Simple closed curves are typically non-separating on high genus surfaces

joint work with E. Goujard, P. Zograf, A. Zorich

Vincent Delecroix

CNRS - Université de Bordeaux

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection

multicurve: disjoint union of simple closed curves on S

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection
multicurve: disjoint union of simple closed curves on S
simple multicurve: distinct component are not homotopic

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection
multicurve: disjoint union of simple closed curves on S
simple multicurve: distinct component are not homotopic

We always consider multicurves up to homotopy.

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection
multicurve: disjoint union of simple closed curves on S
simple multicurve: distinct component are not homotopic

We always consider multicurves up to homotopy.
multicurve = simple multicurve + multiplicity on each component

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection
multicurve: disjoint union of simple closed curves on S
simple multicurve: distinct component are not homotopic

We always consider multicurves up to homotopy.
multicurve = simple multicurve + multiplicity on each component

topological type: an orbit under of multicurves under $\mathrm{MCG}(S)$

Multicurves and simple closed curves

S: topological surface of genus $g \geq 0$ and n punctures

simple closed curve: non-contractible curve on S without auto-intersection
multicurve: disjoint union of simple closed curves on S
simple multicurve: distinct component are not homotopic

We always consider multicurves up to homotopy.
multicurve = simple multicurve + multiplicity on each component

topological type: an orbit under of multicurves under $\mathrm{MCG}(S)$

Types of simple closed curves

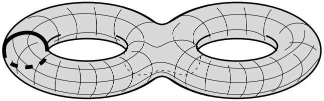
type $\eta_{nsep,(g,n)}$: non-separating curve of genus g

Types of simple closed curves

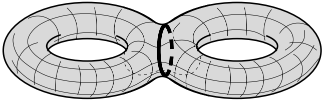
type $\eta_{nsep,(g,n)}$: non-separating curve of genus g
type $\eta_{sep,(g_1,n_1),(g_2,n_2)}$: separating in components of type (g_1, n_1) and (g_2, n_2)

Types of simple closed curves

type $\eta_{nsep,(g,n)}$: non-separating curve of genus g
type $\eta_{sep,(g_1,n_1),(g_2,n_2)}$: separating in components of type (g_1, n_1) and (g_2, n_2)



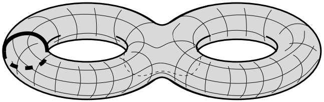
$\eta_{nsep,(2,0)} = \eta_{nsep,2}$



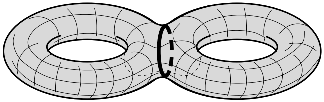
$\eta_{sep,(1,0),(1,0)} = \eta_{sep,1,1}$

Types of simple closed curves

type $\eta_{nsep,(g,n)}$: non-separating curve of genus g
type $\eta_{sep,(g_1,n_1),(g_2,n_2)}$: separating in components of type (g_1, n_1) and (g_2, n_2)

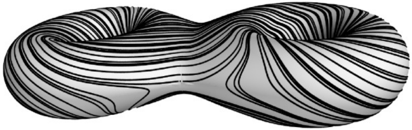


$\eta_{nsep,(2,0)} = \eta_{nsep,2}$



$\eta_{sep,(1,0),(1,0)} = \eta_{sep,1,1}$

What is the type of the following curve?



Asymptotic counting with respect to the type

Theorem (Mirzakhani '08)

For any type η of multicurve, there exists a positive rational constant $c(\eta)$ such that for any metric on S, as $L \to \infty$ we have

$$\#\{\textit{multicurves of type } \eta \textit{ and length} \leq L\} \sim B(\textit{metric}) \cdot \frac{c(\eta)}{b_{g,n}} \cdot L^{6g-6},$$

where $B(\textit{metric})$ is (implicitely) defined as

$$\#\{\textit{multicurves of length} \leq L\} \sim B(\textit{metric}) \cdot L^{6g-6},$$

and we have $\sum_\eta c(\eta) = \int_X B(X) d\mu_{WP}(X) = b_{g,n}$.

The flat torus case $(g, n) = (1, 1)$

$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

The flat torus case $(g, n) = (1, 1)$

$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

simple (multi)curves = primitive integral vector in $\mathbb{R}^2$ up to ± 1

The flat torus case $(g, n) = (1, 1)$

$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

simple (multi)curves = primitive integral vector in $\mathbb{R}^2$ up to ± 1
multicurves = integral vectors

The flat torus case $(g, n) = (1, 1)$

$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

simple (multi)curves = primitive integral vector in $\mathbb{R}^2$ up to ± 1
multicurves = integral vectors
mapping class group action = $\mathrm{PSL}(2, \mathbb{Z})$

The flat torus case $(g, n) = (1, 1)$

$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

simple (multi)curves = primitive integral vector in $\mathbb{R}^2$ up to ± 1
multicurves = integral vectors
mapping class group action = $\mathrm{PSL}(2, \mathbb{Z})$
topological types = multiplicity: $(u, v) \mapsto \gcd(u, v)$

The flat torus case $(g, n) = (1, 1)$

$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

simple (multi)curves = primitive integral vector in $\mathbb{R}^2$ up to ± 1
multicurves = integral vectors
mapping class group action = $\mathrm{PSL}(2, \mathbb{Z})$
topological types = multiplicity: $(u, v) \mapsto \gcd(u, v)$

With respect to the L^2-norm

$$\#\{\text{multicurves of length} \leq L\} \sim \frac{\pi}{2} \cdot L^2$$

and

$$c(k) = \frac{1}{4k^2} \qquad b_{1,1} = \frac{\pi^2}{24}$$

Separating vs non-separating in high genus

Theorem (Mirzakhani '08)

$$\#\{\textit{multicurves of type } \eta \textit{ and length} \leq L\} \sim B(\textit{metric}) \cdot \frac{c(\eta)}{b_g} \cdot L^{6g-6}.$$

Theorem (DGZZ'19)

For $n = 0$ (no puncture), as $g \to \infty$ we have

$$\frac{\sum_{g_1+g_2=g} c(\eta_{sep,g_1,g_2})}{c(\eta_{nsep,g})} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g}.$$

g	2	3	4
$\frac{\text{sep}}{\text{non-sep}}$	$\frac{1}{48} \simeq 0.021$	$\frac{5}{1776} \simeq 0.0028$	$\frac{605}{790992} \simeq 0.00076$

Counting ribbon graphs (aka combinatorial maps)

$\mathcal{R}_{g,n}(b_1, \ldots, b_n)$: ribbon graphs of genus g, no vertex of degree 1, n faces labeled from 1 to n and perimeters b_1, b_2, ..., b_n.

Counting ribbon graphs (aka combinatorial maps)

$\mathcal{R}_{g,n}(b_1, \ldots, b_n)$: ribbon graphs of genus g, no vertex of degree 1, n faces labeled from 1 to n and perimeters b_1, b_2, ..., b_n.

$$\widetilde{N}_{g,n}(b_1, b_2, \ldots, b_n) := \lim_{L \to \infty} \frac{1}{L^{6g-6+2n}} \# \mathcal{R}_{g,n}(Lb_1, Lb_2, \ldots, Lb_n).$$

Counting ribbon graphs (aka combinatorial maps)

$\mathcal{R}_{g,n}(b_1, \ldots, b_n)$: ribbon graphs of genus g, no vertex of degree 1, n faces labeled from 1 to n and perimeters b_1, b_2, ..., b_n.

$$\widetilde{N}_{g,n}(b_1, b_2, \ldots, b_n) := \lim_{L \to \infty} \frac{1}{L^{6g-6+2n}} \#\mathcal{R}_{g,n}(Lb_1, Lb_2, \ldots, Lb_n).$$

Theorem (Kontsevich'92, Norbury'11)

For $(b_1, \ldots, b_n)$ such that $b_1 + b_2 + \ldots + b_n \equiv 0 \mod 2$, the numbers $\widetilde{N}_{g,n}(b_1, b_2, \ldots, b_n)$ coincide with a homogeneous symmetric polynomial $N_{g,n}(b_1, b_2, \ldots, b_n)$ in the b_i^2 of degree $6g - 6 + 2n$ with rational coefficients.

Counting ribbon graphs (aka combinatorial maps)

Theorem (Kontsevich'92,Norbury'11)

For $(b_1, \ldots, b_n)$ such that $b_1 + b_2 + \ldots + b_n \equiv 0 \mod 2$, the numbers $\widetilde{N}_{g,n}(b_1, b_2, \ldots, b_n)$ coincide with a homogeneous symmetric polynomial $N_{g,n}(b_1, b_2, \ldots, b_n)$ in the b_i^2 of degree $6g - 6 + 2n$ with rational coefficients.

$\mathcal{C}_{g,n}$: integer compositions of $3g - 3 + n$ into n non-negative parts
For $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathcal{C}_{g,n}$ we define the *correlator* $\langle \mathbf{d} \rangle_{g,n}$ as

$$N_{g,n}(b_1, b_2, \ldots, b_n) =: \frac{1}{2^{5g-6+2n}} \sum_{\mathbf{d} \in \mathcal{C}_{g,n}} \frac{\langle \mathbf{d} \rangle_{g,n}}{d_1! d_2! \cdots d_n!} b_1^{2d_1} b_2^{2d_2} \cdots b_n^{2d_n}.$$

Counting ribbon graphs (aka combinatorial maps)

Theorem (Kontsevich'92,Norbury'11)

For $(b_1, \ldots, b_n)$ such that $b_1 + b_2 + \ldots + b_n \equiv 0 \mod 2$, the numbers $\widetilde{N}_{g,n}(b_1, b_2, \ldots, b_n)$ coincide with a homogeneous symmetric polynomial $N_{g,n}(b_1, b_2, \ldots, b_n)$ in the b_i^2 of degree $6g - 6 + 2n$ with rational coefficients.

$\mathcal{C}_{g,n}$: integer compositions of $3g - 3 + n$ into n non-negative parts
For $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathcal{C}_{g,n}$ we define the *correlator* $\langle \mathbf{d} \rangle_{g,n}$ as

$$N_{g,n}(b_1, b_2, \ldots, b_n) =: \frac{1}{2^{5g-6+2n}} \sum_{\mathbf{d} \in \mathcal{C}_{g,n}} \frac{\langle \mathbf{d} \rangle_{g,n}}{d_1! d_2! \cdots d_n!} b_1^{2d_1} b_2^{2d_2} \cdots b_n^{2d_n}.$$

Algebraic geometry note: the polynomials $N_{g,n}$ are part of Kontsevich's proof of Witten conjecture. We have

$$\langle \mathbf{d} \rangle_{g,n} = \int_{\overline{\mathcal{M}_{g,n}}} \psi_1^{d_1} \psi_2^{d_2} \cdots \psi_n^{d_n}$$

Explicit formula in the unicellular case ($n = 1$)

We have

$$\langle 3g-2 \rangle_{g,1} = \frac{1}{24^g \cdot g!}.$$

In other words

$$N_{g,1}(b_1) = \frac{1}{2^{5g-6+2n}} \frac{1}{(3g-2)!} \frac{1}{24^g \cdot g!} b_1^{6g-4}.$$

Explicit formula in the unicellular case ($n = 1$)

We have

$$\langle 3g - 2 \rangle_{g,1} = \frac{1}{24^g \cdot g!}.$$

In other words

$$N_{g,1}(b_1) = \frac{1}{2^{5g-6+2n}} \frac{1}{(3g-2)!} \frac{1}{24^g \cdot g!} b_1^{6g-4}.$$

Note: equivalent to the Lehman-Walsh'72, Harer-Zagier'86 formulas for the exact counting of unicellular maps.

Asymptotic formula in the bicellular case ($n = 2$)

Let us introduce

$$h(\mathbf{d}) = \frac{1}{24^g \cdot g!} \cdot \frac{(6g-1)!!}{\prod_{i=1}^{n}(2d_i+1)!!}$$

Asymptotic formula in the bicellular case ($n = 2$)

Let us introduce

$$h(\mathbf{d}) = \frac{1}{24^g \cdot g!} \cdot \frac{(6g-1)!!}{\prod_{i=1}^{n}(2d_i+1)!!}$$

Theorem (DGZZ'19)

For any $\mathbf{d} \in \mathcal{C}_{g,2}$ *we have* $1 - \frac{2}{6g-1} \leq \frac{\langle \mathbf{d} \rangle_{g,2}}{h(\mathbf{d})} \leq 1$.

Asymptotic formula in the bicellular case ($n = 2$)

Let us introduce

$$h(\mathbf{d}) = \frac{1}{24^g \cdot g!} \cdot \frac{(6g-1)!!}{\prod_{i=1}^{n}(2d_i+1)!!}$$

Theorem (DGZZ'19)

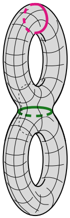
For any $\mathbf{d} \in \mathcal{C}_{g,2}$ *we have* $1 - \frac{2}{6g-1} \leq \frac{\langle \mathbf{d} \rangle_{g,2}}{h(\mathbf{d})} \leq 1$.

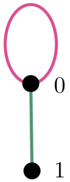
Note: generalized in Aggarwal'20 for correlators with $n \geq 3$.

From simple multicurves to stable graphs

stable graph:

Decorated graph dual to a multicurve and forgetting the embedding in the surface

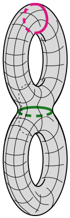


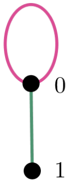


From simple multicurves to stable graphs

stable graph:

Decorated graph dual to a multicurve and forgetting the embedding in the surface



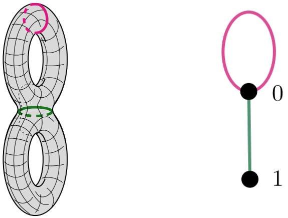


$\{$topological types of simple multicurves$\} \simeq \{$stable graphs$\}$

From simple multicurves to stable graphs

stable graph:

Decorated graph dual to a multicurve and forgetting the embedding in the surface



$\{$topological types of simple multicurves$\} \simeq \{$stable graphs$\}$
$\{$topological types of multicurves$\} \simeq \{$weighted stable graphs$\}$

The coefficient $c(\eta)$ and Kontsevich polynomials $N_{g,n}$

For each stable graph Γ (dual to a multicurve η) we associate a polynomial with variables $(b_e)_{e\in E(\Gamma)}$ and define

$$P_\Gamma(\underline{b}) = A_{g,n}\frac{1}{2^{|V(\Gamma)|-1}}\cdot\frac{1}{|\mathrm{Aut}(\Gamma)|}\cdot\prod_{e\in E(\Gamma)} b_e\cdot\prod_{v\in V(\Gamma)} N_{g_v,n_v}(\underline{b}_v).$$

where $A_{g,n} = \dfrac{2^{2g-3+n}}{(6g-6+2n)\cdot(6g-7+2n)!}$.

The coefficient $c(\eta)$ and Kontsevich polynomials $N_{g,n}$

For each stable graph Γ (dual to a multicurve η) we associate a polynomial with variables $(b_e)_{e\in E(\Gamma)}$ and define

$$P_\Gamma(\underline{b}) = A_{g,n}\frac{1}{2^{|V(\Gamma)|-1}} \cdot \frac{1}{|\mathrm{Aut}(\Gamma)|} \cdot \prod_{e\in E(\Gamma)} b_e \cdot \prod_{v\in V(\Gamma)} N_{g_v,n_v}(\underline{b}_v).$$

where $A_{g,n} = \dfrac{2^{2g-3+n}}{(6g-6+2n)\cdot(6g-7+2n)!}$.

Theorem (Mirzakhani '08, DGZZ '19)

For η is a simple multicurve and associated stable graph Γ we have

$$c(\eta) = \mathcal{Y}(P_\Gamma) \qquad \textit{where } \mathcal{Y} : \prod_{i=1}^{k} b_i^{m_i} \mapsto \prod_{i=1}^{k} m_i!.$$

$c(\eta)$ for simple closed curves

Theorem (Mirzakhani '08, DGZZ '19)

$$c(\eta) = \mathcal{Y}(P_\Gamma) \qquad \textit{where}\, \mathcal{Y} : \prod_{i=1}^{k} b_i^{m_i} \mapsto \prod_{i=1}^{k} m_i!.$$

Non-separating curve

$$c(\eta_{nsep,g}) = \frac{1}{A_{g,n}} \frac{1}{2} \mathcal{Y}\left(bN_{g-1,2}(b,b)\right)$$

Separating curve

$$c(\eta_{sep,g_1,g_2}) = \frac{1}{A_{g,n}} \frac{1}{\mathrm{Aut}} \mathcal{Y}\left(bN_{g_1,1}(b)N_{g_2,1}(b)\right)$$

Mirzakhani's curve counting theorem

$$\{\text{multicurves on } S\} = \{\text{integral points in } \mathcal{ML}(S)\}$$

Mirzakhani's curve counting theorem

$$\{\text{multicurves on } S\} = \{\text{integral points in } \mathcal{ML}(S)\}$$

It follows that as $L \to \infty$

$$\{\text{all multicurves of length} \leq L\} \sim B(\text{metric}) \cdot L^{6g-6}.$$

Mirzakhani's curve counting theorem

$$\{\text{multicurves on } S\} = \{\text{integral points in } \mathcal{ML}(S)\}$$

It follows that as $L \to \infty$

$$\{\text{all multicurves of length} \leq L\} \sim B(\text{metric}) \cdot L^{6g-6}.$$

The fact that for each type of multicurves η its proportion $c(\eta)/b_g$ exists and is positive relies on the ergodic action of $\mathrm{MCG}(S)$ on $\mathcal{ML}(S)$ (Masur'85).

Mirzakhani's curve counting theorem

$$\{\text{multicurves on } S\} = \{\text{integral points in } \mathcal{ML}(S)\}$$

It follows that as $L \to \infty$

$$\{\text{all multicurves of length} \leq L\} \sim B(\text{metric}) \cdot L^{6g-6}.$$

The fact that for each type of multicurves η its proportion $c(\eta)/b_g$ exists and is positive relies on the ergodic action of $\mathrm{MCG}(S)$ on $\mathcal{ML}(S)$ (Masur'85).
The explicit formula for $c(\eta)$ can be proven via Weil-Petersson volumes (Mirzakhani'08) or square-tiled surface counting (DGZZ'19).

Asymptotics of 2-correlators

Witten conjecture (Kontsevich theorem) states that the correlators $\langle \mathbf{d} \rangle_{g,n}$ satisfy recurrence relations.

Asymptotics of 2-correlators

Witten conjecture (Kontsevich theorem) states that the correlators $\langle \mathbf{d} \rangle_{g,n}$ satisfy recurrence relations. They restrict to simple ones for 2-correlators $\langle \mathbf{d} \rangle_{g,2}$ (Zograf'18).

Asymptotics of 2-correlators

Witten conjecture (Kontsevich theorem) states that the correlators $\langle \mathbf{d} \rangle_{g,n}$ satisfy recurrence relations. They restrict to simple ones for 2-correlators $\langle \mathbf{d} \rangle_{g,2}$ (Zograf'18).

One can then prove by induction

$$1 - \frac{2}{6g-1} \leq \frac{\langle \mathbf{d} \rangle_{g,2}}{h(\mathbf{d})} \leq 1.$$

From asymptotics of 2-correlators to $c(\eta)$

Recall that 1-correlators and 2-correlators are respectively the coefficients of $N_{g,1}(b_1)$ and $N_{g,2}(b_1, b_2)$.

From asymptotics of 2-correlators to $c(\eta)$

Recall that 1-correlators and 2-correlators are respectively the coefficients of $N_{g,1}(b_1)$ and $N_{g,2}(b_1, b_2)$.

From the formulas $c(\eta_{nsep,g}) = \frac{1}{A_{g,n}} \frac{1}{2} \mathcal{Y}\left(bN_{g-1,2}(b,b)\right)$ and $c(\eta_{sep,g_1,g_2}) = \frac{1}{A_{g,n}} \frac{1}{\mathrm{Aut}} \mathcal{Y}\left(bN_{g_1,1}(b)N_{g_2,1}(b)\right)$, we deduce asymptotics for $c(\eta_{nsep,g})$ and $c(\eta_{sep,g_1,g_2})$.

Further remarks

- (weak) generalization to multicurves with more components using Aggarwal'20 (DGZZ'20)
- for generic hyperbolic metric, the separating systole has order $2\log(g)$ (Mirzakhani'13, Nie-Wu-Xue'20)