and its horizontal length grows exponentially along $G$.

**Definition 4.1.** Given a curve $\alpha$, we will say that $\alpha$ is $(n, L)$-horizontal at $t \in I$ if there exist an $\epsilon_B$-short curve $\gamma$ on $X_t = \tilde{G}(t)$ and a leaf $\lambda$ in $\lambda_G$ such that the following statements hold (see Figure 4).

(H1) In the universal cover $\tilde{X}_t \cong \mathbb{H}$, there exists a collection of lifts $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\}$ of $\gamma$ and a lift $\lambda$ of $\lambda$ intersecting each $\tilde{\gamma}_i$ at a point $p_i$ (the points $p_i$ are indexed by the order of their appearances along $\tilde{\lambda}$) such that $d_{\tilde{H}}(p_i, p_{i+1}) \geq L$ for all $i = 1, \ldots, n - 1$.

(H2) There exists a lift $\tilde{\alpha}$ of $\alpha$ such that $\tilde{\alpha}$ intersects $\tilde{\gamma}_i$ at a point $q_i$ with $d_{\tilde{H}}(p_i, q_i) \leq \epsilon_B$ for each $i$.

We will call $\gamma$ an anchor curve for $\alpha$ and $\tilde{\alpha}$ an $(n, L)$-horizontal lift of $\alpha$.

Set $G(t) = X_t$. The main result of this section is the following.

**Theorem 4.2.** There are constants $n_0$, $L_0$ and $s_0$ such that the following holds. Suppose that a curve $\alpha$ is $(n_s, L_s)$-horizontal at $s \in I$ with $n_s \geq n_0$ and $L_s \geq L_0$. Then we have the
following propositions.

(I) For any \( t \geq s + s_0 \), \( \alpha \) is \((n_t, L_t)\)-horizontal at \( t \), with
\[
    n_t \succ n_s, \quad \text{and} \quad L_t \geq L_s.
\]

(II) Furthermore, for any \( A \), if \( d_C(G(t), X_s, X_t) \geq A \), then
\[
    \log \frac{n_t}{n_s} \succ A \quad \text{and} \quad L_t n_t \succ e^{1-s} L_s n_s.
\]

Definition 4.3 (Sufficiently Horizontal). Let \((n_0, L_0)\) be the constants given by Theorem 4.2. A curve \( \alpha \) will be said to be sufficiently horizontal at \( t \in I \) if it is \((n, L)\)-horizontal for some \( n \geq n_0 \) and \( L \geq L_0 \).

Example 4.4. Definition 4.1 is a bit technical and warrants some justification. To stay sufficiently horizontal along a Thurston geodesic \( G(t) \), we require the curve \( \alpha \) to fellow-travel \( \lambda \) both geometrically and topologically for a long time. The following example illustrates why these requirements are necessary. Namely, we will show that the weaker version of geometric fellow-traveling does not always persist along a Thurston geodesic.

Referring to the example in Subsection 2.11, for any \( \epsilon \), there exists a Thurston geodesic \( G(t) = X_t \) and a curve \( \gamma \) such that a leaf \( \lambda \) of \( \lambda_G \) intersects \( \gamma \) and, for \( t > 0 \), \( \ell_{X_t}(\gamma) \asymp \epsilon e^t + 2 e^{-e^t} \). Consider the following two points along \( G(t) \):
\[
    X = G(\log \log(1/\epsilon)) \quad \text{and} \quad Y = G(\log 1/\epsilon).
\]

On \( Y \), let \( \alpha \) be the shortest curve that intersects \( \gamma \) with twist \( \gamma(\alpha, Y) = 0 \). It was shown in Subsection 2.11 that
\[
    \text{twist}_\gamma(\lambda, Y) \asymp \frac{1}{\epsilon}.
\]
This implies \( \text{twist}_\gamma(\alpha, \lambda) \asymp 1/\epsilon \), so \( \alpha \) intersects \( \lambda \) at an angle close to \( \pi/2 \) in \( Y \). Furthermore, since \( \ell_Y(\gamma) \asymp 1 \), we have \( \ell_Y(\alpha) = O(1) \). That is, every large enough segment of any lift of \( \alpha \) to the universal cover \( \tilde{Y} \) intersects a lift of \( \lambda \) at a near right angle. Therefore, in \( \tilde{Y} \), no lift of \( \alpha \) will fellow-travel any lift of \( \lambda \).
On the other hand, it was also shown in Subsection 2.11 that
\[ \ell_X(\gamma) \gtrsim \epsilon \log(1/\epsilon). \]

For a given \( L \), we can choose \( \epsilon \) small enough such that the collar neighborhood of \( \gamma \) in \( X \) has width at least \( L \). Since \( \alpha \) and \( \lambda \) both pass through this collar, in the universal cover \( \tilde{X} \), there exists a lift \( \tilde{\alpha} \) and a lift \( \tilde{\lambda} \) that are \( O(1) \)-close for \( L \)-length. In other words, \( \alpha \) and \( \lambda \) fellow-travel in \( \tilde{X} \) but they do not in \( \tilde{Y} \).

**Remark 4.5.** For a given \( t \in I \), it is possible that there are no sufficiently horizontal curves at \( t \). For instance, when the stump of \( \lambda_G \) is a curve that does not intersect any \( \epsilon_B \)-short curve. But this is the only problem, since if the stump of \( \lambda_G \) intersects an \( \epsilon_B \)-short curve \( \gamma \), then any sequence of curves converging to the stump will eventually be sufficiently horizontal, with anchor curve \( \gamma \). In Section 5, we will show that one can always find a sufficiently horizontal curve after moving a bounded distance in \( C(S) \) (Proposition 5.10).

The next proposition will show that the condition (H2) of Definition 4.1 can be obtained by just assuming that \( \tilde{\alpha} \) stays \( \epsilon_B \)-close to the segment \([p_1, p_n]\) in \( \tilde{\lambda} \). A priori, even if \( \tilde{\alpha} \) is \( \epsilon_B \)-close to \([p_1, p_n]\), then the distance between \( p_i \) and \( q_i \) may still be large if \( \tilde{\gamma}_i \) is nearly parallel to \( \tilde{\lambda} \) or \( \tilde{\alpha} \).

**Proposition 4.6.** There are constants \( n_0 \) and \( L_0 \) such that, for any hyperbolic surface \( X \) and constants \( n \geq n_0 \) and \( L \geq L_0 \), the following statement holds. Suppose that \( \gamma \) is an \( \epsilon_B \)-short curve in \( X \), \( \lambda \) is a complete simple geodesic in \( X \), and \( n \) lifts \( \{\tilde{\gamma}_i\} \) of \( \gamma \) and a lift \( \tilde{\lambda} \) are chosen to satisfy (H1). If \( \alpha \) is a curve in \( X \) that has a lift \( \tilde{\alpha} \) which stays, up to a bounded multiplicative error, \( \epsilon_B \)-close to the segment \([p_1, p_n]\) in \( \tilde{\gamma}_i \), then there exists indices \( l \) and \( r \) with \( r - l > n \) such that \( \tilde{\alpha} \) intersects \( \tilde{\gamma}_i \) at a point \( q_i \) and \( d_H(p_i, q_i) \leq \epsilon_B \) for all \( i = l, \ldots, r \).

**Proof.** Recall the standard collar \( U(\gamma) \) is a regular neighborhood of \( \gamma \) in \( X \) that is an embedded annulus with boundary length \( \approx 1 \). Let \( \delta \) be the distance between \( \gamma \) and the boundary of \( U(\gamma) \). We have (see Section 3)
\[ \delta \gtrsim \log(1/\ell_X(\gamma)). \]

Let \( L_0 \) satisfy inequality
\[ \epsilon_B e^{-L_0} < \delta. \]

The distance between \( \tilde{\alpha} \) and \( \tilde{\lambda} \) is a convex function that essentially either increases or decreases exponentially fast. Hence, we can choose a segment \( \tilde{\pi} \) of \( \alpha \) and index \( i_0 \) such that \( \tilde{\pi} \) is within \( \epsilon_B e^{-L} \)-Hausdorff distance of \( \tilde{X} = [p_{i_0}, p_{n-i_0}] \). The index \( i_0 \) can be chosen independent of \( n \) or \( L \) because the distance between \( p_i \) and \( p_{i+1} \) is at least \( L \). Let \( n_0 \geq 2i_0 + 2 \).

Let \( U_i \) be the \( \delta \)-neighborhood of \( \tilde{\gamma}_i \). By the choice of \( \delta \), \( U_i \) and \( U_j \) are disjoint for \( i \neq j \). Since \( \epsilon_B e^{-L} < \delta \), endpoints of \( \tilde{\pi} \) are contained in \( U_{i_0} \) and \( U_{n-i_0} \). Also, for \( i_0 < i < n - i_0 \), \( U_i \) separates \( U_{i_0} \) and \( U_{n-i_0} \) in \( \mathbb{H}^2 \). Hence \( \tilde{\pi} \) intersects every \( \tilde{\gamma}_i \). Consider such an \( i \) and, for simplicity, set \( \tilde{\gamma} = \tilde{\gamma}_i \), \( U = U_i \), \( p = p_i \) and \( q = q_i \). To prove the Proposition, we need to show that \( d_H(p, q) \leq \epsilon_B \). We refer to Figure 5 in the following. Assume, for contradiction, that \( d_H(p, q) > \epsilon_B \). Let \( \phi \) be a hyperbolic isometry with axis \( \tilde{\gamma} \) and translation length \( \ell_X(\gamma) \). Then, since \( \gamma \) is \( \epsilon_B \)-short, up to replacing \( \phi \) with \( \phi^{-1} \), the point \( o = \phi(p) \) is strictly between \( p \) and \( q \).

Since \( \tilde{\lambda} \) and \( \phi(\tilde{\lambda}) \) are disjoint, for some boundary component of \( U \), that we will denote by \( \partial U \), the following holds. For \( p' = \tilde{\lambda} \cap \partial U \) and \( q' = \tilde{\alpha} \cap \partial U \), the point \( o' = \phi(p') \) of intersection of \( \partial U \) and \( \phi(\tilde{\lambda}) \) is between \( p' \) and \( q' \). The curve \( \partial U \) is equidistant to \( \tilde{\gamma} \) and the distance function
$d_{\mathbb{H}}$ is convex along this curve, therefore
\[ d_{\mathbb{H}}(q', p') \geq d_{\mathbb{H}}(o', p') = \ell_X(\partial U) \geq 1. \]

Let $r$ be the closest point on $\tilde{\lambda}$ to $q'$. The point $r$ is contained in one of the quadrants at $p'$, hence $d_{\mathbb{H}}(q', r) \geq 1$ by Lemma 3.6. But for sufficiently large $L_0$, this will contradict that $q'$ is $\epsilon_B e^{-L_0}$ close to $\tilde{\lambda}$. Hence $d_{\mathbb{H}}(p, q) \leq \epsilon_B$ and we are done.

**Proof of Theorem 4.2.** Let $n_s \geq n_0$ and $L_s \geq L_0, n_0, L_0$ to be determined later. Let $\lambda$ be an $(n_s, L_s)$-horizontal curve at $X_t$. As in the definition, we have an anchor curve $\gamma$, a lift $\tilde{\alpha}$ of $\alpha$, a lift $\tilde{\lambda}$ of a leaf of $\lambda_{\mathbb{H}}$ and $n_s$-lifts $\{\tilde{\gamma}_i\}$ of $\gamma$, such that $d_{\mathbb{H}}(p_i, q_i) \leq \epsilon_B$ and $d_{\mathbb{H}}(p_i, p_{i+1}) \geq L_s$, where $p_i$ is the intersection of $\tilde{\gamma}_i$ with $\tilde{\lambda}$, and $q_i$ the intersection of $\tilde{\gamma}_i$ with $\tilde{\alpha}$.

Throughout the proof, we will add several conditions on $n_0, s_0$ and $L_0$. Let $n_0$ and $L_0$ be at least as big as the corresponding constants obtained in Proposition 4.6.

Let $e$ be the point on $\lambda$ to which $\tilde{\alpha}$ is closest. To be able to apply Proposition 3.5 to the curve $\gamma$, we need $e$ to have a distance of at least $2\epsilon_B$ from $p_i$. Assuming $L_0 > 4\epsilon_B$, we have $c$ is $2\epsilon_B$-close to at most one $p_i$. That is, we can choose indices $l$ and $r$, with $(l, r) = (1, n_s - 1)$ or $(l, r) = (2, n_s)$, such that $c$ has a distance at least $2\epsilon_B$ from both $p_l$ and $p_r$.

See Figure 6 for the following. Applying a Möbius transformation if necessary, we can assume that the center of the disk $o$ is the midpoint between $p_l$ and $p_r$. Let $\lambda_+$ and $\lambda_-$ be, respectively, the endpoints of $\tilde{\lambda}$ determined by the rays $\overrightarrow{o p_l}$ and $\overrightarrow{o p_r}$. Let $\tilde{\alpha}_+$ be the endpoint of $\tilde{\alpha}$ closest to $\tilde{\lambda}_+$. Let $\phi$ be the hyperbolic isometry with axis $\tilde{\gamma}_i$ and translation length $\ell(\phi) = \ell_s(\gamma)$. Let $k$ be the constant of Proposition 3.5 and let

\[ \tilde{\lambda}' = \phi^k(\tilde{\lambda}), \quad \tilde{\lambda}_+ = \phi^k(\tilde{\lambda}_+) \quad \text{and} \quad p'_l = \phi^k(p_l). \]

We have that $\tilde{\alpha}_+$ is sandwiched between $\tilde{\lambda}_+$ and $\tilde{\lambda}_+$ and $d_{\mathbb{H}}(p_l, p'_l) \geq \epsilon_B$. Similarly, by considering the hyperbolic isometry $\psi$ with axis $\tilde{\gamma}_r$, we can sandwich $\tilde{\alpha}_-$ between $\tilde{\lambda}_-$ and $\tilde{\lambda}_-$ with $d_{\mathbb{H}}(p_r, p'_r) \geq \epsilon_B$, where

\[ \tilde{\lambda}'' = \psi^k(\tilde{\lambda}), \quad \tilde{\lambda}_-'' = \psi^k(\tilde{\lambda}_-) \quad \text{and} \quad p'_r = \psi^k(p_r). \]

Let $s_0 \geq 0, t \geq s + s_0$ and $f: X_s \to X_t$ be an optimal map, that is, an $e^{t-s}$-Lipschitz map. Since $\lambda_{\mathbb{H}}$ is in the stretch locus of $f$, there is a lift $\tilde{f}: \tilde{X}_s \to \tilde{X}_t$ of $f$ such that $\tilde{f}(\tilde{\lambda}) = \tilde{\lambda}$ and $\tilde{f}(\tilde{\lambda}_\pm) = \tilde{\lambda}_\pm$. 

**Figure 5.** If $d_{\mathbb{H}}(p, q) \geq \epsilon_B$, then $q$ is far from $\tilde{\lambda}$, which is a contradiction.
Note that $\epsilon > 0$ is small enough so that $\epsilon B$ is close to $\lambda$. Consider the sector $V_-$ between the rays $o \lambda_-$ and $o \lambda_r$ and the sector $V_+$ between the rays $o \lambda_r$ and $o \lambda_r'$. The geodesic $\alpha'$ connecting $\tilde{f}(\alpha_+)$ and $\tilde{f}(\alpha_-)$ stays in a bounded neighborhood of the union $V_+$ and $V_-$. Note that

$$d_H(\tilde{f}(p_{i+1}), \tilde{f}(p_i)) \geq L_0 e^{t-s} \quad \text{and} \quad d_H(\tilde{f}(p_{i}), \tilde{f}(p'_i)) \leq e^{t-s} \epsilon B.$$  

Also, the distance between intersecting geodesics increases exponentially fast. Hence

$$d_H(\alpha', \tilde{f}(p_{i+1})) \leq d_H(\tilde{f}(p_{i}), \tilde{f}(p'_i)) \leq e^{-L_0 e^{t-s} e^{t-s}} \epsilon B \leq \epsilon B.$$  

The last inequality holds as long as $L_0 \geq 1$. Similarly, $d_H(\alpha', \tilde{f}(p_{i-1})) \leq \epsilon B$. 

We next show that the projection to $X_i$ of any long enough piece of the segment of $\lambda$ between $\tilde{f}(p_{i-1})$ and $\tilde{f}(p_i)$ intersects a lift of an $\epsilon_B$-short curve.

**Claim 4.8.** For any $l \leq i \leq r - 3$, let $\omega$ be the arc connecting $\tilde{f}(p_{i-1})$ and $\tilde{f}(p_{i+3})$, and let $\omega$ be the projection of $\omega$ to $X_i$. Then $\omega$ intersects an $\epsilon_B$-short curve.

**Proof.** Recall the dual constant $\delta_B > 0$ to $\epsilon_B$, which is a lower bound for the length of any curve that intersects an $\epsilon_B$-short curve.

If $\omega$ is not simple, then $\lambda$ is a lift of a closed curve $\lambda$ and $\omega$ wraps around $\lambda$. By definition $\lambda$ intersects an $\epsilon_B$-short curve in $X_s$ which implies that $\ell_X(\lambda) \geq \delta_B$ and $\ell_X(\lambda) \geq \delta_B e^{t-s}$. If $t - s \geq s_0 > \log(\epsilon_B / \delta_B)$, then $\ell_X(\lambda) \geq \epsilon_B$ and $\lambda$ has to intersect an $\epsilon_B$-short curve in $X_i$. Thus, $\omega$ will also intersect an $\epsilon_B$-short curve.

Now assume that $\omega$ is simple. Let $\gamma' = f(\gamma)$. Note that $\gamma'$ is not necessarily a geodesic in the metric $X_i$. If $\omega$ misses all the curves of length at most $\epsilon_B$, then it is contained in a pair
of pants $P$ in $X_t$ with boundary lengths at most $\epsilon_B$. Since endpoints of $\omega$ lie on $\gamma'$, $\gamma' \cap P$ is non-empty.

First, we assume that $\gamma'$ does not intersect $\partial P$. Then $\gamma' \subset P$ and it is homotopic to a boundary component of $P$. That is, the geodesic representative $\gamma^*$ of $\gamma'$ in $X_s$ is $\epsilon_B$-short. The arc $f^{-1}(\omega)$ is a geodesic in $X_s$ and intersects $\gamma$ (which is a geodesic in $X_s$) at least three times, and so, by Lemma 3.3, $f^{-1}(\omega)$ intersects $f^{-1}(\gamma^*)$ at least once. This implies $\omega$ intersects $\gamma^*$ which proves the claim.

Now assume that $\gamma'$ intersects $\partial P$. For $j = 0, 1, 2$, let $\omega_j$ be the subarc of $\omega$ coming from projecting $[\tilde{f}(p_{i+j}), \tilde{f}(p_{i+j+1})]$ to $X_t$. Let $D_0$ and $K_0$ be the constants of Lemma 3.2. Note that if a subarc of $\lambda$ intersects $\gamma'$, then the arc length between two consecutive intersections is at least $\delta_B e^{t-\gamma} \geq \delta_B e^s$. Assuming $s_0 \geq \log \frac{D_0}{\delta_B}$ we have, for each $j$, $\ell_t(\omega)$ is at least $L_0 e^{t-\gamma}$, while $\ell_t(\gamma')$ is at most $e_B e^{t-\gamma}$. Let $L_0$ be bigger than $K_0(\epsilon_B + 1)$. Then, by Lemma 3.2, at least one of $\omega_j$ has

$$\ell_t(\omega_j) \leq K_0 \cdot (\ell_t(\gamma') + 1) < L_0 e^{t-\gamma}$$

which is impossible and hence $\omega$ intersects one of the pants curves.

Claim 4.8 implies that, for some $\epsilon_B$-short curve (call it $\gamma_t$) and some $n_t \geq n_s$, the projection of $[\tilde{f}(p_{i+1}), \tilde{f}(p_{i-1})]$ to $X_t$ intersects $\gamma_t$ at least $n_t$ times where the arc length between every two intersection points is at least $L_t = L_s e^{t-\gamma}$. And Claim 4.7 implies that there is a large segment of $\alpha$ that remains $O(\epsilon_B)$-close to the segment $[\tilde{f}(p_{i+1}), \tilde{f}(p_{i-1})]$. Applying Proposition 4.6, we conclude that $\alpha$ is $(n_t, L_t)$-horizontal, which proves part (I).

We now prove part (II) of the Theorem 4.2. Suppose

$$d_{c(S)}(X_s, X_t) \geq A. \quad (10)$$

We need to show that $n_t$ lifts of an $\epsilon_B$-short curve intersect the segment $[\tilde{f}(p_{i+1}), \tilde{f}(p_{i-1})]$, where $\log(n_t/n_s) \geq A$ such that any two consecutive intersections are at least $L_t \geq L_s n_s e^{t-\gamma}/n_t$ away.

Let $P$ be an $\epsilon_B$-short pants decomposition on $X_t$ and $m = \min_{\beta \in P} i(\beta, \gamma)$. From equation (1), we have

$$A \geq \log m.$$ 

Note that, for small values of $A$, part (II) follows from Part (I). Hence, we assume that $A$ is large, which implies in particular that $\gamma$ intersects every curve in $P$. Even though part (II) seems to be more general, this last condition is used in an essential way in the proof of Part (II) and the proof does not naturally extend to prove part (I).

We also have

$$m \geq \ell_{X_s}(\gamma) \geq e^{t-\gamma} \geq e^{t-\gamma}. \quad (11)$$

Cut the segment $[\tilde{f}(p_i), \tilde{f}(p_r)]$ into $mn_s$ equal pieces and let $\tilde{\omega}$ be one of them. Denote the projection of $\tilde{\omega}$ to $X_t$ by $\omega$. We would like to show that $\omega$ intersects a curve in $P$. As in the proof of Claim 4.8, if $\omega$ is not simple, then it wraps around a simple closed $\lambda \in \lambda_G$ and assuming $s_0 \geq \log(\epsilon_B/\delta_B)$, it has to intersect a curve in $P$. Hence we assume that $\omega$ is simple.

Assume for a contradiction that $\omega$ is disjoint from $P$. Then $\omega \subset P$ for some pair of pants $P$ with $\epsilon_B$-short boundaries. It follows from Lemma 3.1 that there is $\beta \subset \partial P$ such that $U(\beta)$ contains an endpoint of $\omega$ and such that

$$\ell_{X_s}(\omega) \geq 2(i(\omega, r_\beta) \ell_{X_s}(\beta) + \ell_{X_s}(r_\beta)).$$
Then $\gamma$ intersects $\beta$ at least $m$ times. Pick any of the subarcs $\sigma$ of $\gamma$ that connect both boundary components of $U(\beta)$. Then,
\[i(\omega, \tau_{\beta}) \lesssim i(\omega, \sigma) + i(\sigma, \tau_{\beta}) \lesssim \frac{1}{m} i(\omega, \gamma) + i(\sigma, \tau_{\beta}).\] (12)
The last inequality holds because $\omega$ intersects every component of $\gamma^* \cap U(\beta)$ essentially the same number of times. Also,
\[i(\sigma, \tau_{\beta}) \ell_{t}(\beta) + \ell_{X_{t}}(\tau_{\beta}) \lesssim \ell_{X_{t}}(\sigma) \lesssim \frac{2\epsilon_B e^{\ell_{\beta} - s}}{m} + \frac{2\epsilon_B e^{\ell_{\beta} - s}}{m}.\] (13)
But $e^{\ell_{\beta} - s} \succ m$. Hence, for some uniform constant $C$
\[\ell_{X_{t}}(\omega) \left(1 - \frac{2\epsilon_B}{\delta_B e^{\ell_{\beta} - s}}\right) \lesssim C e^{\ell_{\beta} - s} \frac{m}{m}.\] (15)
The expression in parentheses on the left-hand side is strictly positive since we have assumed $s_0 > \log(\epsilon_B/\delta_B)$. Finally, if we choose $L_0$ such that
\[L_0 \left(1 - \frac{\epsilon_B}{\delta_B e^{s_0}}\right) > C,\]
then equation (15) contradicts
\[\ell_{X_{t}}(\omega) \geq \frac{L_0 e^{\ell_{\beta} - s}}{m}.\]
Contradiction proves that $\omega$ intersects some curve in $P_\beta$.

There are at least $m(n_s - 4)$ such subsegments in $[\tilde{f}(p_{l+1}), \tilde{f}(p_{r-1})]$ and each intersects a lift of a curve in $P$. If we choose every other segment, then we can guarantee that the distance along $\lambda$ between these intersection points is larger than $L_0 = e^{\ell_{\beta} - s}L_0$. Color these segments according to which curve in $P$ their projection to $X_{t_{\beta}}$ intersects and let $\beta$ be the curve used most often. Then the number $n_t$ of segments intersecting a lift of $\beta$ satisfies $n_t \succ mn_s$. Applying Proposition 4.6 finishes the proof.

5. Shadow to the curve graph

To show that the shadow of a Thurston geodesic to the curve graph is a quasi-geodesic, we construct a retraction from the curve graph to the image of the shadow sending a curve $\alpha$ to the shadow of the point in the Thurston geodesic where $\alpha$ is balanced.

5.1. Balanced time for curves

Let $n_0$ and $L_0$ be the constants of Theorem 4.2.

**Definition 5.1 (Balanced time).** Let $G: [a, b] \to T(S)$ be a Thurston geodesic segment. For any curve $\alpha$, let
\[t_\alpha = \inf\{t \in [a, b] \mid \alpha \text{ is } (n_0, L_0)\text{-horizontal at } G(t)\}\]
Let $t_\alpha = b$ if the above set is empty. We refer to $t_\alpha$ as the balanced time of $\alpha$ along $G$. 

Recall the shadow map $\pi: \mathcal{T}(S) \to \mathcal{C}(S)$ from Section 2.12. The following theorem asserts that the shadow map is a coarse Lipschitz map.

**Theorem 5.2.** Let $G: [a, b] \to \mathcal{T}(S)$ be a Thurston geodesic and let $\pi: \mathcal{T}(S) \to \mathcal{C}(S)$ be the shadow map. Suppose $\alpha$ and $\beta$ are disjoint curves with $t_\beta \geq t_\alpha$. Then $\pi \circ G([t_\alpha, t_\beta])$ has uniformly bounded diameter in $\mathcal{C}(S)$.

In the following, we develop some notions that will be used to prove Theorem 5.2.

Let $X$ be a hyperbolic surface. A rectangle $R$ in $X$ is the image of a continuous map $\phi: [0, a] \times [0, b] \to X$ such that $\phi$ is a homeomorphism on the interior of $[0, a] \times [0, b]$ and the image of each boundary segment of $[0, a] \times [0, b]$ is a geodesic arc in $X$.

**Definition 5.3.** Let $\gamma$ be a simple closed geodesic on $X$, $\omega$ be a geodesic arc and $R$ be a rectangle given by $\phi: [0, a] \times [0, b] \to X$. We say $R$ is an $(n, L)$-corridor generated by $\gamma$ and $\omega$ (Figure 7) if the following conditions are satisfied.

(i) Edges $\{0\} \times [0, b] \text{ and } \{a\} \times [0, b]$ are mapped to subarcs of $\omega$.
(ii) There are $0 = t_1 < \cdots < t_n = b$ such that each $[0, a] \times \{t_i\}$ is mapped to a subarc of $\gamma$, for all $i = 1, \ldots, n$.
(iii) Arcs $\phi([t_i, t_{i+1}] \times \{0\})$ and $\phi([t_i, t_{i+1}] \times \{a\})$ have lengths at least $L$.

**Lemma 5.4.** Let $X$ be a hyperbolic surface and $\gamma$ be an $\epsilon_B$-short curve on $X$. For any constants $n$ and $L$, let $\omega$ be a simple geodesic arc (possibly closed) with endpoints on $\gamma$ such that

$$i(\gamma, \omega) \geq C(n, L) = (6|\chi(X)| + 1)n \left\lfloor \frac{L}{\delta_B} \right\rfloor + 3|\chi(X)| + 1.$$

Then there exists an $(n, L)$-corridor generated by $\gamma$ and $\omega$.

**Proof.** Fix $n$ and $L$, and let $C = C(n, L)$. Let $i(\omega, \gamma) = N \geq C$. The closure of each connected component of $X \smallsetminus \{\gamma \cup \omega\}$ is a surface with a piecewise geodesic boundary. Let $Q$ be a complementary component. We refer to points in the boundary of $Q$ where two geodesic pieces meet as an angle. Define the total combinatorial curvature of $Q$ to be

$$\kappa(Q) = \chi(Q) - \frac{\# \text{ of angles in } \partial Q}{4}.$$

We can represent $X$ combinatorially with all angles having value $\pi/2$ to obtain

$$\sum_Q \kappa(Q) = \chi(X).$$

![Figure 7. Corridor generated by $\gamma$ and $\omega$.](image-url)
Note that, for every component $Q$ that is not a rectangle, $\kappa(Q) < -\frac{1}{2}$ and hence, the number of components that are not rectangles is bounded by $2|\chi(X)|$. In fact, the number of angles that appear in non-rectangle components is at most $12|\chi(X)|$, because the ratio of the number of angles to Euler characteristic is maximum in the case of a hexagon. Since the total number of angles is $4N - 4$ (there are only two angles at the first and the last intersection points), the number of rectangles is at least

$$\frac{(4N - 4) - 12|\chi(X)|}{4} = N - 3|\chi(X)| - 1 \geq (6|\chi(X)| + 1)n \left\lceil \frac{L}{\delta_B} \right\rceil.$$

We will say two rectangle components can be joined if they share an arc of $\gamma$. A maximal sequence of joined rectangles is a sequence $\{Y_1, \ldots, Y_M\}$ of rectangles in $X \setminus \{\omega \cup \gamma\}$ such that $Y_i$ and $Y_{i+1}$ can be joined for $i = 1, \ldots, (s - 1)$ and such that $Y_1$ and $Y_s$ share a boundary with a non-rectangle component. The number of edges of rectangles that share with a non-rectangle component is at most two more than the number of angles of non-rectangle components (again coming from the first and last intersection points of $\gamma$ and $\omega$). That is, the number of maximal sequences of joined rectangles is at most

$$2 \cdot (\# \text{ of rectangles}) \leq \frac{2 \cdot (\# \text{ of rectangles})}{12|\chi(X)| + 2}.$$

Therefore, there must be at least one maximal sequence of joined rectangles $\{Y_1, Y_2, \ldots, Y_M\}$ where $M \geq n[L/\delta_B]$. For each $i = 1, \ldots, M$, the sides of $Y_i$ coming from arcs of $\omega$ have endpoints on $\gamma$ and thus are at least $\delta_B$ long. Therefore, the union $\bigcup_{j=1}^M Y_i$ is an $(n, L)$-corridor for $\gamma$ after letting $t_i$ be the point that maps to the intersection number $i \cdot [L/\delta_B]$. \hfill $\square$

**Proposition 5.5.** Let $n_0$ and $L_0$ be the constants from Theorem 4.2. There exists a constant $n_1$ such that, for any $n \geq n_1$ and $L \geq L_0$, if $\alpha$ is $(n, L)$-horizontal at $G(t) = X_t$, then any curve $\beta$ disjoint from $\alpha$ is either $(n_0, L_0)$-horizontal at $X_t$ or $d_{\mathcal{C}(S)}(X_t, \beta) = O(1)$.

**Proof.** Let $n_1 = 3C(n_0, L_0)$ (see Lemma 5.4). Also let $n \geq n_1$ and $L \geq L_0$.

Suppose that $\alpha$ is $(n, L)$-horizontal at $X_t$. Let $\gamma$ be an $\epsilon_B$-short curve on $X_t$, $\tilde{X}$ be a lift of a leaf of $\mathcal{C}^\alpha$, $\tilde{\alpha}$ be a lift of $\alpha$ and $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\}$ be $n$-lifts of $\tilde{\gamma}$ together satisfying Definition 4.1. Choose a most central segment $\tilde{\omega} \subset \tilde{\alpha}$ between $\tilde{\gamma}_1$ and $\tilde{\gamma}_n$ such that $\tilde{\omega}$ intersects $C(n_0, L_0)$ lifts of $\gamma$, including two intersections coming from the endpoints of $\tilde{\omega}$. Let $\omega$ be the projection of $\tilde{\omega}$ to $X_t$. If $i(\omega, \gamma) < C(n_0, L_0)$, then $\omega = \alpha$ and we are done since $d_{\mathcal{C}(S)}(X_t, \beta) \leq d_{\mathcal{C}(S)}(\gamma, \alpha) \leq \log C(n_0, L_0) = O(1)$.

Otherwise, by Lemma 5.4, there exists $(n_0, L_0)$-corridor $R$ generated by $\gamma$ and $\omega$. Let $\phi: [0, a] \times [0, b] \to X_t$ be the map whose image is $R$ satisfying the conditions of Definition 5.3.

Lift $\phi$ to the map

$$\tilde{\phi}: [0, a] \times [0, b] \to \tilde{X}_t \quad \text{with} \quad \tilde{\phi}(\{0\} \times [0, b]) = \tilde{\omega}.$$

Note that $\tilde{\phi}(\{a\} \times [0, b])$ is a translate of a subarc $\tilde{\omega}'$ of $\tilde{\alpha}$ by an isometry of $\mathbb{H}^2$ fixing $\tilde{X}$. Also, $\tilde{\omega}'$ intersects the same number of $\tilde{\gamma}_i$, and $\tilde{\omega}'$ and $\tilde{\omega}$ intersect. But $n \geq 3C(n_0, L_0)$ and $\tilde{\omega}$ was central. Thus $\tilde{\omega}'$ is still between $\tilde{\gamma}_1$ and $\tilde{\gamma}_n$ and hence $\tilde{\omega}'$ is close to $\tilde{\alpha}$.

Since $\beta$ is disjoint from $\alpha$, if $\beta$ intersects $R$ it has to enter from the edge $\phi([0, a] \times \{0\})$, travel through the corridor and exit from the edge $\phi([0, a] \times \{b\})$. Therefore, there must exist a lift $\tilde{\beta}$ of $\beta$ passing through $\phi([0, a] \times [0, b])$ intersecting every $\phi([0, a] \times \{t\})$. That is, $\tilde{\beta}$ intersects $n_0$ of $\gamma_t$ at a distance at most $\epsilon_B$ from $\tilde{\lambda}$. Thus, by Proposition 4.6 $\beta$ is $(n_0, L_0)$-horizontal at $X_t$ if $\beta$ intersects $R$.

Now suppose that $\beta$ is disjoint from $R$. Fix a parameterization $\psi: [0, c] \to X_t$ for $\omega$. Let $\omega_1 = \phi(\{0\} \times [0, b])$ and $\omega_2 = \phi(\{a\} \times [0, b])$. The parameterization $\psi$ traverses $\omega_1$ or $\omega_2$ either
in the same or opposite direction as \( \phi \) and either traverses \( \omega_1 \) before \( \omega_2 \) or vice versa. We assume that \( \psi \) traverses \( \omega_2 \) in the same direction and speed as \( \phi \) and \( \psi \) traverses \( \omega_1 \) before \( \omega_2 \) (the proofs in the other cases are similar). Let

\[
0 \leq s < t < s + b < t + b \leq c
\]

be such that \( \omega_1 = \psi([s, s + b]) \) and \( \omega_2 = \psi([t, t + b]) \). Let \( \omega' = \psi([s, t]) \) and \( \gamma' = \phi([0, a] \times \{0\}) \).

Consider the curve \( \eta \) that is a concatenation of \( \omega' \) and \( \gamma' \). Topologically, \( \eta \) is a non-trivial simple closed curve with \( i(\eta, \gamma) \leq i(\omega, \gamma) = C(n_0, L_0) \). Since \( \beta \) is disjoint from \( R \), it is disjoint from \( \eta \).

Therefore,

\[
d_{\mathcal{C}(S)}(X_t, \beta) \leq d_{\mathcal{C}(S)}(\gamma, \eta) = O(1).
\]

This concludes the proof of the proposition.

\[\square\]

**Proof of Theorem 5.2.** The proof now follows from Proposition 5.5 and Theorem 4.2.

Let \( n_1 \) be as in Proposition 5.5 and \( L_0 \) be as in Theorem 4.2. Let \( s > t_\alpha \) be the first time in \([a, b]\) that \( \alpha \) is \((n_1, L_0)\)-horizontal at \( s \) (let \( s = b \) if this never happens). By Theorem 4.2, \( \pi \circ \mathcal{G}([t_\alpha, s]) \) has uniformly bounded diameter in \( \mathcal{C}(S) \). If \( s \geq t_\beta \), then we are done. Otherwise, \( s < t_\beta \) and, for any \( t \in [s, t_\beta] \), by Proposition 5.5, \( d_{\mathcal{C}(S)}(X_t, \beta) = O(1) \). Therefore, \( \pi \circ \mathcal{G}([s, t_\beta]) \) also has uniformly bounded diameter in \( \mathcal{C}(S) \).

\[\square\]

5.2. Retraction

**Theorem 5.6.** Given a Thurston geodesic \( \mathcal{G} : [a, b] \to T(S) \), the map \( \mathcal{C}(S) \to \pi \circ \mathcal{G}([a, b]) \subset \mathcal{C}(S) \) taking a curve \( \alpha \) to \( \pi(X_{t_\alpha}) \) is a coarse Lipschitz retraction.

Before proving Theorem 5.6, we show how to derive Theorem 1.2. First we give a precise definition of reparameterized quasi-geodesic.

Fix a constant \( K > 0 \). We will call a path \( \phi : [a, b] \to X \) in a metric space \( X \) a \( K \)-quasi-geodesic if, for all \( a \leq s \leq t \leq b \),

\[
\frac{1}{K}(t - s) - K \leq d_X(\phi(s), \phi(t)) \leq K(t - s) + K.
\]

We will say \( \phi \) is a reparameterized \( K \)-quasi-geodesic if there is an increasing function \( h : [0, n] \to [a, b] \) such that \( \phi \circ h \) is a \( K \)-quasi-geodesic. Furthermore, for all \( i \in [0, n - 1] \), we have \( \text{diam}_X([\phi(h(i)), \phi(h(i + 1))]) \leq K \). In the case that \( h \) is not onto, we also require that \( \text{diam}_X([\phi(a), \phi(h(0))]) \leq K \) and \( \text{diam}_X([\phi(h(n)), \phi(b)]) \leq K \). A collection \( \{\phi_i\}_{i \in I} \) of reparameterized quasi-geodesics is uniform if there is a constant \( K \) that works for the collection.

The following is a restatement of Theorem 1.2.

**Theorem 5.7.** The collection of \( \{\pi \circ \mathcal{G} : [a, b] \to \mathcal{C}(S)\} \) ranging over Thurston geodesics \( \mathcal{G} : [a, b] \to T(S) \) is a uniform family of reparameterized quasi-geodesics in \( \mathcal{C}(S) \).

**Proof.** This argument is standard [4] and follows easily from Theorem 5.6.

Let \( \mathcal{G} : [a, b] \to T(S) \) be a Thurston geodesic. Let \( \alpha \in \pi \circ \mathcal{G}(a) \) and \( \alpha' \in \pi \circ \mathcal{G}(b) \) be two curves. Choose a geodesic \( \alpha = \alpha_0, \ldots, \alpha_n = \alpha' \) in \( \mathcal{C}(S) \). Let \( t_i \) be the balanced time of \( \alpha_i \) along \( \mathcal{G} \), and let \( t_{n+1} = b \). By Theorem 5.6, \( \text{diam}_{\mathcal{C}(S)}([\mathcal{G}(t_i), \mathcal{G}(t_{i+1})]) \) is uniformly bounded. The times \( t_j \) may not occur monotonically along \([a, b]\), but, for each \( 0 \leq i \leq n \), there exists \( j \geq i \), such that \( t_j \leq t_i \leq t_{j+1} \), and \( \text{diam}_{\mathcal{C}(S)}([\mathcal{G}(t_j), \mathcal{G}(t_{j+1})]) \leq \text{diam}_{\mathcal{C}(S)}([\mathcal{G}(t_j), \mathcal{G}(t_{j+1})]) = O(1) \). Thus, there is a sequence \( 0 = i_0 < i_1 < \cdots < i_k = n \), such that \( t_{i_{j+1}} > t_{i_j} \) and \( \text{diam}_{\mathcal{C}(S)}([\mathcal{G}(t_{i_j}), \mathcal{G}(t_{i_{j+1}})]) \) is uniformly bounded. We will call such a sequence admissible and choose one with minimal length \( k \). For simplicity, we will relabel each \( t_{i_j} \) by \( t_j \).
Now let $h: [0, k] \to [a, b]$ be defined by sending each subinterval $[j, j + 1]$ to $[t_j, t_{j+1}]$ by a linear map for all $j = 0, \ldots, k - 1$. By Theorem 5.6, $\text{diam}_{C(S)}([G(\alpha), G(t_0)]) = O(1)$ and $\text{diam}_{C(S)}([G(t_k), G(b)]) = O(1)$. Set $G_i = G \circ h(i)$. By construction, $\text{diam}_{C(S)}([G_i, G_{i+1}]) = O(1)$ for all $i \in [0, k - 1]$ and $k \leq n = d_{C(S)}(\alpha, \beta)$. Therefore, for all $0 \leq i \leq i' \leq k$, we have

$$d_{C(S)}(G_i, G_{i'}) < i' - i.$$

The only thing remaining to check is that the lower bound for the definition of quasi-geodesic, that is, for all $0 \leq i \leq i' \leq k$, we will show

$$i' - i \leq d_{C(S)}(G_i, G_{i'}) + 2.$$

It is enough to prove this for $i, i' \in \{0, \ldots, k\}$. For $0 \leq i < i' \leq k$, let $\beta \in \pi \circ G_i = \pi \circ G(t_i)$ and $\beta' \in \pi \circ G_{i'} = \pi \circ G(t_{i'})$. Let $m = d_{C(S)}(\beta, \beta')$ and choose a geodesic $\beta = \beta_0, \ldots, \beta_m = \beta'$ in $C(S)$. Let $s_i$ be the balance time of $\beta_i$ along $G$. After choosing a subsequence, we may assume that $s_i$ appear monotonically along $[a, b]$ and that $t_i < s_0$ and $s_m < t_{i'}$. We can modify the admissible sequence

$$t_0 < \cdots < t_i < \cdots < t_{i'} < \cdots < t_k$$

by

$$t_0 < \cdots < t_i < s_0 < \cdots < s_m < t_{i'} < \cdots < t_k,$$

which is still admissible. By minimality of $k$, we must have $i' - i \leq m + 2$ which proves what we want. \hfill \Box

The proof of Theorem 5.6 requires some technical results about hyperbolic surfaces. Given a hyperbolic surface $X$, consider a simple closed geodesic $\gamma$ that is $\epsilon_B$-short and a simple geodesic $\lambda$ on $X$. Assuming that $\lambda$ intersects $\gamma$ many times, we would like to find a simple closed curve $\alpha$ with a uniformly bounded intersection number with $\gamma$ that is $(n_0, L_0)$-horizontal. The argument here is somewhat delicate since there are essentially two possible situations: either $\lambda$ twists around a relatively short curve $\alpha$ or $\alpha$ is somewhat longer and a long subsegment of it stays close to $\lambda$ in the universal cover.

We find the appropriate curve $\alpha$ by applying surgery between $\lambda$ and $\gamma$ such that $\alpha$ contains a long subsegment of $\lambda$. But we also need to have some control such that after pulling $\alpha$ tight, it still stays close to $\lambda$. The following two lemmas will give the needed control.

In the following, orient the curve $\gamma$ so that the left-hand side and the right-hand side of $\gamma$ are defined. We will say an arc $\omega$ with endpoints on $\gamma$ hits $\gamma$ on opposite sides if the two endpoints of $\omega$ are on different sides of $\gamma$; otherwise, $\omega$ hits $\gamma$ on the same side.

**Lemma 5.8.** Let $\gamma$ and $\lambda$ be as above and let $\alpha = \eta \cup \omega$ be a closed curve in $X$ that is obtained from concatenation of a subarc of $\eta$ of $\gamma$ and a subarc $\omega$ of $\lambda$. Also, assume that $\eta$ hits $\omega$ on opposite sides and $L = \ell_X(\omega) \geq 4\epsilon_B$. Then, $\alpha^\ast$, the geodesic representative of $\alpha$ in $X$, stays in an $O(\epsilon_B)$-neighborhood of $\alpha$.

**Proof.** This is a well-known fact in hyperbolic geometry. We sketch the proof here. Consider the lift of $\alpha$ to $\mathbb{H}^2$ as a concatenation of segments $\eta_i$ and $\omega_i$ that are lifts of $\eta$ and $\omega$, respectively. Since $\omega$ is a subarc of a complete simple geodesic in $X$, the segments $\omega_i$ lie on complete geodesics $\tilde{\omega}_i$ in $\mathbb{H}^2$ that are disjoint. The condition that $\eta$ hits $\omega$ on opposite sides means that $\overline{\omega}_{i+1}$ does not backtrack along $\omega_i$.

Fixing $o$, the center of segment $\omega_0$, as the center of the Poincaré disk, the Euclidean distance between and endpoints of geodesics $\tilde{\omega}_i$ in $\partial \mathbb{H}^2$ form a Cauchy sequence (in fact, they decrease exponentially fast) and hence they converge. Namely, the visual angle at $o$ of the endpoint of $\tilde{\omega}_1$ is at most $O(\epsilon_B e^{-L/2})$ and the visual angle at $o$ between the endpoints of $\tilde{\omega}_i$ and $\tilde{\omega}_{i+1}$ being the integral of the Killing form $\omega$. \hfill \Box
decrease exponentially with $|i|$. Hence the lift of $\gamma^*$ starts and ends near the endpoints of $\tilde{\omega}_0$ with a visual angle of $O(\epsilon_B e^{-L/2})$. Therefore, an $O(\epsilon_B)$-neighborhood of lift of $\gamma^*$ contains $\tilde{\omega}_0$.

This completes the proof. \hfill \Box

**Lemma 5.9.** Let $\beta$ and $\beta'$ be simple closed curves in $X$ (possibly $\beta = \beta'$) with lengths longer than $\delta_B$ and let $\eta$ be a geodesic segment that is disjoint from both. Let $\gamma$ and $\gamma'$ be two segments of length $O(1)$ connecting the endpoints of $\eta$ to $\beta$ and $\beta'$, respectively, so the curve $\alpha$ obtained by the concatenation

$$\beta \cup \gamma \cup \eta \cup \gamma' \cup \beta' \cup \gamma' \cup \eta \cup \gamma$$

is simple. Let $\alpha^*$ be the geodesic representative of $\alpha$. Then, in the universal cover, any lift of $\eta$ is contained in a bounded neighborhood of the union of a lift of $\alpha^*$, a lift of $\beta$ and a lift of $\beta'$.

**Proof.** The lemma is non-trivial because two copies of $\eta$ are used and they may backtrack each other. It is essential that there is a lower bound on the lengths of $\beta$ and $\beta'$ and the lemma essentially follows from Lemma 3.7.

Consider a lift of $\alpha$ to the universal cover. Ignoring the lifts of $\gamma$ and $\gamma'$ which have bounded length, we consider the segments $\beta_i \cup \eta_i \cup \beta_i' \cup \eta_i$, where $\eta_i$ and $\eta_i'$ are lifts of $\eta_i$, $\beta_i$ is a lift of $\beta$, $\beta_i'$ are lifts of $\beta'$ and endpoints of every segment are in a uniformly bounded neighborhood of an endpoint of the next segment.

The segments $\eta_i$ and $\overline{\eta}_i$ lie on geodesics $\lambda_i$ and $\overline{\lambda}_i$ that are disjoint. In fact, there is an isometry of $H^2$, associated to the curve $\beta'$, whose axis contains $\beta_i'$ and sends $\eta_i$ to $\overline{\eta}_i$. Let $\delta_0$ be large enough such that the $\delta_0$-neighborhood of $\beta'$ contains the standard collar $U(\beta')$. Then $\delta_0$ is a universal constant since there is a lower bound on the length of $\beta'$. Let $U_{\delta_0}(\beta')$ be the $\delta_0$-neighborhood of $\beta'$ and $U_i$ be the lift that contains $\beta_i'$. There is a universal lower bound on the length of the boundary of $U_{\delta_0}(\beta')$, which means there is a lower bound on the distance between the intersection points of $\eta_i$ and $\overline{\eta}_i$ with $U_i$. It now follows from Lemma 3.7 that there is a lower bound for the distance between the subsegments of $\eta_i$ and $\overline{\eta}_i$, that are outside of $U_i$. That is, if $\eta_i$ is not near $\beta_i$, it cannot be too close to $\overline{\eta}_i$ and hence $\eta_i$ and $\overline{\eta}_i$ do not fellow-travel for a long time outside of a uniform neighborhood of $\beta_i'$. A similar statement is true for $\overline{\eta}_i$, $\eta_{i+1}$ and $\beta_i$.

Since $H$ is Gromov hyperbolic, the lift of $\alpha^*$ is contained in a uniform neighborhood of segments $\beta_i \cup \eta_i \cup \beta_i' \cup \eta_i$. In fact, each point in any of these segments is either close to the lift of $\alpha^*$ or close to a point in some other segment. We have shown that $\eta_i$ and $\overline{\eta}_i$ do not fellow-travel for a large subsegment. This means that any point in $\eta_i$ is close to either $\beta_i$, $\beta_i'$, or to the lift of $\alpha^*$. \hfill \Box

We would like to show that, at every time $t$, there is a curve $\alpha$ which has balanced time $t_{\alpha} = t$ and whose distance in $C(S)$ from the shadow of $X_t$ is uniformly bounded. The next proposition shows that a coarse version of this statement holds.

**Proposition 5.10.** Let $G: [a, b] \to T(S)$ be a Thurston geodesic segment. For any $t \in [a, b]$, if $\lambda_G$ intersects some short curve on $X_t = G(t)$, then there exists an $(n_0, L_0)$-horizontal curve $\alpha$ on $X_t$ such that $i(\alpha, \gamma) = O(1)$ for any $\epsilon_B$-short curve $\gamma$. Furthermore, $d(\gamma(S), X_{t_n}, X_t) = O(1)$.

**Proof.** First, we will construct $\alpha$. Let

$$N_0 = n_0 \left\lceil \frac{L_0}{\delta_B} \right\rceil + K,$$

where $K$ is the additive error coming from Lemma 5.9.
If $\lambda_G$ has a closed leaf $\lambda$ that intersects every short curve on $X_1$ at most $5N_0$ times, then $\alpha = \lambda$ has the desired properties. Otherwise, we can fix a leaf $\lambda$ in the stump of $\lambda_G$ that intersects some $\epsilon_B$-short curve more than $5N_0$ times.

Fix a segment $\omega$ of $\lambda$ such that $\omega$ has endpoints on a short curve $\gamma$ with $i(\gamma, \omega) = 5N_0$, and $\omega$ intersects all other short curves at most $5N_0$ times. Orient $\gamma$ such that we can talk about the two sides of $\gamma$. We will show that, applying a surgery between $\omega$ and $\gamma$, we can obtain a simple closed curve $\alpha$ that still intersects $\gamma$ and stays close to $\lambda$ for a long time. Unfortunately, this process is delicate and depending on the intersection pattern of $\gamma$ and $\omega$, we may have to apply a different surgery. The conclusion will follow if either of the following two cases occur.

Case (1): There is a subarc $\eta$ of $\omega$ that hits $\gamma$ on opposite sides with $i(\eta, \gamma) \geq N_0$, and the endpoints of $\eta$ can be joined by a segment of $\gamma$ that is disjoint from the interior of $\eta$.

In this case, the geodesic representative $\alpha$ of the concatenation of $\eta$ and a segment of $\gamma$ is $(n_0, L_0)$-horizontal by Lemma 5.8 (see the left-hand side of Figure 8).

Case (2): There is a subarc $\eta$ of $\omega$ and a closed curve $\beta$ disjoint from $\eta$, such that $i(\eta, \gamma) \geq N_0$ and $\ell_X(\beta) \geq \delta_B$, and the endpoints of $\eta$ are close to the same point on $\beta$. Furthermore, each endpoint of $\eta$ can be joined to a nearby point on $\beta$ by a segment of $\gamma$ that is disjoint from $\beta$ and the interior of $\eta$.

In this case (see the right-hand side of Figure 8), let $\alpha$ be the curve obtained by closing up $\eta$ with $\beta$ and one or two subarcs of $\gamma$. If $\eta$ twists around $\beta N_0$ times, then $\beta$ is $(n_0, L_0)$-horizontal by Proposition 4.6; otherwise, by Lemma 5.9, $\alpha$ has a segment that intersects $\gamma$ $N_0$ times and stays close to $\gamma$, in which case $\alpha$ is $(n_0, L_0)$-horizontal by Proposition 4.6.

Case (3): There is a subarc $\eta$ of $\omega$ and there are two closed curves $\beta$ and $\beta'$ that are disjoint from $\eta$, such that $i(\eta, \gamma) \geq N_0$, $\ell_X(\beta) \geq \delta_B$ and $\ell_X(\beta') \geq \delta_B$, and the two endpoints of $\eta$ are close to $\beta$ and $\beta'$. Furthermore, there exists a segment of $\gamma$ joining one endpoint of $\eta$ to $\beta$ and a segment of $\gamma$ joining the other endpoint of $\eta$ to $\beta'$, such that both segments are disjoint from $\beta$, $\beta'$ and the interior of $\eta$.

In this case (see Figure 9), let $\alpha$ be the curve obtained by gluing two copies of $\eta$, $\beta$, $\beta'$ and a few sub-arcs of $\gamma$. If $\eta$ twists around either $\beta$ or $\beta' N_0$ times, then either $\beta$ or $\beta'$ is $(n_0, L_0)$-horizontal by Proposition 4.6; otherwise, by Lemma 5.9, $\alpha$ has a segment that intersects $\gamma$ $N_0$ times and stays close to $\eta$, in which case $\alpha$ is $(n_0, L_0)$-horizontal by Proposition 4.6.

We now show that at least one of these three cases happens.

Let $p_0$ and $q_0$ be the endpoints of $\omega$. Let $p_{-1}$ and $p_1$ be the adjacent intersection points along $\gamma$ to $p_0$ and $q_{-1}$ and $q_1$ be the adjacent intersection points along $\gamma$ to $q_0$. By relabeling if necessary, we may assume that $\omega$ passes from $p_0$ to $p_1$ and then to $p_{-1}$. Assume that $\omega$ passes from $q_0$ to $q_1$ and then to $q_{-1}$. We allow the possibility $p_{-1} = q_0$ and $q_{-1} = p_0$, or $p_{-1} = q_{-1}$.

Let $\omega_0$ be the subarc of $\omega$ from $p_0$ to $p_1$ and let $\omega_1$ be the subarc from $p_1$ to $p_{-1}$. Suppose $\omega_0 \cup \omega_1$ intersects $\gamma$ at least $2N_0$ times. In this situation, we have several possibilities that will yield case (1) or (2). If $\omega_0 \cup \omega_1$ hits $\gamma$ on opposite sides, as in the left-hand side of Figure 8,
then we are in case (1) with \( \eta = \omega_0 \cup \omega_1 \). Otherwise, one of \( \omega_i \) hits \( \gamma \) on opposite sides and the other one hits \( \gamma \) on the same side. Assume that \( \omega_0 \) hits \( \gamma \) on opposite sides, as in the right-hand side of Figure 8. We are again in case (1) if \( \omega_0 \) intersects \( \gamma \) at least \( N_0 \) times. If not, let \( \beta \) be the closed curve obtained from closing up \( \omega_0 \) with an arc of \( \gamma \). Since \( \omega_0 \) has endpoints on \( \gamma \) and \( \beta \) stays close to \( \omega_0 \) by Lemma 5.8, \( \ell_X(\beta) \geq \delta_B \). We are now in case (2) with \( \eta = \omega_1 \). The dotted line in the right-hand side of Figure 8 represents the closed curve obtained from this surgery.

Similarly, let \( \omega'_0 \) be the subarc of \( \omega \) from \( q_0 \) to \( q_1 \) and let \( \omega'_1 \) be the arc from \( q_1 \) to \( q_−1 \). As above, we are done if \( \omega'_0 \cup \omega'_1 \) intersects \( \gamma \) at least \( 2N_0 \) times.

Since \( \omega \) intersects \( \gamma \) \( 5N_0 \) times, if neither \( \omega_0 \cup \omega_1 \) nor \( \omega'_0 \cup \omega'_1 \) intersects \( \gamma \) at least \( 2N_0 \) times, then the arc \( \eta \) from \( p−1 \) to \( q_−1 \) must have at least \( N_0 \) intersections with \( \gamma \). If \( \eta \) hits \( \gamma \) on opposite sides, then we are in case (1). Otherwise, at least one of \( \omega_0 \), \( \omega_1 \), or \( \omega'_0 \cup \omega'_1 \) hits \( \gamma \) on opposite sides. Close this arc to obtain a closed curve \( \beta \) which has length at least \( \delta_B \).

Similarly, let \( \beta' \) be a closed curve obtained from closing up either \( \omega'_0 \), \( \omega'_1 \) or \( \omega'_0 \cup \omega'_1 \). We are now in case (3). In Figure 9, we have illustrated the situation when \( \omega_0 \cup \omega_1 \) forms \( \beta \) and \( \omega'_0 \) forms \( \beta' \).

It remains to show \( d_{C(S)}(X_{t_\alpha}, X_t) = O(1) \). By definition, \( t_\alpha \leq t \) and \( \alpha \) is \( (n_0, L_0) \)-horizontal on \( X_{t_\alpha} \). Assume \( t − t_\alpha \geq s_0 \), where \( s_0 \) is the constant of Theorem 4.2. Let \( \gamma_\alpha \) be an anchor curve for \( \alpha \) at time \( t_\alpha \). The assumption implies \( d_{C(S)}(X_t, \alpha) = O(1) \), so it is enough to show \( d_{C(S)}(\gamma_\alpha, \alpha) = O(1) \). Define \( D = d_{C(S)}(\gamma_\alpha, \alpha) \). Recall that \( D \leq \log_2 i(\alpha, \gamma_\alpha) + 1 \).

Let \( \lambda \) and \( \tilde{\alpha} \) be as in Definition 4.1, let \( \omega \) and \( \tau \) be the segments of \( \lambda \) and \( \tilde{\alpha} \) which are at most \( \epsilon_B \) Hausdorff distance apart and which intersect \( n_0 \) lifts of \( \gamma_\alpha \). We may assume \( i(\alpha, \gamma_\alpha) \) is large enough such that \( \tau \) projects to a proper subarc of \( \alpha \), that is, the length of \( \tau \) is smaller than the length of \( \alpha \). Let \( \tilde{f} \) be the lift of an optimal map \( f : X_{t_\alpha} \to X_t \). By the proof of Theorem 4.2, up to a multiplicative error \( \tilde{f}(\omega) \) intersects \( n_0 2^D \) lifts of a short curve \( \gamma' \) on \( X_t \). Moreover, a segment \( \tau' \) of the geodesic representative of \( \tilde{f}(\tilde{\alpha}) \) is \( \epsilon_B \)-close to \( \tilde{f}(\lambda) \) and also intersects the \( n_0 2^D \) lifts of \( \gamma' \) up to a bounded error. The length of \( \alpha \) on \( X_\tau \) is bigger than the length of \( \tau' \), hence \( i(\alpha, \gamma') \geq n_0 2^D \). But \( i(\alpha, \gamma') = O(1) \) by assumption, therefore \( D \) must be bounded.

**Proof of Theorem 5.6.** By Theorem 5.2, \( \pi \) is a coarse Lipschitz map. Let \( \alpha \) be any short curve on \( X_t \) and \( \lambda_G \) be the maximally stretched lamination. We will show \( \text{diam}_{C(S)}([X_t, X_{t_\alpha}]) = O(1) \).

If (the stump of) \( \lambda_G \) is a short curve, it may not intersect any other short curve at \( X_t \). Let \( s \) be the first time \( \lambda_G \) intersects some short curve \( \gamma \) in \( X_s \). Since \( \lambda_G \) is a short curve in the interval \([t, s]\), we have \( \text{diam}_{C(S)}([X_t, X_s]) \geq d_{C(S)}(\alpha, \gamma) = O(1) \).

We now show \( \text{diam}_{C(S)}([X_{t_\alpha}, X_s]) = O(1) \). Since \( \lambda_G \) intersects a short curve \( \gamma \) on \( X_s \), by Proposition 5.10, there exists a curve \( \beta \) on \( X_s \) with \( i(\beta, \gamma) = O(1) \) and \( \text{diam}_{C(S)}([X_{t_\beta}, X_s]) = O(1) \). We have \( d_{C(S)}(\alpha, \beta) \leq d_{C(S)}(\alpha, \gamma) + d_{C(S)}(\gamma, \beta) = O(1) \), which implies by Theorem 5.2 that \( \text{diam}_{C(S)}([X_{t_\alpha}, X_{t_\beta}]) = O(1) \). The conclusion follows by the triangle inequality.
6. Examples of geodesics

In this section, we construct several examples of geodesics in the Thurston metric demonstrating various possible behaviors, proving Theorems 1.1 and 1.4 from the introduction. The main idea in all these examples is that it is possible for the maximally stretched lamination associated to some Thurston geodesic to be contained in some subsurface $W$ where the lengths of all curves disjoint from $W$ (including $\partial W$) stay constant along the geodesic. This contrasts the behavior of a Teichmüller geodesic, where in a similar situation the length of $\partial W$ would get short along the geodesic [20]. Our construction can be made to be very general. However, in the interest of simplicity, we make an explicit construction when $W$ is a torus with one boundary component.

For a constant $\ell$, let $T(S_{1,1}, \ell)$ be the space of hyperbolic structures on a torus with one geodesic boundary where the length of the boundary curve is $\ell$. Note that we can equip $T(S_{1,1}, \ell)$ with the Thurston metric as usual (see [10] for details).

Let $\mu$ be any irrational measured lamination on $S_{1,1}$. There is a unique way to complete $\mu$ to a complete lamination $\lambda$, such that the complement are two ideal triangles, by adding two bi-infinite leaves both tending to the cusp in one direction and wrapping around $\mu$ in the other. Hence, for any $U_0 \in T(S_{1,1})$, there exists a unique stretch path from $U_0$ with $\mu$ the stump of the maximally stretched lamination. We will denote by stretch($U_0, \lambda, t$) this stretch path.

**Proposition 6.1.** There is a constant $\ell_0$ such that the following holds. For any $U_0 \in T(S_{1,1})$, any irrational measured lamination $\mu$ on $S_{1,1}$ and the associated stretch path

$$U_t = \text{stretch}(U_0, \lambda, t), \quad U_t \in T(S_{1,1}),$$

where $\lambda$ is the unique complete lamination containing $\mu$ as its stump, there is a bi-infinite Thurston geodesic $W_t \in T(S_{1,1}, \ell_0)$, $t \in \mathbb{R}$, where the stump of the maximally stretched lamination is still $\mu$ and, for any other curve $\alpha$ in $S_{1,1}$,

$$\ell_{W_t}(\alpha) \geq \ell_{U_t}(\alpha).$$

**Proof.** We will refer the reader to Figure 10 for this proof.

Choose $U_0 \in T(S_{1,1})$ and represent $\mu$ as a geodesic lamination on $U_0$. Let $\lambda$ be the unique completion of $\mu$ and let $A$ and $B$ be the ideal triangles in the complement of $\lambda$ in $U_0$. Each $A$ and $B$ has two sides $a$ and $b$ coming from the two leaves of $\lambda$ tending toward the cusp, and a third side that wraps around the stump $\mu$. There is an involution of $U_0$ fixing $\lambda$ and switching $A$ with $B$. Hence, the two anchor points of $A$ at $a$ and $b$ are glued, respectively, to the two anchor points of $B$ at $a$ and $b$. See the upper right-hand side of Figure 10.

Now we double this picture. Let $U_0^+ = U_0$ and let $U_0^-$ be an orientation-reversing copy of $U_0^+$. We also label $A = A^+, B = B^+, a = a^+, b = b^+$ and $\mu = \mu^+$ and we label the associated objects in $U_0^-$ by $A^-, B^-, a^-, b^-$ and $\mu^-$. Cut $U_0^\pm$ open along $a^\pm$ and $b^\pm$. Via a reflection map, glue $b^+$ of $A^+$ to $a^-$ of $A^-$ such that the anchor point of $A^+$ in $b^+$ is glued to the anchor point of $A^-$ in $a^-$. Similarly, via a reflection map, glue $b^-$ to $a^+$, $a^+$ to $b^-$ and $a^+$ to $a^-$ gluing the corresponding anchor points. Note that the third sides of $A^\pm$ and $B^\pm$ wrap about $\mu^\pm$ in $U^\pm$. This yields a genus two surface $\tilde{U}_0$ with a geodesic lamination $\tilde{\lambda}$ that contains $\mu^\pm$ as its stump and has two extra leaves each wrapping about $\mu^+$ in one direction and wrapping about $\mu^-$ in the other direction.

In each $A^\pm$ or $B^\pm$, there is a geodesic arc connecting the midpoint of $a^\pm$ to the midpoint of $b^\pm$. Because the gluing maps were reflections, the angles of these arcs with $a^\pm$ and $b^\pm$ match and the four arcs glue together to form a separating geodesic $\gamma$ in $\tilde{U}_0$. Let $\ell_0$ be the length of $\gamma$. By construction, $\ell_0$ is independent of the irrational lamination $\mu$ and $U_0$.

Let $\tilde{U}_t = \text{stretch}(\tilde{U}_t, \tilde{\lambda}, t)$. Then $\tilde{U}_t$ is also obtained from the doubling of $U_t$ as above. Along the stretch paths, the length of geodesic arcs connecting the midpoint of $a^\pm$ to the midpoint
We can double the stretch path on a punctured torus to obtain a stretch path in a surface of genus two. The length of the curve $\gamma$ remains unchanged along the stretch path.

The proposition now holds where $W_t$ is the subsurface of $\tilde{U}_t$ with boundary $\gamma$. The stretch map from $U_s$ to $U_t$ doubles to an $e^{t-s}$-Lipschitz homeomorphism between $\tilde{U}_s$ and $\tilde{U}_t$ that fixes $\gamma$ pointwise. Hence, the length of curves grow by at most a factor of $e^{t-s}$ both from $\tilde{U}_s$ to $\tilde{U}_t$ and from $W_s$ to $W_t$.

To see the last assertion in the proposition, we note that there is a contraction map from $W_t$ to $U_t$. Consider the restriction of $A^-$ and $B^-$ in $W_t$ and foliate it with horocycles perpendicular to the boundary. Then collapse these regions sending each horocycle to a point. This is a distance-decreasing map: the derivative in the direction tangent to the horocycles is zero and in the direction perpendicular to the horocycles is 1. Since stretch paths preserve these horocycles, this collapsing map commutes with the stretch paths and the image of $W_t$ under the collapsing map is exactly $U_t$. Hence, the length of a curve $\alpha$ in $W_t$ is longer than the length of its image in $U_t$ under the collapsing map, which is longer than the length of the geodesic representative of $\alpha$.

This proposition is the building block for all the examples we construct in this section. Essentially, we can glue $W_t$ to a family of surfaces with desired behavior to obtain various examples.

**Proof of Theorem 1.1.** Let $\mu$ be a simple closed curve and let $U_0$ be a point in $\mathcal{T}(S_{1,1})$ where the length of $\mu$ is $\epsilon$ for some small $\epsilon > 0$. Let $W_t$ be the family obtained by Proposition 6.1. Also, choose $V \in \mathcal{T}(S_{1,1}, \ell_0)$ to be a point in the thick part. Let $\beta$ be a curve of bounded length in $V$ and define $V^n = D_\beta^n V$, where $D_\beta$ is the Dehn twist around the curve $\beta$. (The values of $\epsilon$ and $n$ are to be determined below.)
Let $s$ be the time when $\mu$ has length 1 in $W_s$. Define
\[ X = W_0 \cup V, \quad Y = W_s \cup V \quad \text{and} \quad Z = W_{s/2} \cup V^n. \]
By $\cup$, we mean glue the two surfaces along the boundary and consider them as an element of $T(S_{2,0})$. We mark the surfaces so that they have bounded relative twisting along the gluing curve $\gamma$. (Note that relative twisting is only well defined up to an additive error). We claim that if $\log 1/\epsilon \gg n$, then
\[ d_{Th}(X, Z) = d_{Th}(Z, Y) = s/2. \]
First consider $X$ and $Z$. Indeed, since the length of $\mu$ grows exponentially in $W_t$, $s/2$ is a lower bound for the distance $d_{Th}(X, Z)$. We need to show that the length of any other curve grows by a smaller factor. This is true for any curve contained in $W_t$ by Proposition 6.1. For any other curve $\alpha$, let $\alpha_W$ be the restriction of $\alpha$ to $W_0$ and let $\bar{\alpha}_W$ be the restriction to the complement of $W_0$. Any representative of $\alpha$ in $Z$ is no shorter than the geodesic representative. Therefore, there is a uniform constant $C$ such that
\[ \ell_Z(\alpha) \leq e^{s/2} \ell_X(\alpha_W) + C(\ell_X(\bar{\alpha}_W) + n \ell_X(\beta)i(\alpha, \beta)). \]
But $\ell_X(\bar{\alpha}_W) \geq i(\alpha, \beta)$. Hence, if $e^{s/2} \gg C + n$, then we obtain
\[ e^{s/2} \ell_X(\alpha_W) \geq (C + n)(\ell_X(\bar{\alpha}_W) + \ell_X(\beta)i(\alpha, \beta)). \]
Therefore, for sufficiently large $s$, we have
\[ \ell_Z(\alpha) \leq e^{s/2}(\ell_X(\alpha_W) + \ell_X(\bar{\alpha}_W)) = e^{s/2} \ell_X(\alpha). \]
That is, $\mu$ is the maximally stretched lamination from $X$ to $Z$. The argument for the distance from $Z$ to $Y$ is similar.

Now define
\[ G_t(t) = W_t \cup V, \]
and let $G_2(t)$ be the geodesic obtained by a concatenation of the geodesic connecting $X$ to $Z$ and $Z$ to $Y$. Let $\alpha$ be a curve disjoint from $W_t$ that has a bounded length in $X$. Then, for all $t$,
\[ \ell_{G_t(t)}(\alpha) = \ell_{X}(\alpha) \lesssim 1. \]
But the length of $\alpha$ in $Z = G_2(s/2)$ is of order $n$. That is,
\[ d_{Th}(G_t(t), Z) \gg \log \frac{\ell_Z(\alpha)}{\ell_{G_t(t)}(\alpha)} \lesssim \log n, \]
which can be chosen to be much larger than $D$. Note that a lower bound for the distance in the other direction can also be found by replacing $\alpha$ with $D_{\beta}^{-n}(\alpha)$.

To obtain the second part of Theorem 1.1, we note that the distance from $Y$ to $X$ is only of order $\log \log(1/\epsilon)$ if $\epsilon$ is small enough. We now choose $n$ and $\epsilon$ such that
\[ \log \frac{1}{\epsilon} \gg \log n, \quad \log n \lesssim D \quad D \gg \log \log \frac{1}{\epsilon}. \]
This way, if $s \geq 2D$, then $Z = G_2(s/2)$ has distance at least $D$ to any point on any geodesic connecting $Y$ to $X$. This completes the proof of part 2.

Next, we construct an example showing that the set of short curves in a Thurston geodesic connecting two points is not the same as the set of short curves along the Teichmüller geodesic. This is in contrast with the following theorem.

**Theorem 6.2 [14].** For every $K > 0$ and $\epsilon > 0$ there exists $\epsilon' > 0$ such that whenever $X, Y \in T(S)$ are $\epsilon$-thick and have $K$-bounded combinatorics, then any Thurston geodesic $G$ from $X$ to $Y$ remains in the $\epsilon'$-thick part.
Proof of Theorem 1.4. Let $\phi$ be a pseudo-Anosov map in the mapping class group of $S_{1,1}$ and let $U_0 \in \mathcal{T}(S_{1,1})$ be on the Teichmüller axis of $\phi$. For any $n \in \mathbb{Z}$, the maximally stretched lamination $\mu$ from $U_0$ to $\phi^n(U_0)$ is irrational, hence the stretch path from $U_0$ to $\phi^n(U_0)$ is the unique Thurston geodesic connecting them. Let $U_i = |(U_0, \mu, t)$ and let $U_s = \phi^n(U_0)$. The point $\phi^n(U_0)$ is also on the Teichmüller axis of $\phi$ and the Teichmüller geodesic segment connecting $U_0$ to $\phi^n(U_0)$ stays in a uniform thick part (independent of $n$) of Teichmüller space. From [14], we know that the Teichmüller geodesics connecting these two points also stays in this part and fellow-travels the Teichmüller geodesic.

Let $W_i$ be the family of surfaces in $\mathcal{T}(S_{1,1}, \ell_0)$ obtained from Proposition 6.1 and let $V$ be any point in the thick part of $\mathcal{T}(S_{1,1}, \ell_0)$. Now define

$$\mathcal{G}(t) = W_i \cup V, \hspace{1cm} t \in [0, s].$$

Then $\mathcal{G}(t)$ is a Thurston geodesic in $\mathcal{T}(S_{2,0})$. This is because the length of $\mu$ is growing exponentially and the length of every other curve is growing by a smaller factor. (The argument is an easier version of the arguments in the previous proof and is dropped.)

Since $U_i$ is in the thick part, so is $W_i$ and hence $\mathcal{G}(t)$. Let $X = \mathcal{G}(0)$ and $Y = \mathcal{G}(s)$. But, by taking $n$ large enough, we can ensure that $d_W(X,Y)$ is as large as desired, where $W$ is the subsurface of $S_{2,0}$ associated to $W_i$. It then follows from [20] that the boundary of $W$ is short along the Teichmüller geodesic connecting $X$ to $Y$, in fact, its minimum length is inversely proportional to $d_W(X,Y)$. That is, $\partial W$ has bounded length along the Thurston geodesic $\mathcal{G}(t)$ but is arbitrarily short along the Teichmüller geodesic. This completes the proof of the theorem.

Acknowledgements. Our Key Proposition 5.5 is modeled after [2, Proposition 6.4] where Bestvina–Feighn show the projection of a folding path to the free factor graph is a reparameterized quasi-geodesic which is in turn inspired by the arguments of Masur–Minsky [15]. We would like to thank Université de Rennes and Erwin Schrödinger International Institute for Mathematical Physics for their hospitality. We also thank the referee for carefully reading the paper and providing us all the useful comments.

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