E_2-ALGEBRAS AND THE UNSTABLE HOMOLOGY OF MAPPING CLASS GROUPS

ALEXANDER KUPERS

Abstract. We discuss joint work with Soren Galatius and Oscar Randal-Williams on the application of higher-algebraic techniques to classical questions about the homology of mapping class groups. This uses a new "multiplicative" approach to homological stability – in contrast to the "additive" one due to Quillen – which has the advantage of providing information outside of the stable range.

This is joint work with Søren Galatius and Oscar Randal-Williams, see [GKRW19] (which is based on [GKRW18]). My goal is to explain a "machine" for proving homological stability results through the example of mapping class groups of surfaces.

1. Mapping class groups and their homology

Let $\Sigma_{g,r}$ be a genus g surface with r boundary components, see Fig. 1 for an example. (For technical reasons, we will think of the boundary as being that of a square.) We denote the topological group of diffeomorphisms $\Sigma_{g,r} \to \Sigma_{g,r}$ fixing a neighborhood of the boundary pointwise, in the C^∞-topology, by $\mathrm{Diff}_\partial(\Sigma_{g,r})$.

Definition 1.1. The *mapping class group* of $\Sigma_{g,r}$ is $\Gamma_{g,r} \coloneqq \pi_0(\mathrm{Diff}_\partial(\Sigma_{g,r}))$.

By Teichmüller theory, the path components of $\mathrm{Diff}_\partial(\Sigma_{g,r})$ are contractible if $g > 1$ or $r > 0$ [EE69, ES70] so passing from diffeomorphism groups to mapping class groups does not lose any information (at least to an algebraic topologist): the homomorphism

$$\mathrm{Diff}_\partial(\Sigma_{g,r}) \longrightarrow \Gamma_{g,r}$$

is a homotopy equivalence.

Remark 1.2. Since the inclusion $\mathrm{Diff}_\partial(\Sigma_{g,r}) \hookrightarrow \mathrm{Homeo}_\partial(\Sigma_{g,r})$ is a homotopy equivalence, one may also take topological isotopy classes of homeomorphisms as a definition of the mapping class groups.

Date: March 21, 2021.

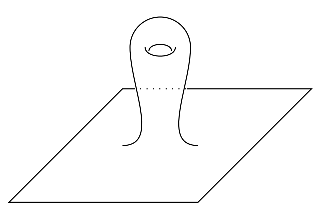

Figure 1. An example of the surface $\Sigma_{1,1}$.

1.1. **Surface bundles and their characteristic classes.** Recall that the *classifying space* BG of a discrete group G is characterised up to homotopy by the conditions that it is homotopy equivalent to a CW-complex, and that $\pi_1(BG) = G$ and $\pi_*(BG) = 0$ otherwise. The above results imply that for $g > 1$ and $r > 0$, the classifying space $B\Gamma_{g,r}$ classifies surface bundles with $\Sigma_{g,r}$ and trivialised boundary bundle over paracompact spaces (otherwise, add the assumptions that the bundles are numerable):

$$\frac{\left\{\begin{matrix}\text{surface bundles } \Sigma_{g,r} \to E \to X \\ \text{with trivialised boundary}\end{matrix}\right\}}{\text{isomorphism}} \longleftrightarrow \frac{\left\{\begin{matrix}\text{continuous maps} \\ X \to B\Gamma_{g,r}\end{matrix}\right\}}{\text{homotopy}}.$$

The map from the right to the left is given as follows: $B\Gamma_{g,r}$ carries a universal bundle $E_{\text{univ}} \to B\Gamma_{g,r}$ and we assign to $f\colon X \to B\Gamma_{g,r}$ the pullback $f^*E_{\text{univ}} \to X$. Roughly, its inverse is given by recording the monodromy.

To effectively classify surface bundles you want characteristic classes: assignments to each $\Sigma_{g,r} \to E \to X$ of a class $\xi(E) \in H^k(X)$, such that $\xi(f^*E) = f^*\xi(E)$. These are in bijection with the cohomology of the classifying space: boundary bundle over paracompact spaces:

$$\left\{\begin{matrix}H^k(-)\text{-valued characteristic} \\ \text{classes of bundles with fiber } \Sigma_{g,r}\end{matrix}\right\} \longleftrightarrow H^k(B\Gamma_{g,r}).$$

The map from the right to the left is given as follows: if $E \to X$ is classified by $f\colon X \to B\Gamma_{g,r}$, then we send $\xi \in H^k(B\Gamma_{g,r})$ to $\xi(E) := f^*\xi \in H^k(X)$.

Example 1.3. The *Miller–Morita–Mumford classes* $\kappa_i \in H^{2i}(B\Gamma_{g,r})$ are given by following characteristic class: given a surface bundle $\pi\colon E \to X$, we take the vertical tangent bundle T_vE (given by $\ker(d\pi\colon TE \to TX)$) is everything is a smooth manifold), take $e(T_vE)^{i+1} \in H^{2i+2}(E)$ and integrate over the fiber to get a class in $H^{2i}(E)$.

1.2. **Homology of mapping class groups.** Studying the cohomology of mapping class groups is roughly the same as studying their homology (through universal coefficient theorems). So what we do know?

Topologists like the case $r = 1$ most, since it easy to map out of. Among many operations, we will use the following:

$$\begin{array}{ccc} B\Gamma_{g,1} & \xrightarrow[\tau]{\text{capping off}} & B\Gamma_{g,0} \\ {\scriptstyle\text{stabilise}}\downarrow{\scriptstyle\sigma} & & \\ B\Gamma_{g+1,1}. & & \end{array}$$

Theorem 1.4 (Harer, Ivanov, Boldsen, Randal-Williams)**.** *We have that*

$$H_d(B\Gamma_{g,1}, B\Gamma_{g-1,1}) = 0 \quad \textit{for } * \leq \tfrac{2}{3}g.$$

This "homological stability" result refers to the relative homology groups of the map σ; a more concrete but equivalent statement is that $\sigma\colon H_d(B\Gamma_{g-1,1}) \to H_d(B\Gamma_{g,1})$ is an isomorphism for $d \leq \frac{2}{3}g - 1$ and a surjection for $d = \frac{2}{3}g$. In words, the homology in fixed homological degree d is independent of the genus g when g is sufficiently large. There is a similar statement for τ, the "capping off" map; it too is an isomorphism in a range tending to ∞ with g.

This result implies that $H_d(B\Gamma_{g,1})$ is equal to $\mathrm{colim}_{g\to\infty} H_d(B\Gamma_{g,1})$, the *stable homology*, in a *stable range*. We know what this stable homology is given by:

Theorem 1.5 (Madsen–Weiss)**.** *We have that*

$$\operatorname*{colim}_{g\to\infty} H_*(B\Gamma_{g,1}) = H_*(\Omega_0^\infty \mathrm{MTSO}(2)).$$

The right side may not be familiar, but for homotopy theorists it is an easily understood object (the infinite loop space of a Thom spectrum of a virtual bundle over $B\mathrm{SO}(2)$). For example, the following is a straightforward computation for those familiar with stable homotopy theory:

Example 1.6. In the stable range, $H^*(B\Gamma_{g,1};\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\ldots]$. That is, it is the graded-commutative algebra generated by the MMM-classes and there are no relations among these. This computation established the *Mumford conjecture*.

Question 1.7. What happens outside the stable range?

There are a few known results (see [GKRW19] for detailed references):

- Computations in genus ≤ 4 either by computer (using ribbon graphs and radial slit diagrams), or algebraic geometric methods (using classification of genus g curves in terms of certain standard models).
- Recent work of Chan–Galatius–Payne finds many unstable classes near the virtual cohomological dimension.

All that is known so far is listed in the diagram below (I have already added in the consequences of [GKRW19], in orange):

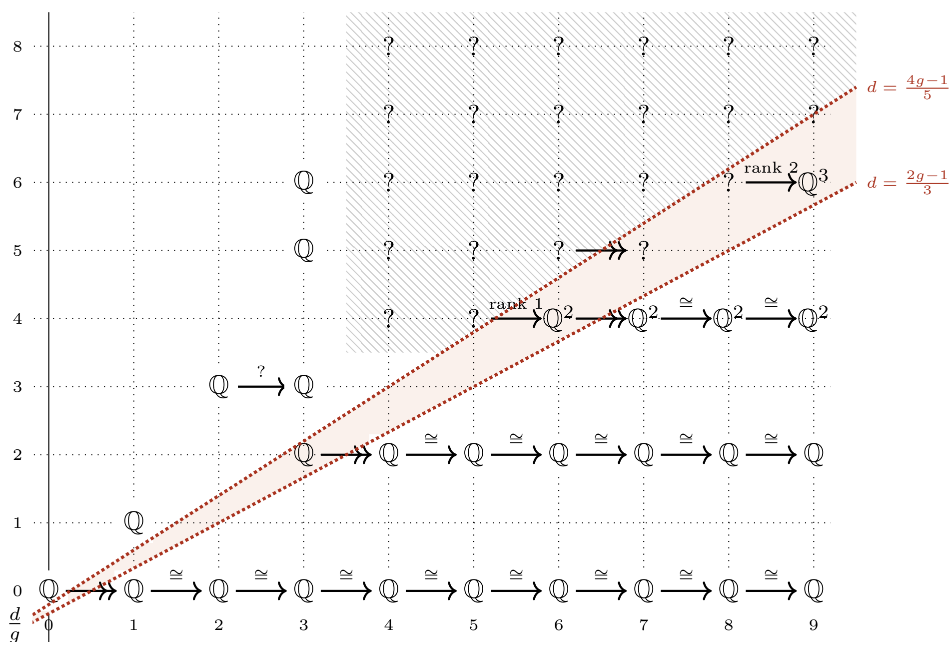

FIGURE 2. A summary of the low-degree low-genus rational homology of $\Gamma_{g,1}$ and the stabilisation maps.

Remark 1.8. $B\Gamma_{g,0}$ is closely related to the algebraic geometer's moduli spaces of curves $\mathcal{M}_g$. In particular, they have the same rational cohomology. This was used in the computations mentioned above, and leads to one of the two applications of unstable cohomology of mapping class groups: intersection theory on $\mathcal{M}_g$.

Our paper adds another tool: the failure of homological stability itself stabilises. This uses that there exists a homology class $k \in H_2(B\Gamma_{3,1};\mathbb{Q})$ dual to $\kappa_1 \in H^2(B\Gamma_{3,1};\mathbb{Q})$ in the sense that $\langle \kappa, k\rangle = 1$ with $\langle -,-\rangle$ the pairing between homology and cohomology.

Theorem 1.9 (Galatius–K.–Randal-Williams). *There is a map*

$$k \cdot -\colon H_{d-2}(B\Gamma_{g-2,1}, B\Gamma_{g-3,1};\mathbb{Q}) \longrightarrow H_d(B\Gamma_{g+1,1}, B\Gamma_{g,1};\mathbb{Q})$$

which is an isomorphism for $* \leq \frac{4g-6}{5}$ *and a surjection for* $* \leq \frac{4g-1}{5}$.

The orange region in Fig. 2 is that between the stable range and this "secondary stability range", and the entries given in it are a consequence of our theorem. In particular, we can use it to compute:

Example 1.10. $H^3(B\Gamma_{4,1};\mathbb{Q}) = 0$.

In the remainder of this talk, we will explain how to prove Theorem 1.9 by exploiting a higher-algebraic structure present on $\bigsqcup_g B\Gamma_{g,1}$.

2. E_2-algebras

The space $\mathbf{R} = \bigsqcup_{g\geq 0} B\Gamma_{g,1}$ has additional structure, which we implicitly used already when considering the stabilisation map σ:

(1) A grading by g, and to keep track of this we think of $\mathbf{R}$ as an object in $\mathsf{Fun}(\mathbb{N},\mathsf{Top})$: $g \mapsto B\Gamma_{g,1}$.
(2) An E_2-algebra structure.

Definition 2.1. $E_2(k)$ be the space of maps $\bigsqcup_k [0,1]^2 \hookrightarrow [0,1]^2$ so that (i) on each square the map is a composition of scaling and translation, (ii) the images of the squares have disjoint interior.

By cutting out the interior of each of the k squares and gluing in surfaces of genus $g_1,\ldots,g_k$, we obtain a surface of genus $g_1+\cdots+g_k$. By taking diffeomorphisms of each of these surfaces and extending them by the identity on the complement of the squares, we obtain a map

$$E_2(k) \times B\Gamma_{g_1,1} \times \cdots \times B\Gamma_{g_k,1} \longrightarrow B\Gamma_{g_1+\cdots+g_k,1}.$$

Example 2.2. Picking a point in $E_2(2)$ as in Fig. 3, we get a *multiplication* map

$$B\Gamma_{g_1,1} \times B\Gamma_{g_2,1} \longrightarrow B\Gamma_{g_1+g_2,1}.$$

Taking $g_1 = g$ and $g_2 = 1$, and fixing a point in $B\Gamma_{1,1}$, this yields a model for the stabilisation map

$$\sigma\colon B\Gamma_{g,1} = B\Gamma_{g,1} \times \{*\} \subset B\Gamma_{g,1} \times B\Gamma_{1,1} \longrightarrow B\Gamma_{g+1,1}.$$

This is commutative up to homotopy, by moving the squares such that they switch location. This homotopy is not unique, as we can move the first cube *over* the second or move it *under* the second.

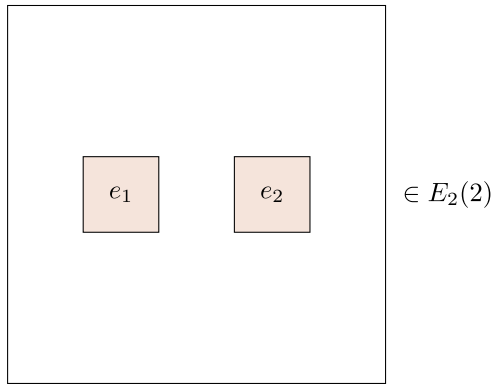

FIGURE 3. An element of $E_2(3)$.

We can combine these for all $g_1, \ldots, g_k$ into a morphism

$$E_2(k) \otimes \mathbf{R}^{\otimes k} \longrightarrow \mathbf{R}$$

in the category $\mathsf{Fun}(\mathbb{N}, \mathsf{Top})$, where $\otimes$ is given by Day convolution. There is an action of the symmetric group $\mathfrak{S}_k$ on $E_2(k)$, and composition maps $E_2(\ell) \times E_2(k_1) \times \cdots \times E_2(k_\ell) \to E_2(k_1 + \cdots + k_\ell)$ which are suitably equivariant. We say that $\{E_2(k)\}_{k \geq 1}$ has the structure of an *operad*, the (non-unital) E_2-operad.

Remark 2.3. Operads encode algebraic structures, this one encoding an algebraic structure which has a multiplication that is somewhat commutative (E_1 being associative and E_∞ being commutative): the "space of multiplications" $E_2(2)$ is not contractible but homotopy equivalent to a circle.

The morphisms described above then assemble to an E_2-algebra structure on $\mathbf{R} \in \mathsf{Fun}(\mathbb{N}, \mathsf{Top})$. Theorem 1.9 will be proven by building a CW-approximation to $\mathbf{R}$ *as an E_2-algebra*.

3. Building E_2-algebra

The category of E_2-algebras is well-understood. We can construct free E_2-algebras (keeping track of some grading if desired):

$$\mathsf{Fun}(\mathbb{N}, \mathsf{Top}) \ni X \longmapsto F^{E_2}(X) \in \mathsf{Alg}_{E_2}(\mathsf{Fun}(\mathbb{N}, \mathsf{Top})).$$

This is the left adjoint to the forgetful functor $U^{E_2} \colon \mathsf{Alg}_{E_2}(\mathsf{Fun}(\mathbb{N}, \mathsf{Top})) \to \mathsf{Fun}(\mathbb{N}, \mathsf{Top})$. The homology of free E_2-algebra with coefficients in a field $\Bbbk$ was computed by F. Cohen [CLM76]:

$$H_{*,*}(F^{E_2}(X); \Bbbk) \cong \text{free Dyer–Lashof algebra on } H_{*,*}(X; \Bbbk)$$

where $H_{g,d}(X) = H_d(X(g))$ (so the first entry is keeping track of the *genus* $g \in \mathbb{N}$ and the second entry is the *homological degree* $d \in \mathbb{Z}$).

Example 3.1. Taking $\Bbbk = \mathbb{Q}$ we have that $H_{*,*}(F^{E_2}(D^{1,0}); \mathbb{Q})$ is the free 1-*Poisson algebra* on a single generator in bidegree $(g, d) = (1, 0)$; this is the free graded-commutative algebra on σ in bidegree $(g, d) = (1, 0)$ and $[\sigma, \sigma]$ in bidegree $(g, d) = (2, 1)$.

We want to import the theory of CW-complexes into the category of E_2-algebras. To do so, we first need the analogue of a cell attachment. To do so we recall a cell attachment is the left pushout diagram below, and the right diagram is the input for a cell attachment to E_2-algebras:

$$\begin{array}{ccc} S^{d-1} & \xrightarrow{e} & X \\ \downarrow & & \downarrow \\ D^d & \longrightarrow & X \cup_e D^d \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} S^{g,d-1} & \xrightarrow{e} & U^{E_2}(\mathbf{R}) \\ \downarrow & & \\ D^{g,d} & & \end{array} .$$

The right term is a diagram in $\mathsf{Fun}(\mathbb{N}, \mathsf{Top})$, with $S^{g,d-1}$ the $(d-1)$-sphere in genus g and similarly for $D^{g,d}$. We can use the adjunction $F^{E_2} \dashv U^{E_2}$ to transform this into a diagram of E_2-algebras and obtain an *E_2-cell attachment* when we take the pushout

$$\begin{array}{ccc} F^{E_2}(S^{g,d-1}) & \xrightarrow{F^{E_2}(e)} & \mathbf{R} \\ \downarrow & & \downarrow \\ F^{E_2}(D^{g,d}) & \longrightarrow & \mathbf{R} \cup_e^{E_2} D^{g,d}. \end{array}$$

A *CW-E_2-algebra* is one obtained by iterated cell attachments (in order of dimension). In topological spaces, you can approximate a space X by a CW-complex. Moreover, there is a relative version: given a map $f\colon X \to Y$ you can attach cells to X and extend the map over these to a weak equivalence to Y. If X and Y are simple spaces and $H_{d'}(Y, X)$ vanishes $d' < d$ you only need cells of dimensions $\geq d$.

We proved a similar (relative) CW-approximation theorem for E_2-algebras. Instead of restricting our attention to simple spaces, we replace the category of spaces Top with the category of simplicial $\mathbb{k}$-modules $\mathsf{sMod}_{\mathbb{k}}$, which of a more algebraic nature. This amounts to replacing $\mathbf{R}$ with the simplicial $\mathbb{k}$-algebra $\mathbf{R}_{\mathbb{k}} \coloneqq \mathbb{k}[\mathrm{Sing}(\mathbf{R})]$ but this suffices since in the end we care about homology, which you can recover as the homotopy groups of $\mathbf{R}_{\mathbb{k}}$. To obtain a bound on the dimensions of the E_2-cells you need to attach in a relative CW-approximation, homology is replaced by E_2-homology: if $f\colon \mathbf{R} \to \mathbf{S}$ is a map in $\mathsf{Alg}_{E_2}(\mathsf{Fun}(\mathbb{N}, \mathsf{sMod}_k))$ then $H^{E_2}_{g',d'}(\mathbf{S}, \mathbf{R})$ vanishes for $g' < g$ or $d' < d$, you only need E_2-cells of dimensions $\geq d$ in genus $\geq g$. In particular, whenever you add an E_2-cell in bidegree (g, d) to your E_2-algebra, there is a long exact sequence in E_2-homology

$$\cdots \longrightarrow H^{E_2}_{g',d'}(\mathbf{R}) \longrightarrow H^{E_2}_{g',d'}(\mathbf{R} \cup^{E_2} D^{g,d}) \longrightarrow \begin{cases} \mathbb{Z} & \text{if } (g', d') = (g, d) \\ 0 & \text{otherwise} \end{cases} \longrightarrow \cdots$$

To make this theory useful, we need to be able to compute E_2-homology of $\mathbf{R}$. This is done using bar constructions, and often implemented in two steps:

(1) For many examples, including the one of mapping class groups, one may compute E_1-homology in terms of a simplicial complex; the *E_k-splitting complexes.*
(2) One may compute the E_2-homology from the E_1-homology: using a *bar spectral sequence.*

4. Secondary homological stability for mapping class groups

Let us return to the example of mapping class groups. We will work with rational coefficients $\Bbbk = \mathbb{Q}$ for the sake of simplicity. We saw there was an E_2-algebra $\mathbf{R}\colon g \mapsto B\Gamma_{g,1}$ in $\mathsf{Fun}(\mathbb{N}, \mathsf{Top})$ of mapping class groups, and form $\mathbf{R}_\mathbb{Q} \in \mathsf{Alg}_{E_2}(\mathsf{Fun}(\mathbb{N}, \mathsf{sMod}_\mathbb{Q}))$, which records the rational homology of mapping class groups. Using the procedure described above we obtain a E_1-splitting complexes whose connectivity can be computed using techniques used to study arc complexes, and a bar spectral sequence may be used to deduce from this that

$$H^{E_2}_{g,d}(\mathbf{R}_\mathbb{Q}) = 0 \qquad \text{for } d < g - 1.$$

This says we only need E_2-cells above a certain line.

Our strategy will be to build a small E_2-algebra $\mathbf{A}$ mapping to $\mathbf{R}$, which captures its behavior in low genus and low degrees, and then invoke CW-approximation. The idea is that since $\mathbf{R}$ will be obtained from $\mathbf{A}$ by only attaching E_2-cells of slope $\frac{g}{d} > \lambda$, $\mathbf{A}$ captures all the homological stability behavior below this line.

Let's see how to get Harer stability, Theorem 1.4, and a weaker version of Theorem 1.9 with slope $\frac{3}{4}$ instead of $\frac{4}{5}$ (in actuality, we use this weak version to deduce $H_3(B\Gamma_{4,1}; \mathbb{Q}) = 0$ and rerun the below argument with this input to get the better slope).

Let us look at our unstable homology chart again, Fig. 2, and reason as follows:

- To get all the generators in $H_{g,0}$ for $g \geq 0$, we need a single E_2-cell in bidegree $(1, 0)$ that we denote as σ. Thus our first step is just the free E_2-algebra $F^{E_2}(\mathbb{Q} \cdot \sigma)$.
- Its homology is the free 1-Poisson algebra, which in particular has $[\sigma, \sigma] \in H_{2,1}(F^{E_2}(\mathbb{Q}\sigma))$ but there is no non-zero class in $H_1(B\Gamma_{2,1}; \mathbb{Q})$. Thus we add an E_2-cell ρ in bidegree $(2, 2)$ to kill it. Thus our second step is $F^{E_2}(\mathbb{Q} \cdot \sigma) \cup^{E_2}_{[\sigma,\sigma]} \mathbb{Q} \cdot \rho$.
- Finally, we compute that $H_{3,2}(F^{E_2}(\mathbb{Q} \cdot \sigma) \cup^{E_2}_{[\sigma,\sigma]} \mathbb{Q} \cdot \rho) = 0$ so our class $\lambda \in H_2(B\Gamma_{3,1}; \mathbb{Q})$ is missing. We add an E_2-cell λ in bidegree $(3, 2)$ to generate k. Thus our final step is

  $$\mathbf{A} \coloneqq F^{E_2}(\mathbb{Q} \cdot \sigma) \cup^{E_2}_{[\sigma,\sigma]} \mathbb{Q} \cdot \rho \cup^{E_2}_0 \mathbb{Q} \cdot \lambda.$$

  In slope $< \frac{3}{4}$, it looks like the free graded-commutative algebra on σ, k (e.g. $[\sigma, \lambda]$ lies in bidegree $(4, 3)$).

We have built $\mathbf{A} \to \mathbf{R}$ so that it is an isomorphism in bidegree (g, d) with $g \leq 3$ and $d \leq 2$. Using a Hurewicz theorem for E_2-homology, we get that $H^{E_2}_{g,d}(\mathbf{R}, \mathbf{A}) = 0$ for $g \leq 3$ and $d \leq 2$. Using the long exact sequence

$$\cdots \longrightarrow H^{E_2}_{g,d}(\mathbf{A}) \longrightarrow H^{E_2}_{g,d}(\mathbf{R}) \longrightarrow H^{E_2}_{g,d}(\mathbf{R}, \mathbf{A}) \longrightarrow \cdots$$

we compute the group $H^{E_2}_{g,d}(\mathbf{R}, \mathbf{A})$ of lowest slope $\frac{d}{g}$ is $(g, d) = (5, 4)$. This can be used to prove that $\mathbf{A}$ and $\mathbf{R}$ exhibit the same homological stability patterns in the range $\frac{d}{g} < \frac{5}{4}$. But we see that below slope $\frac{2}{3}$ everything multiplication by σ is an isomorphism, and after taking the quotient by σ below slope $\frac{3}{4}$ multiplication by λ is an isomorphism.

References

[CLM76] F. R. Cohen, T. J. Lada, and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 533, Springer-Verlag, Berlin-New York, 1976. MR 0436146 5

[EE69] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry **3** (1969), 19–43. MR 276999 1

[ES70] C. J. Earle and A. Schatz, *Teichmüller theory for surfaces with boundary*, J. Differential Geometry **4** (1970), 169–185. MR 277000 1

[GKRW18] S. Galatius, A. Kupers, and O. Randal-Williams, *Cellular E_k-algebras*, https://arxiv.org/abs/1805.07184v2, 2018. 1

[GKRW19] ———, *E_2-cells and mapping class groups*, Publ. Math. Inst. Hautes Études Sci. **130** (2019), 1–61. 1, 3