TRAIN TRACKS AND MEASURED LAMINATIONS ON INFINITE SURFACES
HYPERBOLIC LUNCH
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Definition 0.1. *A surface X is called infinite if its fundamental group $\pi_1(X)$ is infinitely generated.*

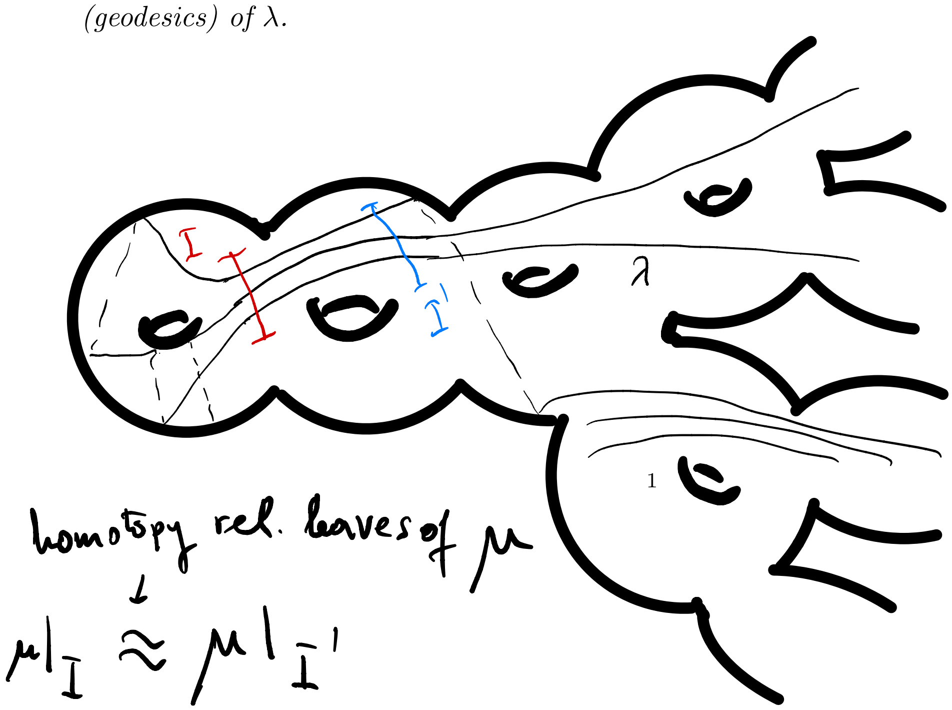
Any infinite Riemann surface X contains a unique hyperbolic metric in its conformal class. All metric properties of infinite Riemann surfaces are with respect to their conformal hyperbolic metric.

We fix a representation $X \equiv \mathbb{D}/\Gamma$ where $\mathbb{D}$ is the unit disk and Γ is a Fuchsian group isomorphic to $\pi_1(X)$.

Definition 0.2. *A geodesic lamination λ on X is a closed subset of X which is foliated by pairwise disjoint, simple and complete geodesics.*

Unlike for finite surfaces, it is not enough to specify a closed subset of an infinite surface to determine a geodesic lamination. This is because some geodesic laminations may cover open subsets of X and there is more than one foliation of such open subsets (think of the unit disk or a funnel).

Definition 0.3. *A measured (geodesic) lamination μ on X supported on a geodesic lamination λ is an assignment of a Radon measure on each hyperbolic geodesic arc I transverse to λ with support in $I \cap \lambda$. The measure is invariant under homotopies preserving leaves (geodesics) of λ.*

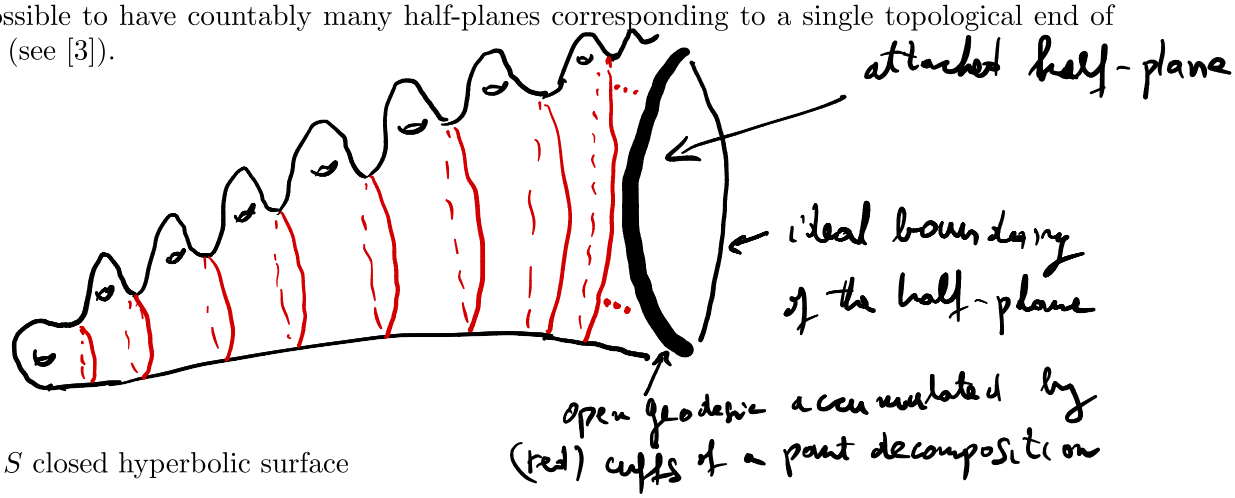


Theorem 0.4 (Alvarez-Rodriguez [1], [3])**.** *Any infinite hyperbolic surface X can be obtained by isometric gluings of countably many geodesic pairs of pants along their boundary geodesics and by attaching at most countably many funnels and geodesic half-planes.*

The first example of a geodesic half-plane inside an infinite Riemann surface is due to Basmajian [2].

A geodesic half-plane in X corresponds to an interval of discontinuity for Γ on S^1 that has trivial stabilizer in Γ.

A funnel or a puncture of X corresponds to a single topological end of X, and no two punctures or funnels can correspond to the same topological end. On the other hand, it is possible to have countably many half-planes corresponding to a single topological end of X (see [3]).



S closed hyperbolic surface

Theorem 0.5 (Thurston, Birman-Series)**.** *The union of all simple geodesics in S has Hausdorff dimension* 1*.*

In particular, a geodesic lamination on S is nowhere dense and of zero area.

Question: What about geodesic laminations on infinite hyperbolic surfaces?

If X contains a funnel or a half-plane then there are geodesic laminations that cover an open subset of X. Therefore the question is interesting only for surfaces which do not have funnels and half-planes.

A Riemann surface X does not contain a funnel or a half-plane if and only if Γ is of the first kind if and only if X is equal to its convex core $\mathcal{C}(X)$.

Theorem 0.6 (Š. [10])**.** *Let X be an infinite Riemann surface and let λ be a geodesic lamination contained in the convex core $\mathcal{C}(X)$ of X. Then λ is nowhere dense.*

If the covering group Γ is of the first kind then the union of the leaves of any geodesic lamination on X is nowhere dense.

There are examples of surfaces with the covering group of the first kind and geodesic lamination with non-zero area.

S_1, S_2 closed hyperbolic surfaces of the same genus and $f : S_1 \to S_2$ a homeomorphism

$\tilde{f} : \tilde{S}_1 \to \tilde{S}_2$ extends to $\pi_1(S_1)$- and $\pi_1(S_2)$-equivariant homeomorphism $\tilde{f} : \partial_\infty \tilde{S}_1 \to \partial_\infty \tilde{S}_2$ of the boundaries at infinity

Since the space of geodesics $G(\tilde{S}_i)$ of the universal covering $\tilde{S}_i$ of S_i is identified $(\partial_\infty \tilde{S}_i \times \partial_\infty \tilde{S}_i) \setminus$ diagonal we obtain a homeomorphism

$$\tilde{f} : G(\tilde{S}_1) \to G(\tilde{S}_2)$$

when S_1, S_2 are finite surface and a funnel corresponds to a puncture there is no induced homeomorphism of the spaces of geodesics; even when two funnels correspond to each other there is no canonical homeomorphism of the spaces of geodesics

for infinite surfaces X_1 and X_2, a homeomorphism $f : X_1 \to X_2$ does not induce a homeomorphism of $G(\tilde{X}_1)$ and $G(\tilde{X}_2)$ when funnels and/or half-planes are present; we show this is the only obstruction

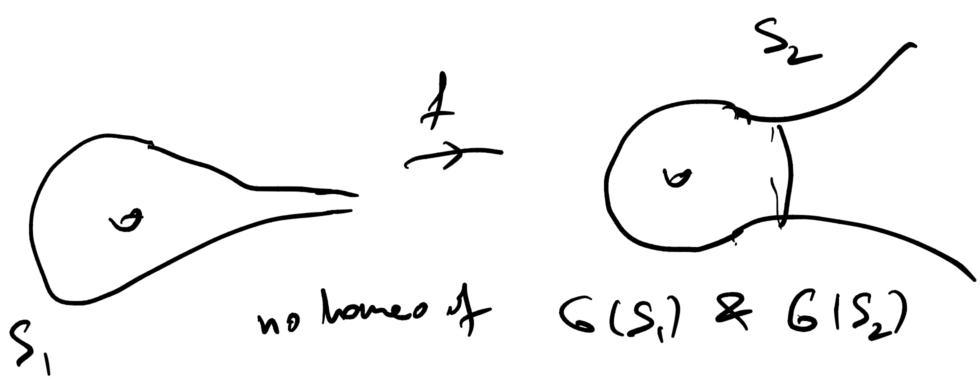
Theorem 0.7 (Š. [10]). *Let X_1 and X_2 be two infinite surfaces with covering groups of the first kind and $f : X_1 \to X_2$ a homeomorphism. Then there exists an equivariant homeomorphism*

$$\tilde{f} : G(\tilde{X}_1) \to G(\tilde{X}_2).$$

The map projects to a homeomorphism

$$f : G(X_1) \to G(X_2)$$

such that closed geodesics of X_1 are mapped onto closed geodesics of X_2 in the homotopy class of the image under $f : X_1 \to X_2$.



Goal: Parametrize the space $ML(X)$ of all measured laminations on X.

Motivation from closed surface case S:

- $ML(S)$ parametrizes the Teichmüller space $T(S)$ of S by earthquakes (Thurston)
- $ML(S)$ is piecewise linear manifold (Thurston, Bonahon)
- train tracks provide manifold charts of $ML(S)$ (Thurston, Penner-Harer, Bonahon)
- Thurston boundary to $T(S)$ is identified with $PML(S)$-projective measured laminations (Thurston)

Assumption: From now on, $X = \mathcal{C}(X)$ or the covering Fuchsian group Γ is of the first kind

we would like to describe measured laminations on X using countably many parameters

Since $X = \mathcal{C}(X)$, X has a geodesic pants decomposition (Alvarez-Rodriguez). In fact, any locally finite topological pants decomposition straightens to a locally finite geodesic pants decomposition (Basmajian-Š. [3]).

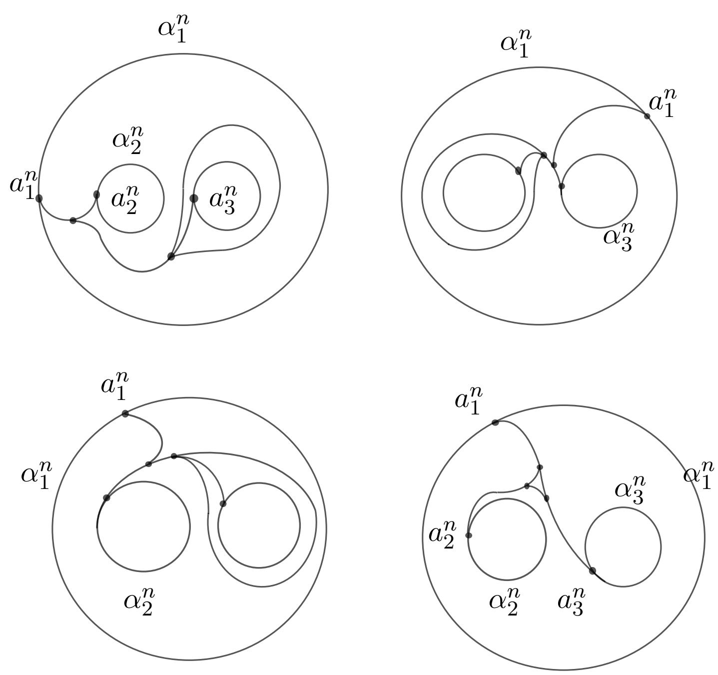


FIGURE 1. The four standard train tracks on a pair of pants with 3 cuffs. The smoothing at each cuff is chosen arbitrarily.

Fix a geodesic pants decomposition $\{P_n\}_n$ of X.

Choose a Dehn-Thurston train track in each P_n which meet each cuff at a fixed basepoint such that the complementary regions are triangles and punctured monogons (Penner-Harer).

Choose smoothing at the basepoints. Figures 1 and 2 show different possibilities for the Dehn-Thurston train tracks in each P_n.

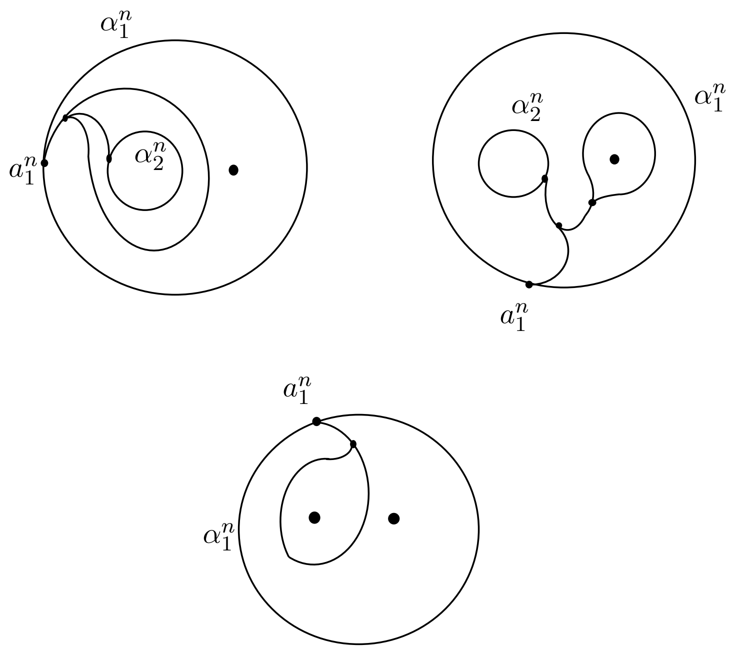


Figure 2. The standard train tracks when pairs of pants have 2 or 1 cuff.

Definition 0.8. *A pants train track Θ on X is obtained from $\{P_n\}_n$ by taking choices of the standard Dehn-Thurston train tracks in each P_n with cuffs being additional edges of Θ and smoothing at the cuffs.*

Different choices of train tracks in each P_n and different choices of smoothing gives rise to an uncountable family of train tracks.

Proposition 0.9. *A bi-infinite edge path γ in Θ determines a unique simple geodesic $g(\gamma)$ in X.*

Definition 0.10. *We say that a geodesic g in X is weakly carried by Θ if there exists an edge path γ such that $g = g(\gamma)$.*

Proposition 0.11. *For any $\mu \in ML(X)$, there is a choice of a pants train track that weakly carries each geodesic of the support of μ.*

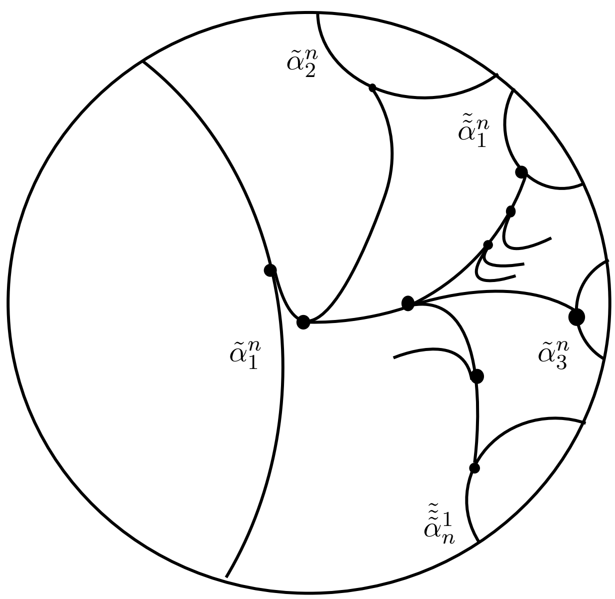


FIGURE 3. The lift of a standard train track. There are four lifts of cuffs connected by finite edge paths to a single vertex of the lift of a cuff on the left side of the figure.

Let $\tilde{\Theta}$ be the lift of Θ to the universal cover of X (see Figure 3).

$E(\tilde{\Theta})$ the set of edges of $\tilde{\Theta}$

for $e \in E(\tilde{\Theta})$, let $G(e)$ be the set of geodesics weakly carried by $\tilde{\Theta}$ that have e in their corresponding bi-infinite edge paths

Define $f_{\tilde{\mu}} : E(\tilde{\Theta}) \to \mathbb{R}$ by

$$f_{\tilde{\mu}}(e) = \tilde{\mu}(G(e)) = \tilde{\mu}(I_e)$$

where I_e is a geodesic arc intersecting all geodesics of $G(e)$ and no other geodesics weakly carried by $\tilde{\Theta}$; $f_{\tilde{\mu}}(e)$ is the weight on e for the measured lamination $\tilde{\mu}$

$f_{\tilde{\mu}}$ satisfies the switch relation at each vertex of $\tilde{\mu}$: the sum of the weights of incoming edges equals to the sum of the weights of the outgoing edges

a map $f : E(\tilde{\Theta}) \to [0, \infty)$ which satisfies the switch relation at each vertex is called an edge-weight system

Theorem 0.12 (Š. [10]). *Let X be an infinite Riemann surface whose covering group is of the first kind and let Θ be a pants train track on X. Let $\mathcal{W}(\tilde{\Theta}, [0, \infty))$ be the set of edge-weight systems for Θ. Then the assignment of edge-weights to the space $ML(X, \Theta)$ of measured lamination weakly carried by Θ is a homeomorphism for the weak* topology on $ML(X, \Theta)$ and the topology of pointwise convergence on $\mathcal{W}(\tilde{\Theta}, [0, \infty))$.*

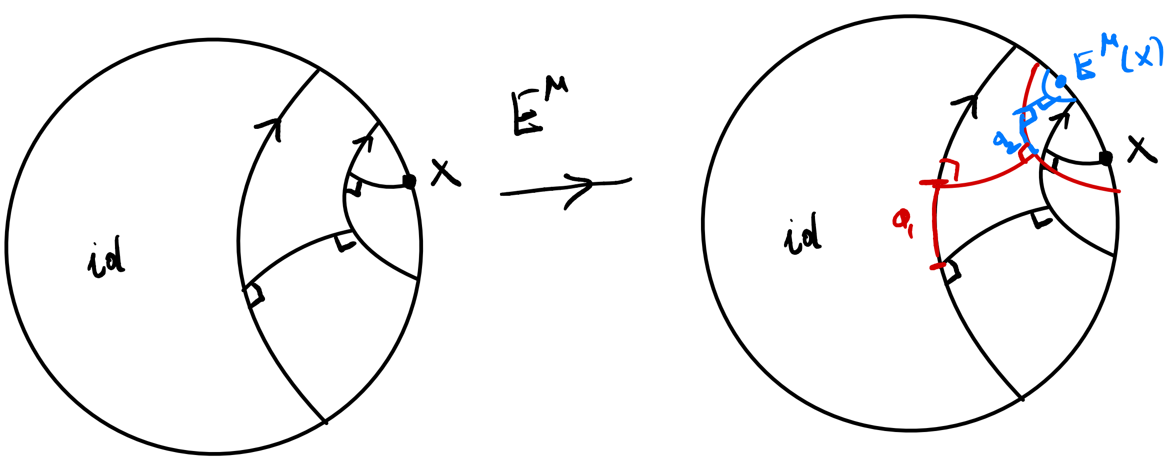
Further motivations:

Theorem 0.13 (Thurston: geology is transitive). *For closed hyperbolic surface S, any homeomorphism $f : S \to S_1$ is homotopic to a unique earthquake $E^\mu : S \to S_1$.*

E^μ is isometry off the support of μ, relative motion of different complementary components is movement to the left given by the amount of the transverse measure μ of the geodesics of the support separating the components (a generalization of positive left twists)

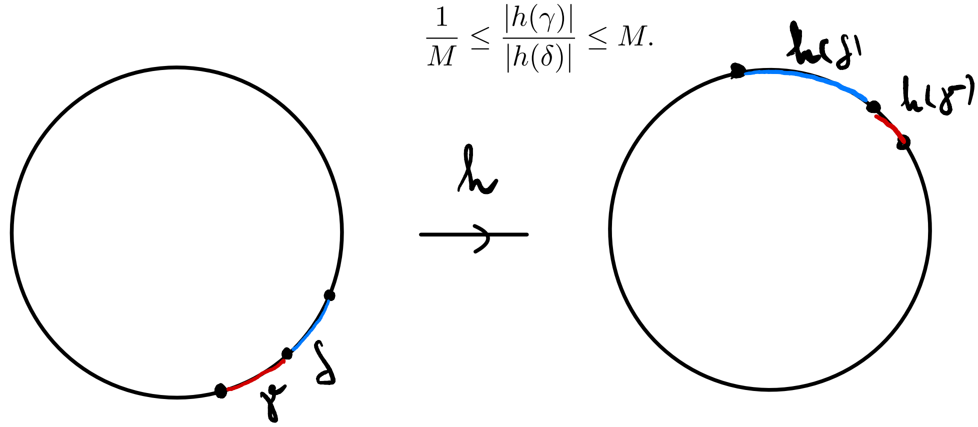
Theorem 0.14 (Thurston). *Any homeomorphism $f : S^1 \to S^1$ (fixing 1, i and -1) is obtained by continuous extension of a unique earthquake $E^\mu : \mathbb{D} \to \mathbb{D}$, where $\mu \in ML(\mathbb{D})$.*

The earthquakes are unique but not every measured lamination $\mu \in ML(\mathbb{D})$ gives an earthquake (open problem is which measured lamination give rise to earthquakes of $\mathbb{D}$)
Example: $\mu = a_1 \mathbf{1}_{g_1} + a_2 \mathbf{1}_{g_2}$, where $\mathbf{1}_g$ is the Dirac measure at g



Definition 0.15. *A orientation preserving homeomorphism $h : S^1 \to S^1$ is called quasisymmetric if there exists $M \geq 1$ such that for any two adjacent arcs $\gamma, \delta \subset S^1$ with equal length $|\gamma| = |\delta|$ we have*

$$\frac{1}{M} \leq \frac{|h(\gamma)|}{|h(\delta)|} \leq M.$$



Definition 0.16. *The universal Teichmüller space $T(\mathbb{D})$ consists of all quasisymmetric maps $h : S^1 \to S^1$ that fix 1, i and -1.*

Theorem 0.17 (Thurston, Š., Gardiner-Hu-Lakic, Epstein-Marden-Markovic, Hu, Š.)**.** *Let $h : S^1 \to S^1$ be an orientation preserving homeomorphism that fixes 1, i and -1 and let $E^\mu : \mathbb{D} \to \mathbb{D}$ be an earthquake with continuous extension $E^\mu|_{S^1} = h$. Then h is quasisymmetric if and only if*

$$\|\mu\|_{Th} := \sup \mu(I) < \infty$$

where the supremum is over all geodesic arcs $I \subset \mathbb{D}$ of hyperbolic length 1.

For $X = \mathbb{D}/\Gamma$, the *Teichmüller space* of X is

$$T(X) = \{h | h : S^1 \to S^1 \text{ quasisymmetric , fix } 1, i, -1 \text{ and conjugates } \Gamma \text{ onto a Fuchsian group}\}$$

Then $T(X) \subset T(\mathbb{D})$, and there is a one to one correspondence between the Teichmüller space $T(X)$ and the space of bounded measured laminations

$$ML_b(X) = \{\mu \in ML(X) | \|\tilde{\mu}\|_{Th} < \infty\}$$

where $\tilde{\mu} \in ML(\mathbb{D})$ is the lift of μ.

Theorem 0.18 (Bonahon-Š. [4])**.** *The Teichmüller space $T(X)$ embeds into the space of bounded geodesic currents of X. The asymptotic rays to the image of the embedding are precisely given by the projective bounded measured laminations $PML_b(X)$ which is Thurston boundary to $T(X)$. The action of the quasiconformal mapping class group $QMCG(X)$ extends by continuity to a homeomorphism of the closure $T(X) \cup PML_b(X)$.*

(For an alternative approach to Thurston boundary for $T(X)$ using the lengths of simple closed geodesics see Š. [12]).

We are motivated by the Teichmüller theory to consider a parametrization of $ML_b(X) \subsetneq ML(X)$.

It is much harder task than parametrization of $ML(X)$ and we need additional assumptions on the geometry of X.

Definition 0.19. *The set of bounded edge-weight systems*

$$\mathcal{W}_b(\tilde{\Theta},[0,\infty)) = \{f \in \mathcal{W}(\tilde{\Theta},[0,\infty)) | \|f\|_\infty < \infty\}.$$

Theorem 0.20 (Š. [10])**.** *Let X be an infinite hyperbolic surface equipped with a geodesic pants decomposition whose cuff lengths are between two positive constants.*

Then

$$ML_b(X,\Theta) = ML(X,\Theta) \cap ML_b(X)$$

is in a one to one correspondence with $\mathcal{W}_b(\tilde{\Theta},[0,\infty))$.

In addition, the bijection

$$ML_b(X,\Theta) \to \mathcal{W}_b(\tilde{\Theta},[0,\infty))$$

is a homeomorphism for the "uniform" weak topology on $ML_b(X,\Theta)$ and the topology induced by the norm $\|\cdot\|_\infty$ on $\mathcal{W}_b(\tilde{\Theta},[0,\infty))$.*

The above theorem holds when X has punctures. In this case the condition of cuff lengths being between two positive constants is not apllied to punctures but it is in force for the geodesic cuffs.

The "uniform" weak* topology (introduced jointly with H. Miyachi [7] in a work on the continuity of the correspondence between $T(X)$ and earthquake measures; and a newer incarnation jointly with F. Bonahon [4])):

$\tilde{\mu}$ realized as a measure on the space of geodesics $G(\tilde{X}) = (\partial_\infty \tilde{X} \times \partial_\infty \tilde{X}) \setminus$ diagonal ; note that $G(\tilde{X})$ is not a compact space

Then

$$\tilde{\mu}_n \to \tilde{\mu}$$

as $n \to \infty$ in the uniform weak* topology if for every continuous function $\xi : G(\tilde{X}) \to \mathbb{R}$ with compact support we have

$$\sup_{\varphi \in \mathrm{Isom}(\tilde{X})} \Big| \iint_{G(\tilde{X})} \xi \circ \varphi d(\tilde{\mu}_n - \tilde{\mu}) \Big| \to 0$$

as $n \to \infty$. When $\tilde{X} \equiv \mathbb{D}$ then $\mathrm{Isom}(\tilde{X}) \equiv \mathrm{Mob}(\mathbb{D})$.

Note that the topology on $ML_b(X)$ induced by $\|\cdot\|_{Th}$ is too strong and the weak* topology is too weak.

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