Shannon's theorem and Poisson boundaries for locally compact groups

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Summary

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joint with Behrang Forghani

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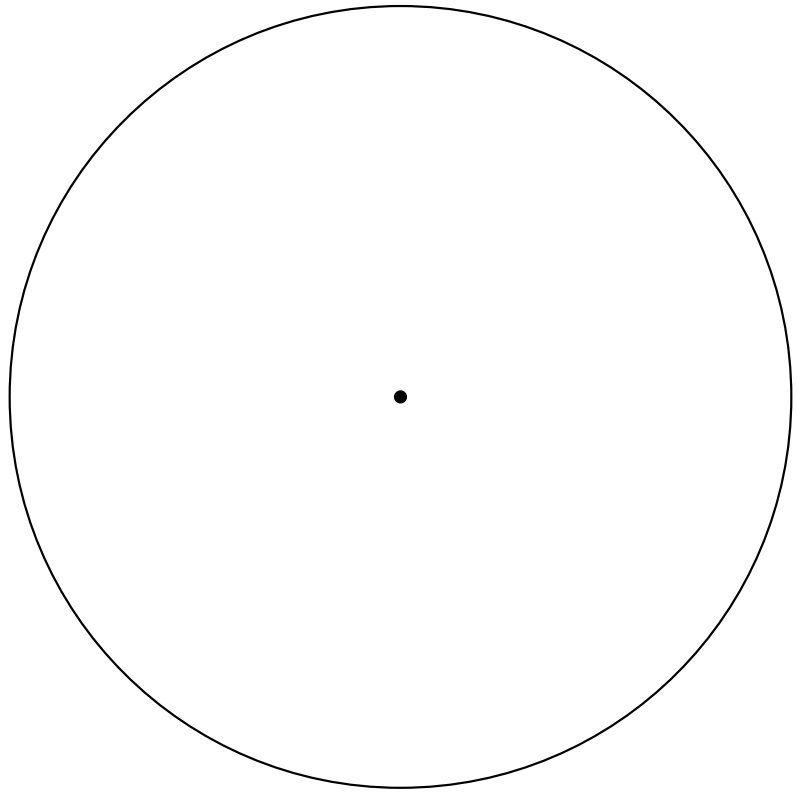

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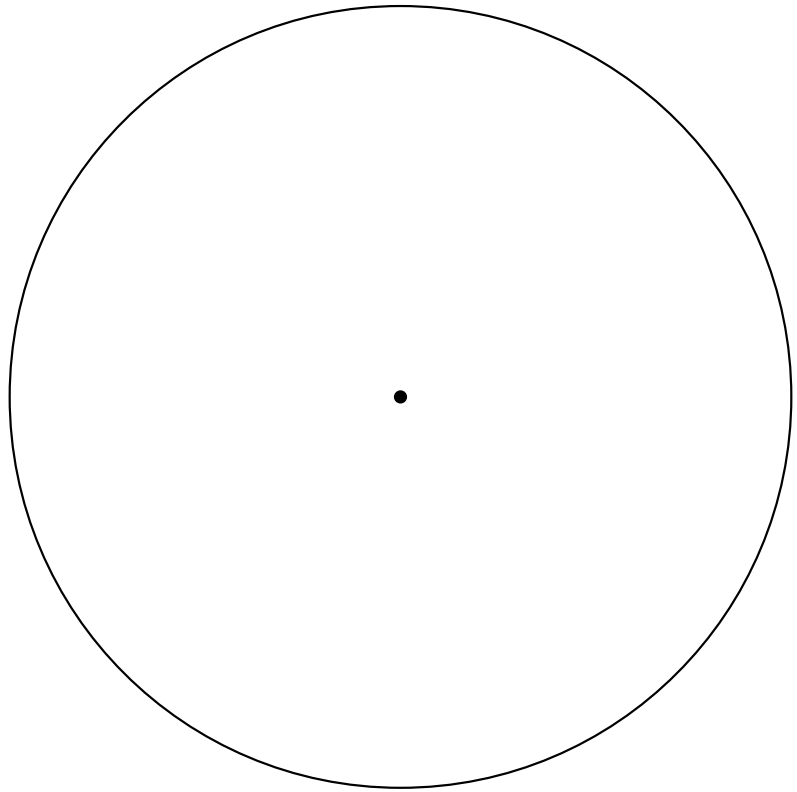

Theorem (Poisson representation)

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is harmonic on $\mathbb{D}$, i.e. $\Delta u = 0$.

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$$P_r(\theta) := \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

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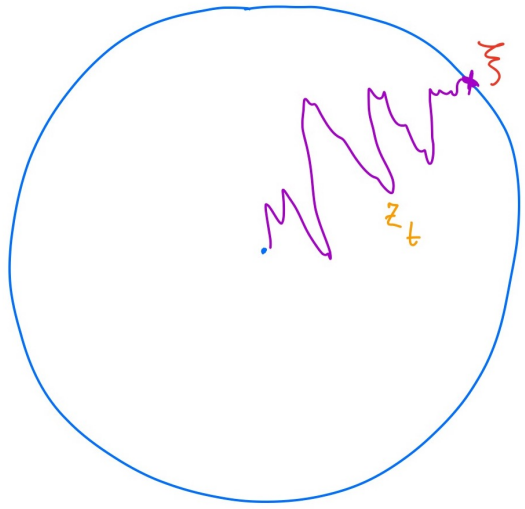

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Question. Can we generalize this to other groups $G \neq PSL_2(\mathbb{R})$?

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$$\nu = \int_G g\nu\, d\mu(g).$$

Random walks and μ-boundaries

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A space (B, ν) is a μ-boundary if there exists a measurable map

$$\mathrm{bnd} : \Omega \to B$$

such that $\mathrm{bnd} = \mathrm{bnd} \circ T$.

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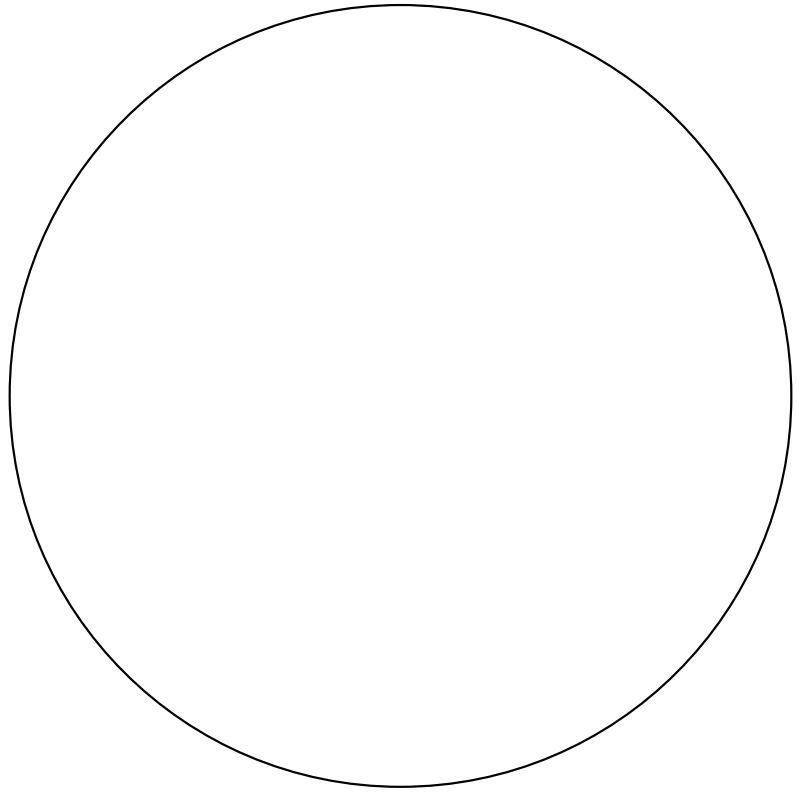

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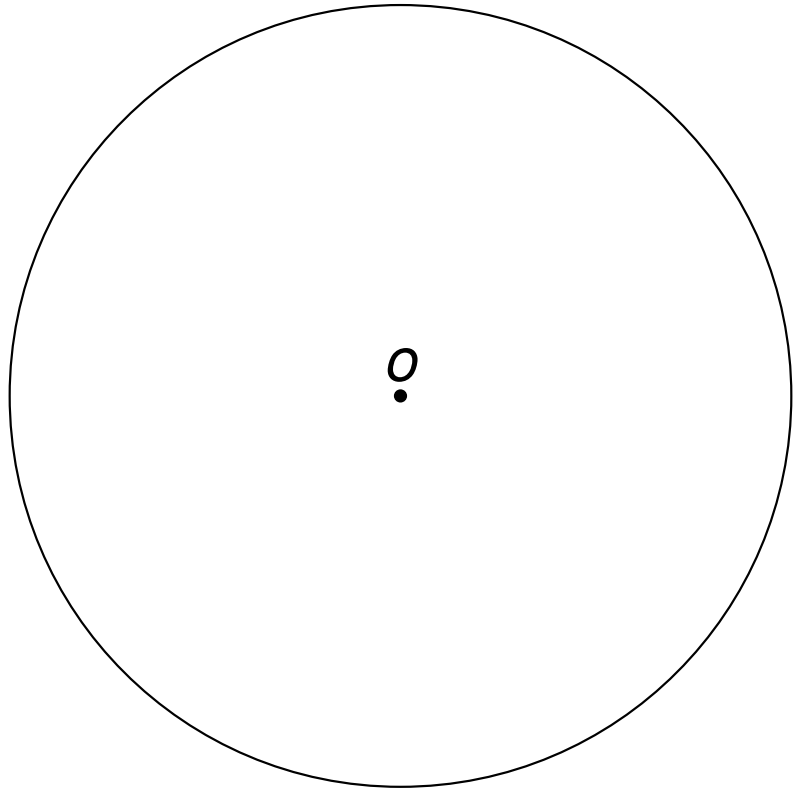

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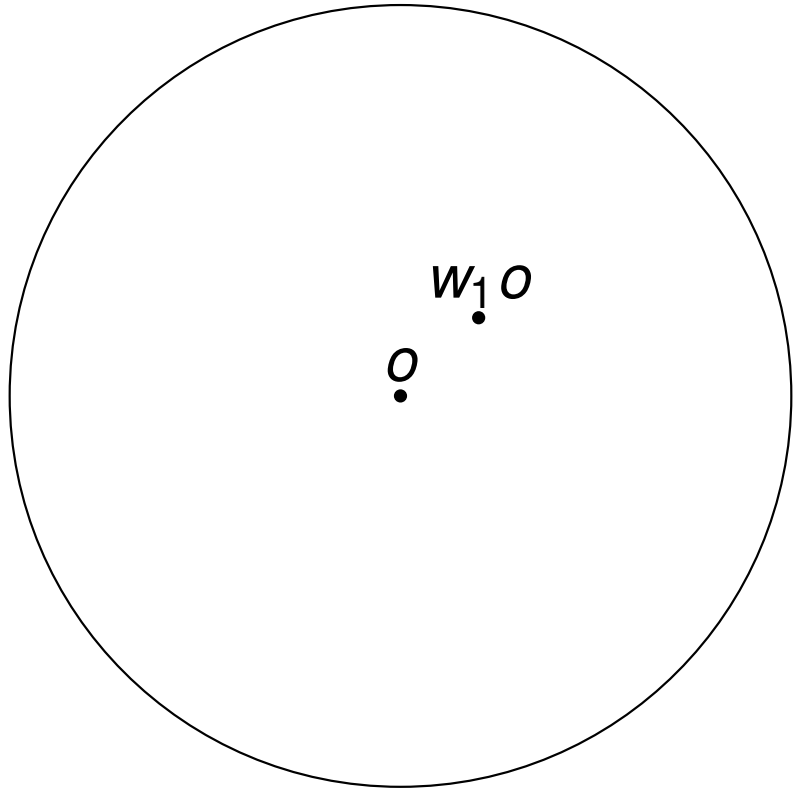

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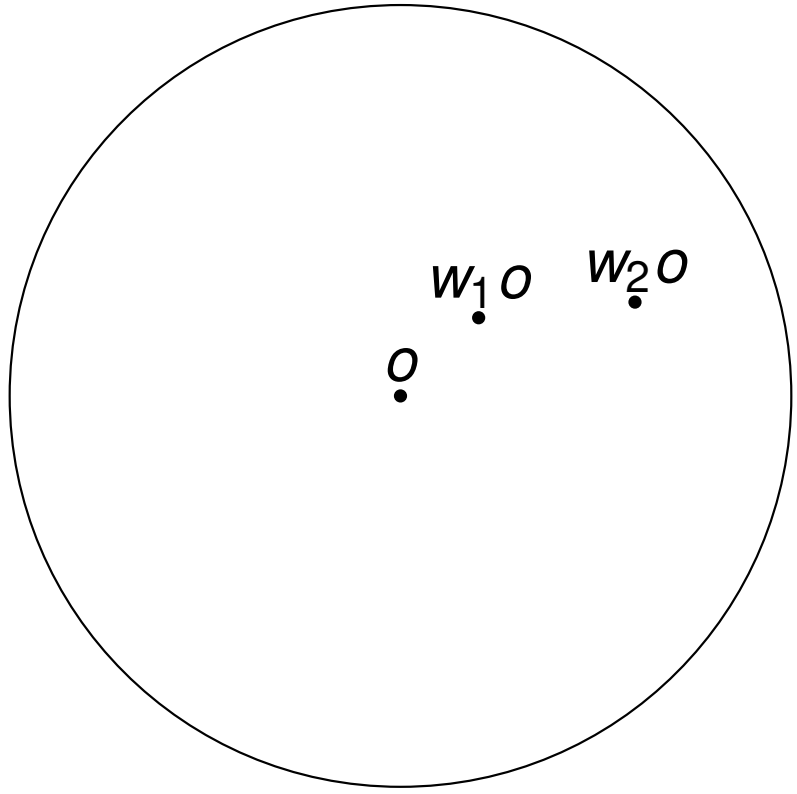

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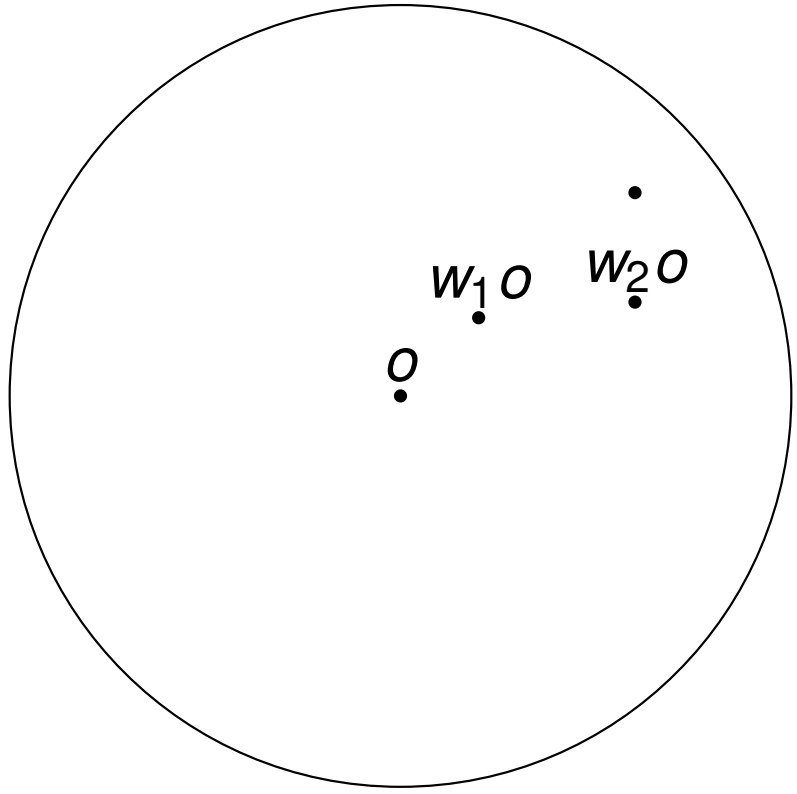

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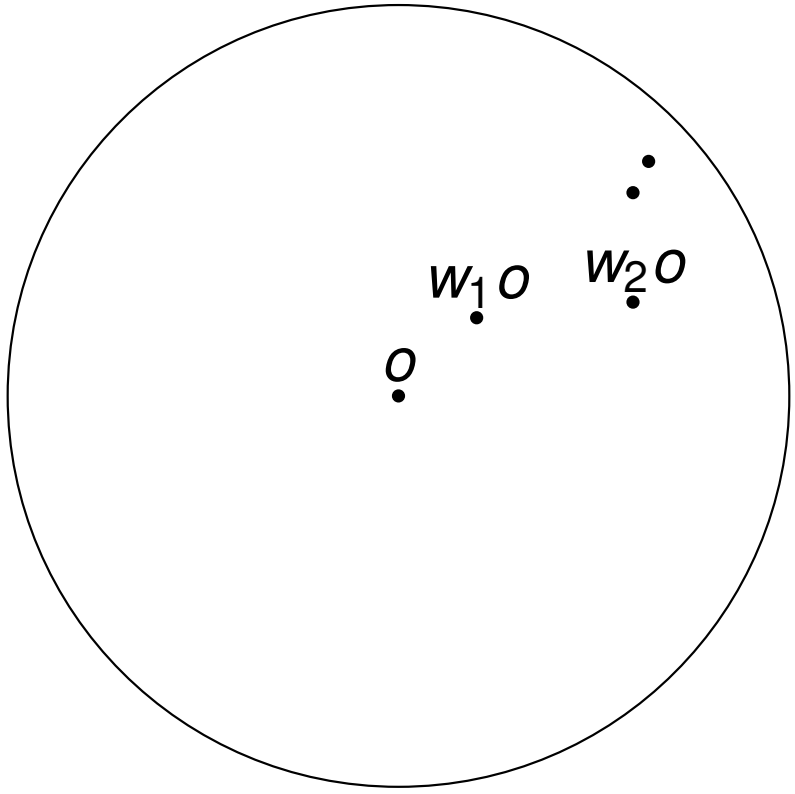

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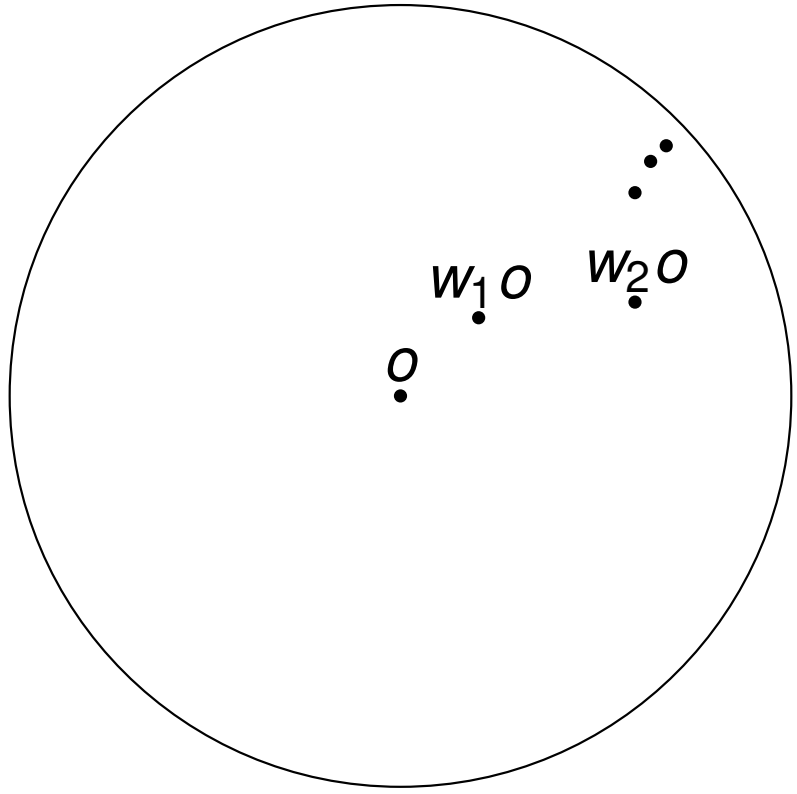

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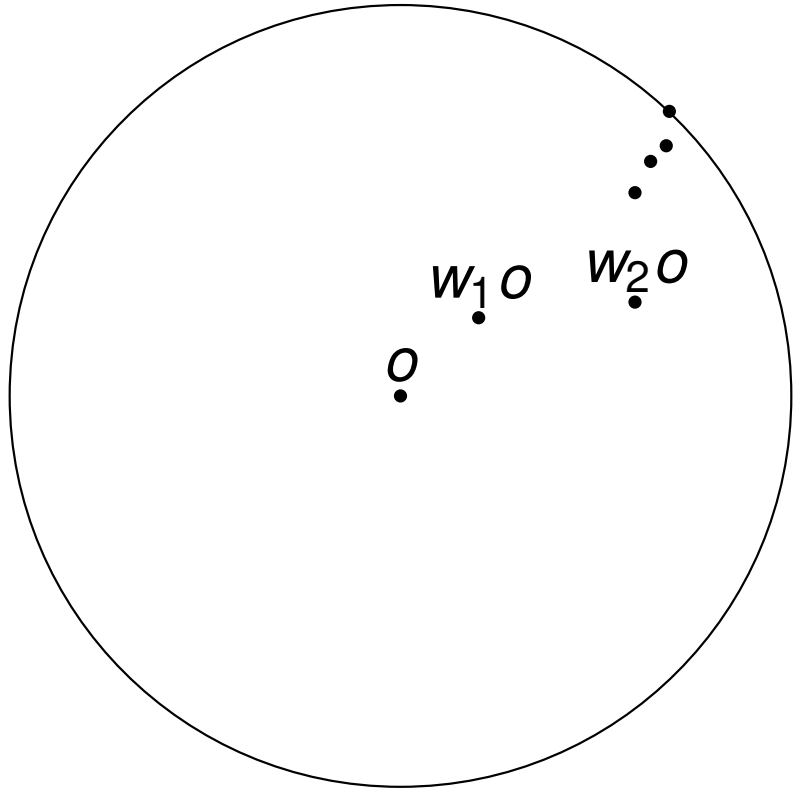

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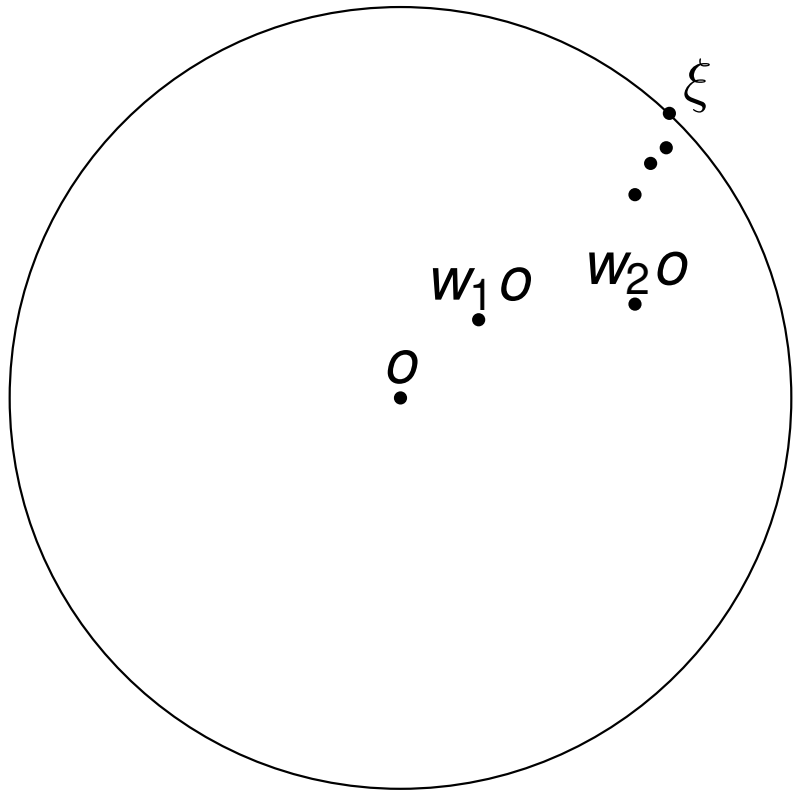

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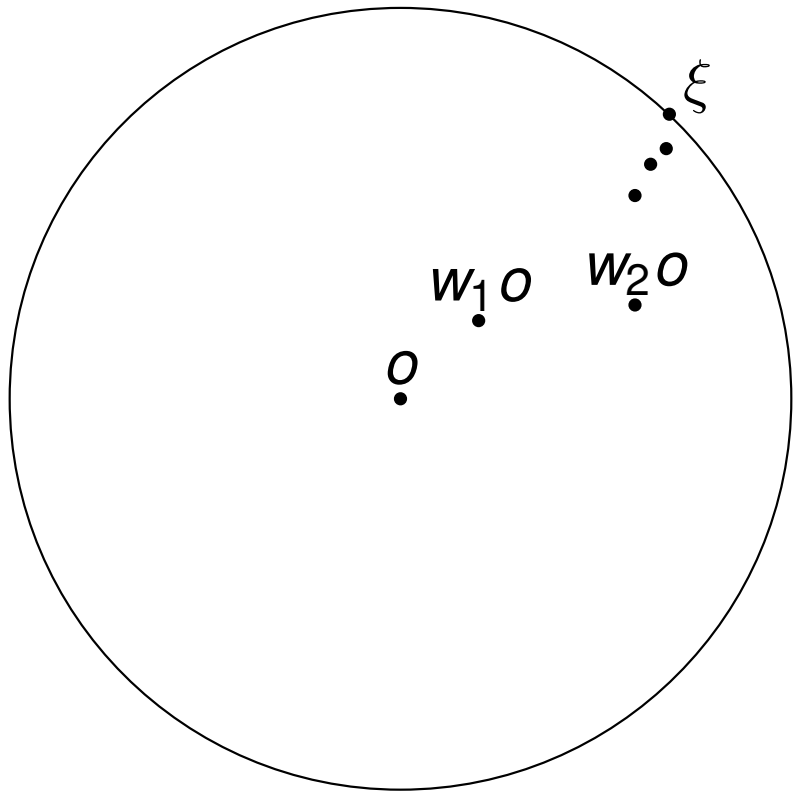

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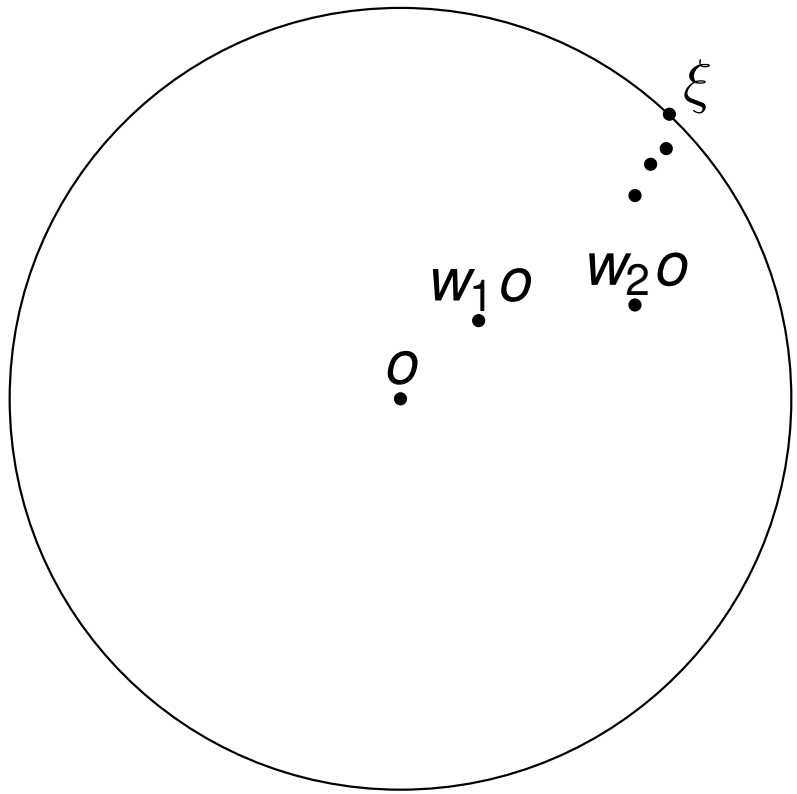

Moreover, $(\partial X, \nu)$ is a μ-boundary, given by the map

$$\Omega \ni \mathrm{bnd}(\omega) := \lim_{n\to\infty} w_n o \in \partial X$$

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A μ-boundary (B, ν) is the Poisson boundary if Φ is an isomorphism

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A μ-boundary (B, ν) is the Poisson boundary if Φ is an isomorphism

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Poisson boundary is trivial (= 1 point) $\Leftrightarrow$ bounded harmonic functions are constant

General Poisson representation - II (Furstenberg)

Let G be a (lcsc) group, μ a probability measure on G, let (B, ν) be a μ-boundary.

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Examples. Abelian groups; nilpotent groups

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Let $G < Isom(X, d)$ be a group, μ a measure on G, ν the hitting measure on ∂X.

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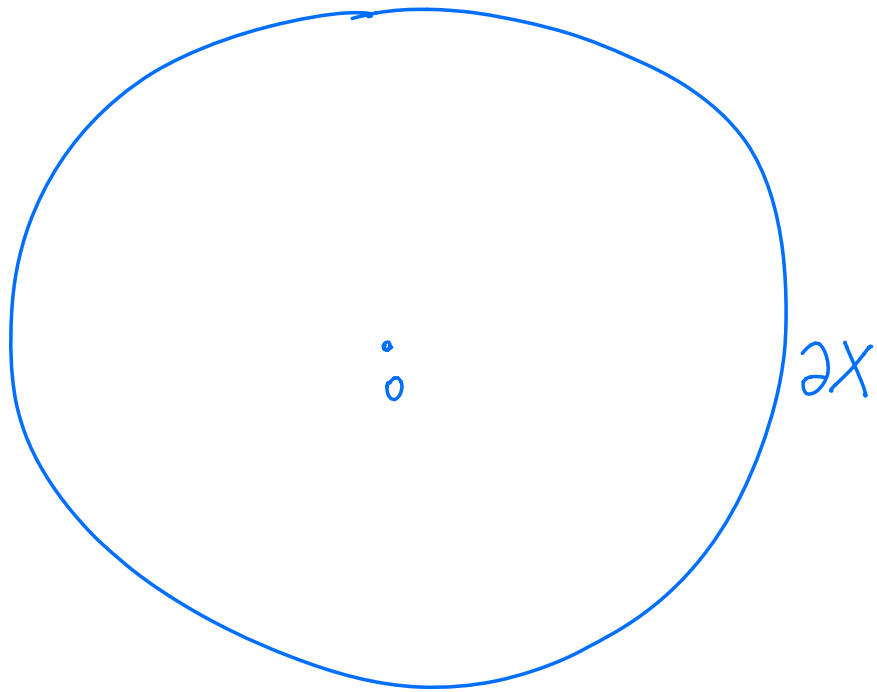

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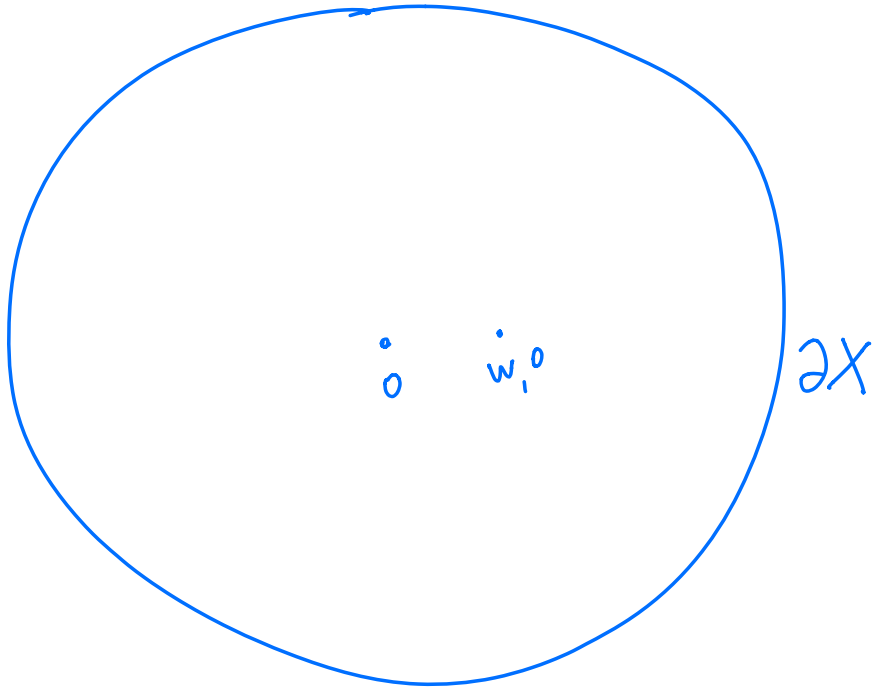

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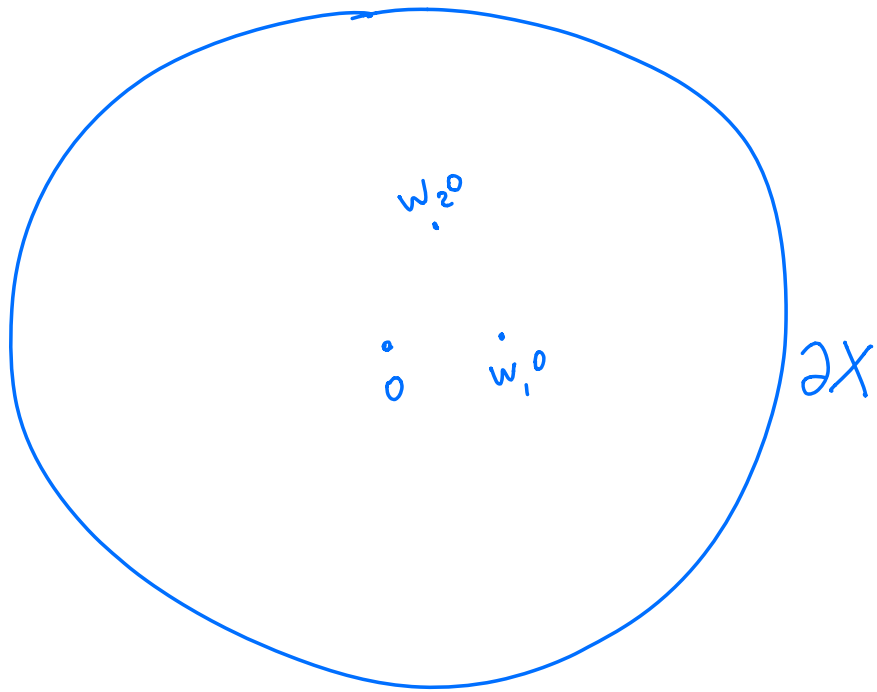

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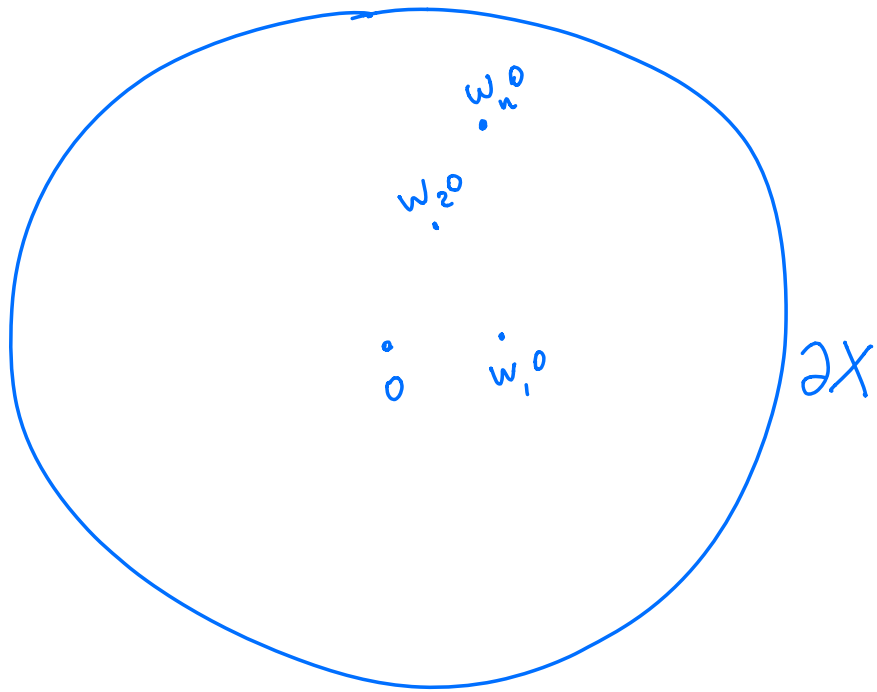

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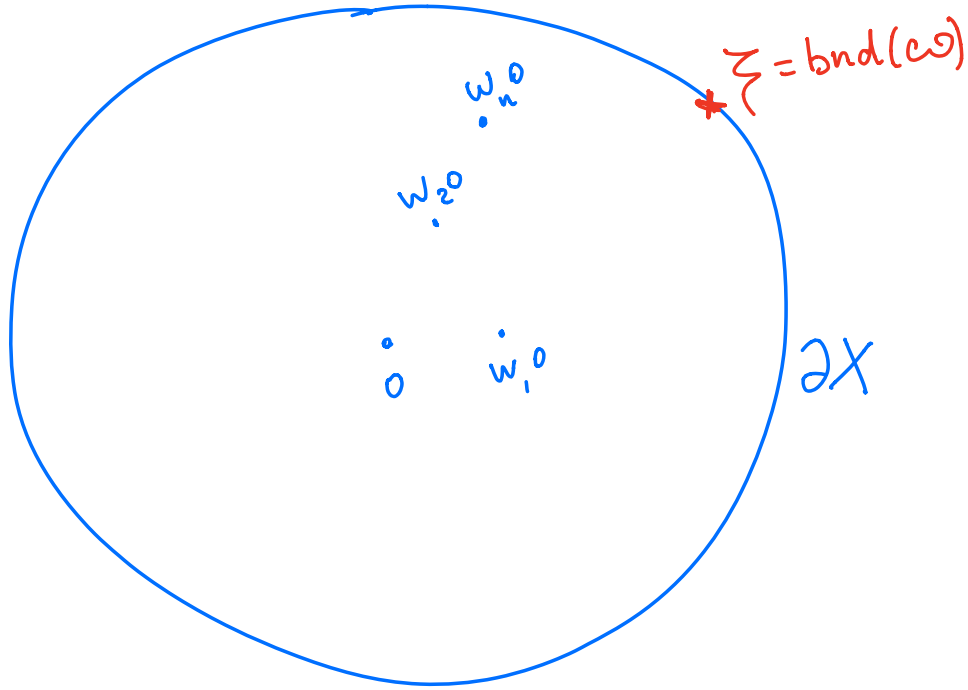

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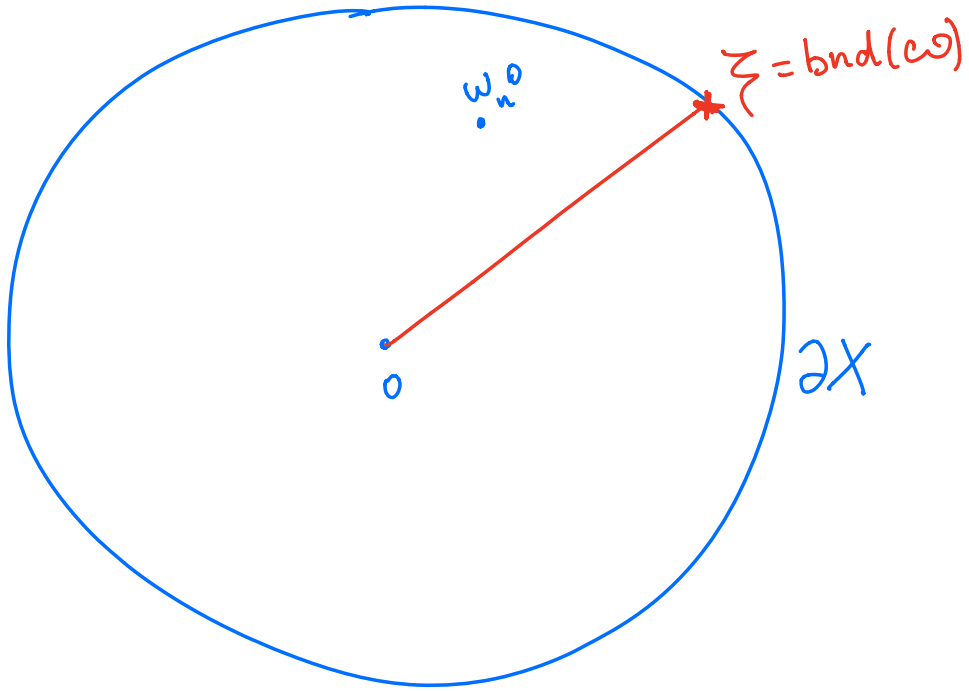

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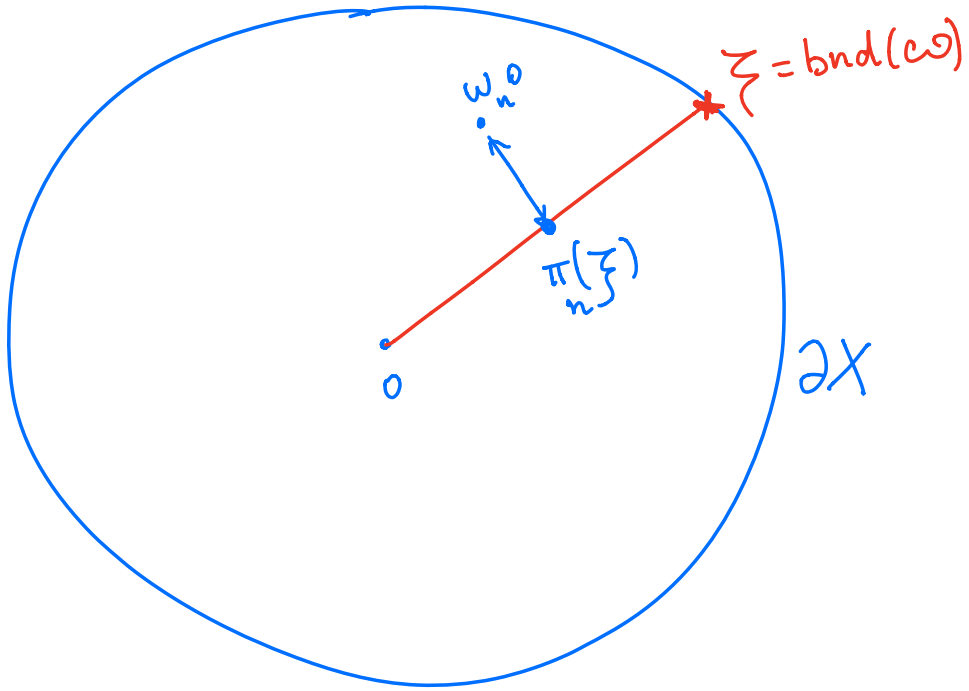

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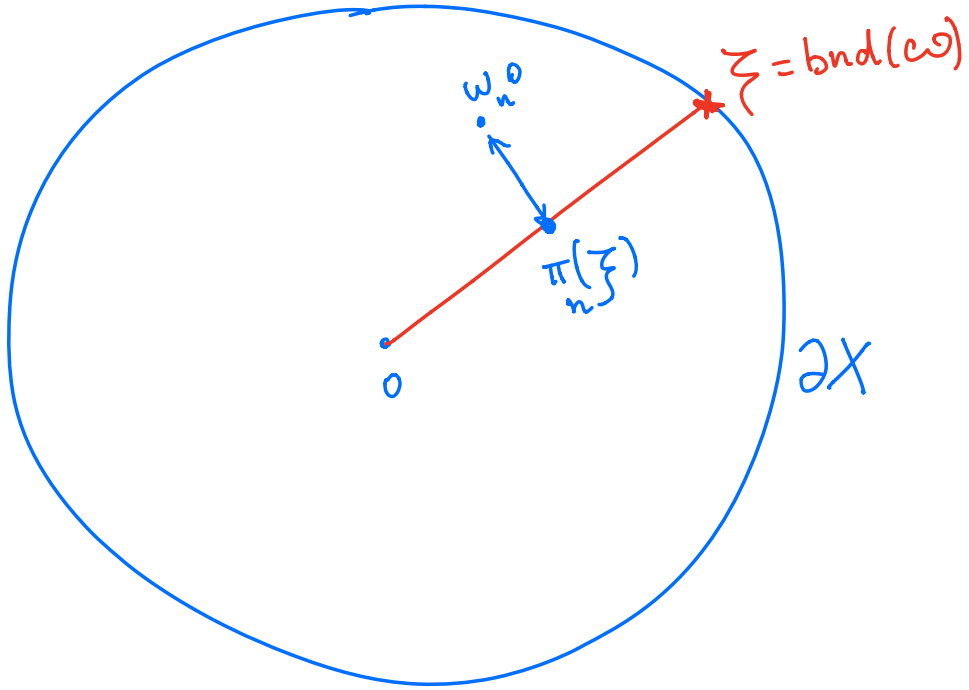
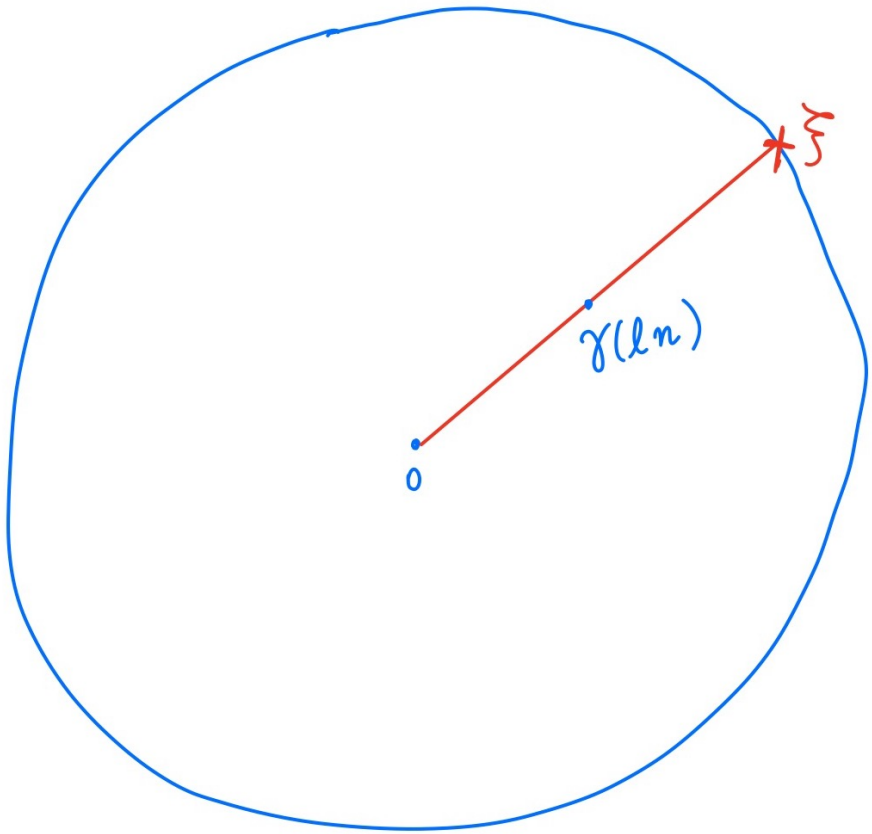


Sublinear tracking and entropy



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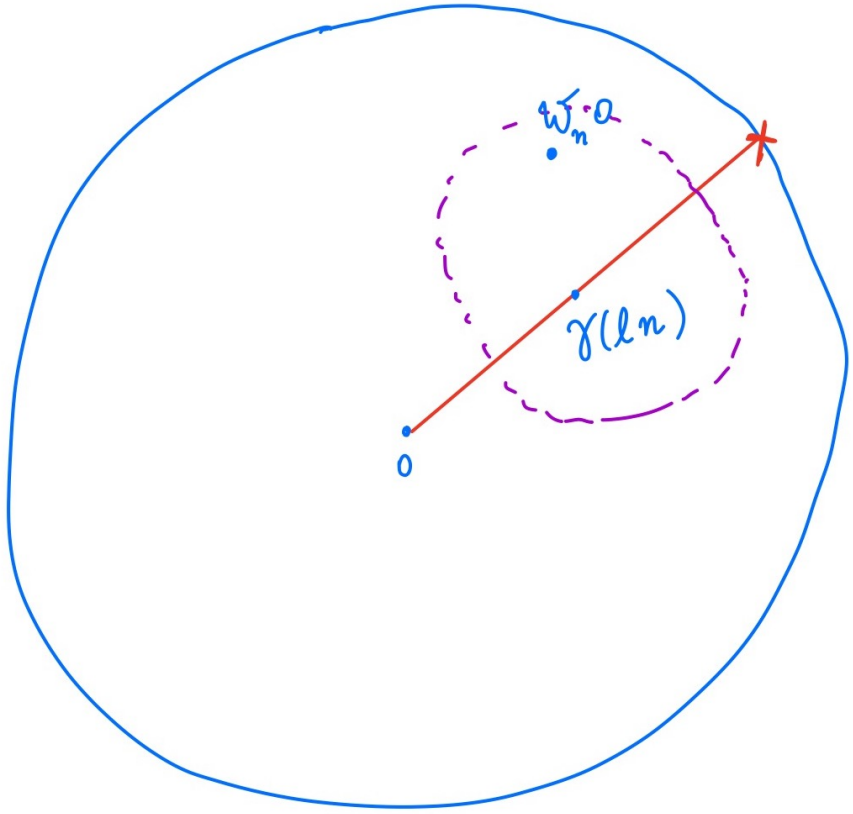

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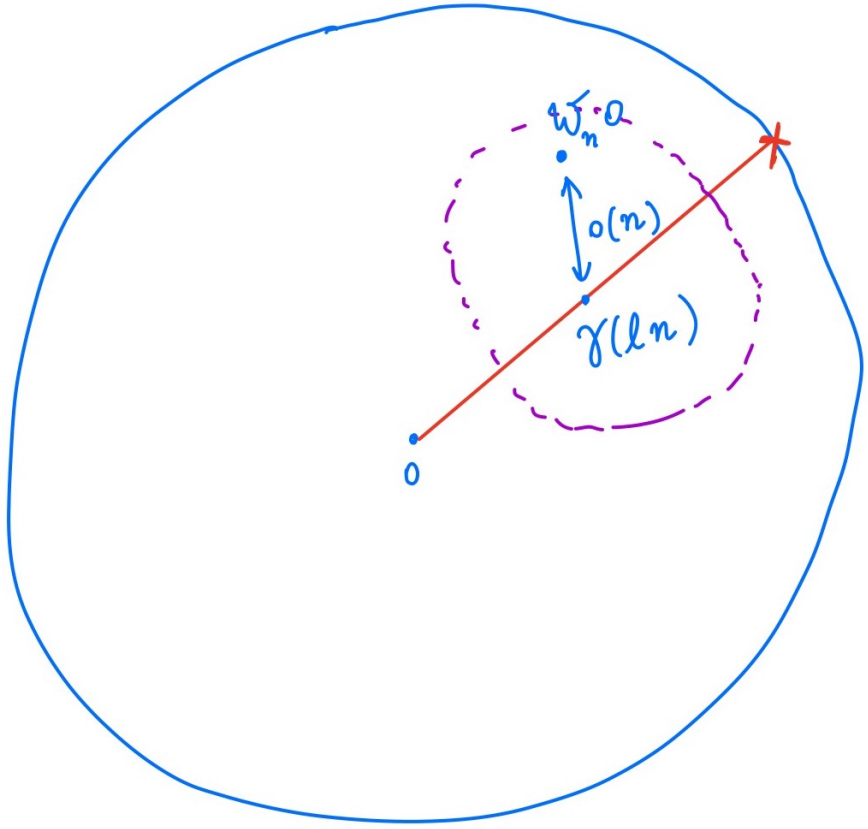

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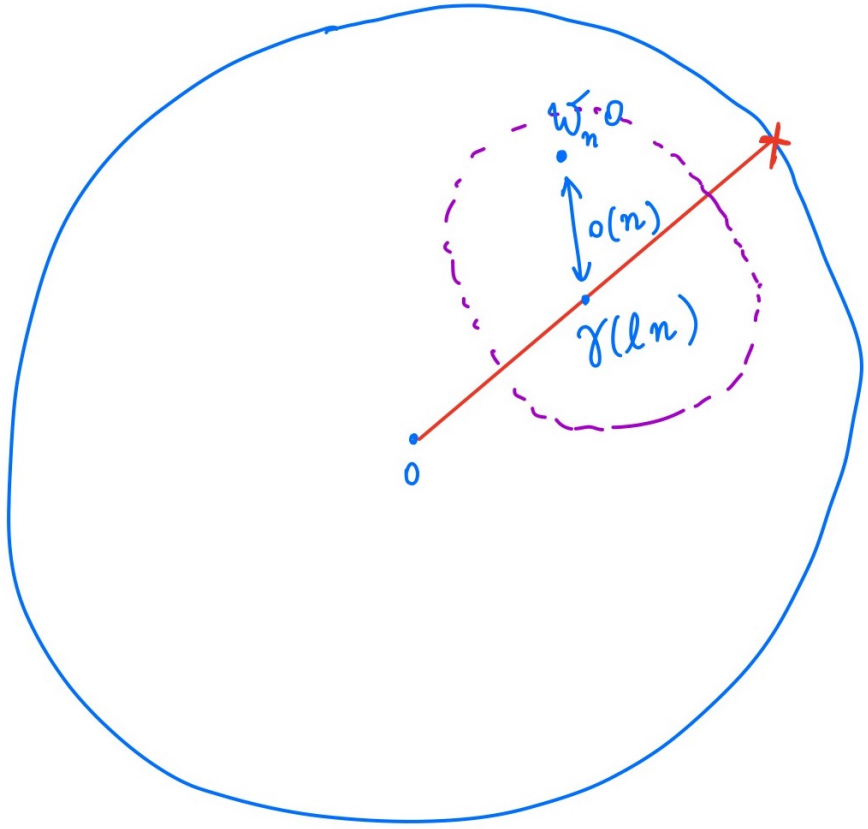

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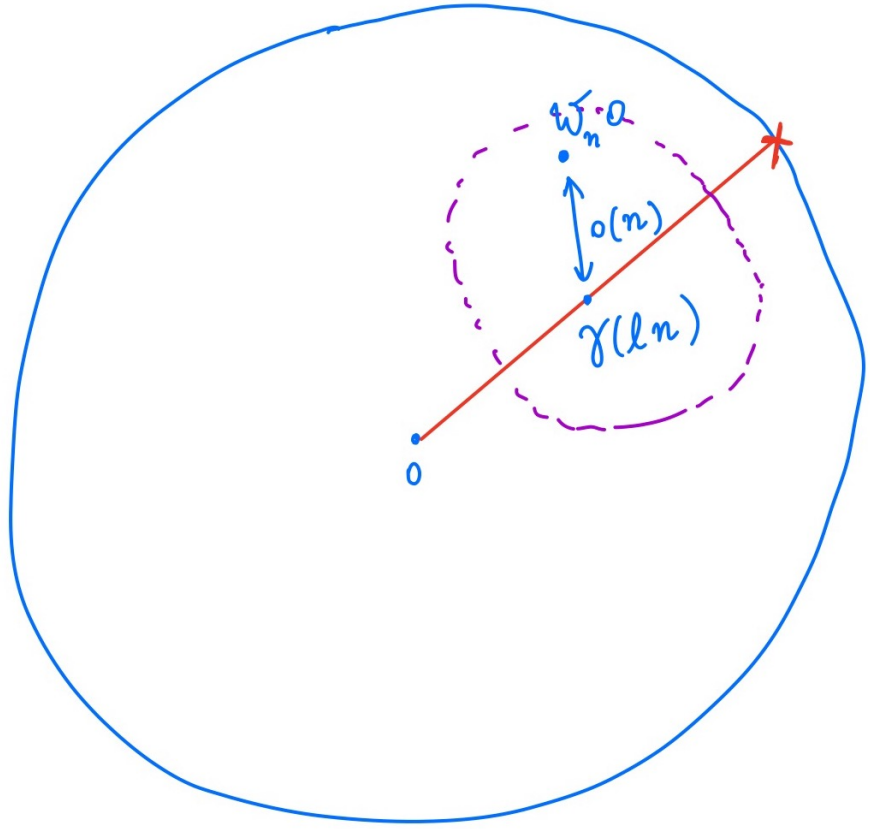

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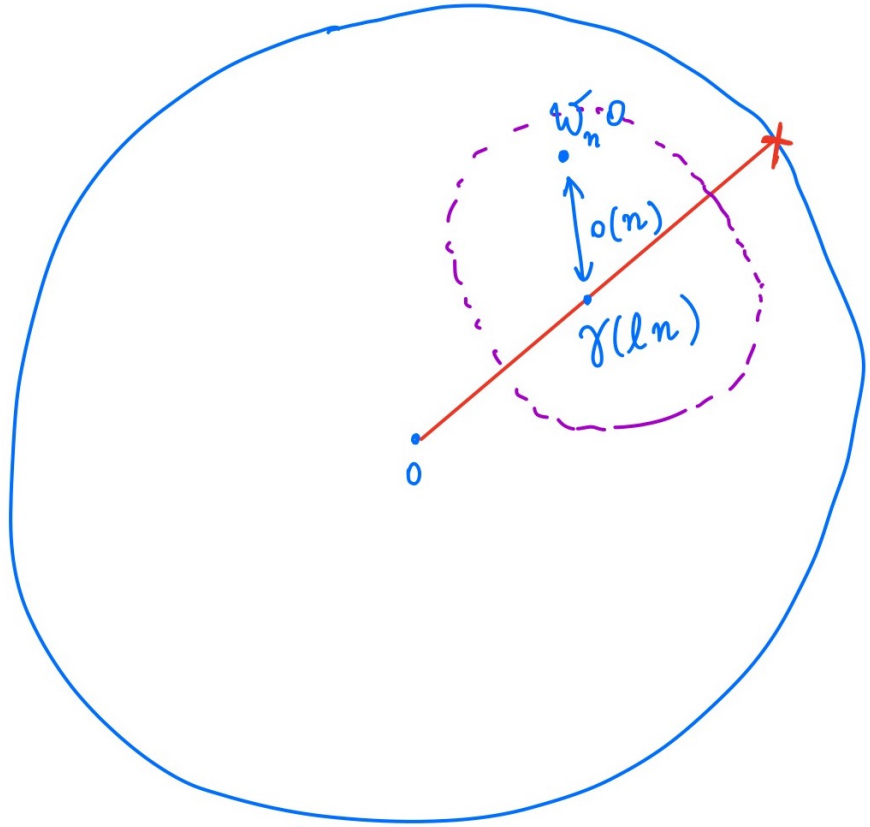

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(Kaimanovich-Woess '02) Ray criterion for invariant Markov operators (intermediate between discrete and lcsc groups)

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- For countable G: [Kaimanovich '94] .

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- For countable G: [Karlsson-Margulis '99].

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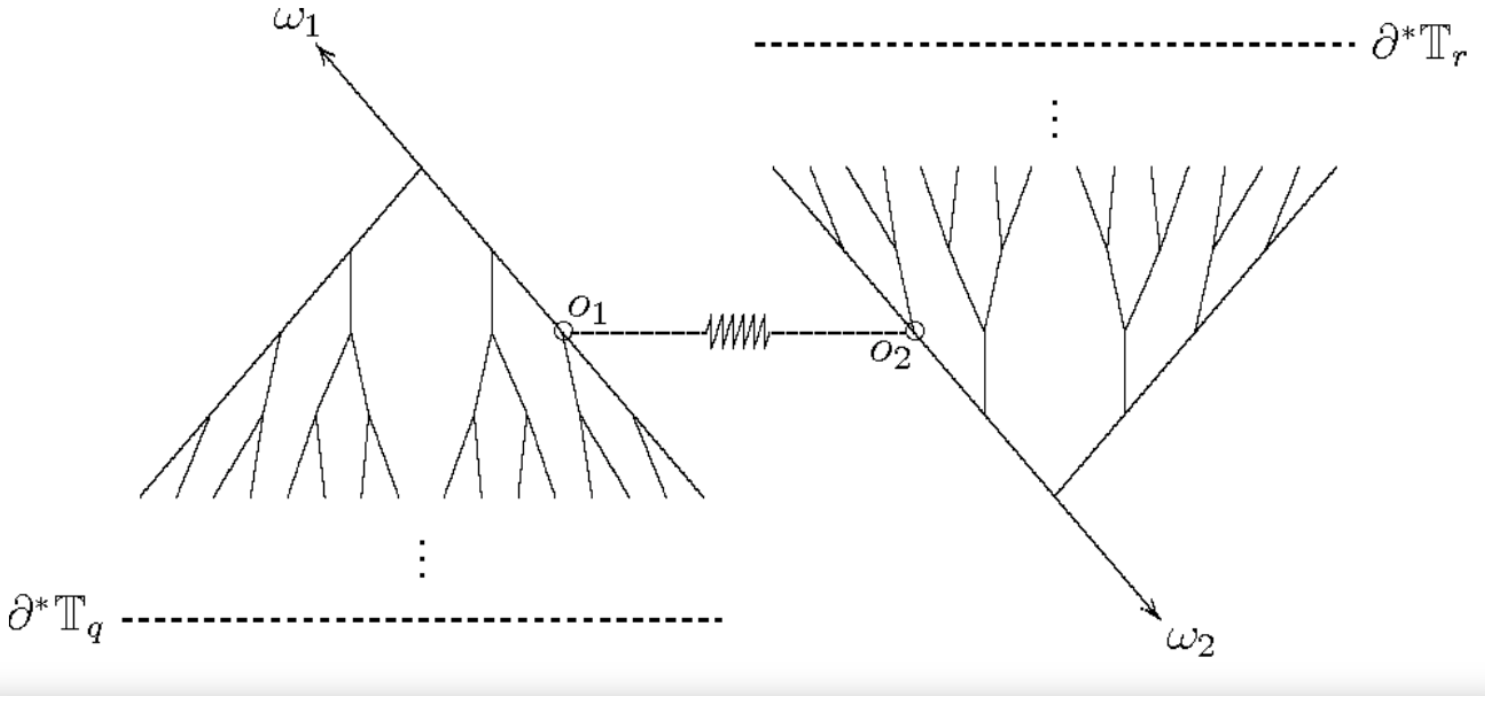

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Question. (Derriennic) Can we generalize to locally compact groups?

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Recall $h(\nu) =$ Furstenberg entropy.

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