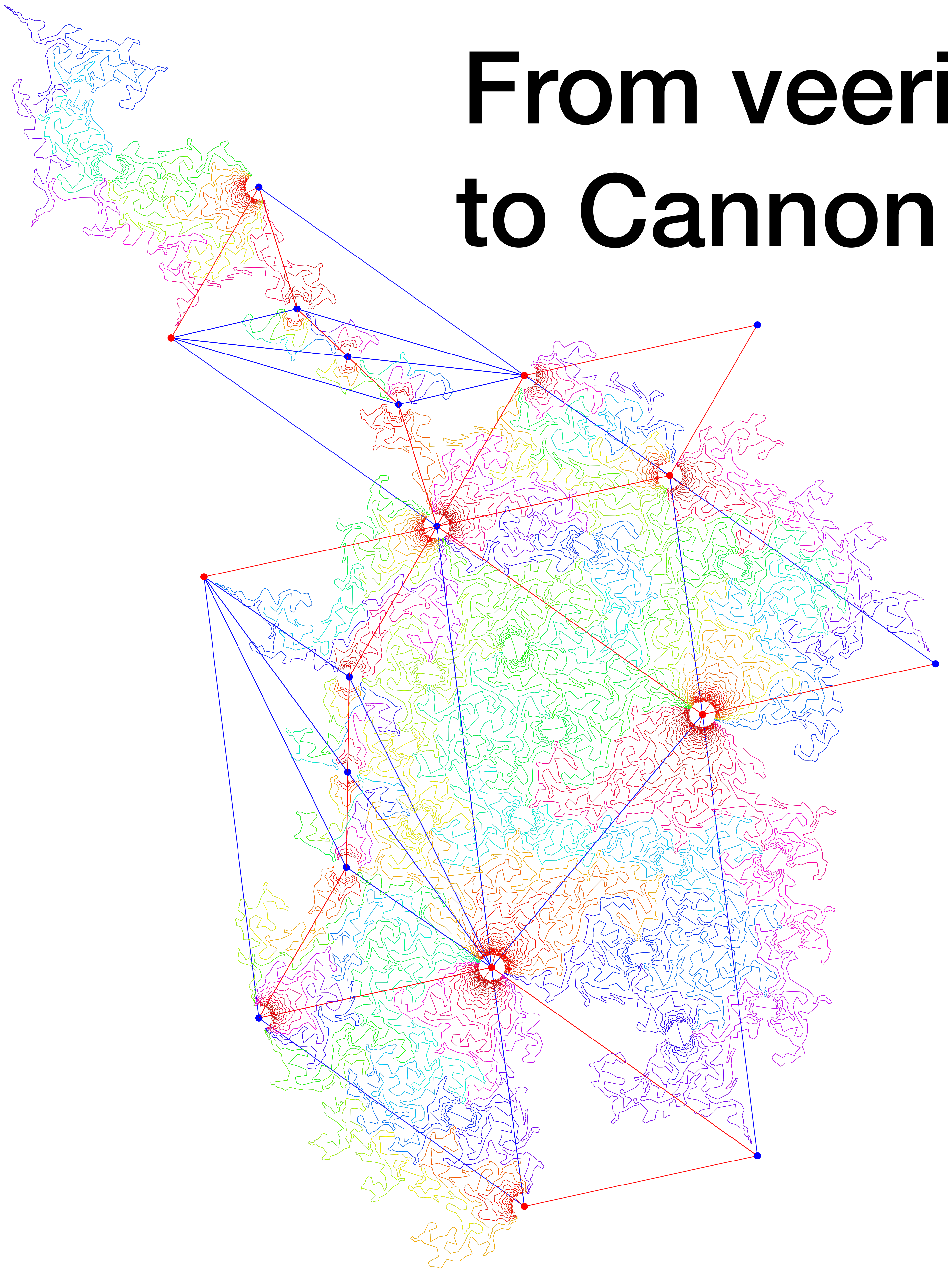


From veering triangulations to Cannon—Thurston maps

Saul Schleimer

University of Warwick

2020-07-15

joint work with
Jason Manning and
Henry Segerman

Outline

Cannon—Thurston maps

Veering triangulations

From the one to the other

Cannon-Thurston maps

Notation: F^2 and M^3 will be connected, oriented, hyperbolic manifolds of finite volume. We use $\tilde{F} \cong \mathbb{H}^2$ and $\tilde{M} \cong \mathbb{H}^3$ for their universal covers.

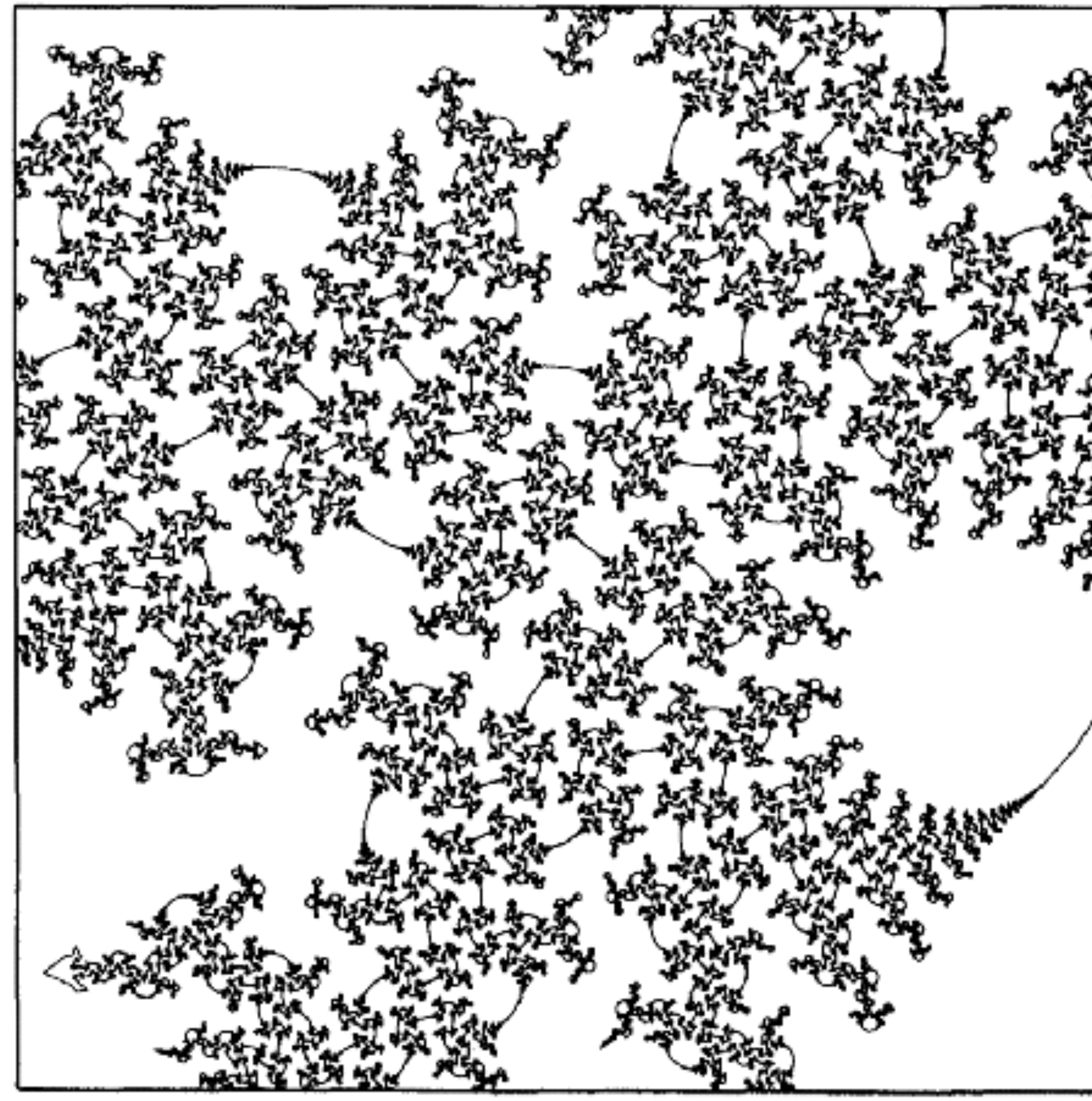


Figure by Bill Thurston, from *Three dimensional manifolds, kleinian groups and hyperbolic geometry*

Let $S^1 \cong \partial\mathbb{H}^2$ and $S^2 \cong \partial\mathbb{H}^3$ be the boundaries at infinity.

Let $\overline{F} = \tilde{F} \cup S^1 \cong \mathbb{B}^2$ and $\overline{M} = \tilde{M} \cup S^2 \cong \mathbb{B}^3$ be the "natural" compactifications.

Cannon-Thurston maps

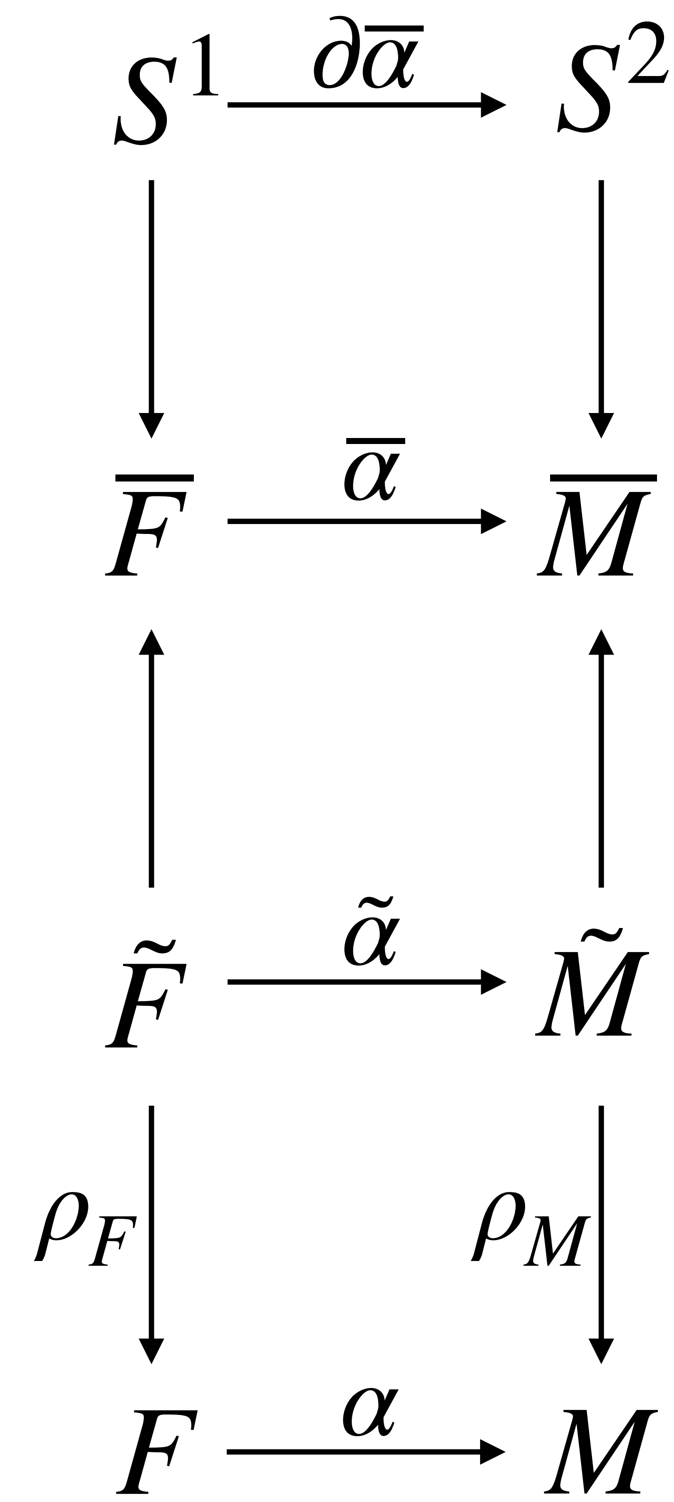
Suppose that $\alpha\colon F \to M$ is π_1-injective.

Let $\rho_F\colon \tilde{F} \to F$ and $\rho_M\colon \tilde{M} \to M$ be the universal covering maps.

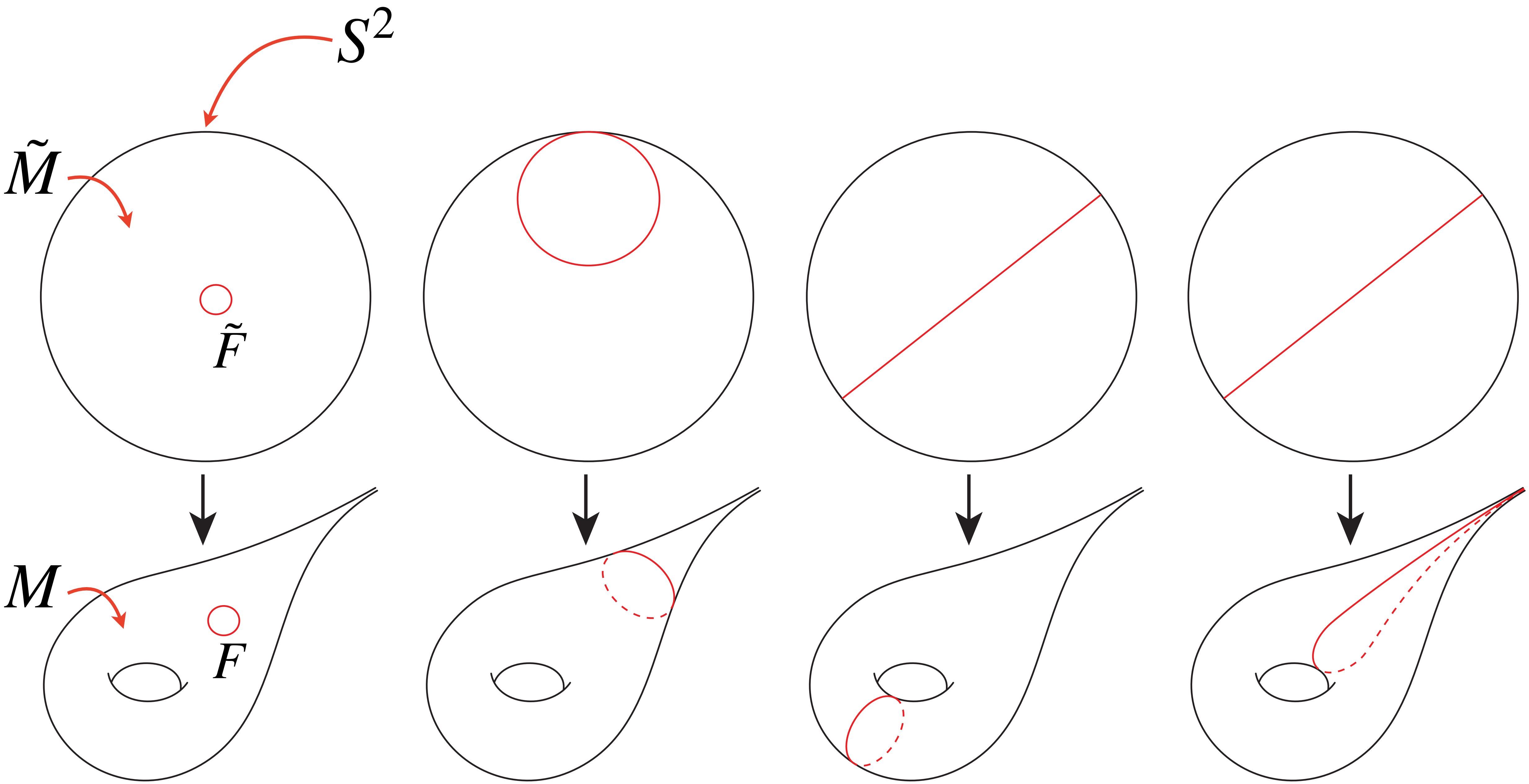
Let $\tilde{\alpha}\colon \tilde{F} \to \tilde{M}$ be an *elevation* of α: a lift of $\alpha \circ \rho_F$.

Suppose that $\tilde{\alpha}$ extends to a continuous map $\bar{\alpha}\colon \overline{F} \to \overline{M}$.

Denote the restriction of $\bar{\alpha}$ to S^1 by $\partial\overline{\alpha}$. This is the *Cannon-Thurston map*.

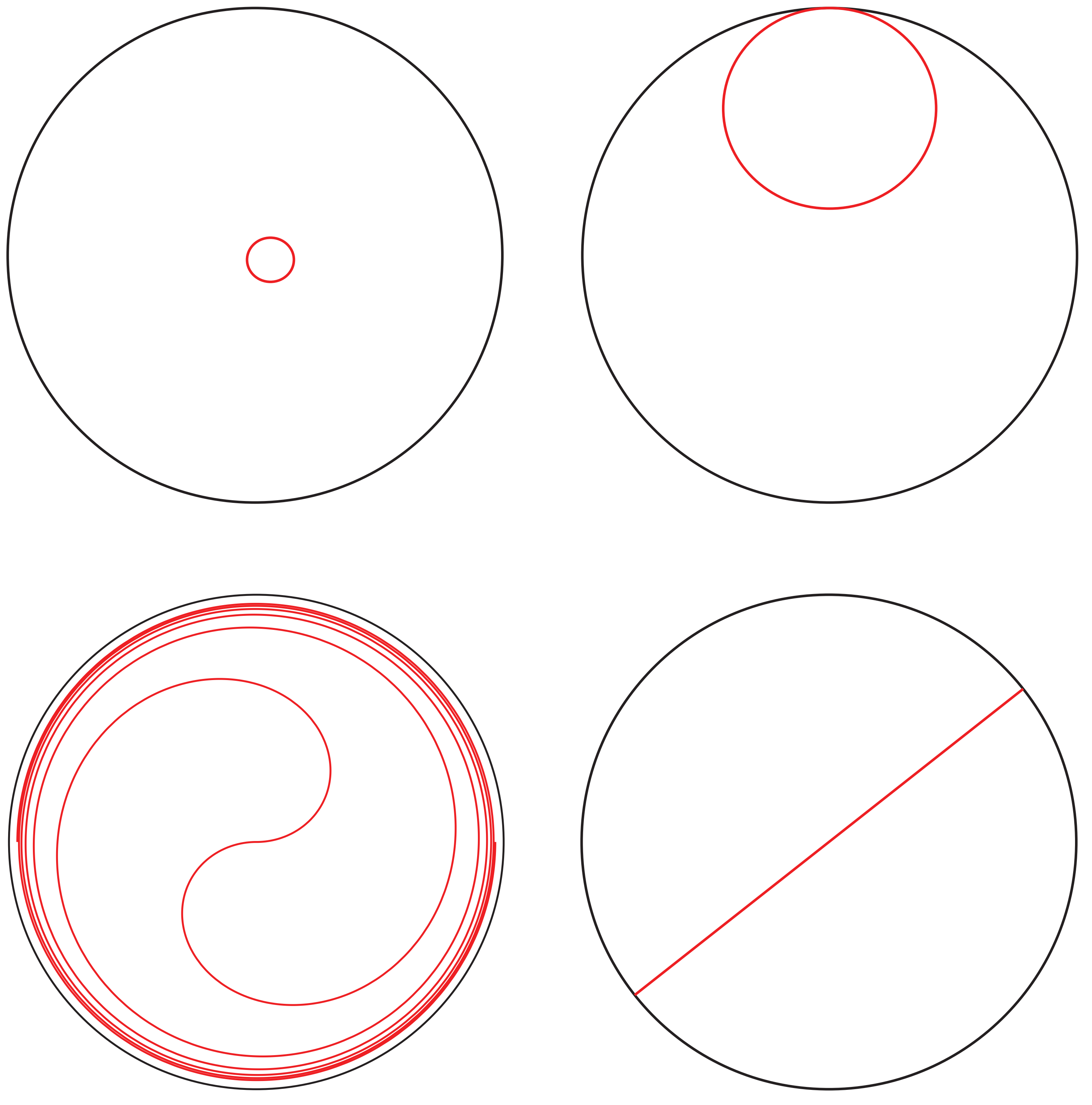


Cannon-Thurston maps



[Cheating a bit]

Cannon-Thurston maps



Cannon-Thurston maps

Let M be a hyperbolic surface bundle and $F \subset M$ be a fibre. Any elevation $\tilde{F} \subset \tilde{M}$ of F is a properly embedded disk.

Theorem (Cannon-Thurston): The inclusion $\alpha : F \to M$ induces a Cannon-Thurston map. (And they deduce that $\partial\alpha$ is sphere-filling.*)

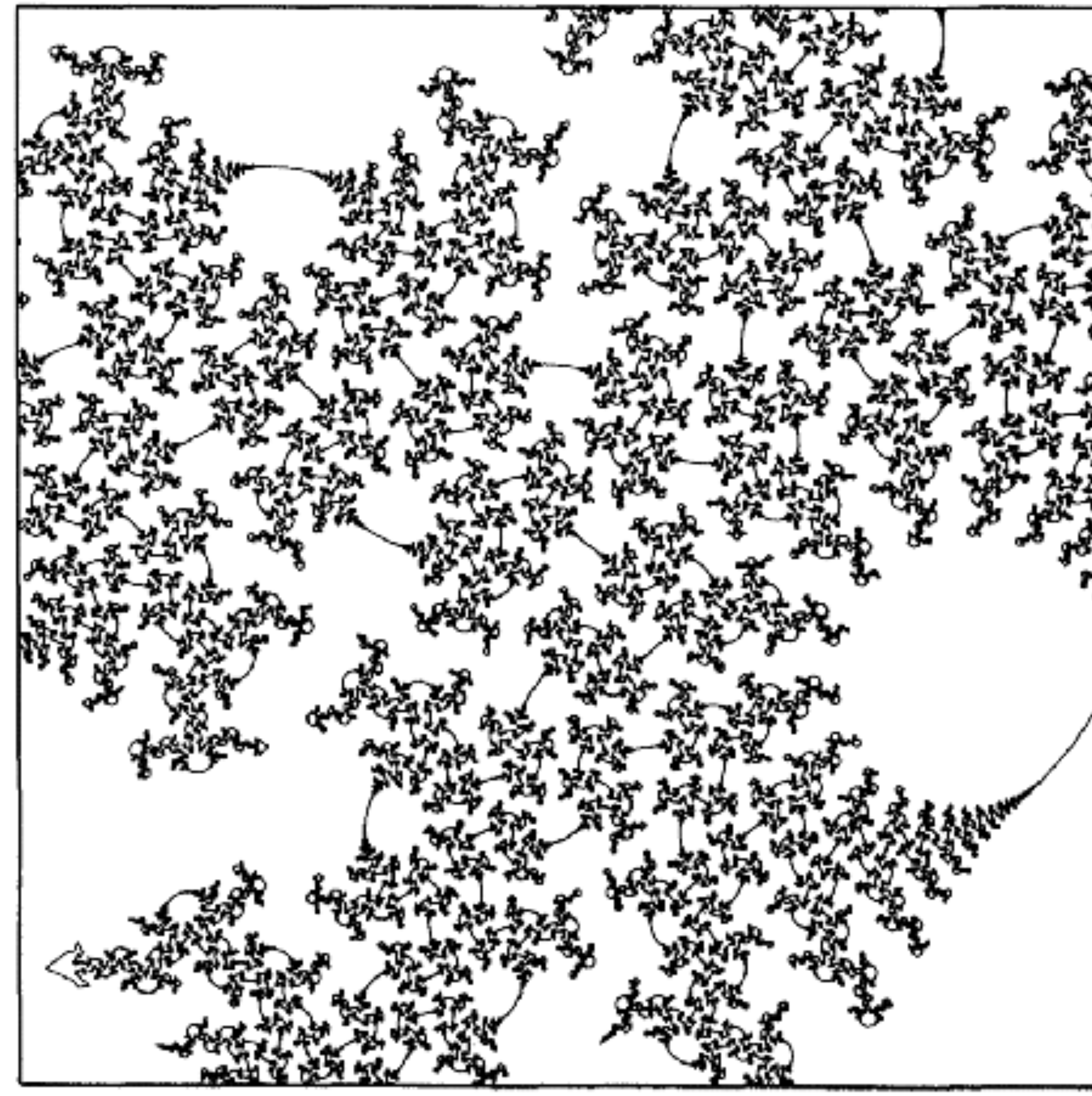
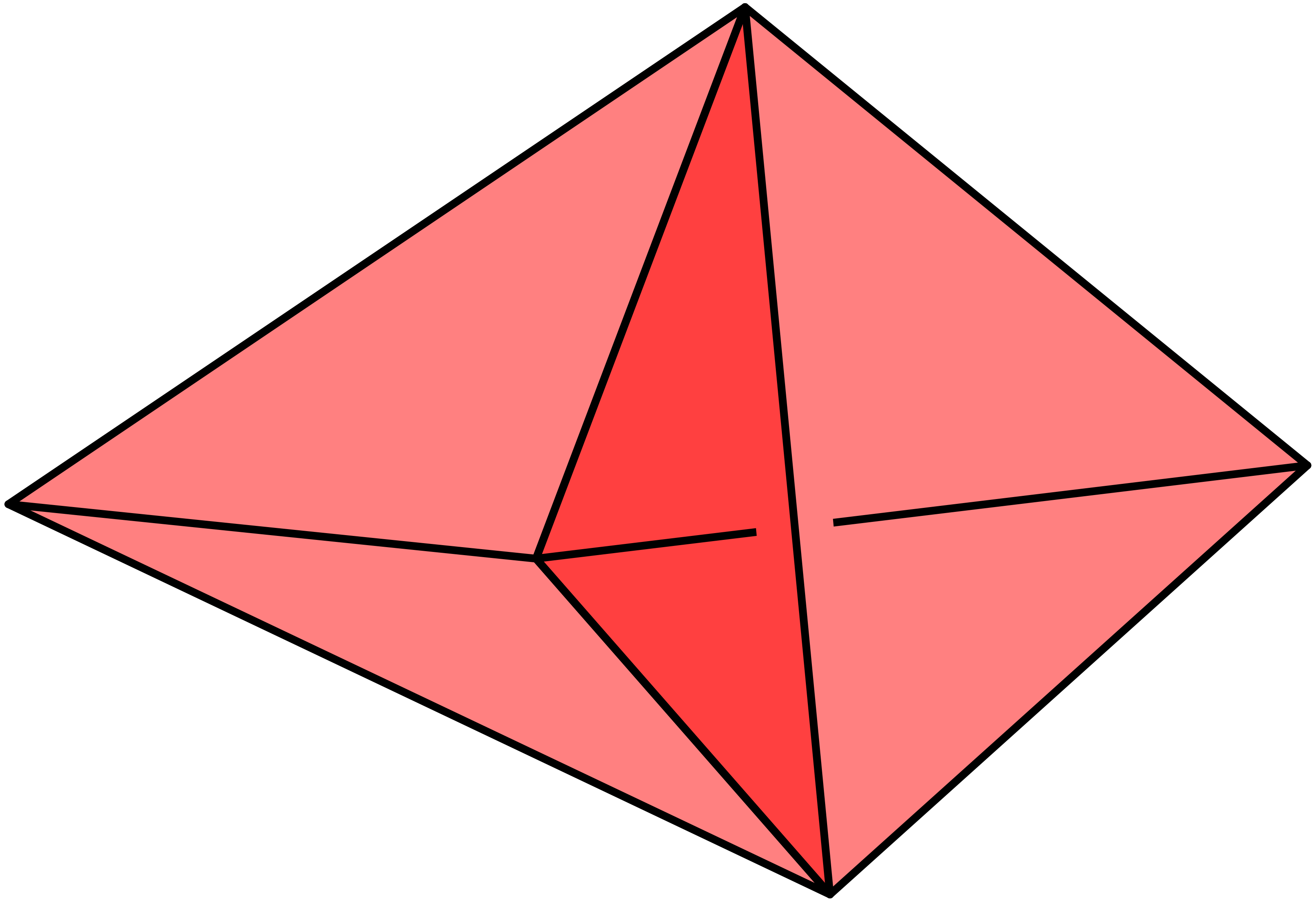


Figure by Bill Thurston, from *Three dimensional manifolds, kleinian groups and hyperbolic geometry*

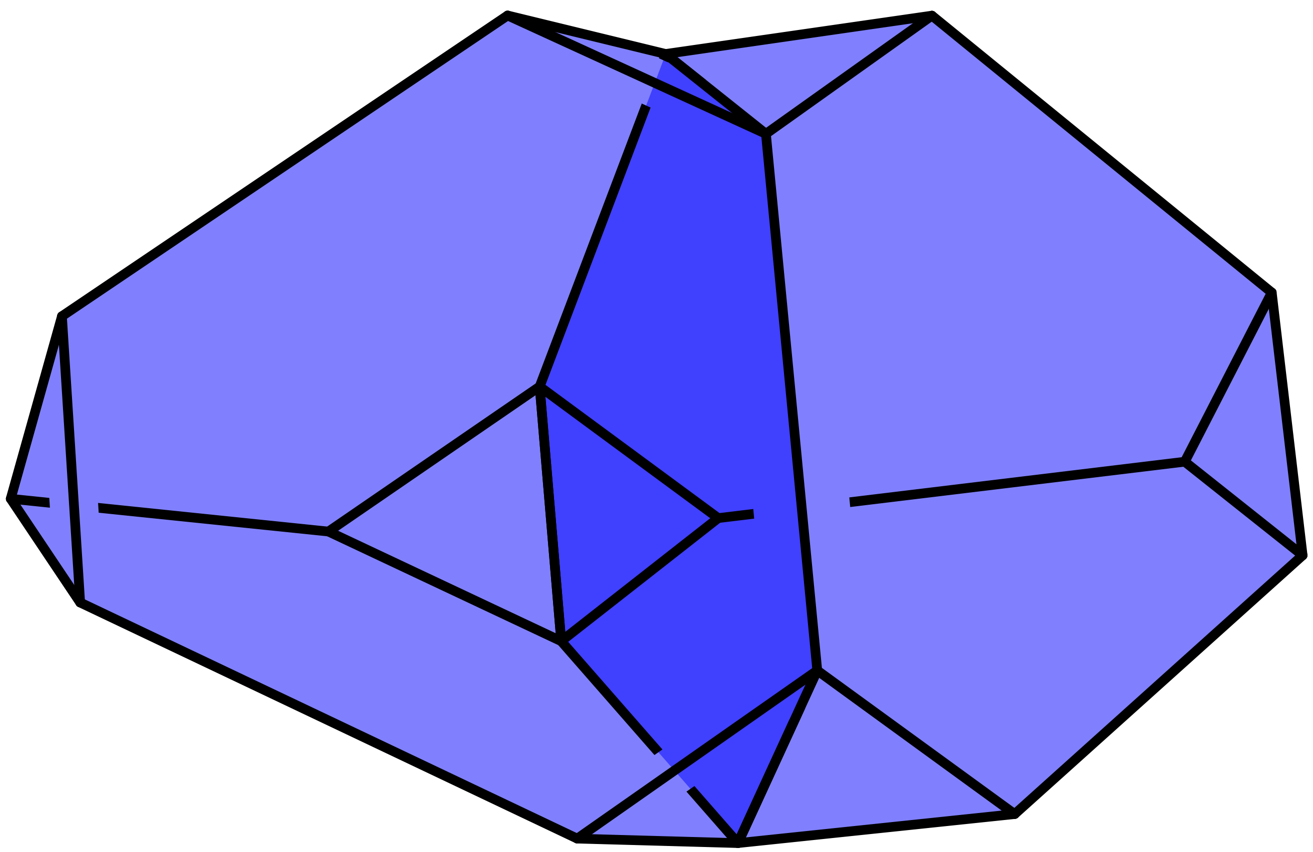
Theorem (Manning-S-Segerman): A veering structure induces a Cannon-Thurston map (even when the manifold is non-fibered).

*Many cases dealt with by many authors: Cannon-Thurston, Minsky, Alperin-Dicks-Porti, McMullen, Bowditch, Mitra.

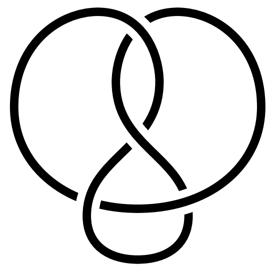
Triangulations

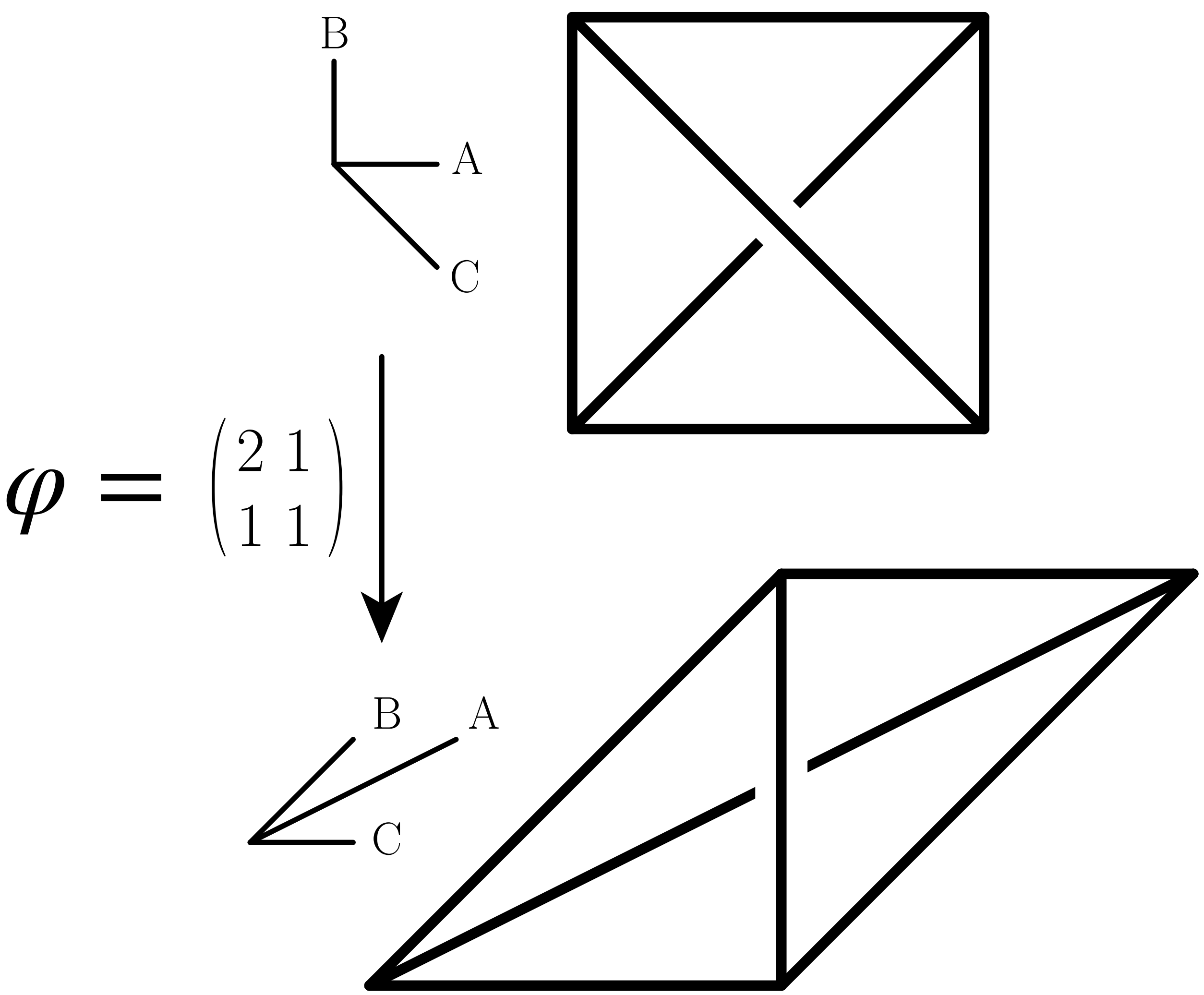


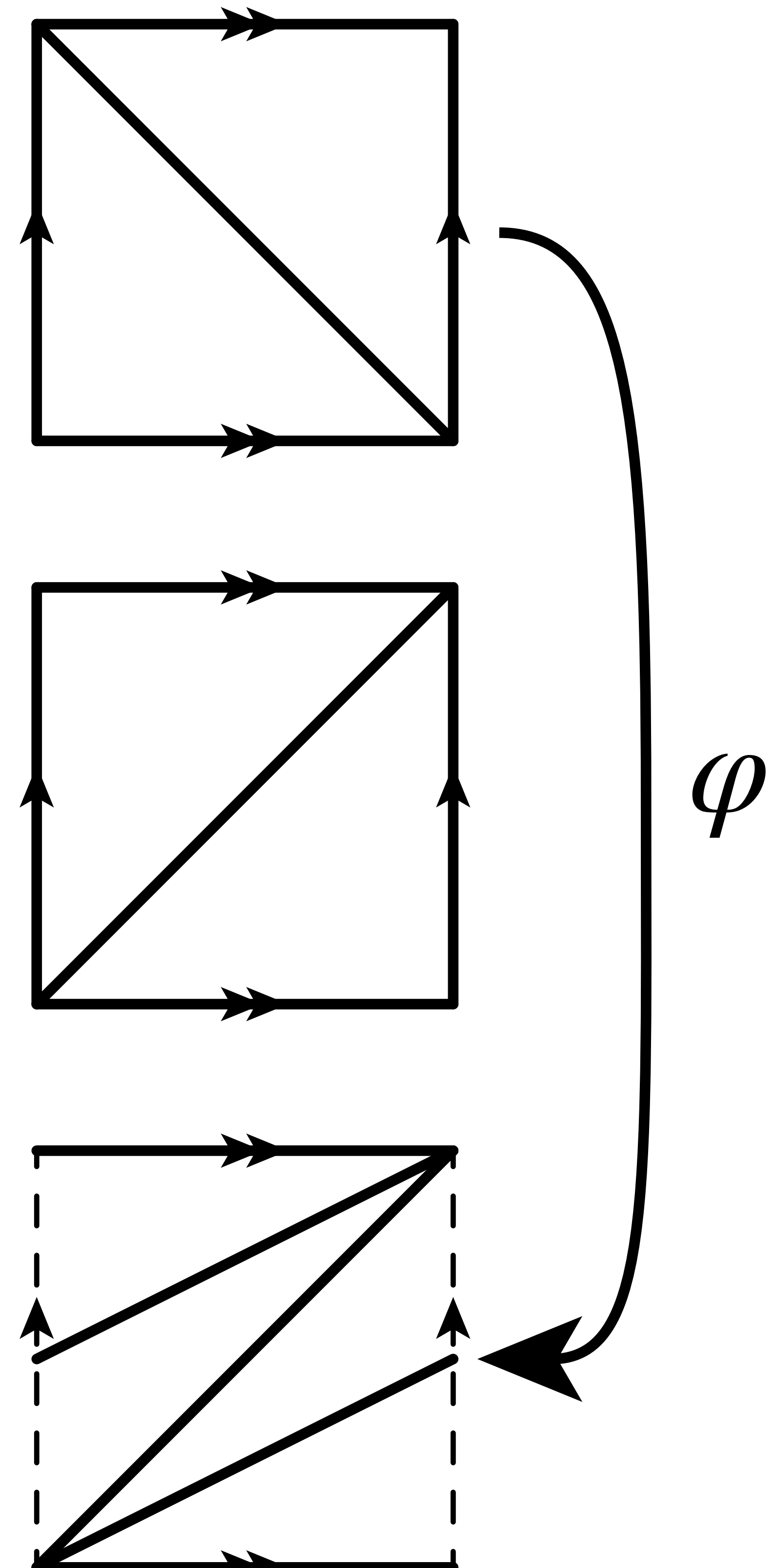
Ideal Triangulations (Thurston)



Example: the figure 8 knot complement

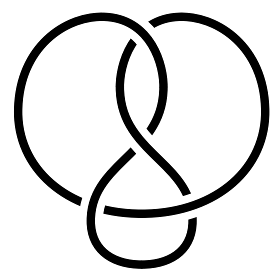
$M = S^3 -$ 

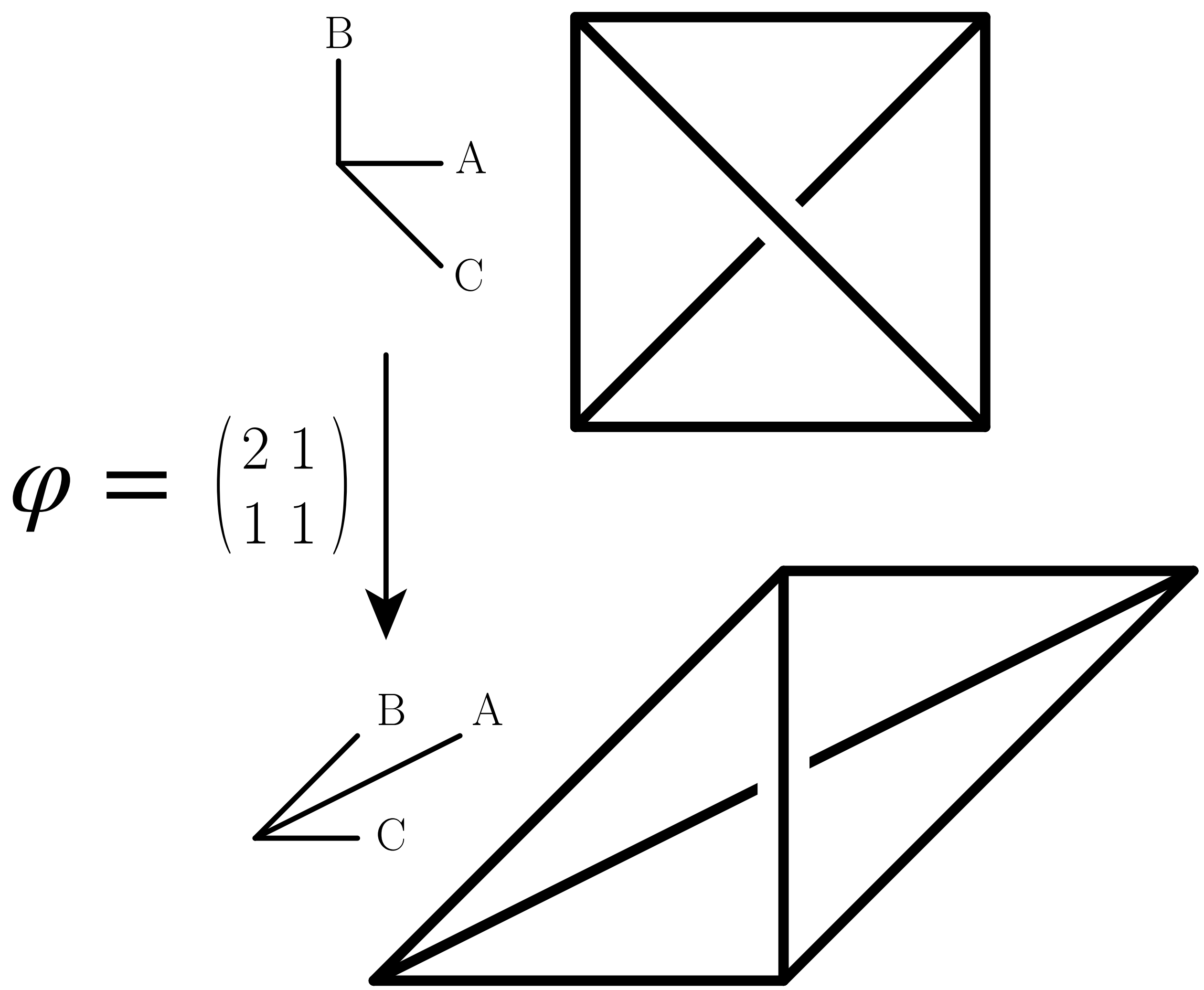


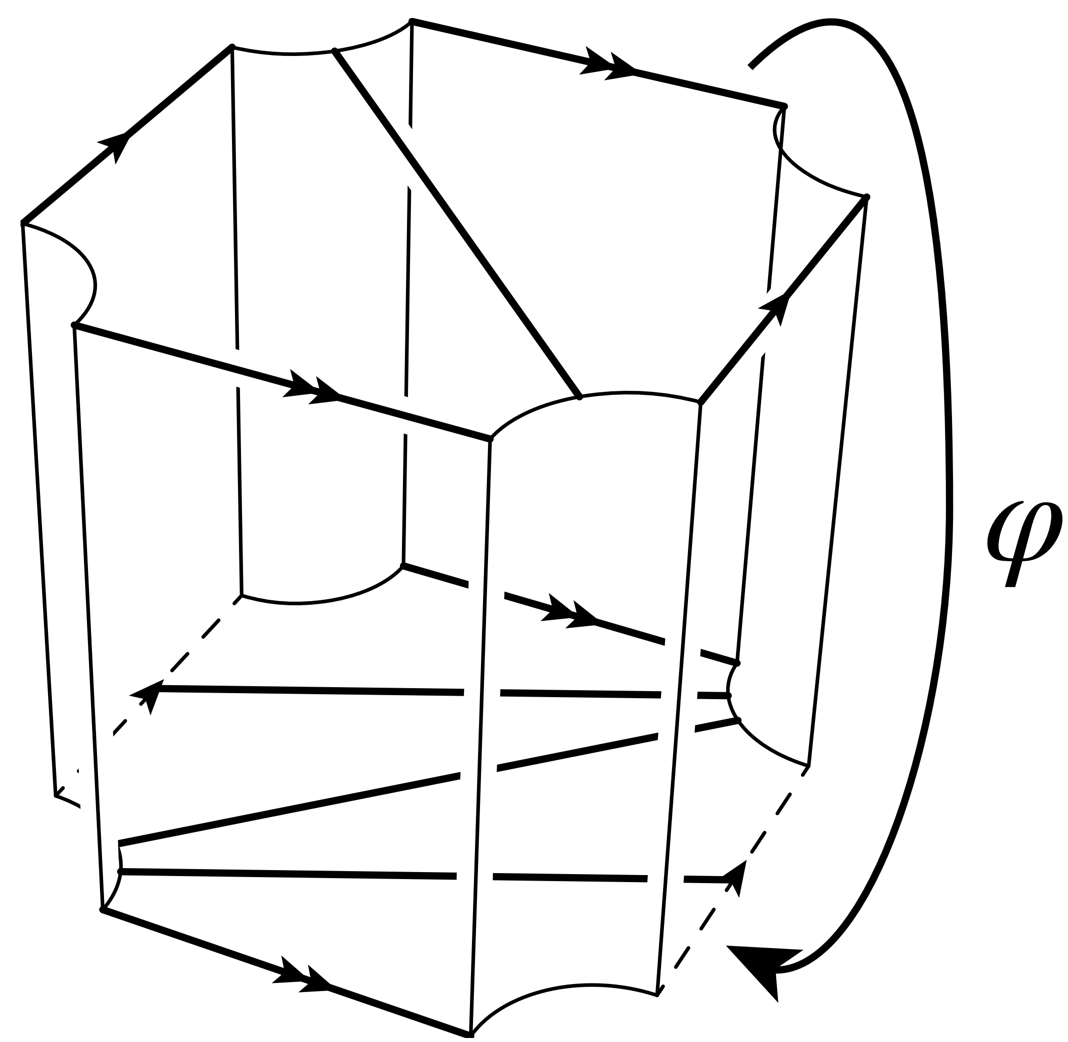


This is a *layered* triangulation

Example: the figure 8 knot complement

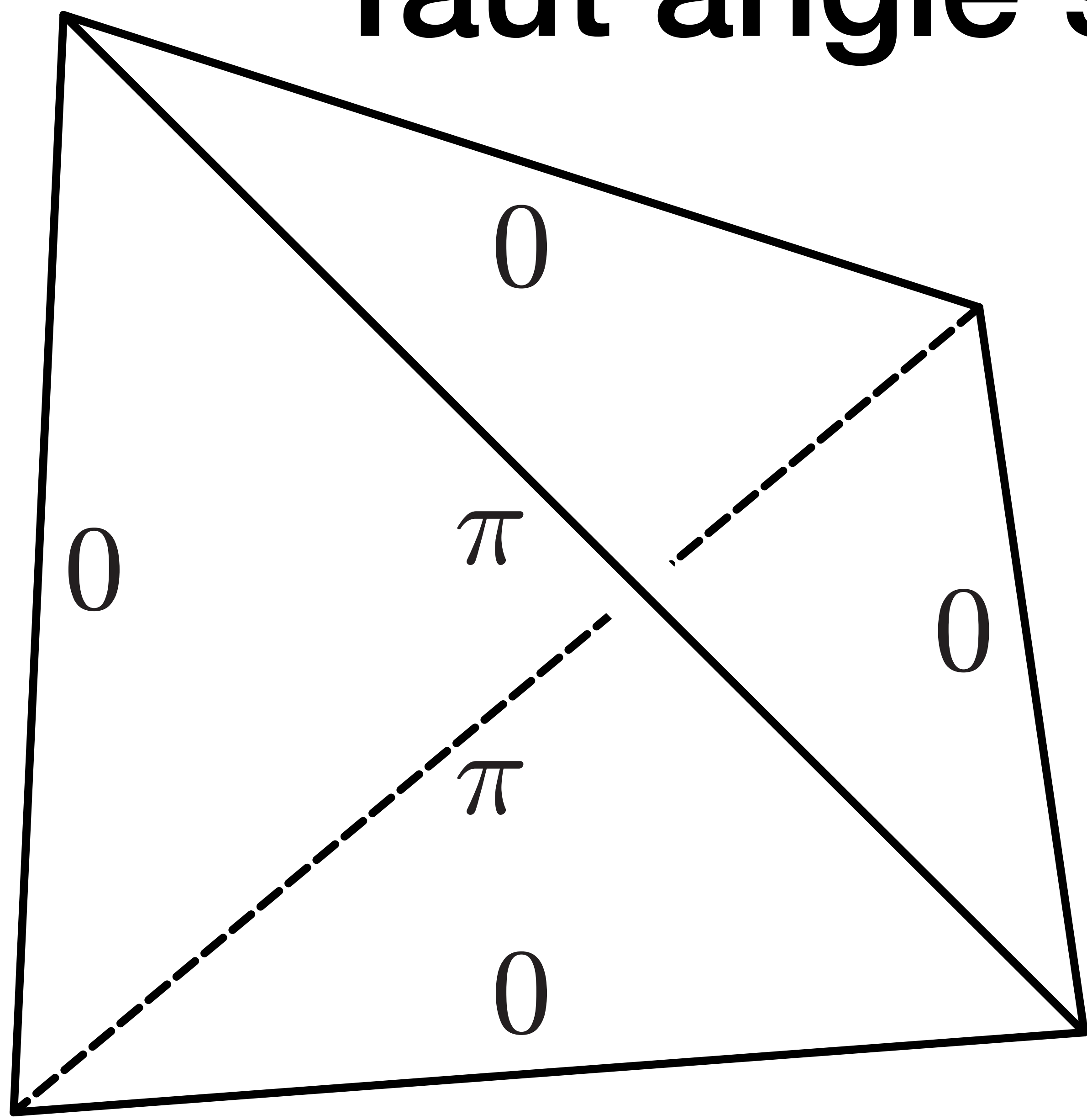
$M = S^3 -$ 





This is a *layered* triangulation
of the *surface bundle* $M \cong (T^2_* \times I)/(x,1) \sim (\varphi(x),0)$

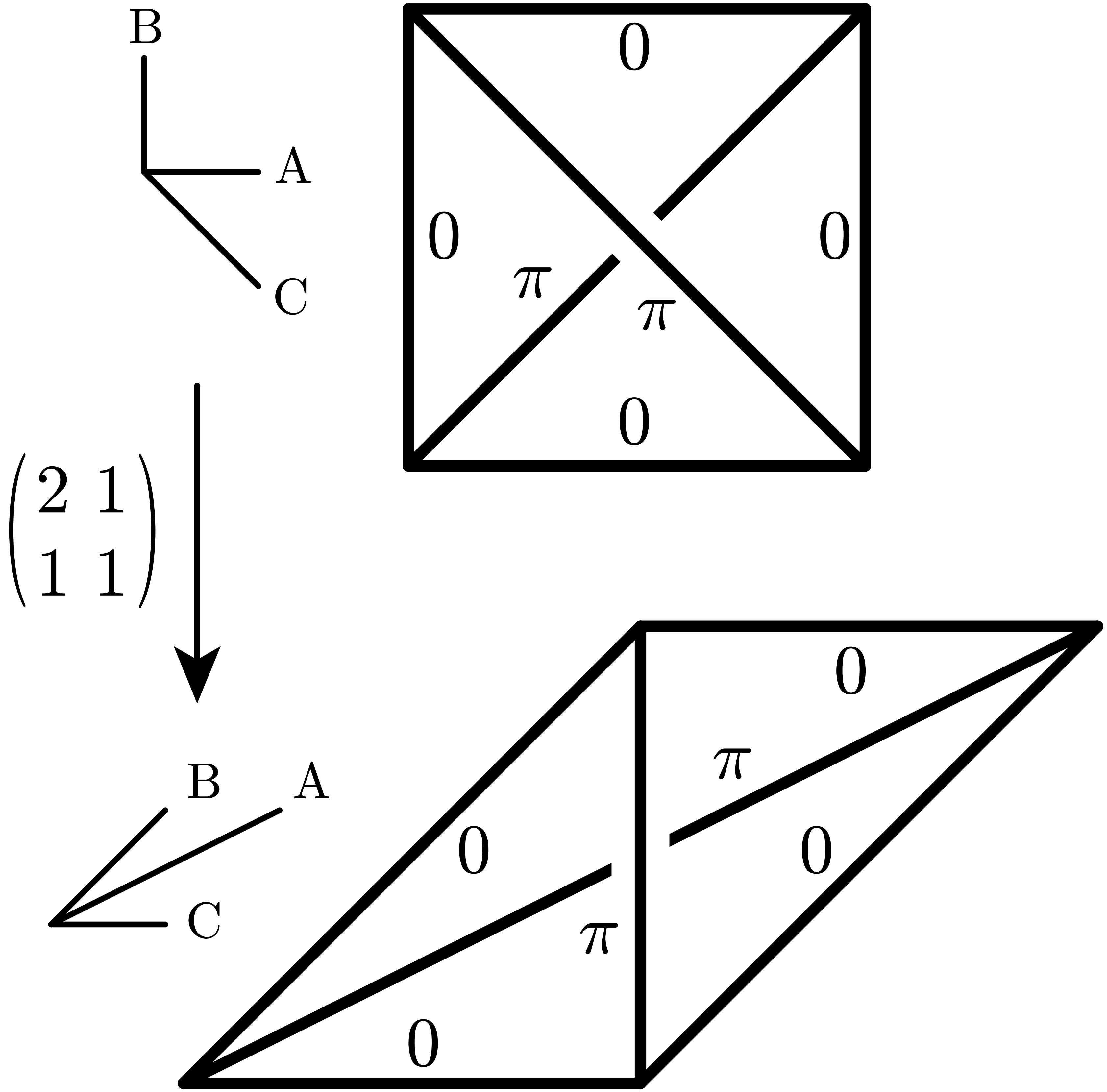
Taut angle structure (Lackenby)



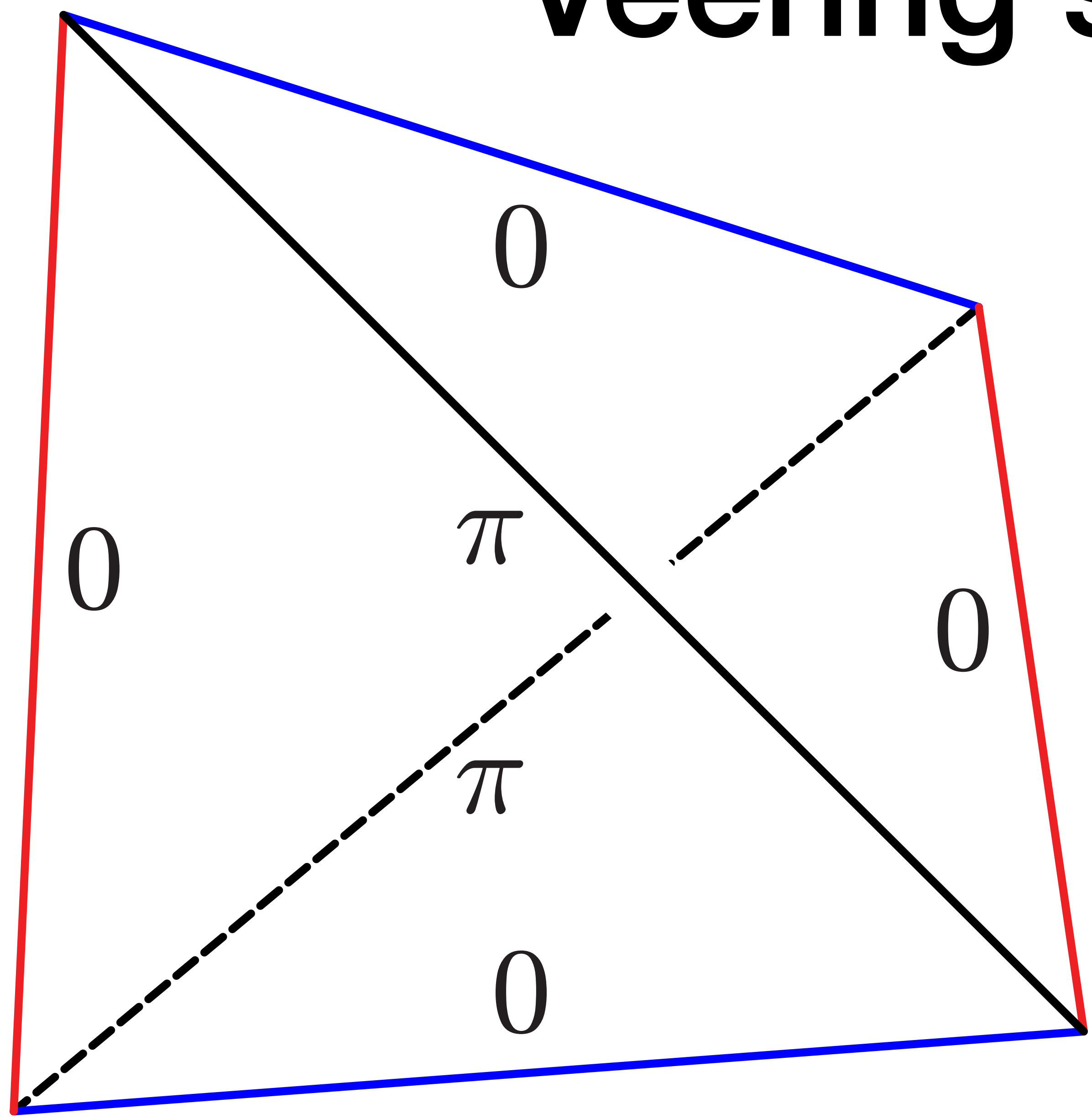
Assign angles 0 and π to the edges of each tetrahedron.

We require that the sum of angles around each edge of the triangulation is 2π.

Example: the figure 8 knot complement

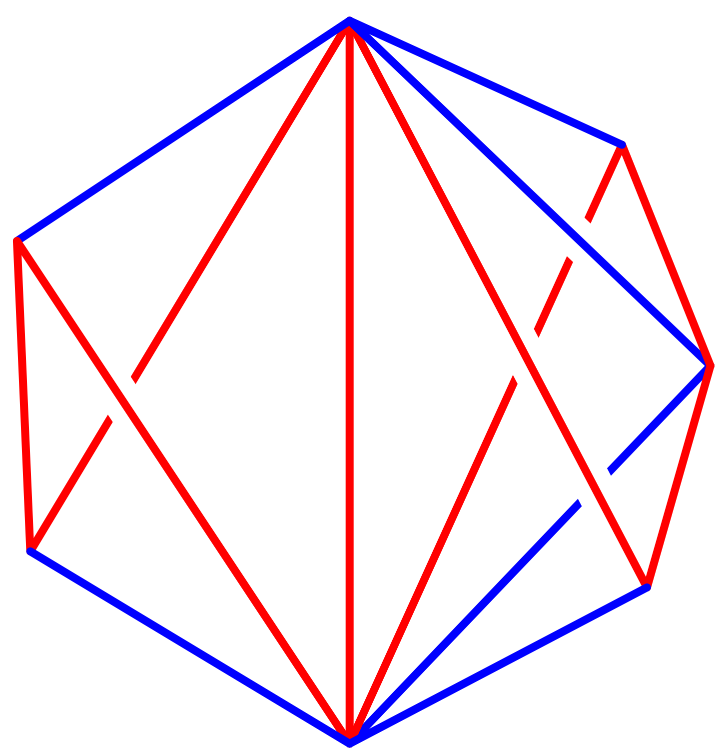


Veering structure (Agol)



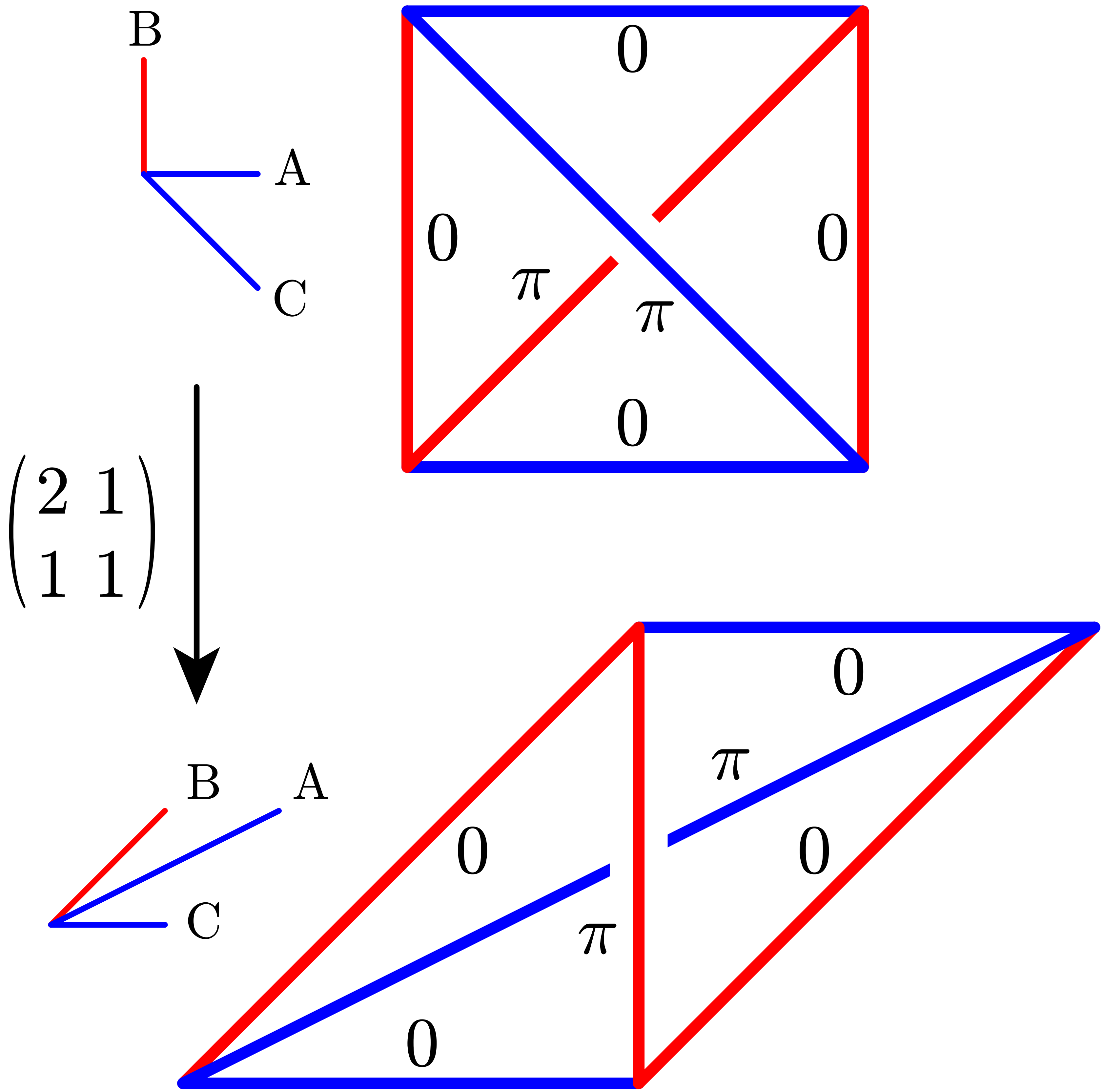
taut angle structure, and:

Each tetrahedron colours its 0 angle edges red or blue.



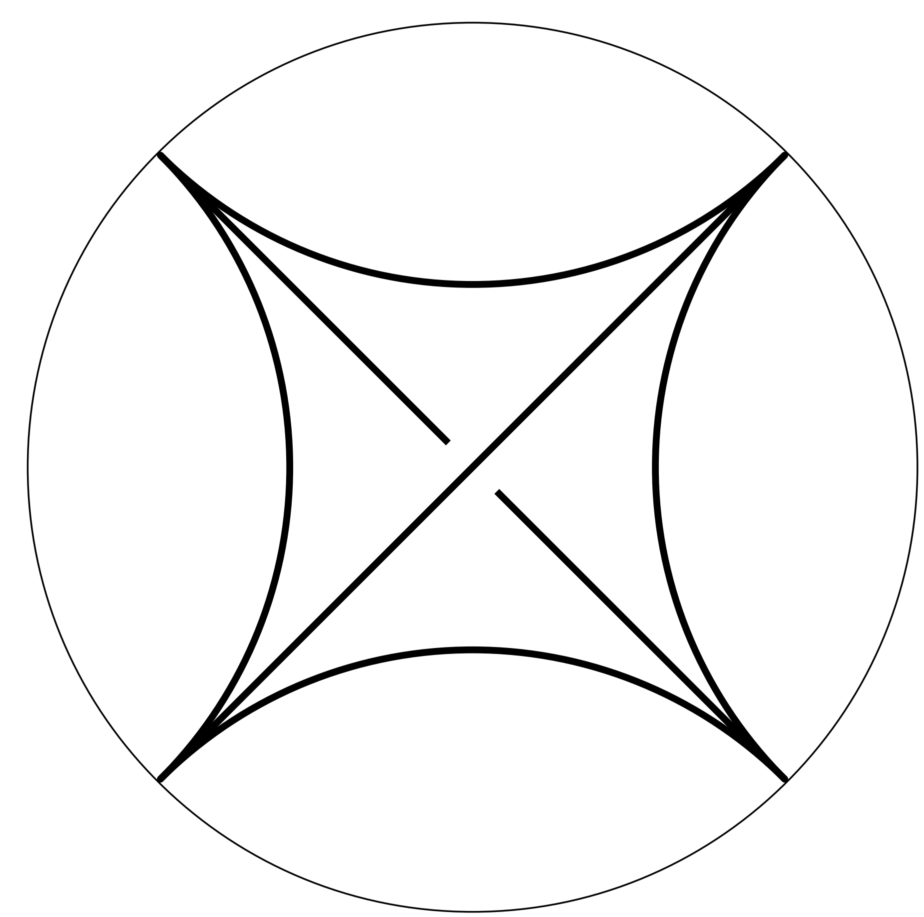
These colours must be consistent for all tetrahedra incident to the edge.

Ex: the figure 8 knot complement



Layers and continents in the universal cover

Taut ideal tetrahedra layer to make larger taut ideal polyhedra: *continents.*



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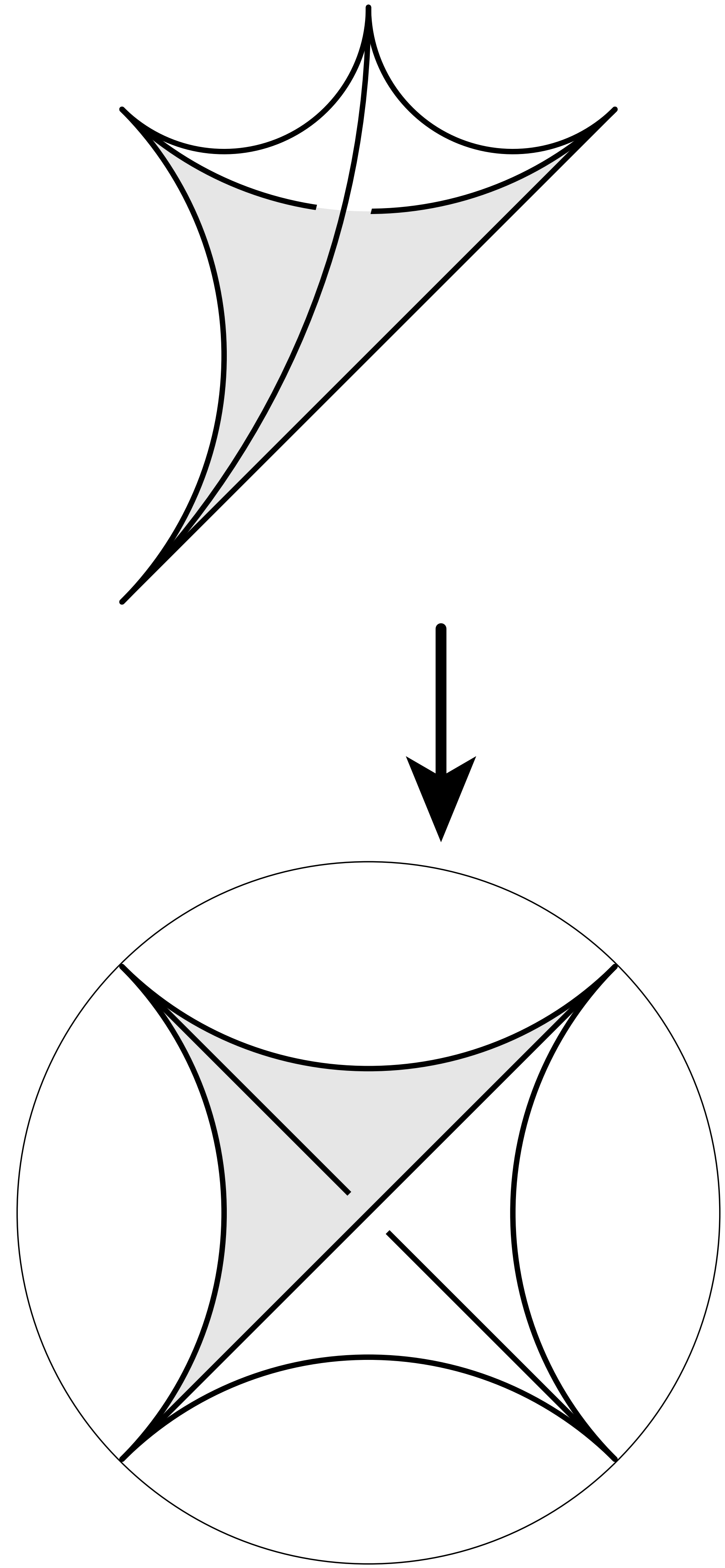

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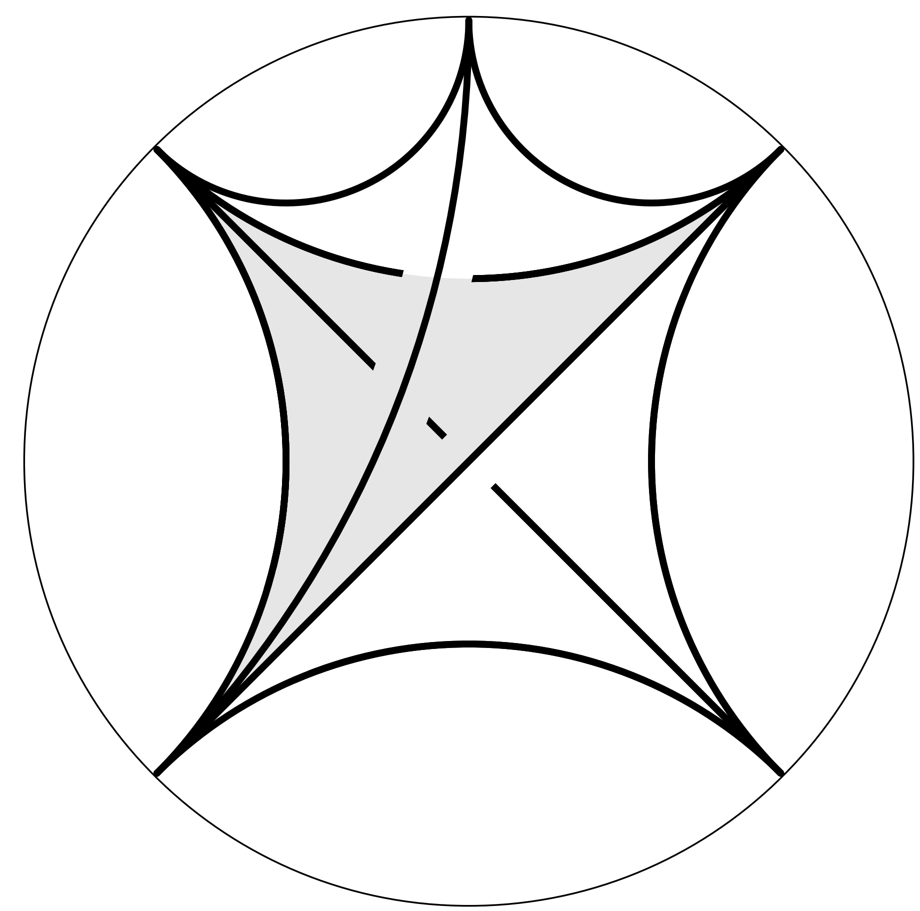

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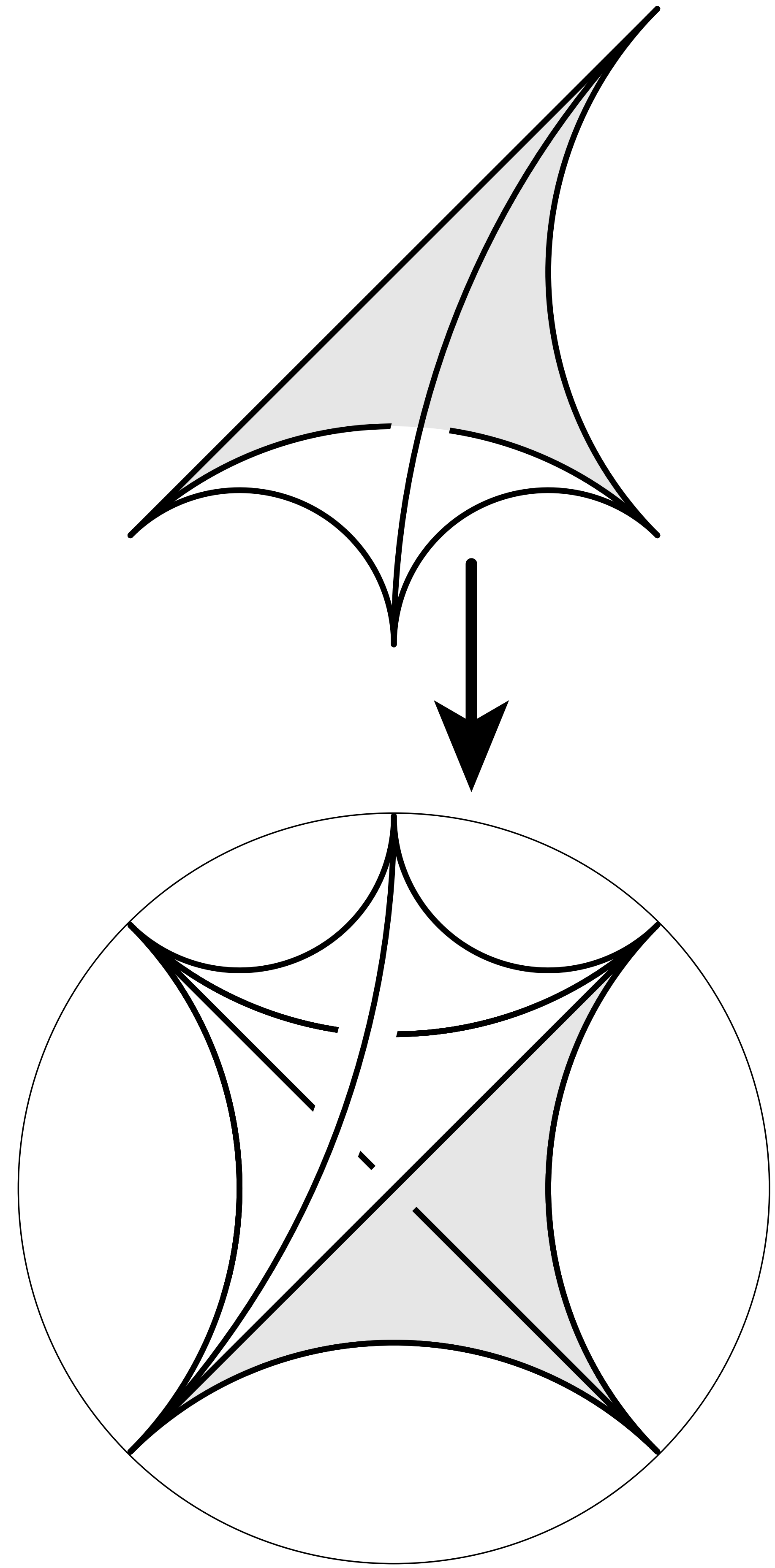

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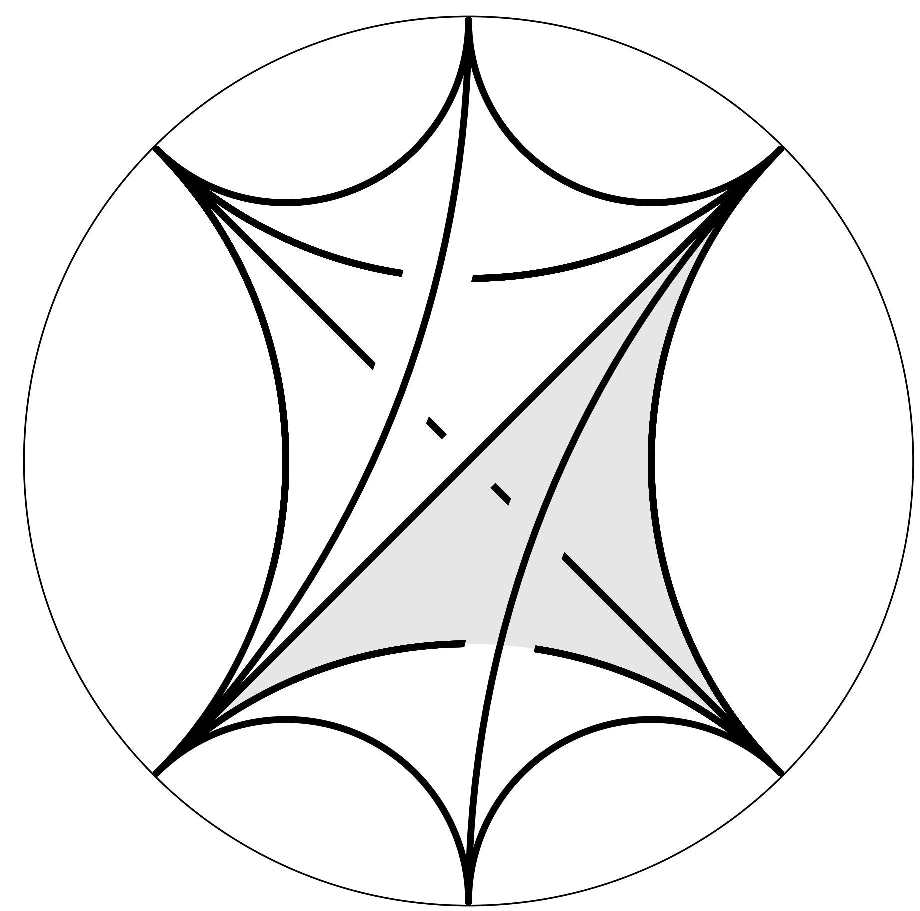

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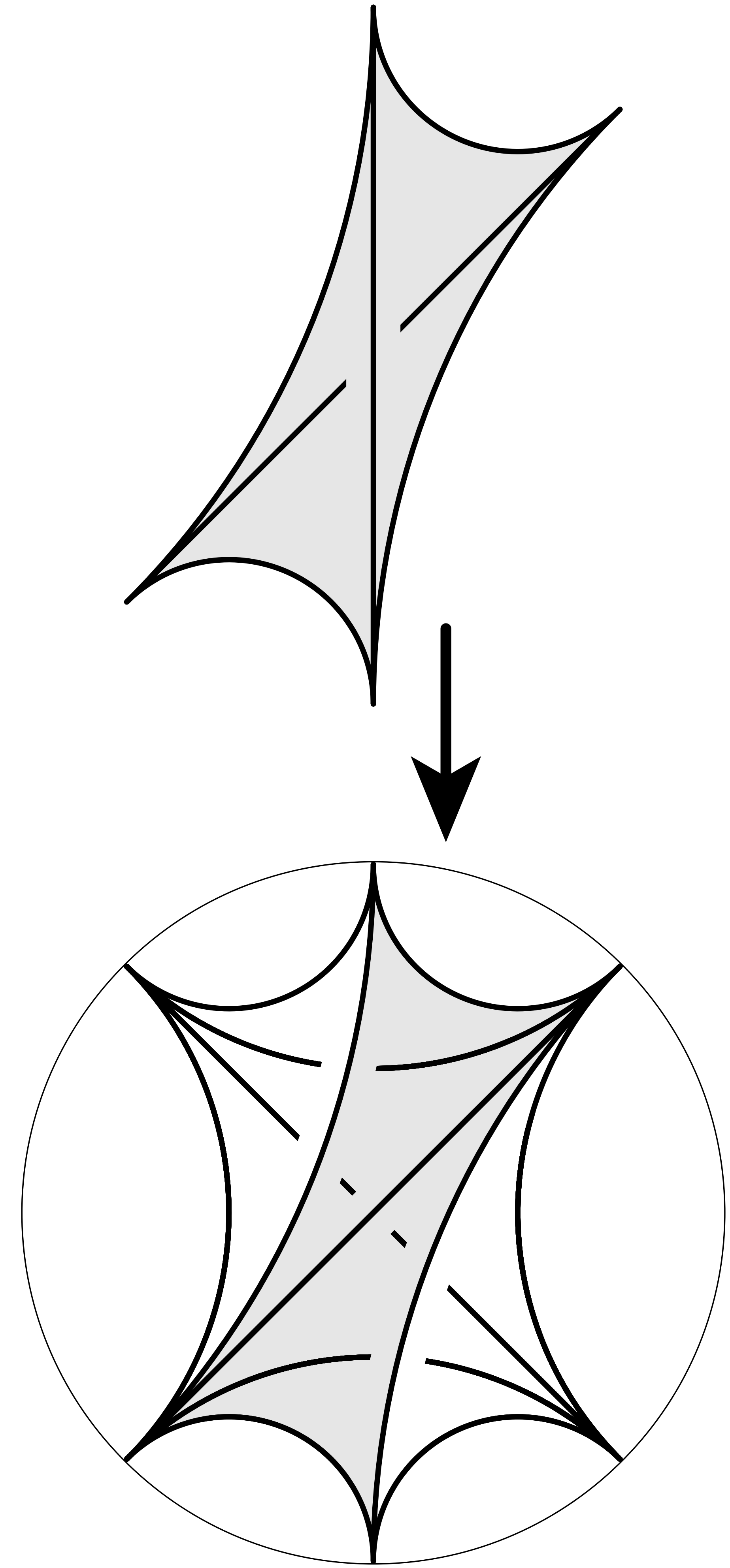

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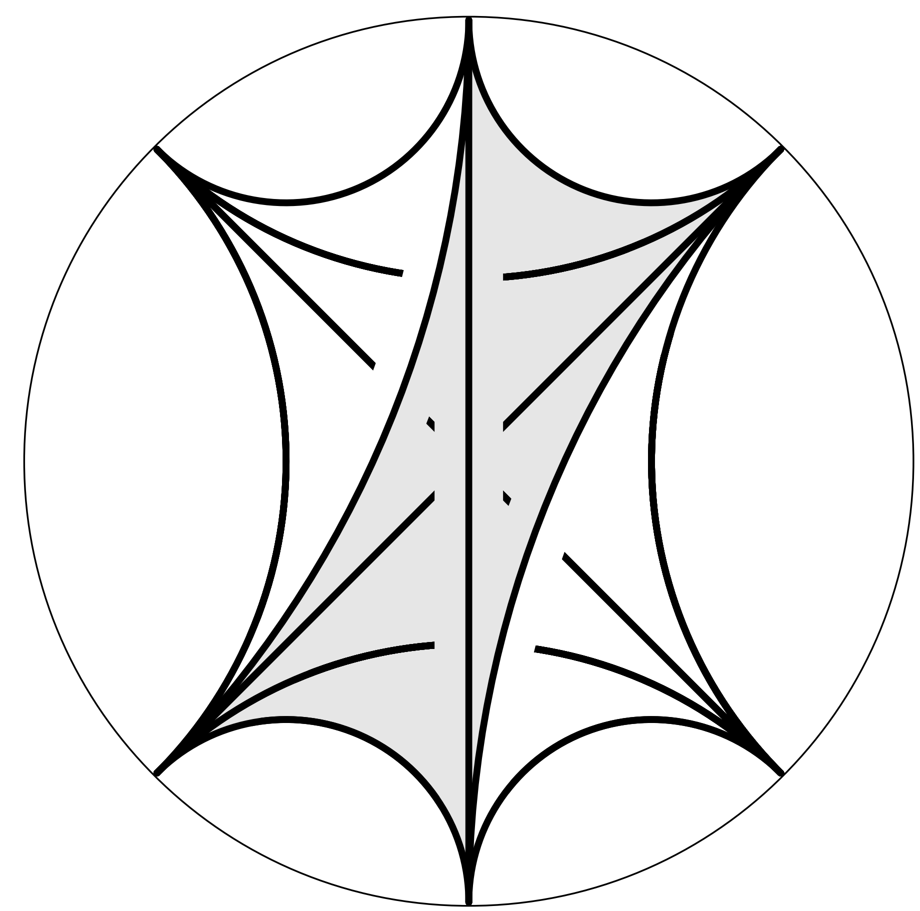

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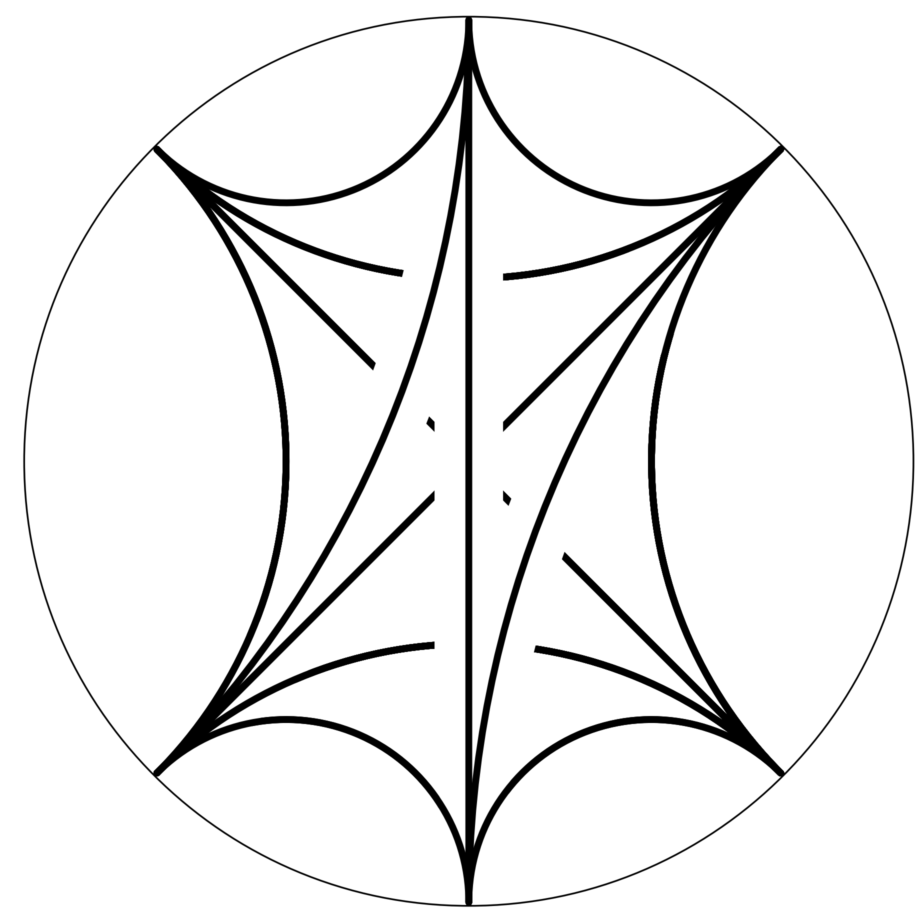
Taut ideal tetrahedra layer to make larger taut ideal polyhedra: *continents.*



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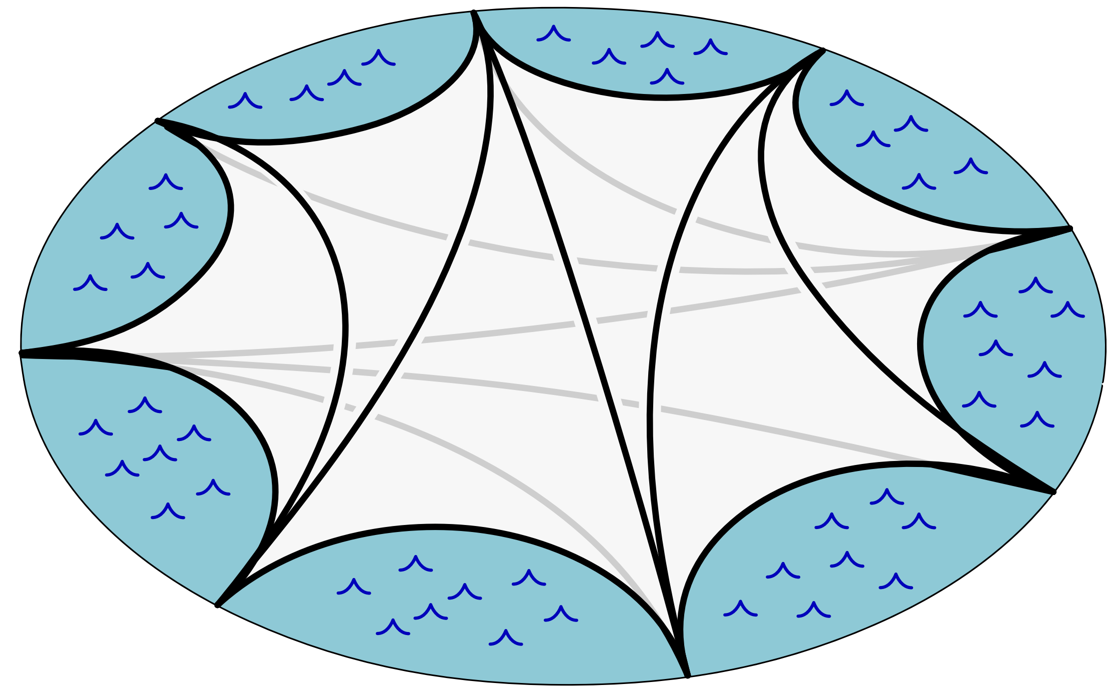


Taut ideal tetrahedra layer to make larger taut ideal polyhedra: *continents.*

This gives a circular order to the vertices of the tetrahedra, *compatible* with the taut angle structure.

Theorem A (S—Segerman): A veering triangulation admits a *unique* compatible circular order on the vertices of the universal cover.

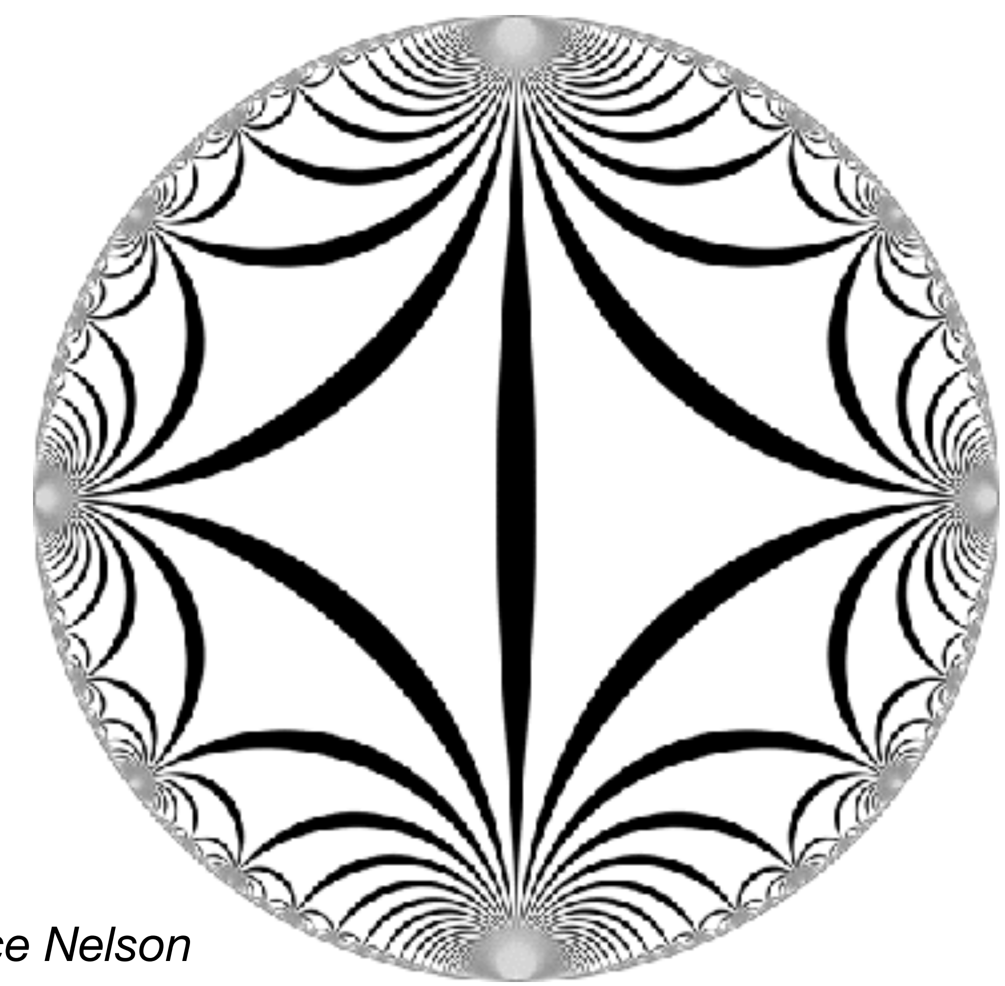


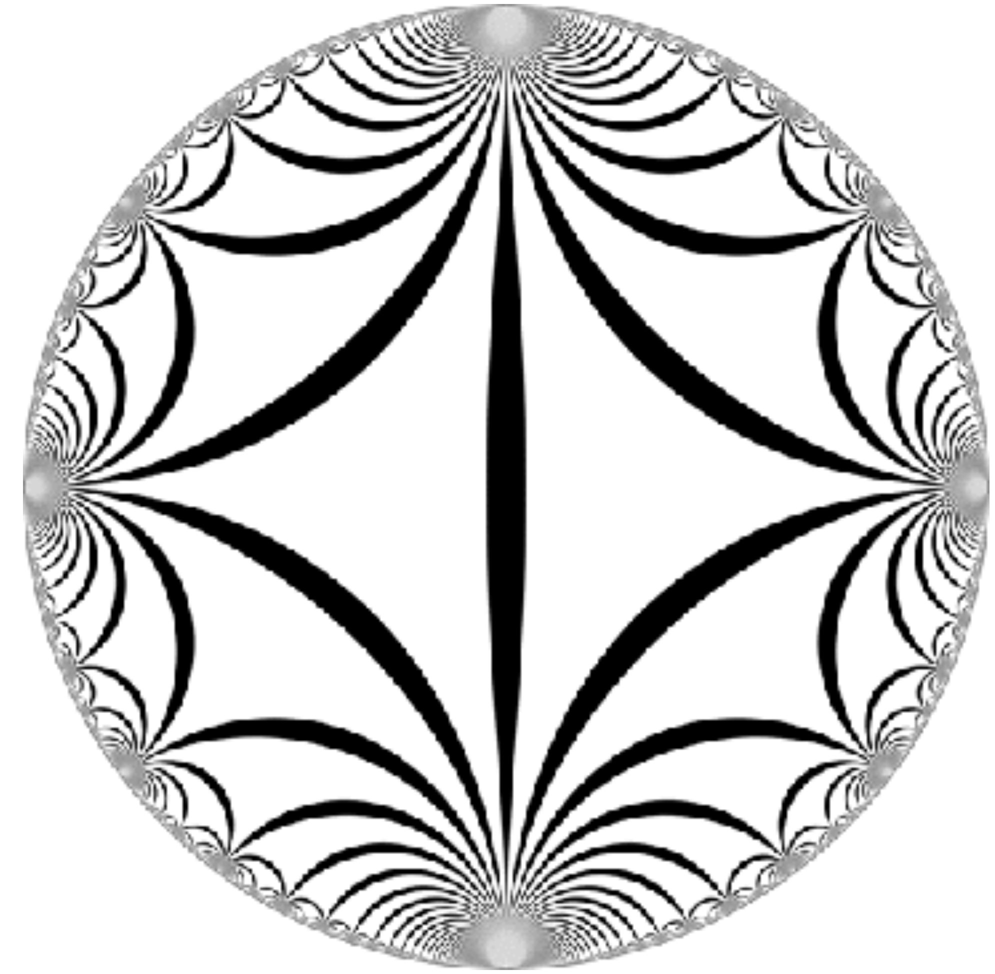
Figure by Roice Nelson

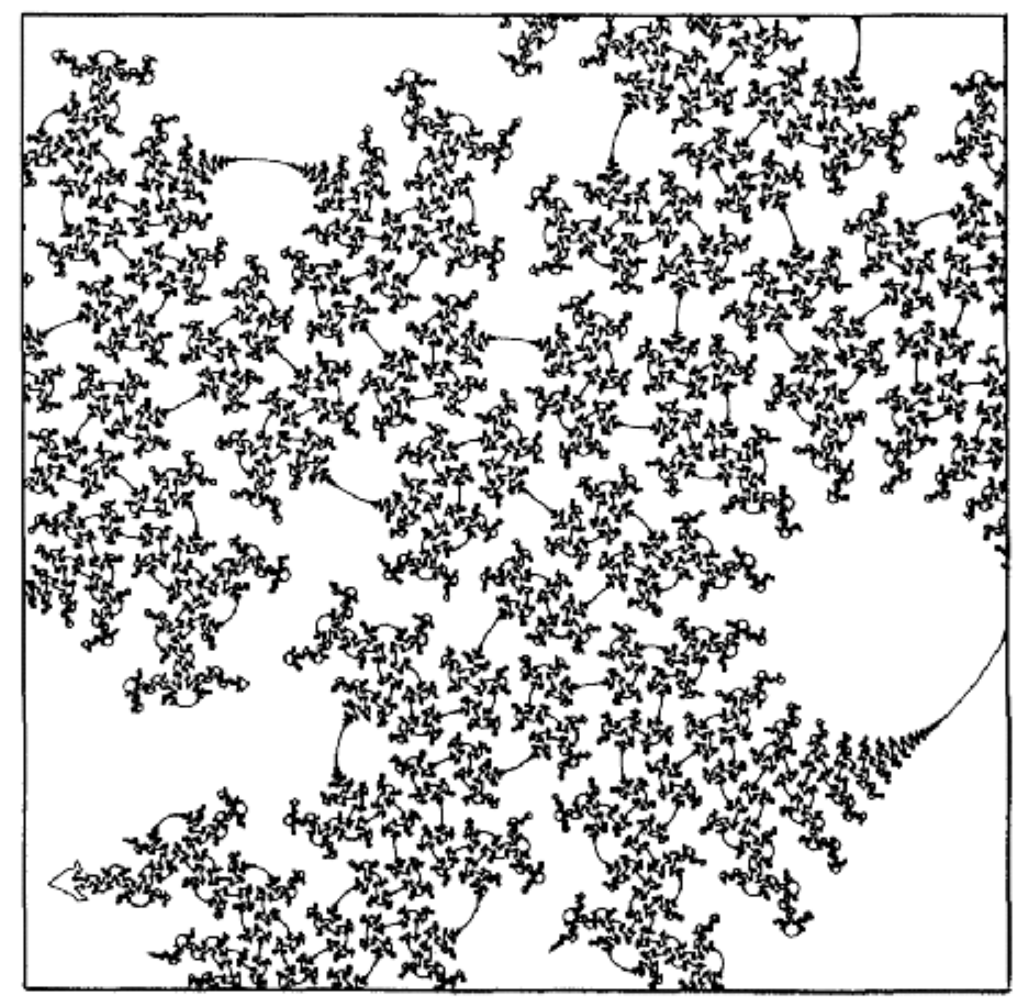
Theorem A (S—Segerman): A veering triangulation admits a *unique* compatible circular order on the vertices of the universal cover.

We complete this circular order to the *veering circle.*

The veering circle is our equivalent to the boundary of the elevation $\partial\tilde{F} \hookrightarrow \partial\tilde{M} \cong \partial\mathbb{H}^3 \cong S^2$ in the usual Cannon-Thurston map.

Theorem B (Manning-S-Segerman): There is a continuous equivariant sphere-filling map from the veering circle to $\partial\tilde{M} \cong \partial\mathbb{H}^3 \cong S^2$.

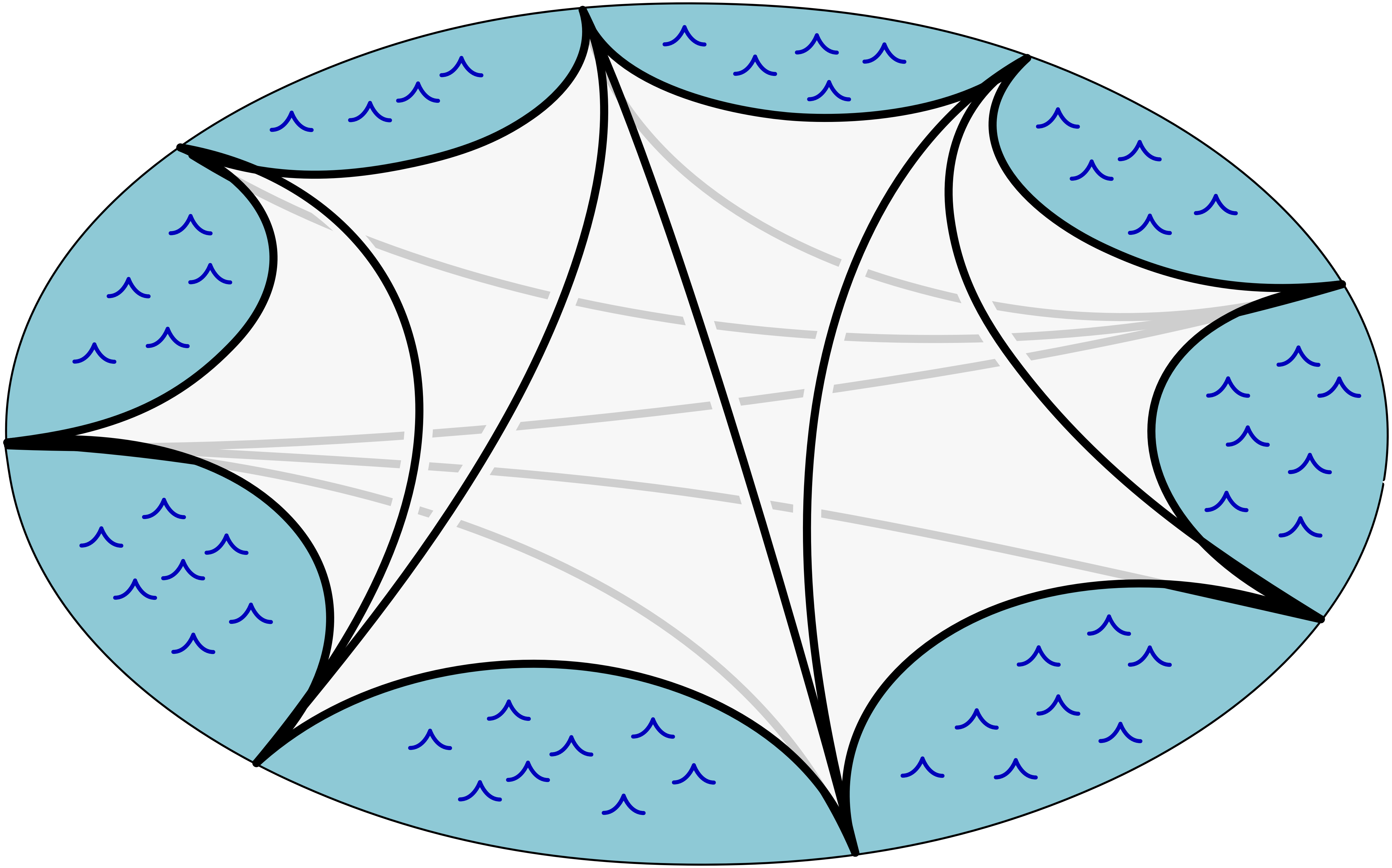


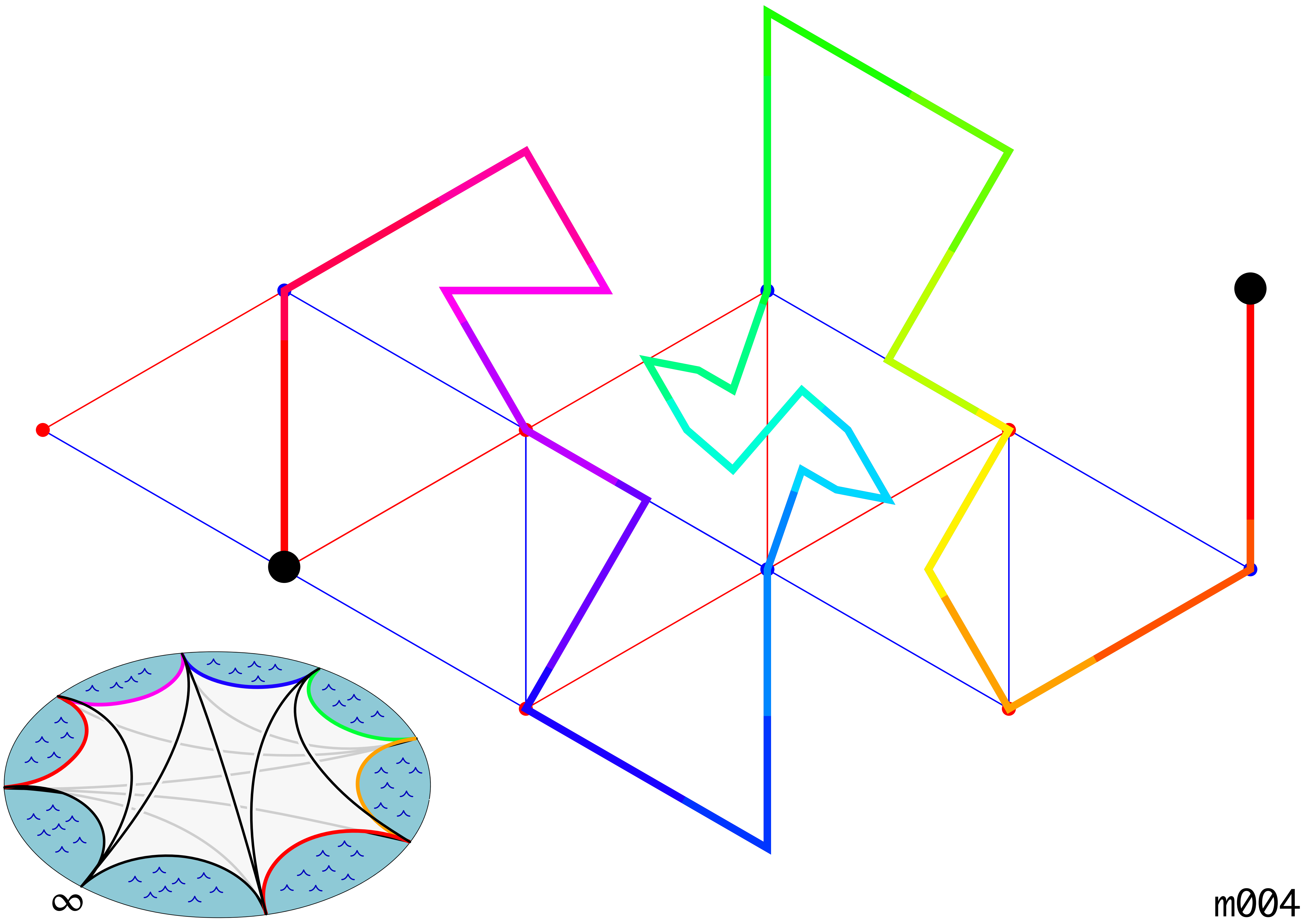


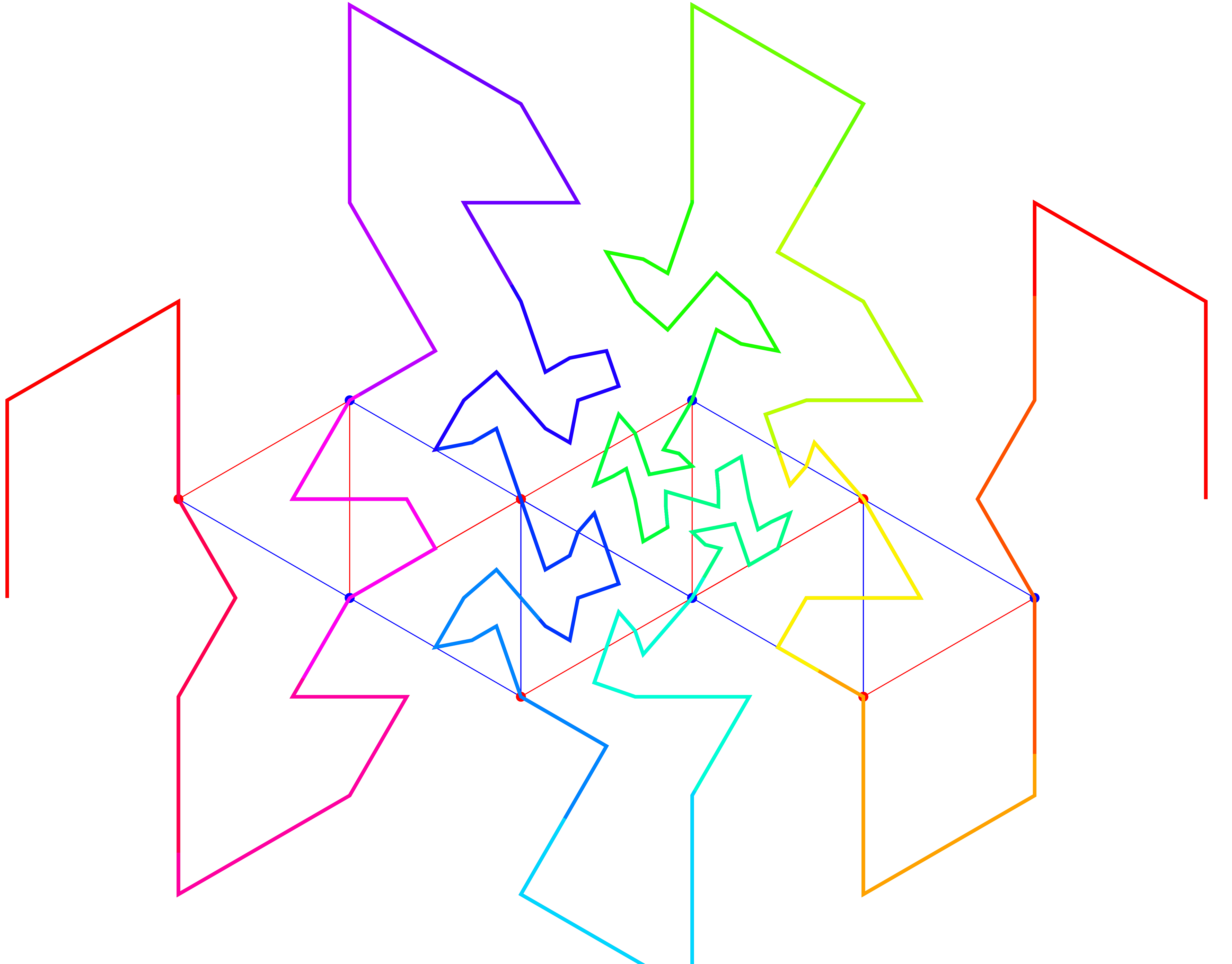
Before saying something about the proof, we can draw some approximations to our Cannon-Thurston maps.

Thurston's original approximation is made from a large disk $D \subset \tilde{F}$: draw ∂D in place of $\partial \tilde{F}$.

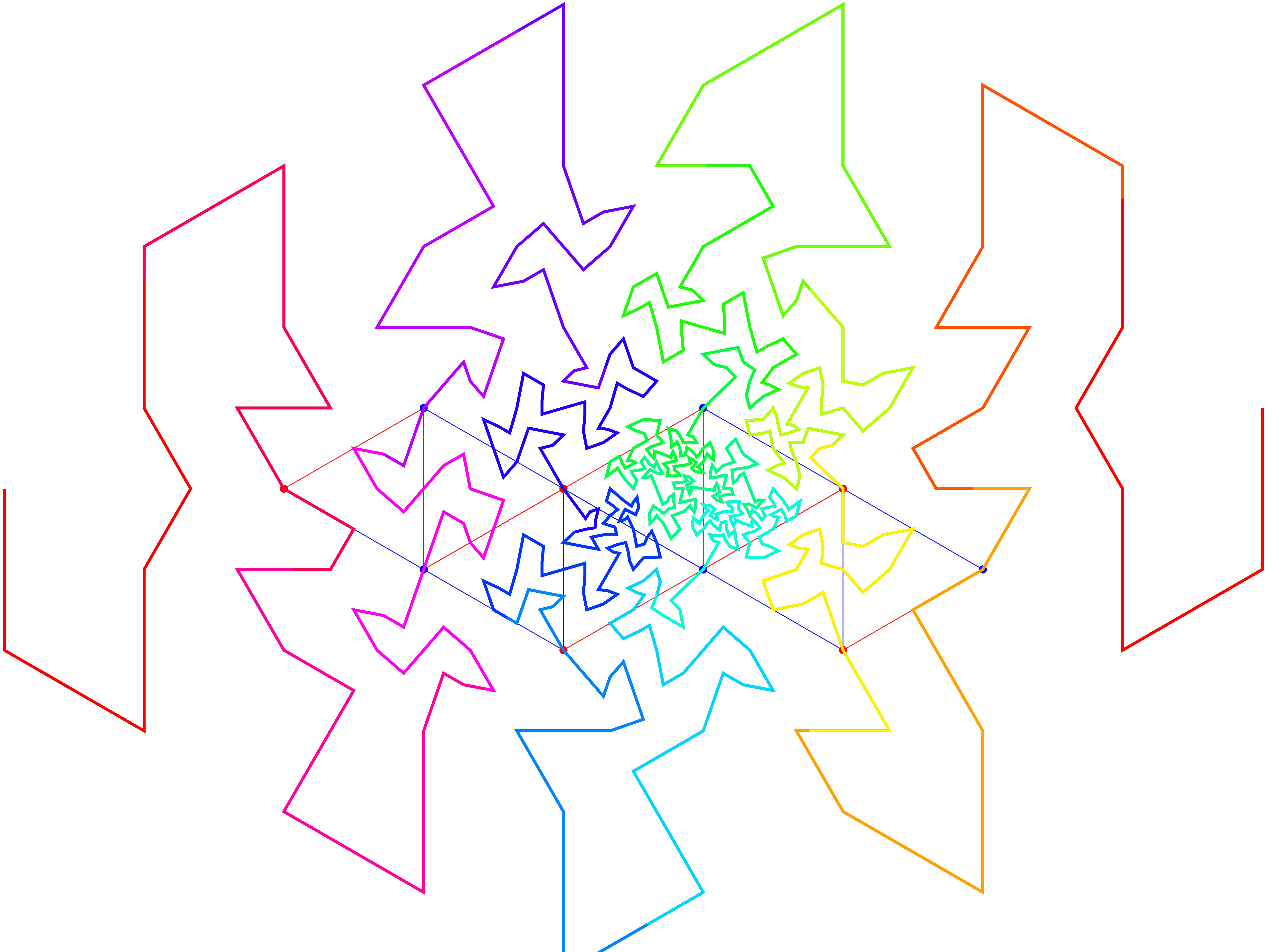
In place of a large disk D, we use a large continent.




∞
m004



m004



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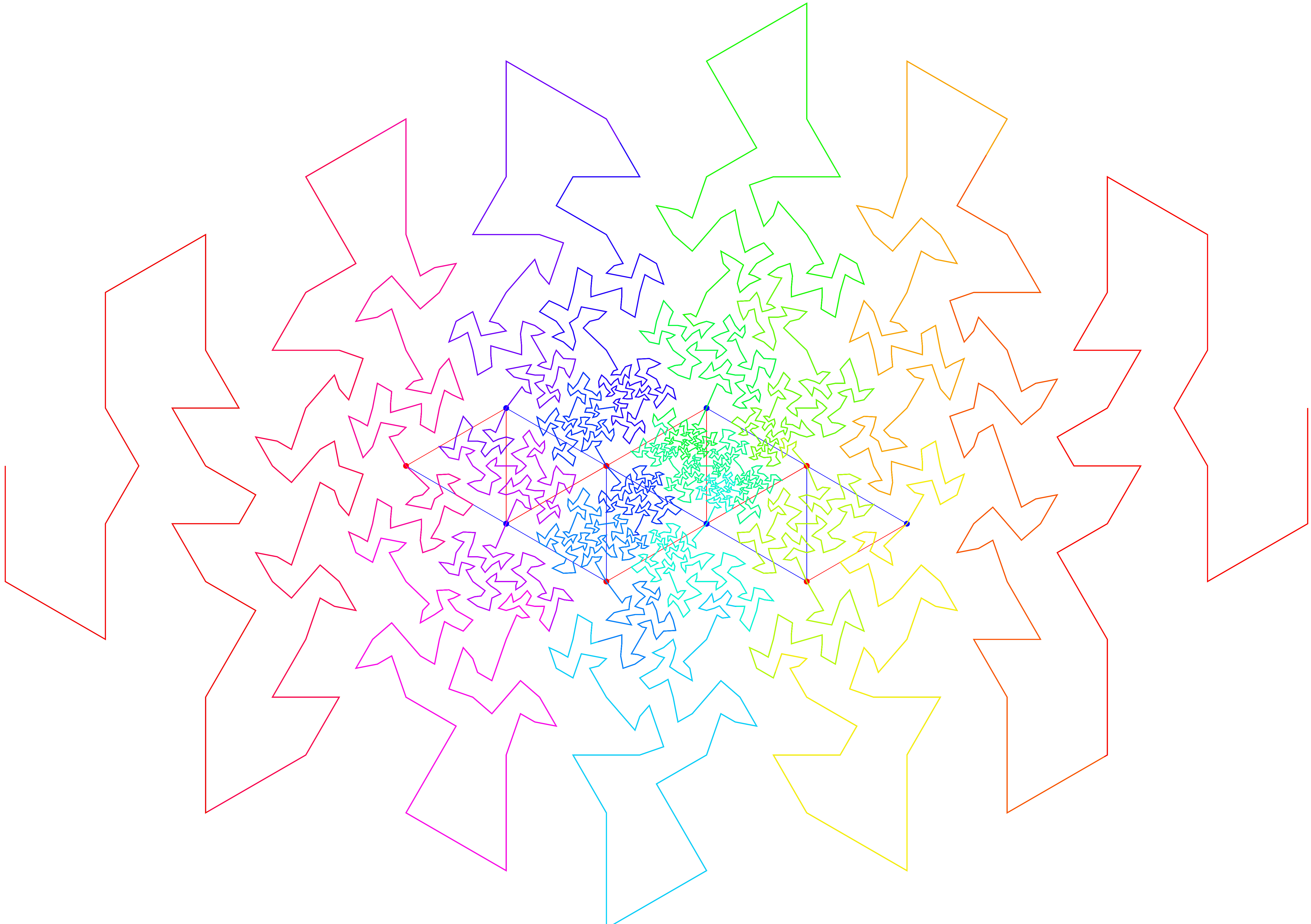

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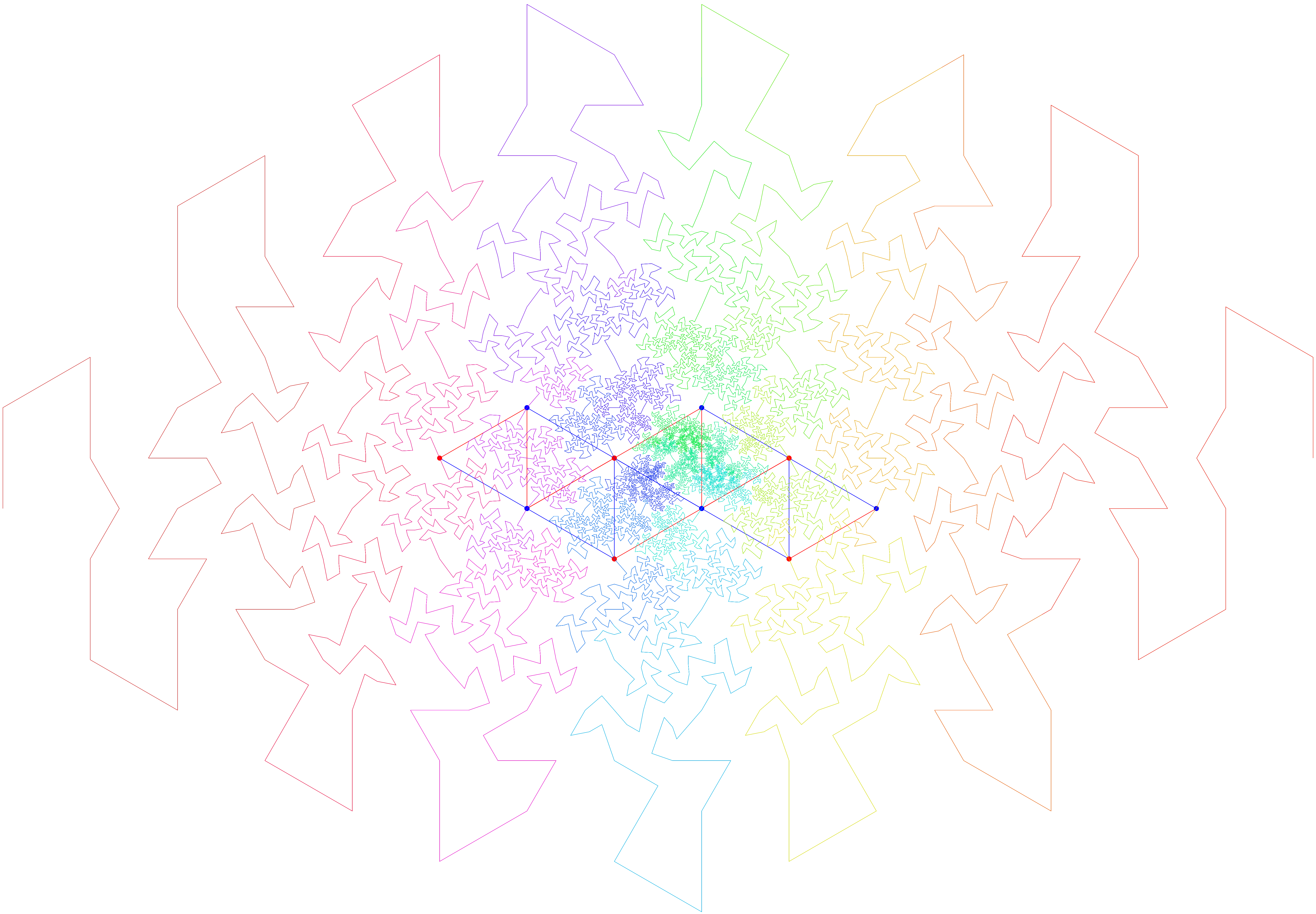
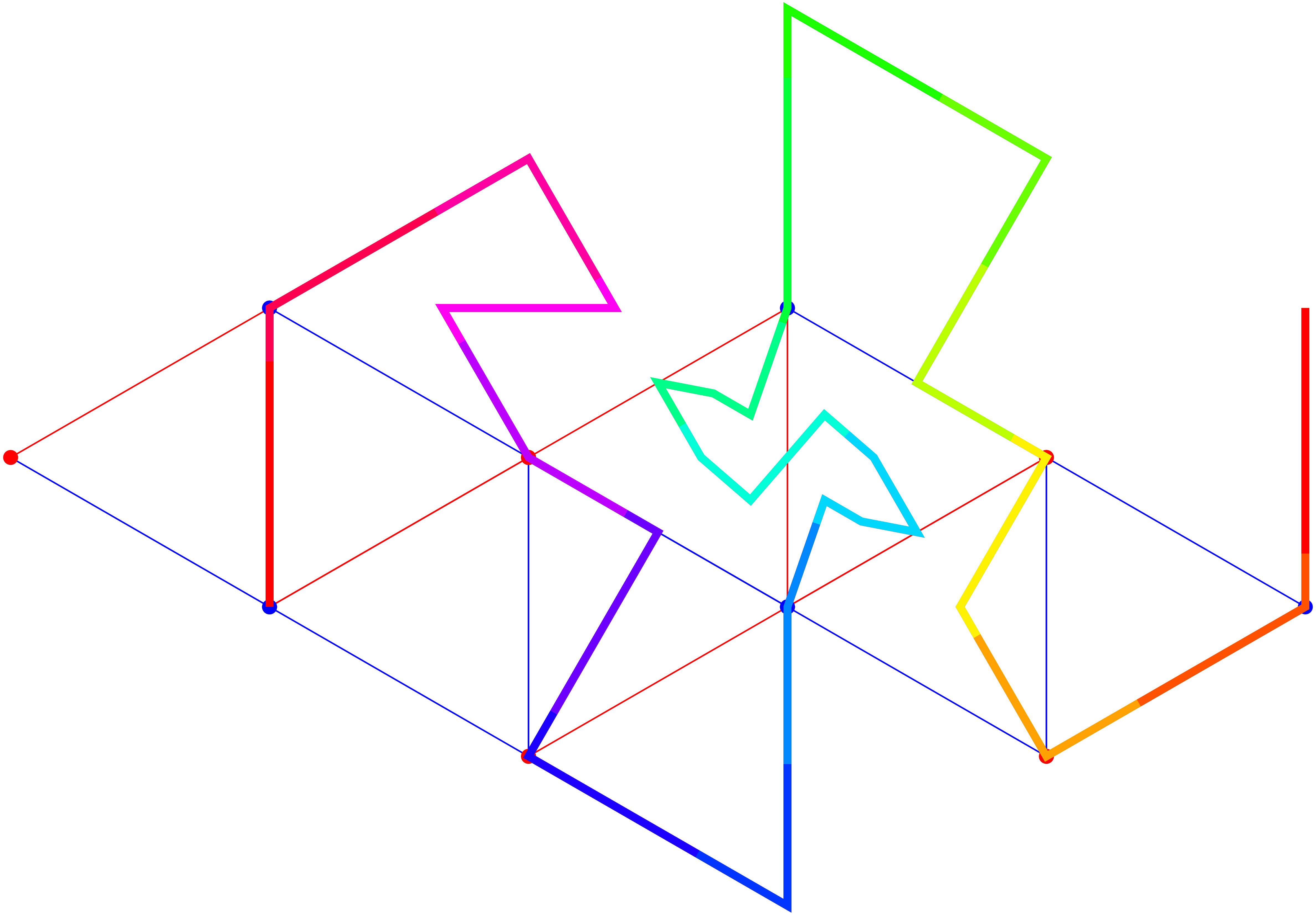


m004

Focus in on one fundamental domain of the boundary torus.



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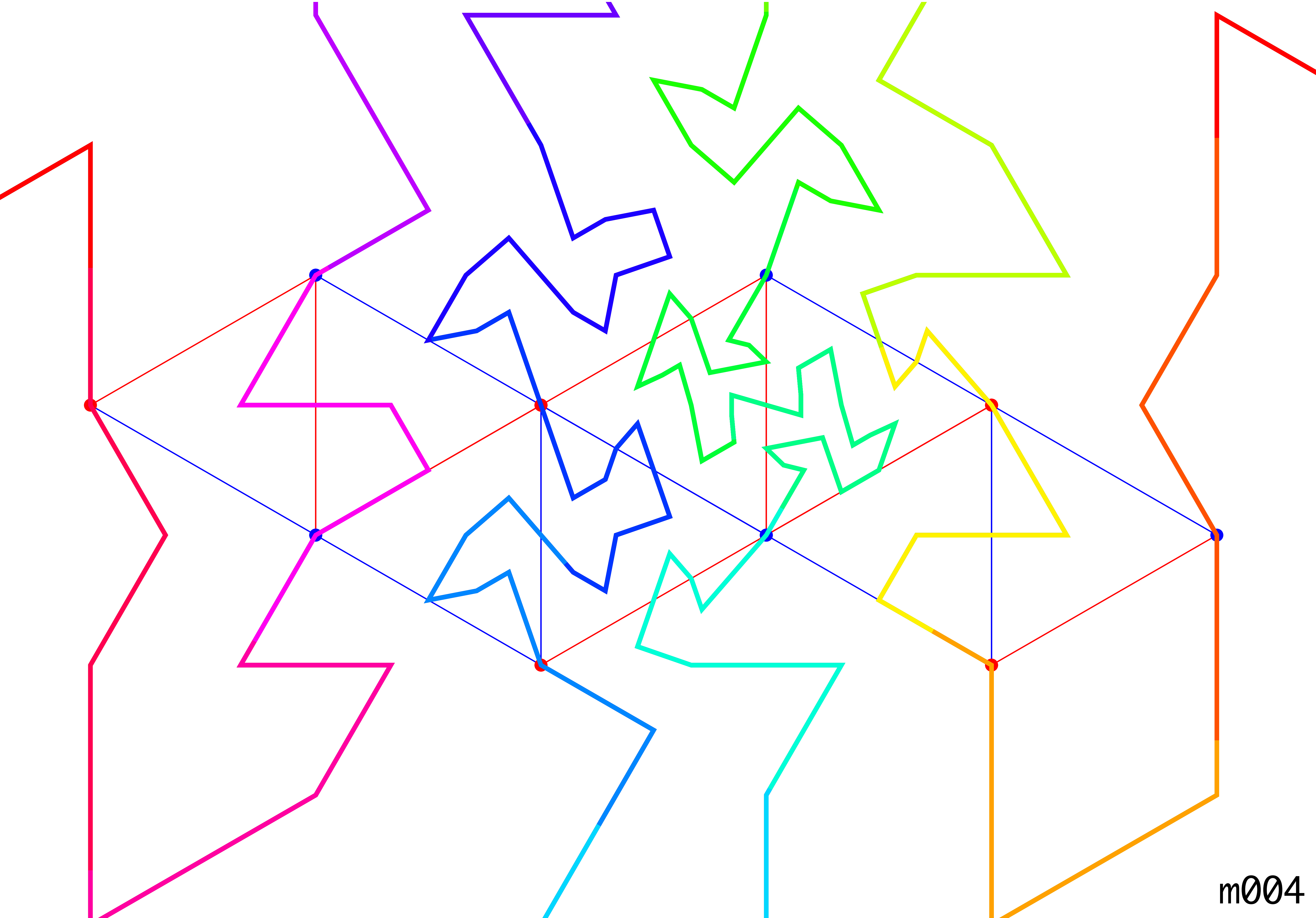

m004

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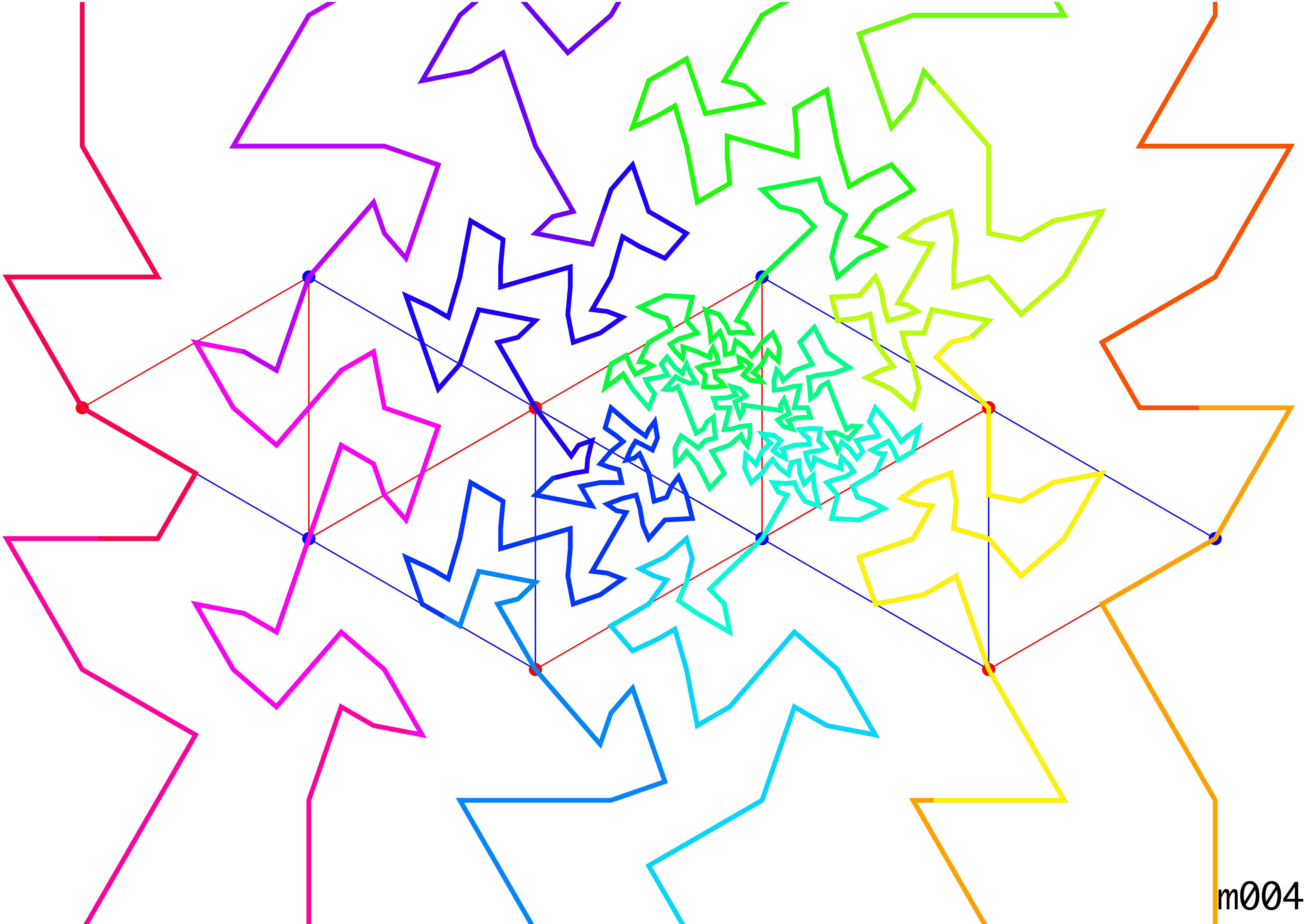
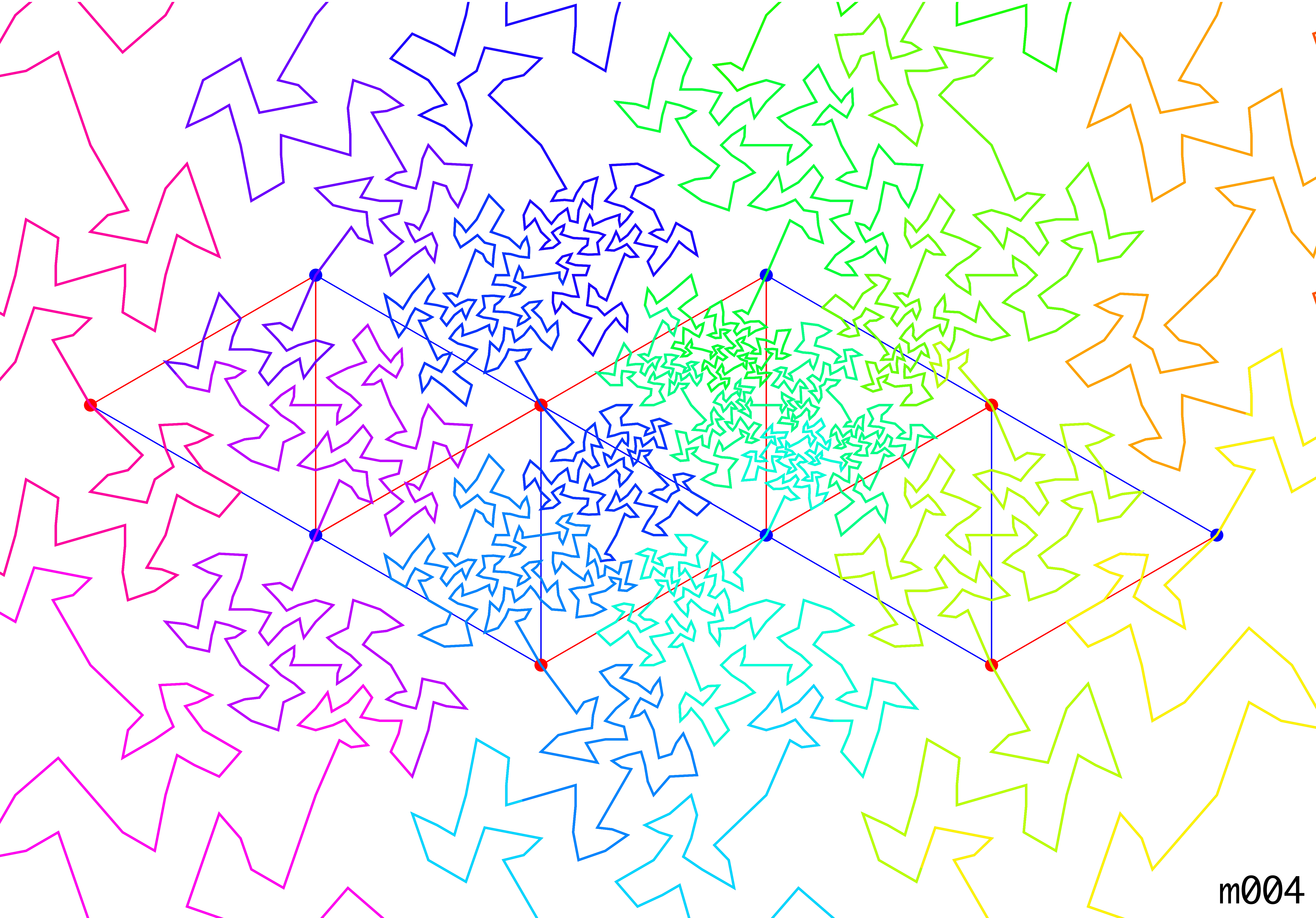


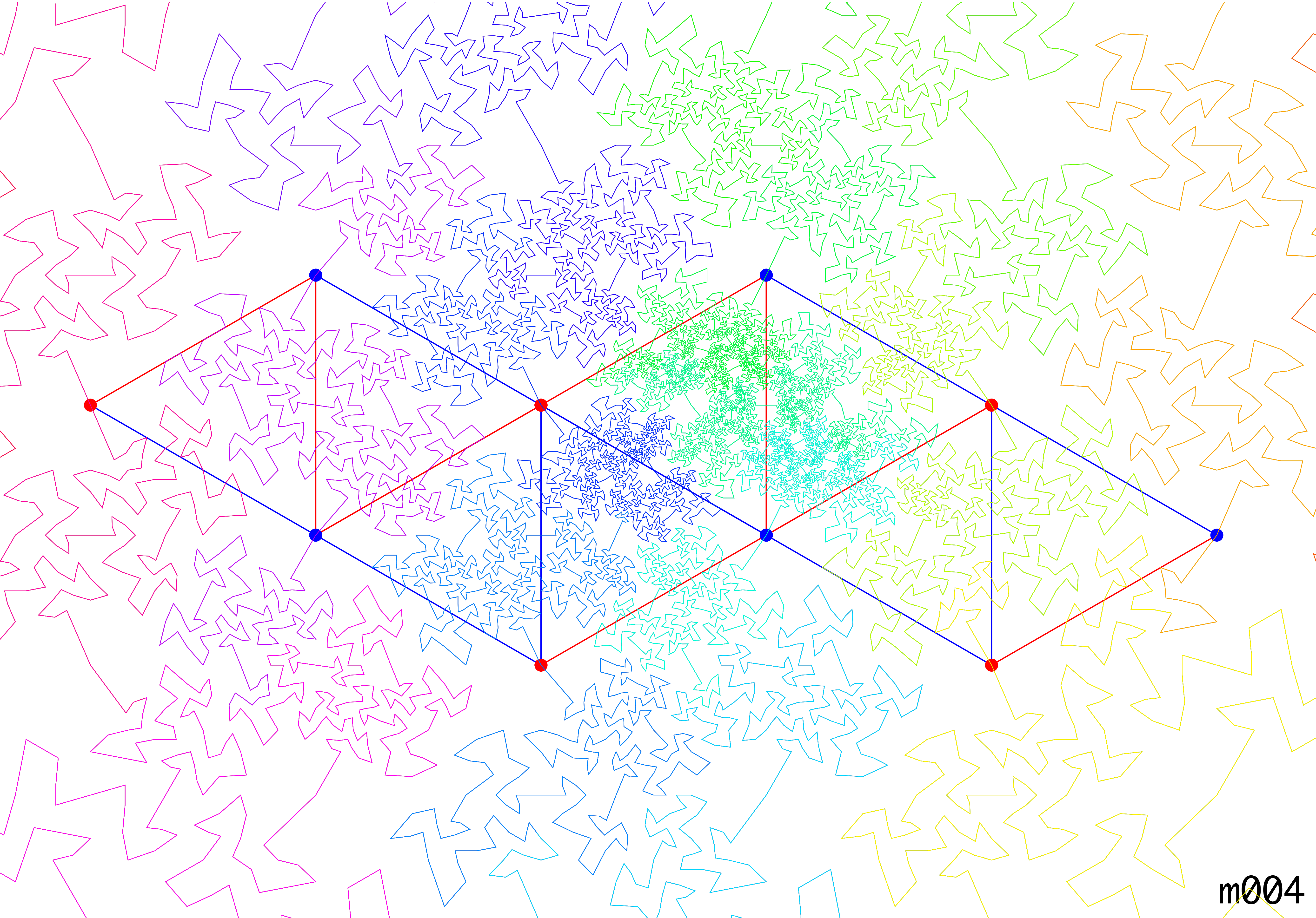
m004

Focus in on one fundamental domain of the boundary torus.



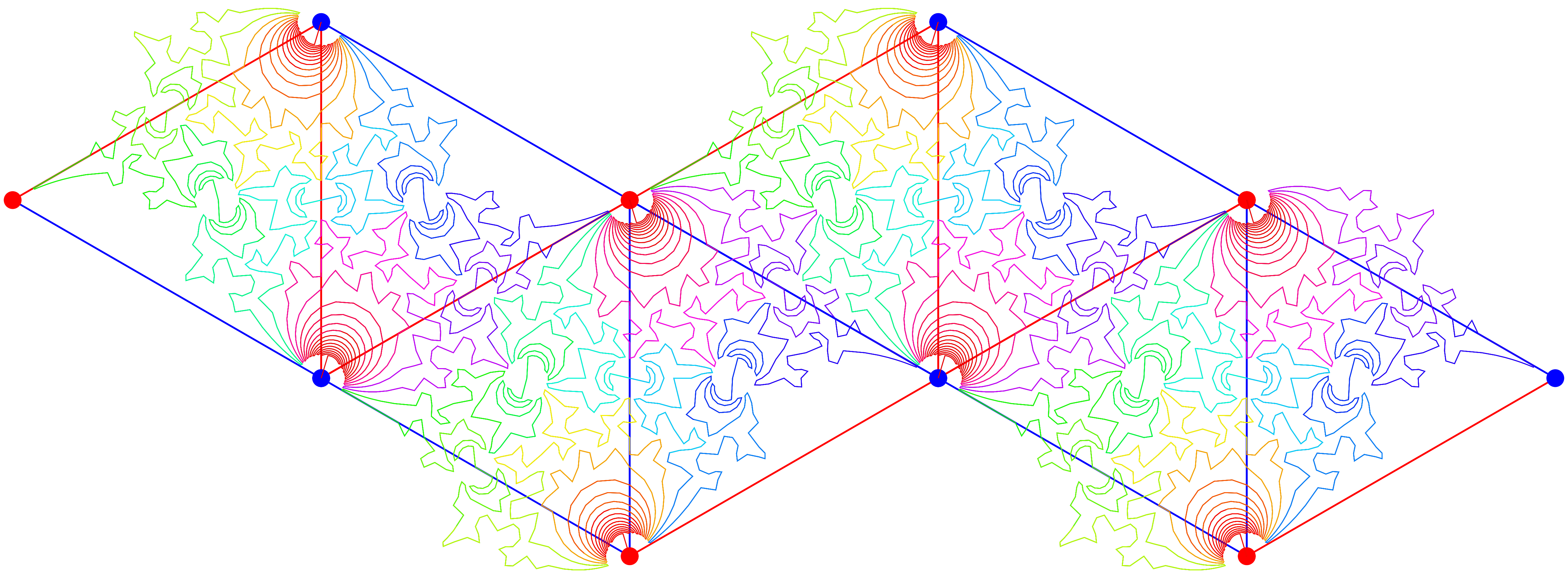
m004

Focus in on one fundamental domain of the boundary torus.



m004

With less naive algorithms we can draw just this fundamental domain, with a more uniform density.



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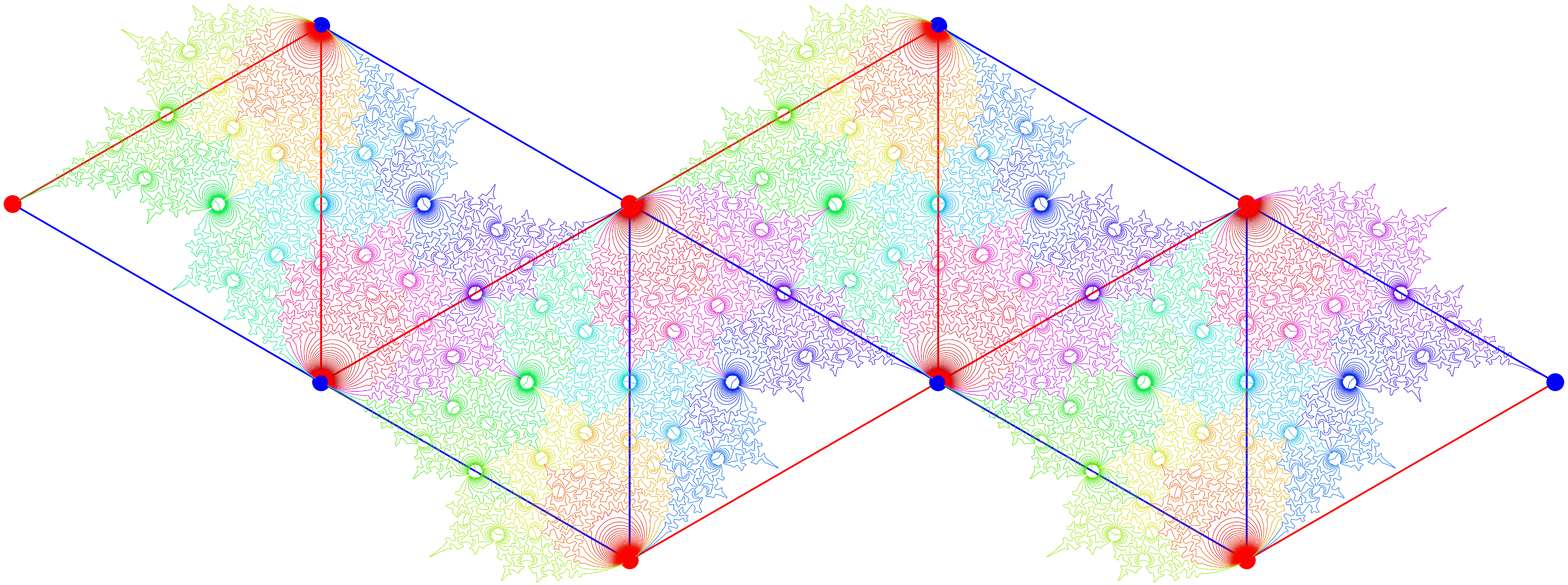

m004

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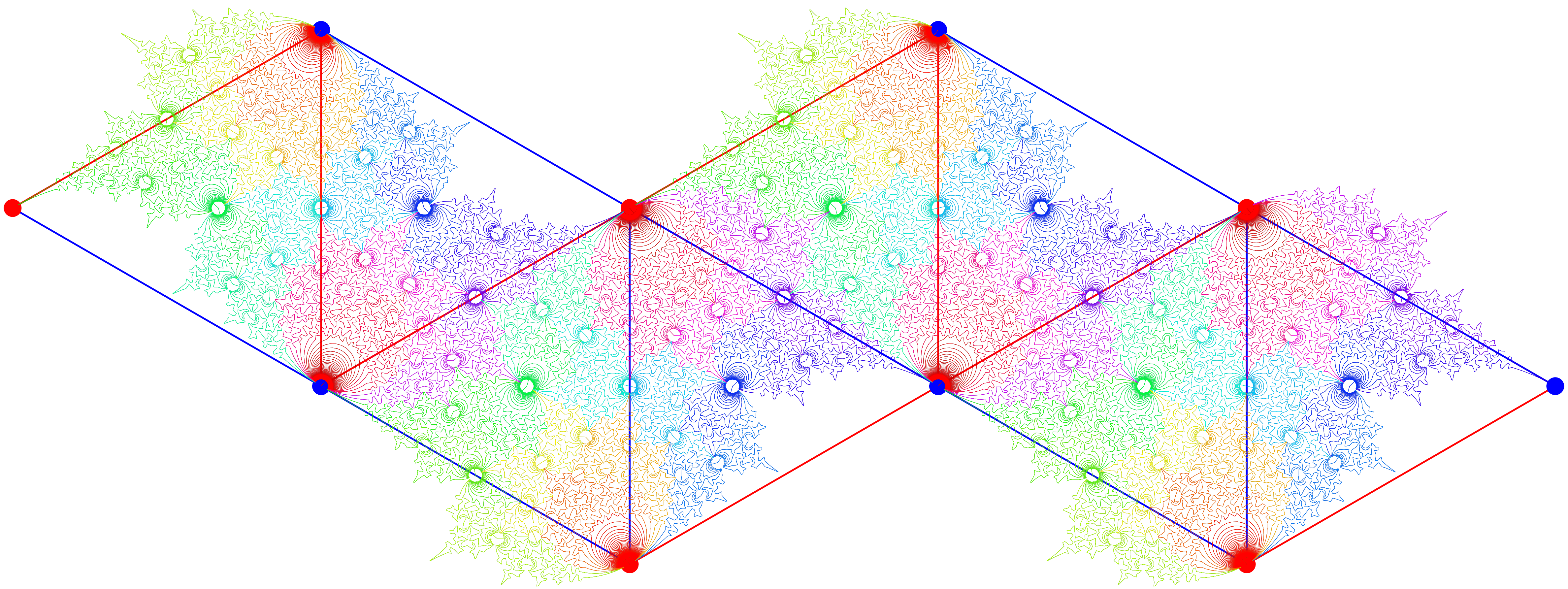
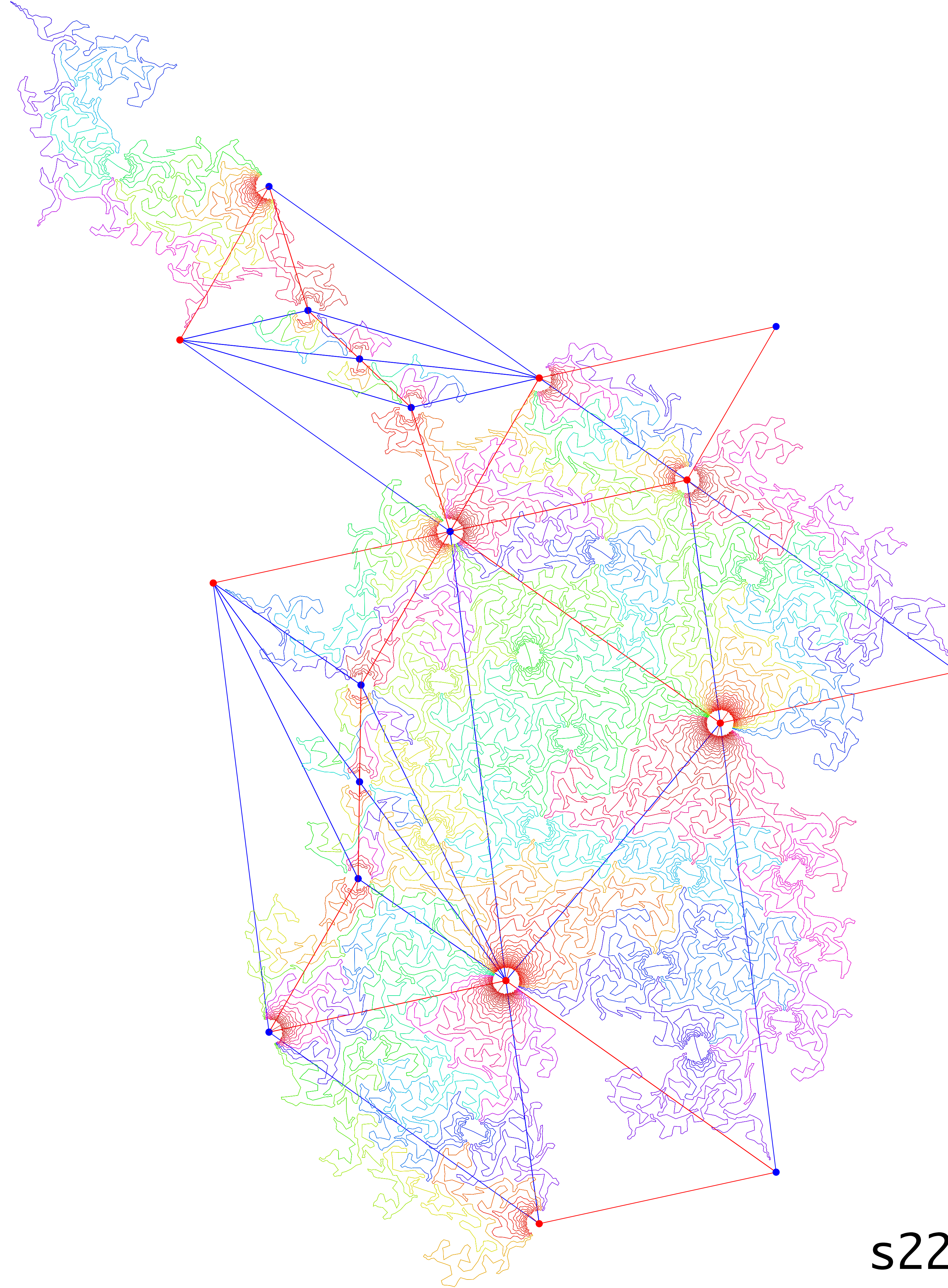


For a fibered manifold, we recover the Cannon-Thurston map built in earlier constructions.

m004

But for non-fibered manifolds, we get something new.

We prove that these Cannon-Thurston maps cannot come from surface subgroups, even virtually.



s227

Proof:

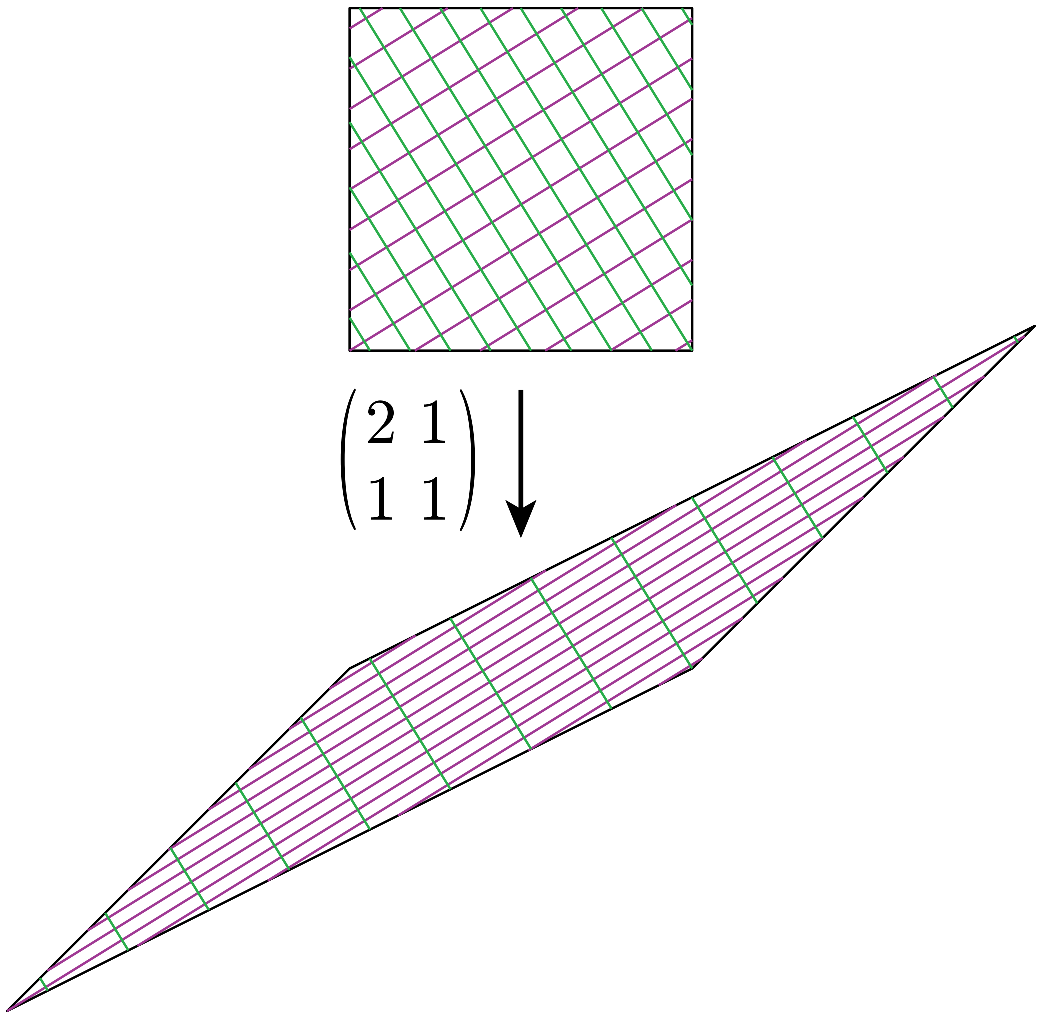
Suppose we have $\phi : S^1 \to S^2$, $\psi : S^1 \to S^2$, a pair of Cannon-Thurston maps. We say that these are *equivalent* if there are homeomorphisms before and after making the diagram commute. From ϕ, we can recover the laminations Λ^ϕ and Λ_ϕ in S^1; they are exactly the non-injectivity loci of ϕ (with some work and a binary choice). Thus we can recover the link space (in our first paper), and so the veering triangulation. This means that the given three-manifolds have commensurable veering triangulations (more work to get from infinite sheeted to finite sheeted here), so one is fibered if and only if the other is.

Q.E.D.

Laminations in the veering circle

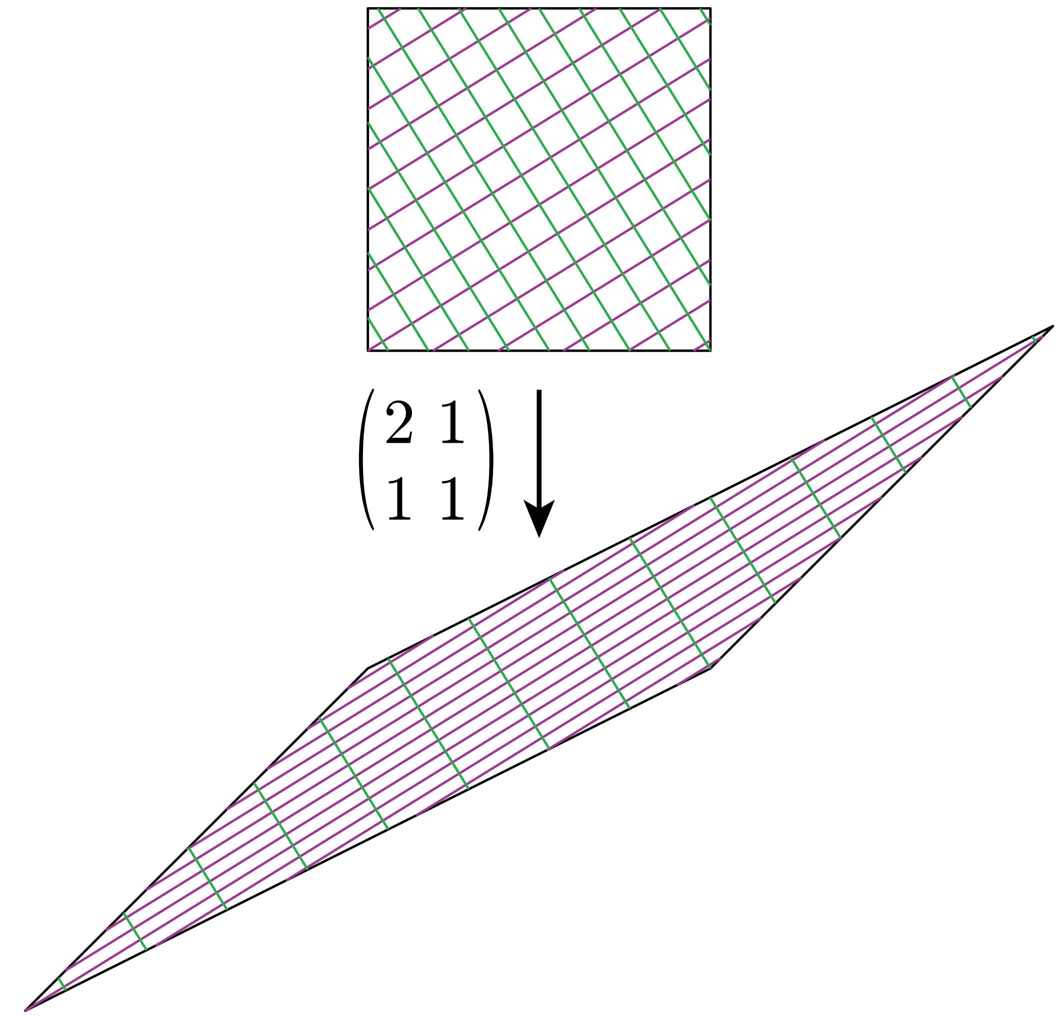
We build our map $S^1 \to S^2$ by collapsing the *upper and lower laminations* in the veering circle.

These are related to the stable and unstable foliations for a pseudo-Anosov map.



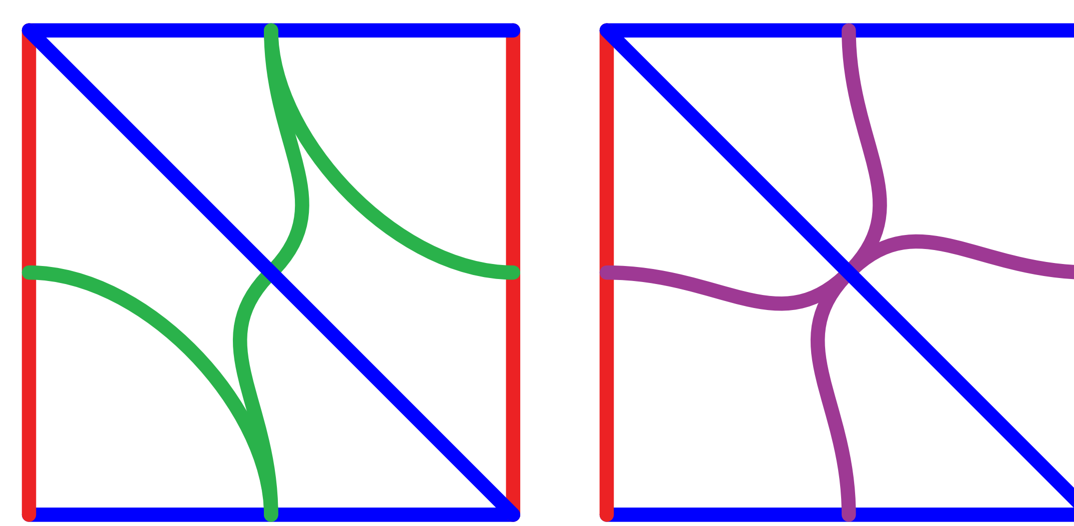
Laminations in the veering circle

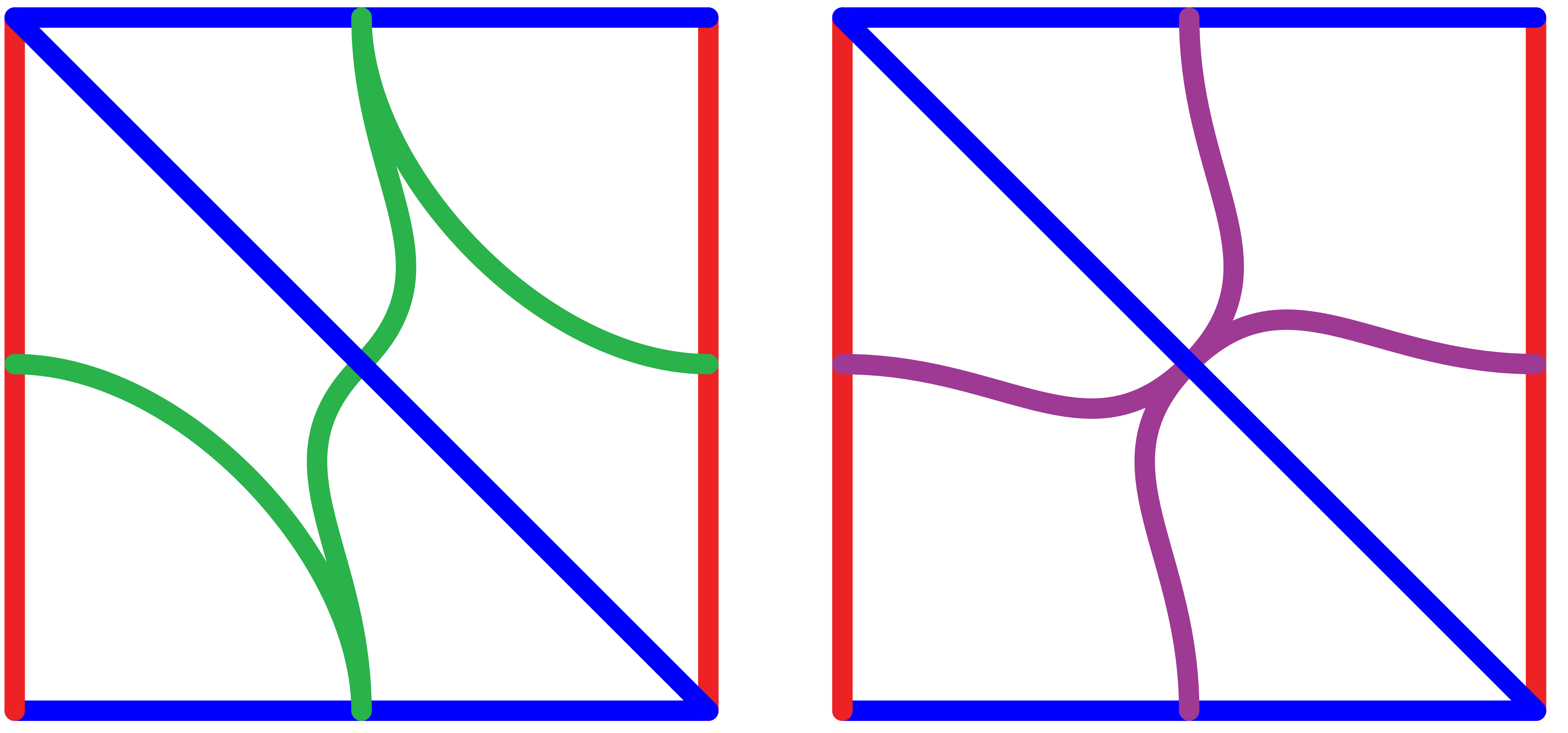
We build our map $S^1 \to S^2$ by collapsing the *upper and lower laminations* in the veering circle.



These are related to the stable and unstable foliations for a pseudo-Anosov map.

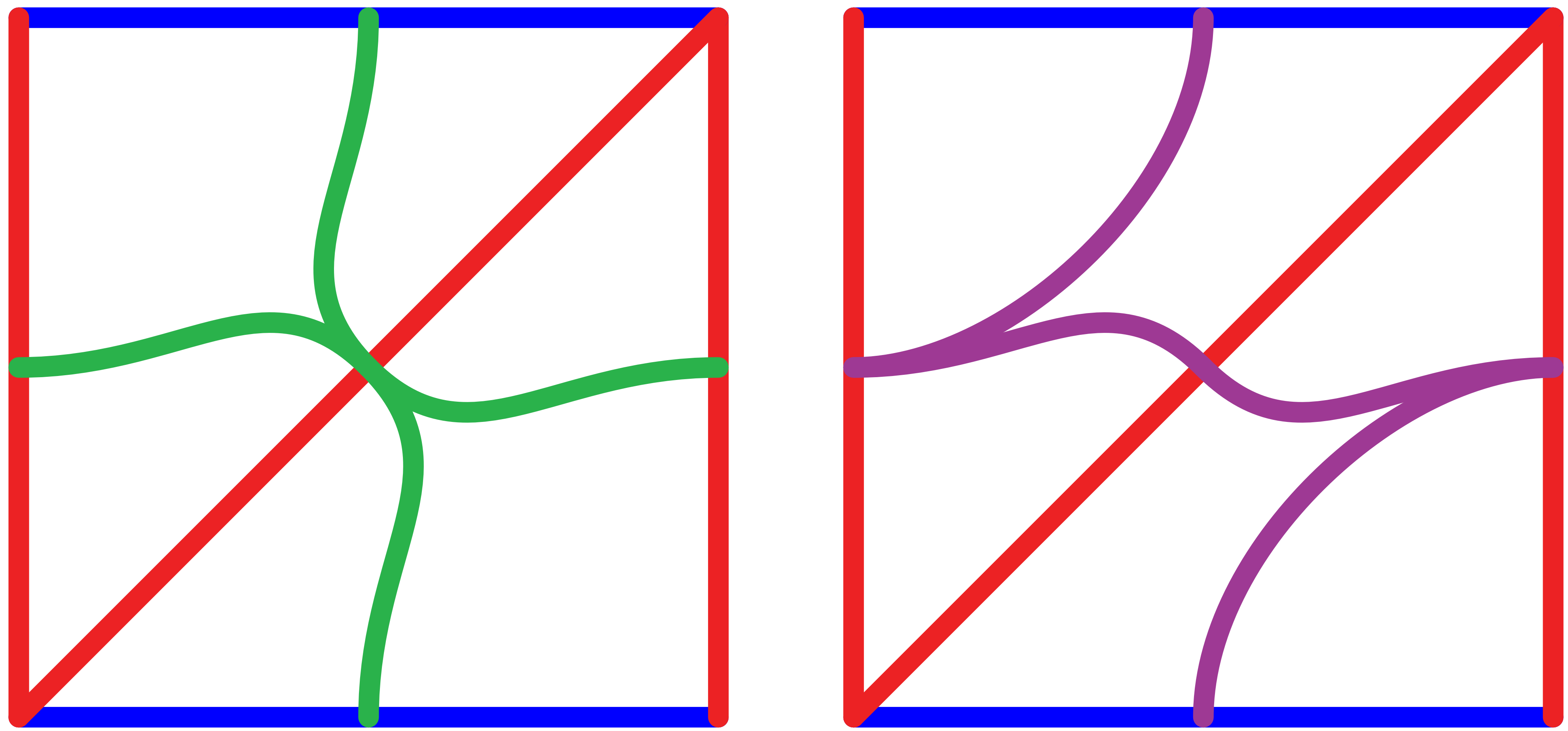
Agol's layered construction uses splitting sequences of train tracks. These give us a hint for where these laminations come from.





upper faces

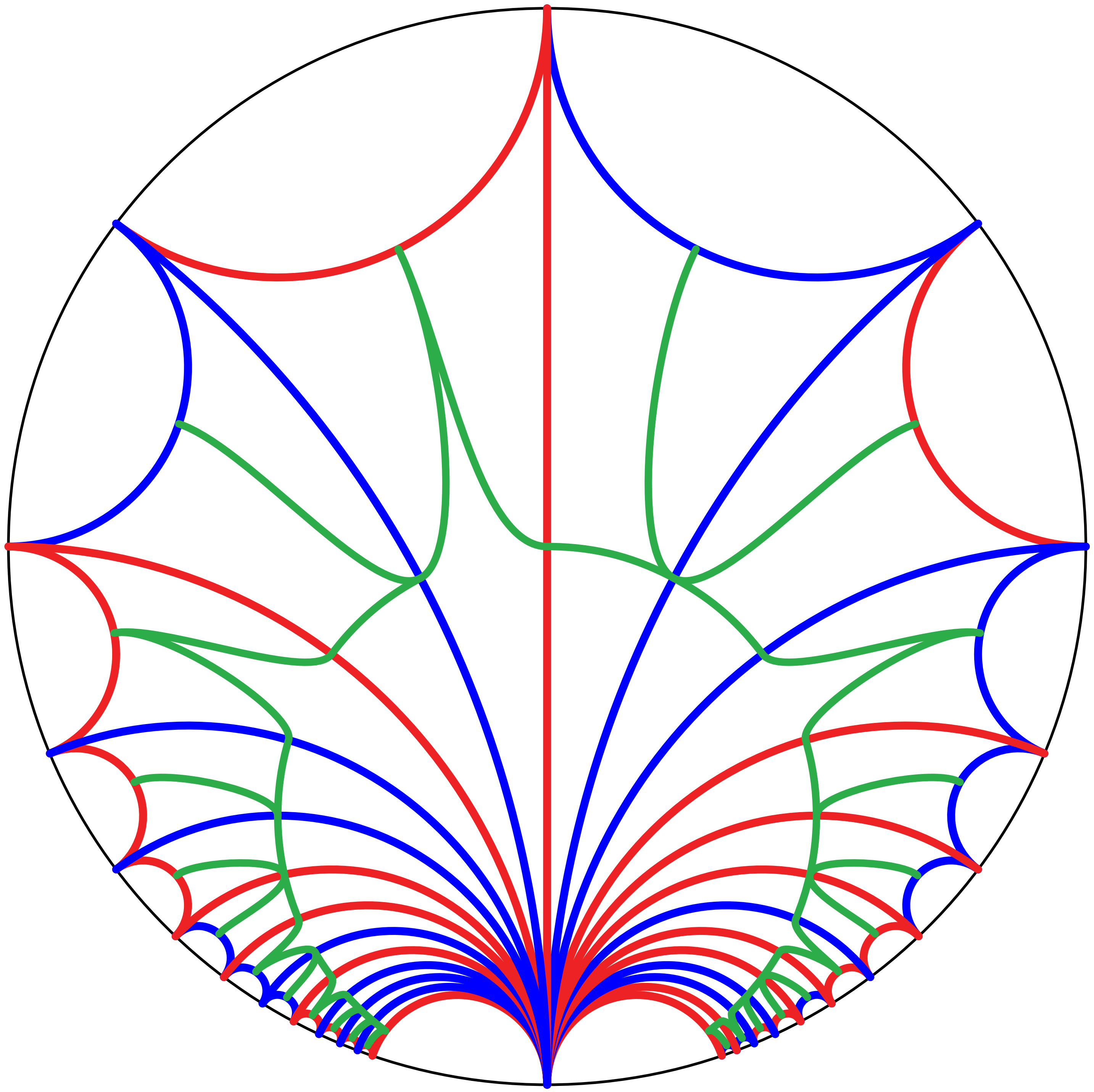
Moving down through a tetrahedron folds the *upper track* (green) and splits the *lower track* (purple).

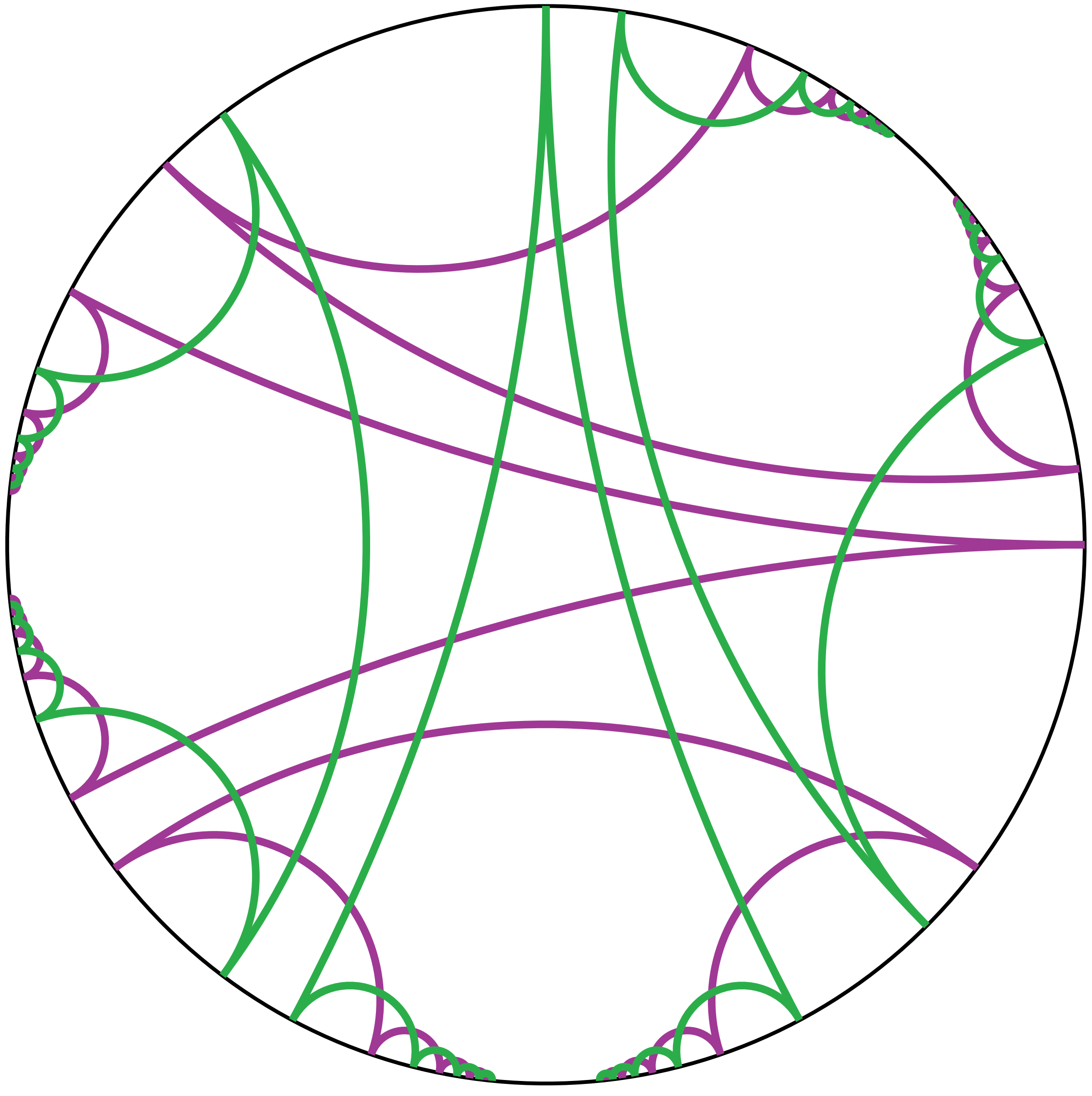


lower faces

Moving down through a tetrahedron folds the *upper track* (green) and splits the *lower track* (purple).

We showed that for any veering triangulation $(\mathscr{T}, \alpha)$ the universal cover $(\widetilde{\mathscr{T}}, \widetilde{\alpha})$ is layered.

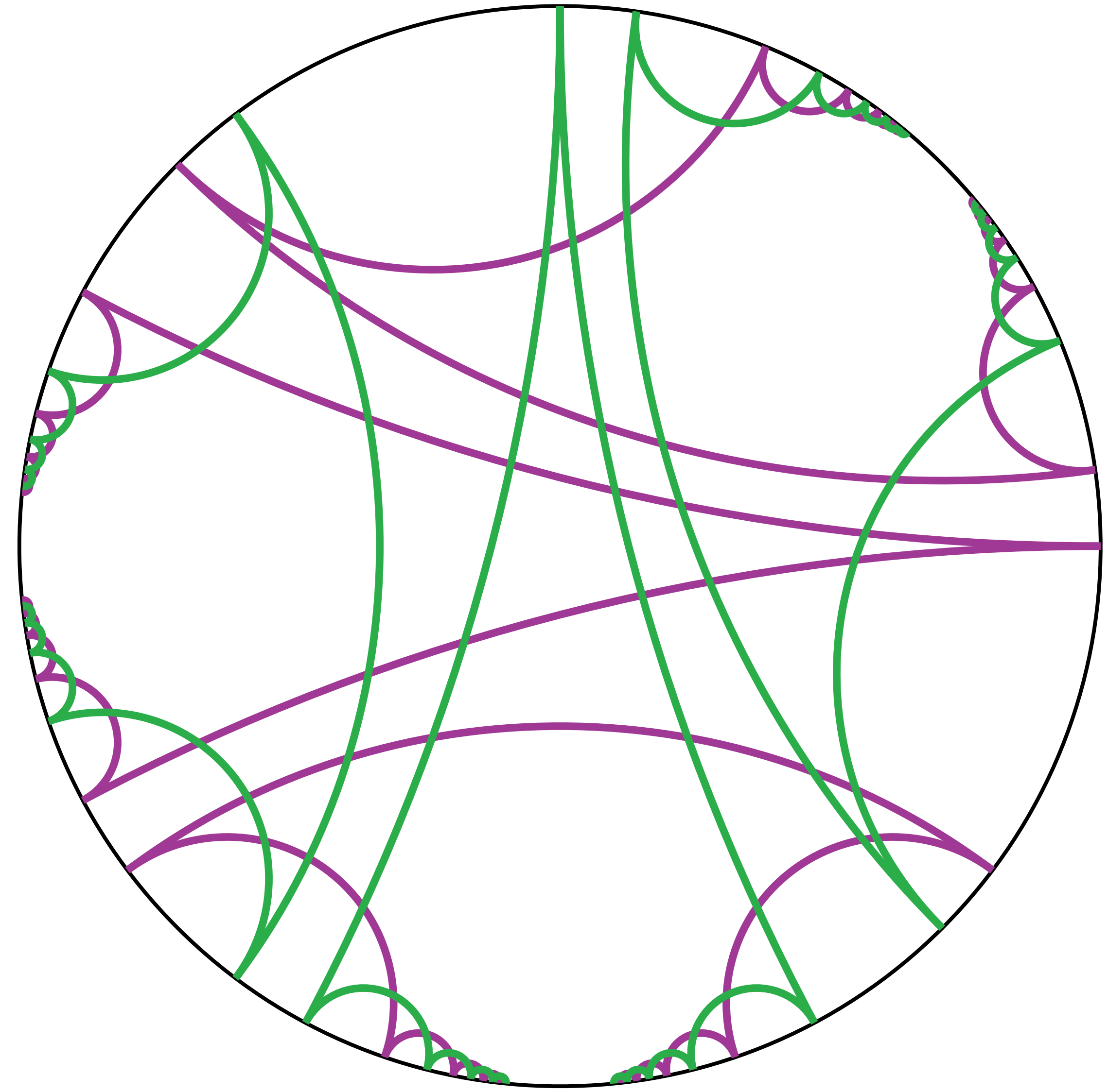




We split train tracks through all layers to generate the upper and lower laminations Λ^{α}, Λ_{α} in the veering circle.

Properties of these laminations:

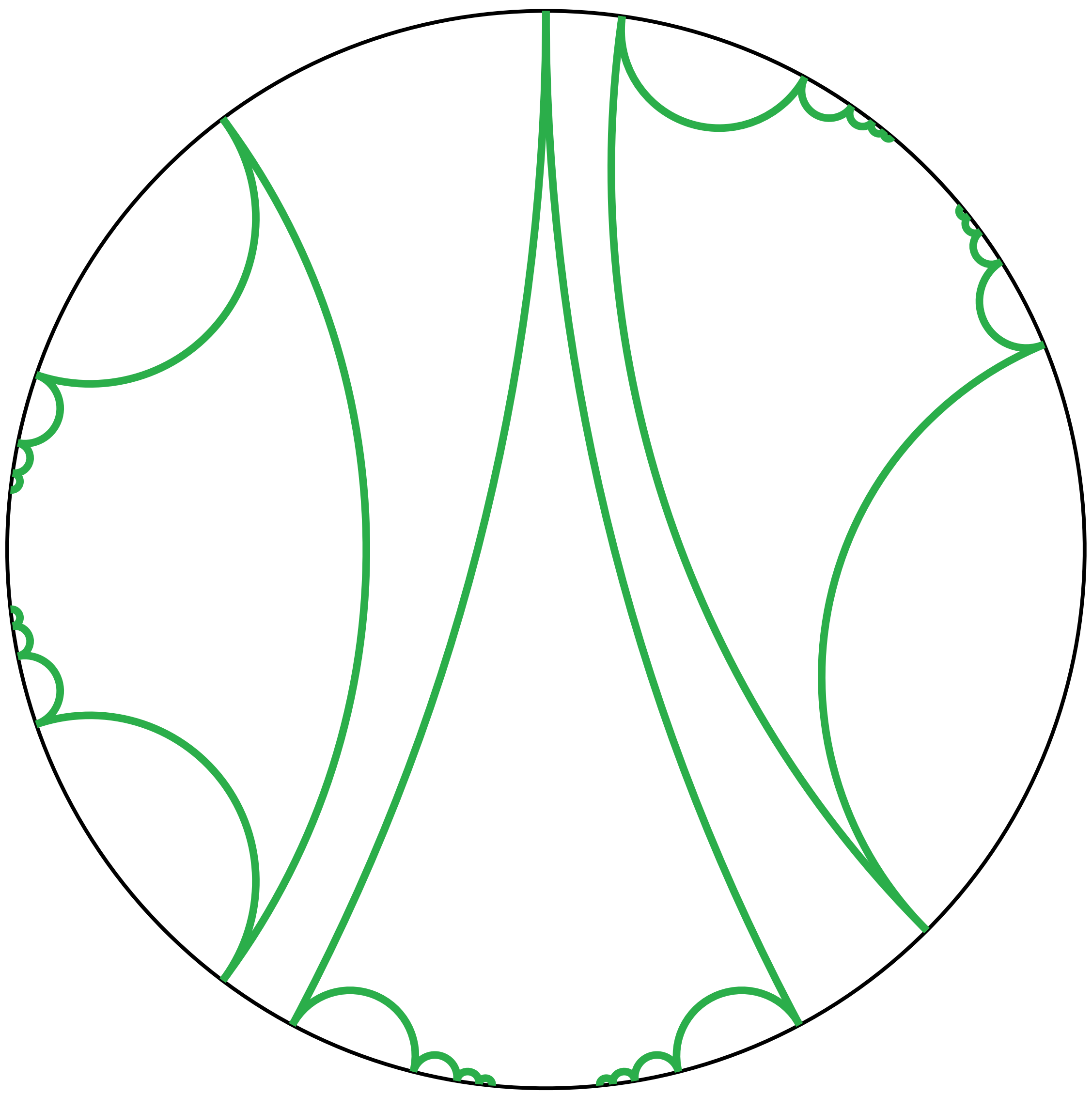
- Leaves of Λ^α share endpoints only around *crowns*, associated to cusps of $\widetilde{\mathscr{T}}$. (Same for Λ_α.)
- Leaves of Λ^α share no endpoints with leaves of Λ_α.
- No isolated leaves in Λ^α or Λ_α.



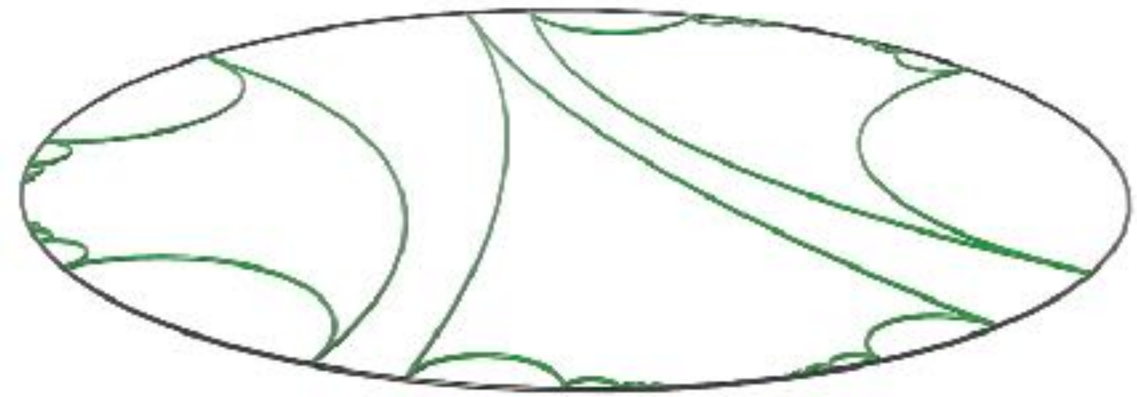
We decompose the veering circle into a collection of disjoint *decomposition elements*. There are three types of these:

- A cusp and the tips of its upper and lower crowns.
- The two endpoints of a non-crown leaf.
- A singleton. (Every other point of the veering circle.)

Project the laminations Λ^{α}, Λ_{α} to the upper and lower hemispheres of a sphere with equator the veering circle.



Project the laminations Λ^{α}, Λ_{α} to the upper and lower hemispheres of a sphere with equator the veering circle.



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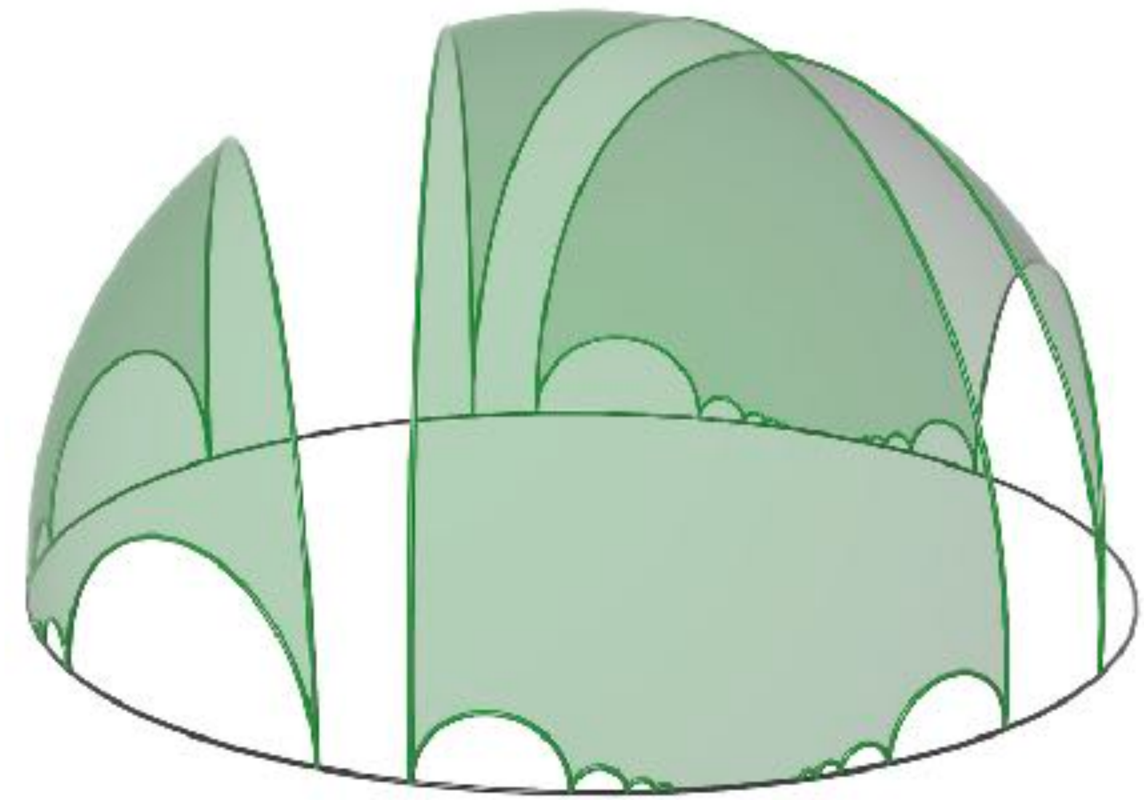

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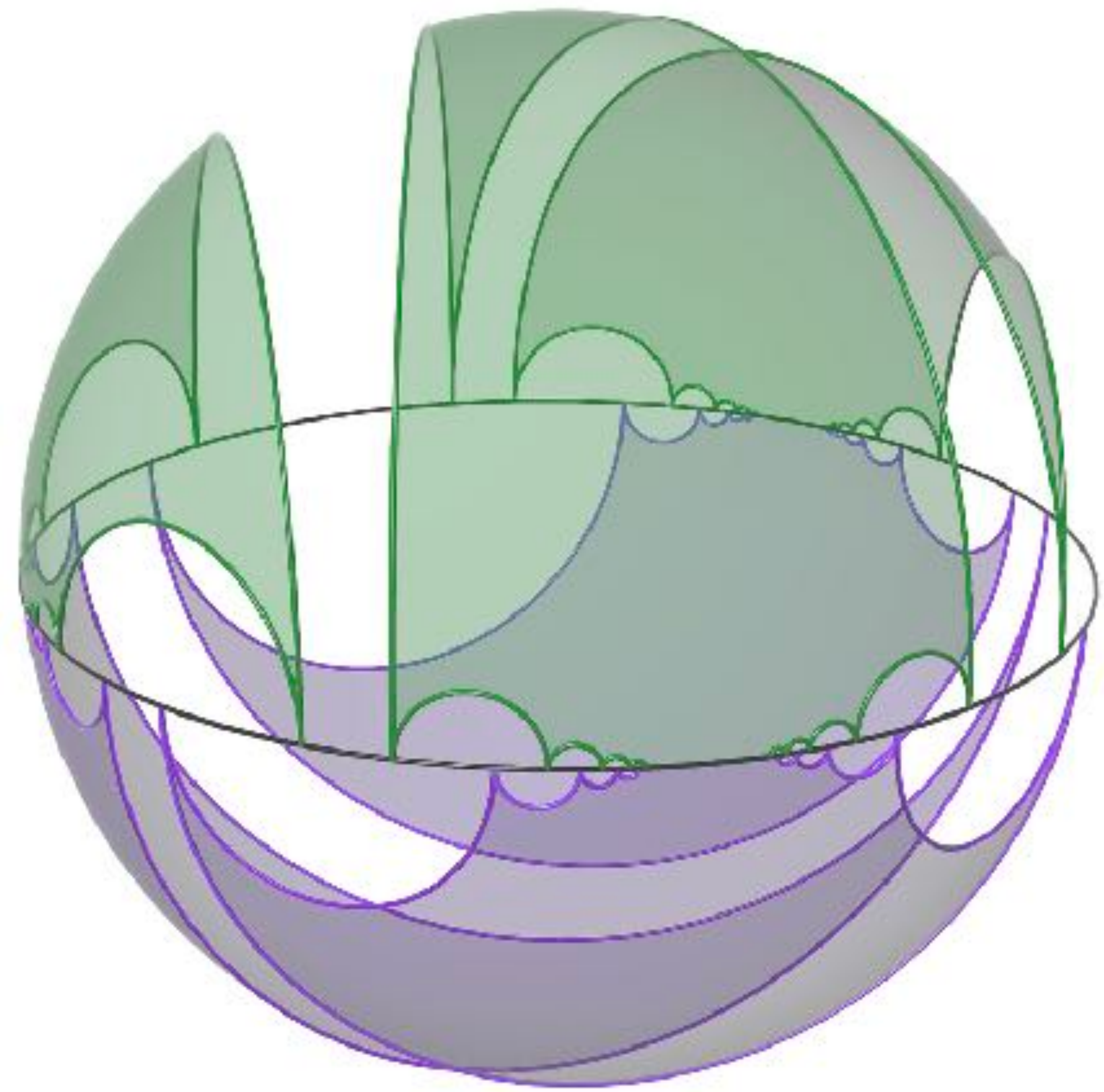


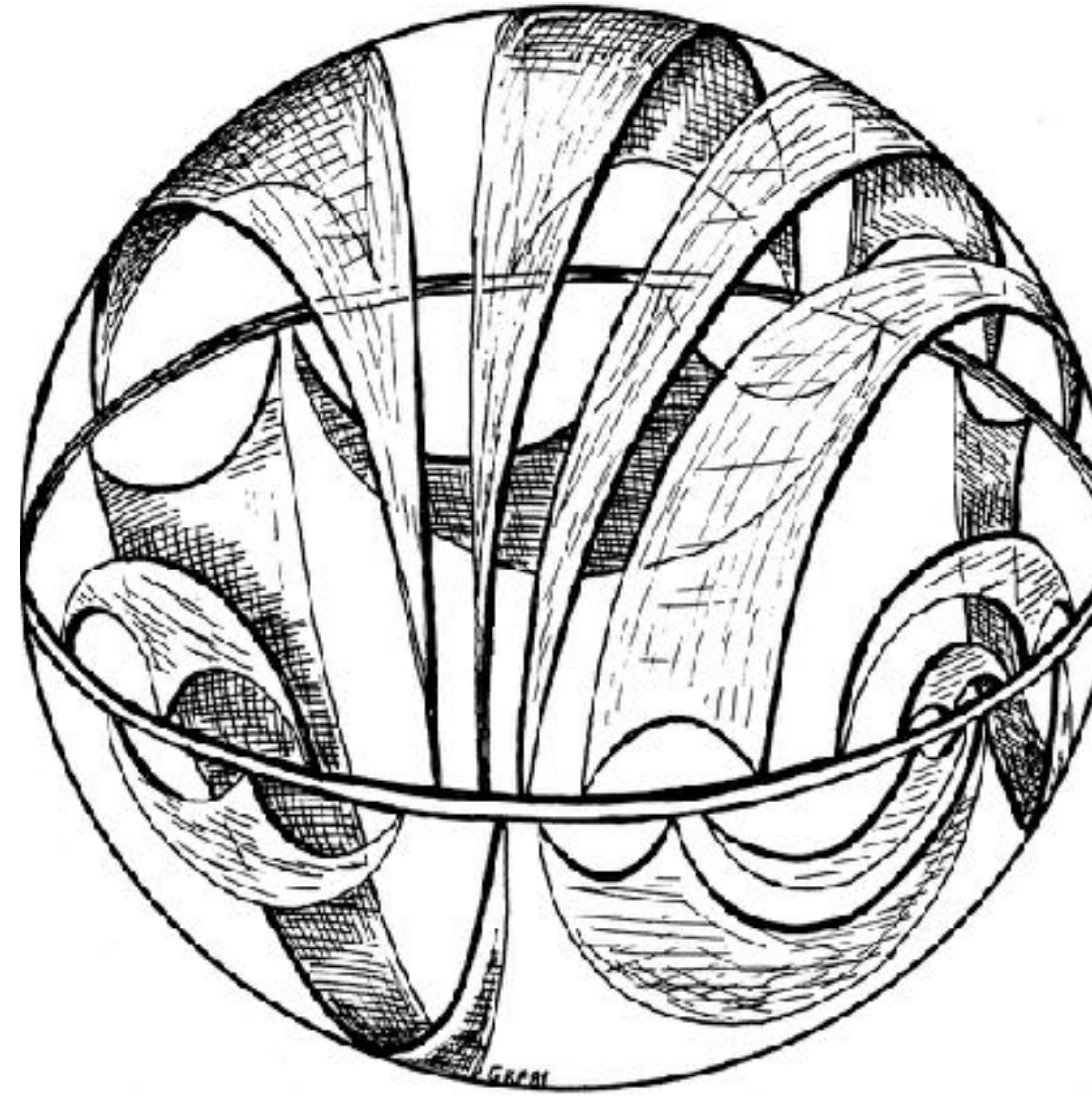
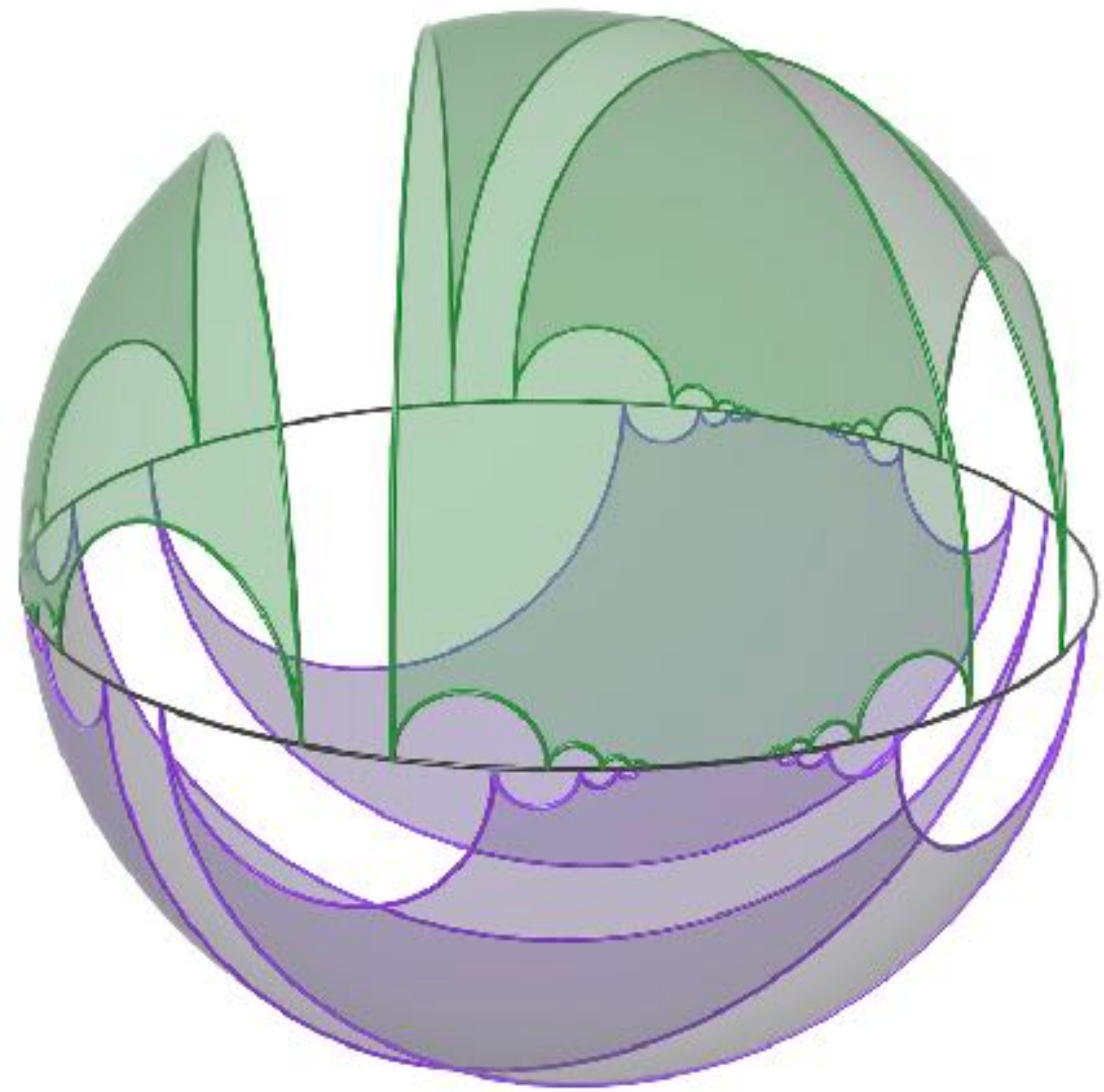


Figure by George Francis, from *Three dimensional manifolds, kleinian groups and hyperbolic geometry* by Bill Thurston

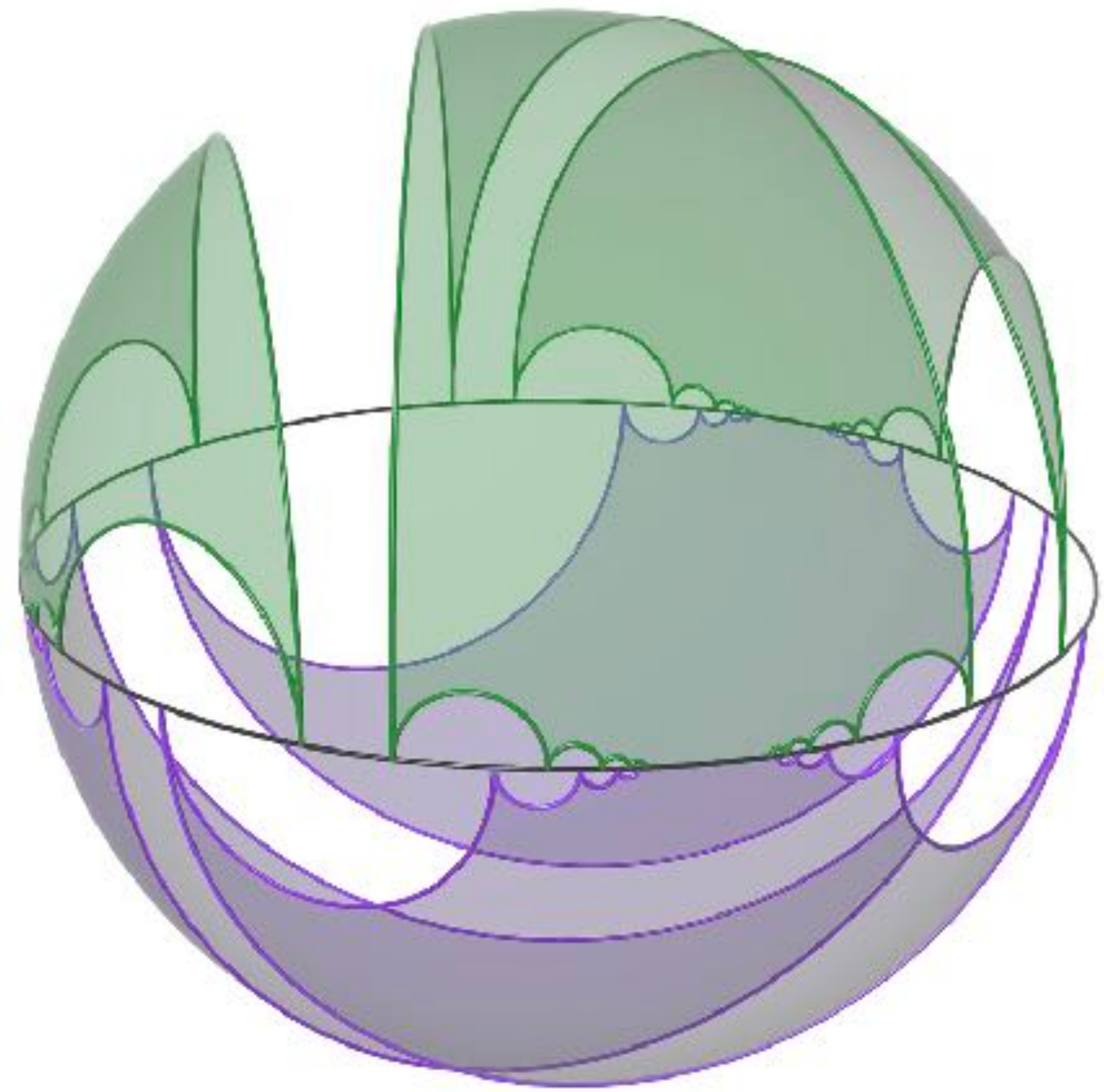
For a closed manifold, the decomposition elements have finite size.

Project the laminations Λ^{α}, Λ_{α} to the upper and lower hemispheres of a sphere with equator the veering circle.



Project the laminations Λ^{α}, Λ_{α} to the upper and lower hemispheres of a sphere with equator the veering circle.

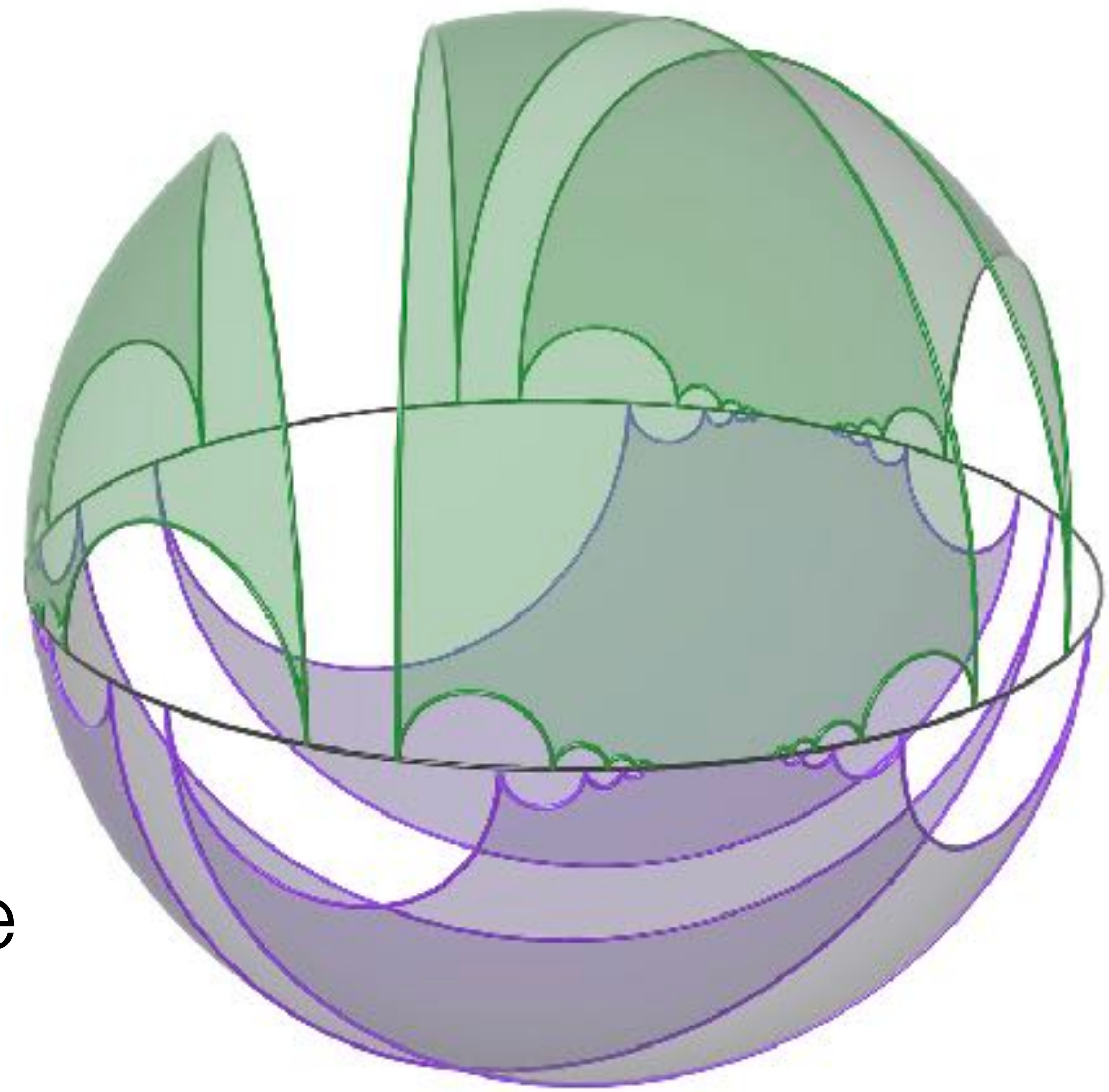
Collapse each decomposition element to a point.



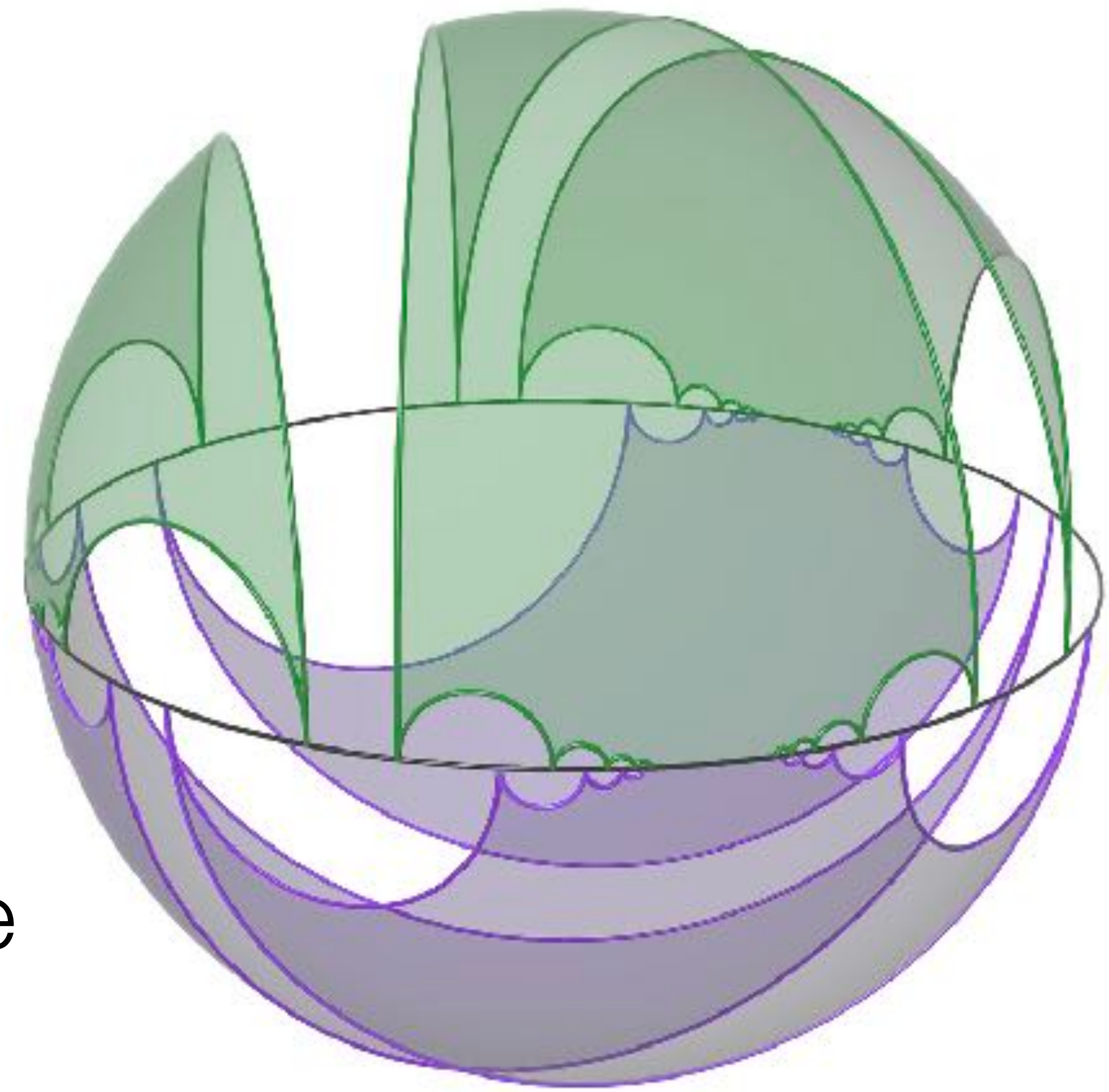
Project the laminations Λ^α, Λ_α to the upper and lower hemispheres of a sphere with equator the veering circle.

Collapse each decomposition element to a point.

A theorem of Moore shows that the quotient is homeomorphic to S^2. We call this the *veering two-sphere*.



Project the laminations Λ^α, Λ_α to the upper and lower hemispheres of a sphere with equator the veering circle.

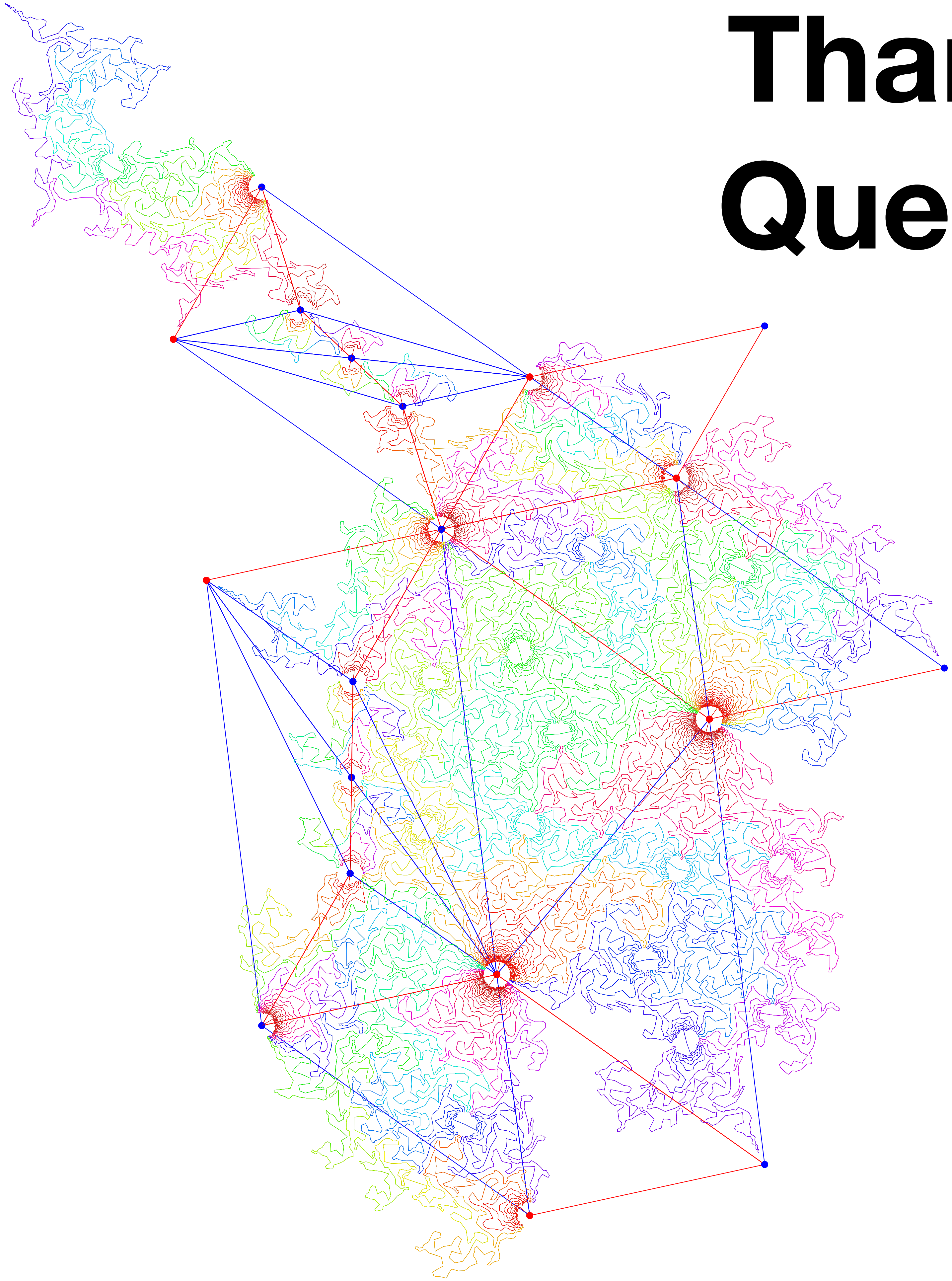


Collapse each decomposition element to a point.

A theorem of Moore shows that the quotient is homeomorphic to S^2. We call this the *veering two-sphere*.

Now we prove that the action of the fundamental group on the veering two-sphere is a convergence group action*. Finally we apply a theorem of Yaman to show that the veering two-sphere is equivariantly homeomorphic to $\partial\mathbb{H}^3$.

Thank you!
Questions?



Thank you!