



Benjamini-Schramm convergence of periodic orbits

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Abstract

We prove a criterion for Benjamini-Schramm convergence of periodic orbits of Lie groups. This general observation is then applied to homogeneous spaces and the space of translation surfaces.

Keywords Benjamini-Schramm convergence · Measure classification · Stabilizers

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1 Benjamini-Schramm convergence

Let $H \subset \mathrm{SL}_N(\mathbb{R})$ be a non-compact semisimple group. Even though $H \subset \mathrm{SL}_N(\mathbb{R})$, we will write e for the identity element in H . The notation I (for the identity matrix) will only be used when the vector space structure of the space of matrices is relevant.

Let $\| \cdot \|$ denote the maximum norm on $\mathrm{Mat}_N(\mathbb{R})$ with respect to the standard basis, and put

$$B^H(e, R) = \{h \in H : \|h - I\| < R \text{ and } \|h^{-1} - I\| < R\}.$$

We also equip H with the right invariant Riemannian metric induced by the Killing form (and a fixed choice of a maximal compact subgroup of H), and let $B_{\mathrm{Rie}}^H(e, r)$ denote the ball of radius r centered at the identity with respect to this metric. Then for every R , there exists $r > 0$ so that

$$B_{\mathrm{Rie}}^H(e, r) \subset B^H(e, R).$$

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Let $r(R)$ denote $1/2$ the supremum of all such r , then $B_{\text{Ric}}^H(e, r(R)) \subset B^H(e, R)$ and $r(R) \rightarrow \infty$ as $R \rightarrow \infty$; indeed it is not difficult to see that $r(R) \geq C \log R$ where $C > 0$ depends on the embedding $H \subset \text{SL}_N(\mathbb{R})$.

Let $\Delta \subset H$ be a discrete subgroup. The injectivity radius of $y \in H/\Delta$ is define as the supremum over all $r > 0$ so that the map $h \mapsto hy$ is injective on $B_{\text{Ric}}^H(e, r)$.

Let $\Delta_n \subset H$ be a sequence of lattices in H . The sequence $\{H/\Delta_n : n \in \mathbb{N}\}$ *Benjamini-Schramm* converges to H if for every $r > 0$ we have

$$\mu_n\left(\{y \in H/\Delta_n : \text{injectivity radius of } y < r\}\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

where μ_n denote the H -invariant probability measure on H/Δ_n for every n .

Throughout, we assume that H acts continuously on X preserving the measure μ ; also assume that $\text{Stab}_H(x)$ is discrete for every $x \in X$.

An orbit $Hx \subset X$ is called *periodic* if $Hx \subset X$ is a closed subset and $\text{Stab}_H(x)$ is a lattice in H .

For a periodic orbit Hx , let μ_{Hx} denote the pushforward of the H -invariant probability measure of $H/\text{Stab}_H(x)$ to Hx .

Proposition 1.1 *Let $\{Hx_n : n \in \mathbb{N}\}$ be a sequence of periodic orbits in X satisfying that*

$$\mu_{Hx_n} \rightarrow \mu \text{ as } n \rightarrow \infty. \tag{1.1}$$

Assume further that for every $R > 0$ there exists a continuous function $f_R : X \rightarrow [0, \infty)$ satisfying the following two properties:

1. $f_R(x) > 0$ for μ -a.e. $x \in X$,
2. if $f_R(x) > 0$ for some $x \in X$, then $\text{Stab}_H(x) \cap B^H(e, R) = \{e\}$.

Then $H/\text{Stab}_H(x_n)$ Benjamini-Schramm converges to H .

Proof Let $R > 0$. Let $Y = Hx \subset X$ be a periodic orbit, and put $\Delta = \text{Stab}(x)$. The map $h\Delta \mapsto hx$ is a homeomorphism from H/Δ onto Y . Let $h\Delta \in H/\Delta$, and write $y = hx \in Y$. Suppose now that $h_1h\Delta = h_2h\Delta$ for some $h_1, h_2 \in B^H(e, R)$. Then $\|h_2^{-1}h_1 - I\| < NR^2$ and

$$h_2^{-1}h_1 \in h\Delta h^{-1} = \text{Stab}_H(y).$$

This and the assumption (2) in the proposition imply that

$$\text{If } f_{NR^2}(y) > 0, \text{ then the injectivity radius of } h\Delta \text{ is at least } r(R); \tag{1.2}$$

recall that $B_{\text{Ric}}^H(e, r(R)) \subset B^H(e, R)$.

Let now $\varepsilon > 0$. In view of our assumption (1) in the proposition, there exists a compact subset $K_\varepsilon \subset X$ so that

$$\mu(K_\varepsilon) > 1 - \varepsilon \text{ and } f_{NR^2}(x) > 0 \text{ for all } x \in K_\varepsilon.$$

Since f is continuous and K_ε is compact, there exists some $\delta > 0$ so that $f_{NR^2}(x) > 0$ for all $x \in \mathcal{N}_\delta(K_\varepsilon)$, where $\mathcal{N}_\delta(K_\varepsilon)$ denotes a finite open covering of the set K_ε with balls of radius δ centered at points in K_ε .

Since $\mathcal{N}_\delta(K_\varepsilon)$ is an open set and $\mu_{Hx_n} \rightarrow \mu$, we conclude that

$$\liminf_n \mu_{Hx_n}(\mathcal{N}_\delta(K_\varepsilon)) \geq \mu(\mathcal{N}_{\delta/2}(K_\varepsilon)) \geq 1 - \varepsilon.$$

This and the fact that $\mathcal{N}_\delta(K_\varepsilon) \subset \{y \in Hx_n : f_{NR^2}(y) > 0\}$ imply: there exists some n_0 so that

$$\mu_{Hx_n}(\{y \in Hx_n : f_{NR^2}(y) > 0\}) > 1 - 2\varepsilon \text{ for all } n > n_0.$$

In consequence, using (1.2) we deduce that

$$\mu_{H/\text{Stab}(x_n)}(\{y \in H/\text{Stab}(x_n) : \text{injectivity radius of } y \text{ is } < r(R)\}) < 2\varepsilon$$

for all $n > n_0$. Since $r(R) \rightarrow \infty$ as $R \rightarrow \infty$, the claim follows. □

In subsequent sections, we discuss two settings where Proposition 1.1 is applicable: the homogeneous setting is discussed in §2 and the space of Abelian differentials in §3; see in particular Theorems 2.2 and 3.1.

2 Homogeneous spaces

Let \mathbf{G} be a connected algebraic group defined over \mathbb{R} , and let $G = \mathbf{G}(\mathbb{R})^\circ$ be the connected component of the identity in the Lie group $\mathbf{G}(\mathbb{R})$.

Let $\Gamma \subset G$ be a lattice. Throughout this section, we assume that Γ is torsion free. Let $X = G/\Gamma$, and let μ_X denote the G -invariant probability measure on X .

Theorem 2.1 *Let the notation be as above. Let $H \subset G$ be a connected semisimple Lie group. Assume that*

$$\bigcap_{g \in G} gHg^{-1} \text{ is a finite group.} \tag{2.1}$$

Let $\{Hx_n : n \in \mathbb{N}\}$ be a sequence of periodic H -orbits in X so that

1. *There exists a compact subset $K \subset X$ with $Hx_n \cap K \neq \emptyset$ for all n .*
2. *For every $H \subset L \subset G$ and any closed orbit Lx , at most finitely many of the orbits Hx_n are contained in Lx .*

Then $H/\text{Stab}_H(x_n)$ Benjamini-Schramm converges to H .

Note that the condition $\bigcap_{g \in G} gHg^{-1}$ is a finite group in the theorem is satisfied for instance if G semisimple and H does not contain any of the simple factors of G .

Theorem 2.2 *Let M be a real or complex hyperbolic d -manifold with $d \geq 3$. Assume that M contains infinitely many properly immersed totally geodesic hypersurfaces $\{V_n : n \in \mathbb{N}\}$. Then $\{V_n\}$ Benjamini-Schramm converges to \mathbb{H}^{d-1} in the real hyperbolic case and to $\mathbb{C}\mathbb{H}^{d-1}$ in the complex case.*

Proof We prove the result for the case real hyperbolic manifold, the complex case is similar.

Let $G = \text{SO}(d, 1)^\circ$, $\Gamma = \pi_1(M)$, and $H = \text{SO}(d - 1, 1)^\circ$. Then V_n lifts to a closed orbit Hx_n in $X = G/\Gamma$ for every n .

Note that $H \subset G$ is a maximal connected subgroup which is not a parabolic subgroup of G . Therefore, the assumptions in Theorem 2.1 are satisfied for G, H , and the orbits $\{Hx_n : n \in \mathbb{N}\}$. The claim thus follows from Theorem 2.1. □

We note that when Γ is arithmetic Theorem 2.1 can be proved using the results in [1, §5]. This condition holds if Γ is an irreducible lattice and the real rank of G is at least two

by Margulis’ arithmeticity theorem [9]. Moreover, it was proved by Corlette and Gromov-Shoen [4, 8] that lattices in $SP(n, 1)$ and F_4^{-20} are arithmetic. While non-arithmetic lattices in $SO(n, 1)$, for all n , and $SU(n, 1)$, for $n = 2, 3$, exist, recent developments, [2, 3, 10], show that the presence of infinitely many totally geodesic hyperplanes¹ in real and complex hyperbolic manifolds of finite volume imply arithmeticity of their fundamental group. Therefore, in all of the interesting cases, the assertion of Theorem 2.1 can be obtained by combining rather deep existing results in the literature. However, the proof we provide here is different and is arguably simpler. In particular, our proof does not rely on the arithmeticity of Γ and further property of congruences lattices; instead, our proof relies only on a special case of an equidistribution theorem of Mozes and Shah [14].

Lemma 2.3 *Let the notation and the assumptions be as in Theorem 2.1. Then for μ_X -a.e. $x \in X$ we have*

$$\text{Stab}_H(x)\{e\}.$$

Proof Let \mathbf{H} denote the Zariski closure of H in \mathbf{G} . Since H is a connected semisimple Lie group, it has finite index in the group $H' := \mathbf{H}(\mathbb{R}) \cap G$.

By Chevalley’s theorem, there exists a finite dimensional (real) representation (ρ, W) of \mathbf{G} and a vector $w \in W$ so that $\mathbf{H} = \{g \in \mathbf{G} : gw = w\}$. In particular, we conclude that

$$H' = G \cap \mathbf{H} = \{g \in G : gw = w\}. \tag{2.2}$$

Let now $x = g_0\Gamma$. Then $\text{Stab}(x) = g_0\Gamma g_0^{-1}$, and $H \cap g_0\Gamma g_0^{-1}$ is nontrivial if and only if there exists some $e \neq \gamma \in \Gamma$ so that $\gamma \in g_0^{-1}Hg_0$. Since $H \subset H'$, we conclude that $\gamma g_0^{-1}w = g_0^{-1}w$. Hence,

$$g_0^{-1} \in \mathbf{F}_\gamma = \{g \in \mathbf{G} : \gamma gw = gw\}.$$

For every $\gamma \in \Gamma$, the set \mathbf{F}_γ is an algebraic variety defined over \mathbb{R} . Moreover, $G = \mathbf{G}(\mathbb{R})^\circ$ is Zariski dense in \mathbf{G} . These and the fact that Γ is countable imply that unless there exists some $e \neq \delta \in \Gamma$ so that

$$\delta gw = gw \text{ for all } g \in G,$$

the lemma holds — indeed in that case $G \setminus \left(\cup_{\gamma \in \Gamma} \mathbf{F}_\gamma\right)$ is a conull subset of G , and for every g in this set we have $\text{Stab}_H(g\Gamma) = \{e\}$.

Assume now to the contrary that $G = \{g \in G : \delta gw = gw\}$ for some nontrivial $\delta \in \Gamma$. Then by (2.2) we have $\delta \in gH'g^{-1}$ for all $g \in G$, hence,

$$\delta \in \bigcap_{g \in G} gH'g^{-1}.$$

Since $[H' : H] < \infty$, there exists some n so that $\delta^n \in gHg^{-1}$ for all $g \in G$. That is, $\delta^n \in \bigcap_{g \in G} gHg^{-1}$. However, Γ is torsion free and $\bigcap_{g \in G} gHg^{-1}$ is a finite group. This contradiction completes the proof. \square

Proof of Theorem 2.1 We may and will assume that $G \subset \text{SL}_N(\mathbb{R})$ for some N . As before, for all subgroups $L \subset G$ and all $R > 0$, let

$$B^L(e, R) = \{g \in L : \|g - I\| < R \text{ and } \|g^{-1} - I\| < R\}$$

¹ The works [2, 3] are indeed more general and allow for properly immersed maximal totally geodesic submanifolds of dimension at least 2.

where $\| \cdot \|$ denotes the maximum norm on $SL_N(\mathbb{R})$ with respect to the standard basis.

Recall that μ_X denotes the G -invariant probability measure on X . First note that by a theorem of Mozes and Shah [14] and our assumptions (1) and (2) in the theorem, we have

$$\mu_{Hx_n} \rightarrow \mu_X \text{ as } n \rightarrow \infty. \tag{2.3}$$

Let dist denote the right invariant Riemannian metric on G induced using the killing form. Let $R > 1$, and put $\text{Stab}(x)_R = \text{Stab}_G(x) \cap B^G(e, R)$; this is a finite set. Define $f_R : X \rightarrow [0, \infty)$ by

$$f_R(x) = \inf\{d(h, g) : h \in B^H(e, R), g \in \text{Stab}(x)_R \setminus \{e\}\}.$$

Since $\text{Stab}_G(g\Gamma) = g\Gamma g^{-1}$ and R is fixed, f_R is continuous. Furthermore, $f_R(x) > 0$ for some $x \in X$ if and only if $B^H(e, R) \cap \text{Stab}_G(x) = \{e\}$. In particular, by Lemma 2.3 we have

$$f_R(x) > 0 \text{ for } \mu_X - \text{a.e. } x \in X.$$

Altogether, we deduce that f_R satisfies the conditions in Proposition 1.1.

The theorem thus follows from Proposition 1.1 in view of (2.3). □

3 The space of Abelian differentials

Let $g \geq 2$, and let \mathcal{T}_g denote the Teichmüller space of complex structure on a compact Riemann surface of genus g . We denote by \mathcal{M}_g the corresponding moduli space, i.e., the quotient of \mathcal{T}_g by the mapping class group, Mod_g .

As it is well-known, Mod_g is not torsion free, however, it has subgroups of finite index which are torsion free — indeed the kernel of the natural map from Mod_g to $\text{Sp}_{2g}(\mathbb{Z}/3\mathbb{Z})$ is torsion free.

We fix, once and for all, a covering map

$$\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_g$$

which corresponds to a torsion free finite index subgroup of Mod_g .

Let $f : \mathbb{H}^2 \rightarrow \mathcal{M}_g$ be an isometric immersion for the Teichmüller metric. Typically, $f(\mathbb{H}^2)$ is dense in \mathcal{M}_g , however, there are situations where $f(\mathbb{H}^2)$ is an algebraic curve in \mathcal{M}_g . In the latter case, the stabilizer Δ of f is a lattice in $\text{Isom}(\mathbb{H}^2)$, and we obtain a *Teichmüller curve*

$$f : V = \mathbb{H}^2/\Delta \rightarrow \mathcal{M}_g.$$

For every $g \geq 2$, the moduli space \mathcal{M}_g contains a dense family of Teichmüller curves which arise as branched cover of flat tori. There are also examples of infinite families of *primitive* Teichmüller curves, i.e., Teichmüller curves which do not arise as a branched cover of flat tori, in \mathcal{M}_g when $g = 2, 3, 4$, [12, 13].

Theorem 3.1 *Let $\{V_n : n \in \mathbb{N}\}$ be an infinite family of Teichmüller curves in \mathcal{M}_g . For every n , let $\widehat{V}_n \rightarrow V_n$ be a lift of V_n to $\widehat{\mathcal{M}}_g$. Then $\{\widehat{V}_n : n \in \mathbb{N}\}$ Benjamini-Schramm converges to \mathbb{H}^2 .*

C. Leininger and A. Wright (independently) have supplied an alternative (and arguably softer) proof of Theorem 3.1. This argument relies on the fact that the length of shortest geodesic on Teichmüller curves tends to infinity, see Proposition 3.2, and is independent

of measure classification theorems. We also thank T. Gelander for helpful communications regarding IRSs.

Here, we present a proof based on Proposition 1.1 and [6] to highlight a unifying theme between the homogeneous setting and the setting at hand.

For every $M \in \mathcal{T}_g$, let $\Omega(M)$ be the g -dimensional space of holomorphic 1-forms on M . By integrating a non-zero form $\omega \in \Omega(M)$ we obtain, away from the zeros of ω , a flat metric $|\omega|$ on M and local charts whose transition functions are translations.

Form a vector bundle over the Teichmüller space \mathcal{T}_g where the fiber over each point is $\Omega(M)$. Let $\Omega\mathcal{T}_g \rightarrow \mathcal{T}_g$ be the complement of the zero section of this vector bundle.

There is a natural action of $GL_2^+(\mathbb{R})$ (and hence of $SL_2(\mathbb{R})$) on $\Omega\mathcal{T}_g$: given a holomorphic 1-form $\omega = \Re(\omega) + i\Im(\omega)$ and $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$,

$$h \cdot \omega = \begin{pmatrix} i \\ i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Re(\omega) \\ \Im(\omega) \end{pmatrix}. \tag{3.1}$$

We let $\Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$ denote the quotient of $\Omega\mathcal{T}_g$ by action of the mapping class group of S_g .

For every $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\sum \alpha_i = 2g - 2$, let $\mathcal{H}(\alpha)$ denote the set of $(M, \omega) \in \Omega\mathcal{M}_g$ where ω has zeros of type α . Then $\Omega\mathcal{M}_g = \bigsqcup \mathcal{H}(\alpha)$.

Let $(M, \omega) \in \mathcal{H}(\alpha)$ and let $\Sigma \subset M$ denote the set of zeroes of ω . Let $\{\gamma_1, \dots, \gamma_k\}$ denote a \mathbb{Z} -basis for the relative homology group $H_1(M, \Sigma, \mathbb{Z})$. (It is convenient to assume that the basis is obtained by extending a symplectic basis for the absolute homology group $H_1(M, \mathbb{Z})$.) We can define a map $\Phi : \mathcal{H}(\alpha) \rightarrow \mathbb{C}^k$ by

$$\Phi(M, \omega) = \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_k} \omega \right)$$

The map Φ (which depends on a choice of the basis $\{\gamma_1, \dots, \gamma_k\}$) is a local coordinate system on (M, ω) . Alternatively, we may think of the cohomology class $[\omega] \in H^1(M, \Sigma, \mathbb{C})$ as a local coordinate on the stratum $\mathcal{H}(\alpha)$. We will call these coordinates *period coordinates*.

The area of a translation surface is given by

$$a(M, \omega) = \frac{i}{2} \int_M \omega \wedge \bar{\omega}.$$

We let $\Omega_1\mathcal{M}_g$ and $\mathcal{H}_1(\alpha)$ denote the locus of unit area 1-forms in $\Omega\mathcal{M}_g$ and $\mathcal{H}(\alpha)$, respectively.

The $SL_2(\mathbb{R})$ -action and the Kontsevich-Zorich cocycle

The action in (3.1) descends to an action of $SL_2(\mathbb{R})$ on $\mathcal{H}_1(\alpha)$. Indeed, write $\Phi(M, \omega)$ as a $2 \times d$ matrix x . The action of $SL_2(\mathbb{R})$ in these coordinates is linear.

Let $\text{Mod}(M, \Sigma)$ be the mapping class group of M fixing each zero of ω . We choose a fundamental domain for the action of $\text{Mod}(M, \Sigma)$, and think of the dynamics on the fundamental domain. Then, the $SL_2(\mathbb{R})$ action becomes

$$x = \begin{pmatrix} \Re(\omega) \\ \Im(\omega) \end{pmatrix} \mapsto hx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Re(\omega) \\ \Im(\omega) \end{pmatrix} A(h, x), \tag{3.2}$$

where $A(h, x) \in \text{Sp}_{2g}(\mathbb{Z}) \times \mathbb{Z}^{m-1}$ is the *Kontsevich-Zorich cocycle*.

Thus, $A(h, x)$ is the change of basis one needs to perform to return the point hx to the fundamental domain. It can be interpreted as the monodromy of the Gauss-Manin connection (restricted to the orbit of $SL_2(\mathbb{R})$).

3.1 Affine measures and manifolds

For a subset $\mathcal{E} \subset \mathcal{H}_1(\alpha)$ we write

$$\mathbb{R}\mathcal{E} = \{(M, t\omega) : (M, \omega) \in \mathcal{E}, t \in \mathbb{R}\} \subset \mathcal{H}(\alpha).$$

An ergodic $SL_2(\mathbb{R})$ -invariant probability measure ν on $\mathcal{H}_1(\alpha)$ is called *affine* if the following hold:

- (i) The support \mathcal{M} of ν is an *immersed submanifold* of $\mathcal{H}_1(\alpha)$, i.e., there exists a manifold \mathcal{N} and a proper continuous map $f : \mathcal{N} \rightarrow \mathcal{H}_1(\alpha)$ so that $\mathcal{M} = f(\mathcal{N})$. The self-intersection set of \mathcal{M} , i.e., the set of points of \mathcal{M} which do not have a unique preimage under f , is a closed subset of \mathcal{M} of ν -measure 0. Furthermore, each point in \mathcal{N} has a neighborhood U such that locally $\mathbb{R}f(U)$ is given by a complex linear subspace defined over \mathbb{R} in the period coordinates.
- (ii) Let $\bar{\nu}$ be the measure supported on $\mathbb{R}\mathcal{M}$ so that $d\bar{\nu} = d\nu da$. Then each point in \mathcal{N} has a neighborhood U such that the restriction of $\bar{\nu}$ to $\mathbb{R}f(U)$ is an affine linear measure in the period coordinates on $\mathbb{R}f(U)$, i.e., it is (up to normalization) the restriction of the Lebesgue measure to the subspace $\mathbb{R}f(U)$.

A suborbifold \mathcal{M} for which there exists a measure ν such that the pair (\mathcal{M}, ν) satisfies (i) and (ii) is said to be *affine invariant submanifold*.

We sometimes write $\nu_{\mathcal{M}}$ to indicate the affine invariant measure ν on affine invariant submanifold \mathcal{M} .

Note that in particular, any affine invariant submanifold is a closed subset of $\mathcal{H}_1(\alpha)$ which is invariant under the action of $SL_2(\mathbb{R})$, and which in period coordinates is an affine subspace. We also consider the entire stratum $\mathcal{H}_1(\alpha)$ to be an (improper) affine invariant submanifold.

3.2 Typical affine stabilizer is trivial

In this section, we prove the following statement:

Proposition 3.2 *Let $(\mathcal{M}, \nu) \subset \mathcal{H}_1(\alpha) \subset \Omega\widehat{\mathcal{M}}_{g,n}$ be an affine invariant submanifold. Assume that \mathcal{M} is not a Teichmüller curve. Then for ν -a.e. $x \in \mathcal{M}$,*

$$\text{Stab}_{SL_2(\mathbb{R})}(x)$$

is trivial.

Recall that the set of self-intersections \mathcal{M}' of \mathcal{M} is a proper closed invariant submanifold of \mathcal{M} , hence, $\dim \mathcal{M}' < \dim \mathcal{M}$, see [6]; in particular, $\nu(\mathcal{M}') = 0$. Therefore, it suffices to prove the proposition for ν -a.e. $x \in \mathcal{M} \setminus \mathcal{M}'$. Let $\widetilde{\mathcal{M}}$ denote the lift of $\mathcal{M} \setminus \mathcal{M}'$ to $\Omega\mathcal{T}_g$.

Fix $\phi \in \widehat{\text{Mod}}(S_g)$ (that is, ϕ is not torsion). Define

$$P(\phi) = \left\{ x \in \widetilde{\mathcal{M}} : A \cdot x = \phi(x), \text{ for some } A \in SL_2(\mathbb{R}) \right\}.$$

We will show, for every $\phi \in \widehat{\text{Mod}}(S)$, $P(\phi)$ is a ν -measure zero subset of $\widetilde{\mathcal{M}}$. Note that, by assumption, $\dim(\widetilde{\mathcal{M}}) > 3$.

Consider $x \in P(\phi)$ and let E_x be the $GL^+(2, \mathbb{R})$ orbit of x . Then E_x can be considered as (an open subset of) the tangent space of the Teichmüller disk \mathbb{H}_x associated to x (the projection of E_x to Teichmüller space). The restriction of Teichmüller metric to \mathbb{H}_x equips \mathbb{H}_x with the hyperbolic metric (up to a factor 2). We observe that ϕ stabilizes \mathbb{H}_x acting on \mathbb{H}_x by an isometry. In fact, we have either (see, for example, [11, Lemma 5.6])

- ϕ acts loxodromically on \mathbb{H}_x and ϕ a pseudo-Anosov element.
- ϕ acts parabolically on \mathbb{H}_x and ϕ is a multi-curve.
- ϕ acts elliptically on \mathbb{H}_x and ϕ has finite order in $\text{Mod}(S)$.

Note that the third case is excluded since we are assuming ϕ is not torsion. We argue each case separately showing that $P(\phi) \cap \widetilde{\mathcal{M}}$ is a ν -measure zero subset of $\widetilde{\mathcal{M}}$.

Remark 3.3 We are in fact proving more than what is stated in Proposition 3.2. Recall that, for $x \in \widetilde{\mathcal{M}}$, the Veech group of x is the subgroup of $\text{PSL}_2(\mathbb{R})$ which stabilizes \mathbb{H}_x setwise. Hence, the proof actually gives that for ν -a.e. $x \in \widetilde{\mathcal{M}}$, the Veech group of x is finite.

ϕ is pseudo-Anosov element

A pseudo-Anosov map ϕ stabilizes only one Teichmüller disk, the one where $\partial\mathbb{H}_x$ contains $F_+(\phi)$ and $F_-(\phi)$; the stable and the unstable foliation associated to ϕ . Therefore, $P(\phi) = T_1\mathbb{H}_x$, the unit tangent bundle over \mathbb{H}_x . Since \mathcal{M} is not a Teichmüller curve, it has a dimension larger than 3. Hence $P(\phi) \cap \widetilde{\mathcal{M}}$ is a ν -measure zero subset of $\widetilde{\mathcal{M}}$.

ϕ is a multi-twist

Let ϕ be a multi-twist around γ , namely

$$\phi = \prod D_{\gamma_i}^{p_i}.$$

Let $\mathbb{R}P(\phi)$ be the subset of $\mathcal{H}(\alpha)$ obtained from points in $P(\phi)$ after scaling. Then, for any $x \in \mathbb{R}P(\phi)$, a measured foliation that is topologically equivalent to $\gamma = \{\gamma_1, \dots, \gamma_k\}$ has to appear in the boundary of \mathbb{H}_x . That is, after a rotation, we can assume $x = (F_-, F_+)$ and $F_+ = \sum c_k \gamma_k$. Furthermore, x has a cylinder decomposition where the modulus of these cylinders are rationally multiples of each other ([11, Lemma 5.6]). That is, there are $r_i \in \mathbb{Q}$ such that

$$r_i \cdot \frac{i(F_-, \gamma_i)}{c_i} = r_j \cdot \frac{i(F_-, \gamma_j)}{c_j},$$

for $1 \leq i, j \leq k$. We also have

$$\sum c_i \cdot i(F_-, \gamma_i) = \text{area}(x).$$

That is, given γ , r_i , F_- and $\text{area}(x)$, we can calculate the values of c_i . Hence, F_+ and subsequently x are uniquely determined by γ , r_i , F_- and $\text{area}(x)$. There are countably many choices for the values r_i and the multi-curve γ . We now show that the dimension of the space of possible measured foliations F_- is half the dimension of $\mathbb{R}\widetilde{\mathcal{M}}$ where $\mathbb{R}\widetilde{\mathcal{M}}$ is the subset of $\mathcal{H}(\alpha)$ obtained from point in $\widetilde{\mathcal{M}}$ after scaling.

For a filling bi-recurrent train-track τ (see [15] for definition and discussion) any admissible weight on τ defines a measured foliation. We then say this measured foliation is carried by τ . The complementary regions of a filling train tracks are n -gons or punctured n -gons. A

foliation carried by τ has a singular point associated to each complementary region of τ . We say τ is of type $\alpha = (\alpha_1, \dots, \alpha_m)$ if τ has m complementary components that are punctured α_i -gons, $i = 1 \dots m$. We denote the space of admissible weights in τ by $W(\tau)$.

Lemma 3.4 *For every $x \in \mathcal{H}(\alpha)$ there are train tracks τ_+ and τ_- of type α such that a neighborhood of $\mathcal{H}(\alpha)$ around x is homeomorphic to $U \times V$ where U, V are open subsets of $W(\tau_+)$ and $W(\tau_-)$ respectively. In fact, the real part of the period coordinates for $\mathcal{H}(\alpha)$ give coordinates for U and the imaginary part of the period coordinates, give coordinates for V .*

Proof Let Δ be a triangulation of x by saddle connections (for example, L^∞ -Delaney triangulations see [7, §3]). Pick a subset \mathcal{B} of the edges of Δ that give a basis for the homology of x relative to the zeros Σ of x . Then the complex numbers $\{\int_\omega x\}_{\omega \in \mathcal{B}}$ give local coordinates for $H(\alpha)$. For every edge ω of Δ , we have

$$i(\omega, F_-) = \Re \left(\int_\omega x \right).$$

In fact, F_- can be constructed, triangle by triangle, from the set of real numbers $\{i(\omega, F_-)\}_{\omega \in \Delta}$. That is there is a train-track τ_- dual to the triangulation Δ (again, see [7, §3] for the construction of such train-tracks) such that $\{\Re(\int_\omega x)\}_{\omega \in \mathcal{B}}$ form an admissible weights on τ_- . At any point $y \in \mathcal{H}(\alpha)$ near x , the triangulation Δ can still be represented by saddle connections and the set $\{\Re(\int_\omega y)\}_{\omega \in \mathcal{B}}$ form an admissible weights on τ_- that is associated to the vertical foliation at y . That is, $\{\Re(\int_\omega y)\}_{\omega \in \mathcal{B}}$, thought of as admissible weights on τ_- give local coordinates for the set of measured foliation that appear as a horizontal foliation of an element of $\mathcal{H}_1(\alpha)$ near x . The same also holds for τ_+ and the vertical foliations. \square

Since $\mathbb{R}\widetilde{\mathcal{M}}$ is an affine sub-manifold of $\mathcal{H}(\alpha)$, it is locally defined by a set of affine equations on period coordinates, see e.g. §3.1 and [5]. That is, there are subspaces $U' \subset U$ and $V' \subset V$, defined by the same set of affine equations, such that a neighborhood of x in $\mathbb{R}\widetilde{\mathcal{M}}$ is naturally homeomorphic to $U' \times V'$. In particular, where U' and V' have half the dimension of $\mathbb{R}\widetilde{\mathcal{M}}$.

Let W be the intersection of $\mathbb{R}P(\phi)$ with this neighborhood. Recall that, fixing the multi-curve γ , rational numbers r_i and the area, every point in W is determined, up to rotation, by a point in U' . Therefore, W is a countable union of set of dimension $\dim(U') + 2$. But

$$\dim(U') + 2 = \frac{1}{2} \dim(\mathbb{R}\widetilde{\mathcal{M}}) + 2 < \dim(\mathbb{R}\widetilde{\mathcal{M}}),$$

where the last inequality follows from the assumption that $\dim(\mathbb{R}\widetilde{\mathcal{M}}) > 4$. That is, $\mathbb{R}P(\phi) \cap \mathbb{R}\widetilde{\mathcal{M}}$ is a countable union of lower dimensional subset of $\mathbb{R}\widetilde{\mathcal{M}}$ and therefore, has $\bar{\nu}$ -measure zero, see §3.1 for the definition of $\bar{\nu}$. Since, $\text{Stab}_{\text{SL}_2(\mathbb{R})}(x)$ does not change after scaling, we have, $P(\phi) \cap \widetilde{\mathcal{M}}$ has ν -measure zero in $\widetilde{\mathcal{M}}$.

3.3 Proof of Theorem 3.1

In this section we prove Theorem 3.1. The proof is based on the following proposition.

Proposition 3.5 *Let $\{E_k : k \in \mathbb{N}\} \subset \mathcal{H}_1(\alpha) \subset \Omega\widehat{\mathcal{M}}_{g,n}$ be a sequence of closed $\text{SL}_2(\mathbb{R})$ orbits each equipped with the $\text{SL}_2(\mathbb{R})$ -invariant probability measure μ_k . Assume further that there exists an affine invariant submanifold $(\mathcal{M}, \nu) \subset \mathcal{H}_1(\alpha)$ so that*

$$\mu_k \rightarrow \nu \text{ as } k \rightarrow \infty. \tag{3.3}$$

Let V_k denote the Teichmüller curve associated to E_k for all k . Then $\{V_k\}$ Benjamini-Schramm converges to \mathbb{H} .

Proof The proof is based on Proposition 1.1. Let us write $E_k = \text{SL}_2(\mathbb{R}) \cdot x_k$. We will show that $\text{SL}_2(\mathbb{R})/\text{Stab}_{\text{SL}_2(\mathbb{R})}(x_k)$ Benjamini-Schramm converges to $\text{SL}_2(\mathbb{R})$ from which the proposition follows.

First note that (\mathcal{M}, ν) is not a closed $\text{SL}_2(\mathbb{R})$ orbit, see [6, Thm. 2.3]. Hence, by Proposition 3.2, we have

$$\text{Stab}_{\text{SL}_2(\mathbb{R})}(x) = \{e\} \text{ for } \nu - \text{a.e. } x \in \mathcal{M}. \tag{3.4}$$

In the remaining parts of the argument, we write $H = \text{SL}_2(\mathbb{R})$ and use the notation in §1. In particular, for all $R > 0$, let

$$B^H(e, R) = \{h \in H : \|h - I\| < R \text{ and } \|h^{-1} - I\| < R\}$$

where $\|\cdot\|$ denotes the maximum norm on $\text{Mat}_2(\mathbb{R})$ with respect to the standard basis. Similarly, for $r > 0$, let $B^H_{\text{Rie}}(e, r)$ denote the ball of radius r centered at the identity with respect to the bi-SO(2)-invariant Riemannian metric on H induced using the Killing form.

For every $x \in \mathcal{H}_1(\alpha)$, let r_x denote $1/2$ of the injectivity radius of x in $\mathcal{H}_1(\alpha)$ with respect to the Teichmüller metric. Then $x \mapsto r_x$ is continuous on $\mathcal{H}_1(\alpha)$; moreover, $h \mapsto hx$ is injective on $B^H_{\text{Rie}}(e, r_x)$.

Let $R > 0$ and for every $x \in \mathcal{M}$, put $B^H_R(x) := \bar{B}^H(e, R) \setminus B^H_{\text{Rie}}(e, r_x)$; note that this is a compact subset of $\text{SL}_2(\mathbb{R})$. Define $f_R : \mathcal{M} \rightarrow [0, \infty)$ by

$$f_R(x) = \min \left\{ \text{dist}_{\text{Teich}}(x, hx) : h \in B^H_R(x) \right\}.$$

Note that f_R is continuous. Indeed, let $y_m \rightarrow y$, and let $h_m \in B^H_R(y_m)$ be so that $f_R(y_m) = \text{dist}_{\text{Teich}}(y, h_m y_m)$. Let $\{f_R(y_{m_i})\}$ be a converging subsequence of $\{f_R(y_m)\}$. Since $B^H_R(y_m)$ converges to $B^H_R(y)$ (in Hausdorff metric on compact sets), there is a subsequence $h_{m_{i_j}} \rightarrow h \in B^H_R(y)$ which implies: $f_R(y) \leq \lim_i f_R(y_{m_i})$. In consequence, $f_R(y) \leq \liminf f_R(y_m)$. To see the opposite direction, let $h \in B^H_R(y)$ be so that $f_R(y) = \text{dist}_{\text{Teich}}(y, hy)$. Let $h_m \in B^H_R(y_m)$ be so that $h_m \rightarrow h$, then $f_R(y_m) \leq \text{dist}_{\text{Teich}}(y, h_m y_m)$ and for every $\varepsilon > 0$ we have $\text{dist}_{\text{Teich}}(y, h_m y_m) \leq \text{dist}_{\text{Teich}}(y, hy) + \varepsilon = f_R(y) + \varepsilon$ so long as m is large enough. Hence $\limsup f_R(y_m) \leq f_R(y) + \varepsilon$. The continuity of f_R follows.

Moreover, in view of (3.4), we have $f_R(x) > 0$ for ν -a.e. $x \in \mathcal{M}$. Finally, since for every x , the map $h \mapsto hx$ is injective on $B^H_{\text{Rie}}(e, r_x)$, we have $\text{Stab}_{\text{SL}_2(\mathbb{R})}(x) \cap B^H_{\text{Rie}}(e, r_x) = \{e\}$. Thus if $f_R(x) > 0$ for some $x \in \mathcal{M}$, then $\text{Stab}_{\text{SL}_2(\mathbb{R})}(x) \cap B^H(e, R) = \{e\}$.

Altogether, we deduce that f_R satisfies the conditions in Proposition 1.1. This and (3.3) imply that Proposition 1.1 applies and yields:

$$\text{SL}_2(\mathbb{R})/\text{Stab}_{\text{SL}_2(\mathbb{R})}(x_k) \text{ Benjamini-Schramm converges to } \text{SL}_2(\mathbb{R}).$$

The proof is complete. □

Proof of Theorem 3.1 Let $\{V_k : k \in \mathbb{N}\} \subset \widehat{\mathcal{M}}_{g,n}$ be a sequence of Teichmüller curves. We will show that for every subsequence $\{V_{k_i}\}$, there exists a further subsequence $\{V_{k_{i_j}}\}$ which Benjamini-Schramm converges to \mathbb{H} the theorem follows from this.

Let $\{V_{k_i}\}$ be a subsequence of $\{V_k\}$. Passing to a further subsequence, which we continue to denote by $\{V_{k_i}\}$, we may assume that the corresponding $\text{SL}_2(\mathbb{R})$ orbits $\{E_{k_i}\}$ lie in $\mathcal{H}_1(\alpha) \subset \Omega \widehat{\mathcal{M}}_{g,n}$ for a fixed α .

Now by [6, Thm. 2.3], see also [6, Cor. 2.5], there exists a subsequence $\{E_{k_{i_j}}\}$ of $\{E_{k_i}\}$, and an affine invariant manifold (\mathcal{M}, ν) , so that $\mu_{k_{i_j}} \rightarrow \nu$ where $\mu_{k_{i_j}}$ denotes the $\mathrm{SL}_2(\mathbb{R})$ -invariant measure on $E_{k_{i_j}}$.

By Proposition 3.5, we have $V_{k_{i_j}}$ Benjamini-Schramm converges to \mathbb{H} ; as we wished to show. \square

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