# Beurling's criterion and systolic inequalities

22 July 2020 14:09

## 1. Extremal length

$X$ — Riemann surface, perhaps with boundary $\to$ conformal structure

Def: A **conformal metric** on X is a measurable function $\rho: TX \to [0,\infty)$ such that

$$\rho(\lambda v) = |\lambda|\rho(v)$$

for every $v \in TX$ and $\lambda \in \mathbb{C}$. $\to$ define lengths, area

angles + size $\to$ Riemannian metric ?

$\Gamma$ — a set of paths or closed curves in X

If $\rho$ is a conformal metric and $\gamma \in \Gamma$, then we define the **length** of $\gamma$ with respect to $\rho$ as

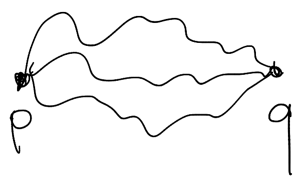
$$\ell(\gamma,\rho) = \int_\gamma \rho = \int \rho(\gamma'(t))\,dt$$

if the integral makes sense, $\ell(\gamma,\rho) = \infty$ otherwise, and

$$\ell(\Gamma,\rho) = \inf\{\, \ell(\gamma,\rho) : \gamma \in \Gamma \,\}$$

is the **length of the path family**.

$\Gamma$ = all paths btw $p$ & $q$



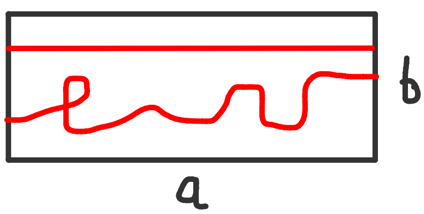

then $\ell(\Gamma,\rho) = d(p,q)$ in metric $\rho$.

Def: The **extremal length** of $\Gamma$ is defined as

$$\mathrm{EL}(\Gamma, X) = \sup_\rho \frac{\ell(\Gamma,\rho)^2}{\mathrm{area}(\rho)} \leftarrow \text{invariant under scalings.}$$

Where the sup is over all conformal metrics $\rho$ on $X$ such that $0 < \mathrm{area}(\rho) < \infty$.

Key example: $X =$ rectangle, $\Gamma =$ all paths joining the two vertical sides




Then the metric $\rho$ realizing the supremum in the definition of $\mathrm{EL}(\Gamma, X)$ is simply the Euclidean metric, which gives

$$\mathrm{EL}(\Gamma, X) = \frac{\ell(\Gamma,\rho)^2}{\mathrm{area}(\rho)} = \frac{a^2}{ab} = \frac{a}{b} = \text{aspect ratio of the rectangle.}$$

We'll prove this as a special case of Beurling's criterion for the extremality of the metric $\rho$.

## 2. Beurling's criterion

Def: A metric $\rho_0$ on $X$ is **evenly covered by shortest paths** of $\Gamma$ if

- There is a non-empty subset $\Gamma_0 \subset \Gamma$ of shortests paths, i.e., such that

$$\ell(\gamma, \rho_0) = \underbrace{\ell(\Gamma, \rho_0)}_{\hookrightarrow \text{ infimum of lengths of curves in } \Gamma} \quad \text{for every } \gamma \in \Gamma_0.$$

- There is a measure $\mu$ on $\Gamma_0$ such that

$$\rho_0^2 = (\rho_0 \text{ along } \gamma) \times d\mu$$

i.e., to integrate against $\rho_0^2$ over $X$, we can apply an interated integral (Fubini).

Example: In the rectangle example, the Euclidean metric $\rho_0$ is evenly covered by the horizontal paths.

$\hookrightarrow d\mu = dy$ = vertical transv. measure.

Theorem (Beurling's criterion): Let $\rho_0$ be a conformal metric on $X$ that is evenly covered by shortest paths of $\Gamma$. Then $\rho_0$ realizes the supremum in the definition of $\mathrm{EL}(\Gamma, X)$, that is,

$$\mathrm{EL}(\Gamma, X) = \frac{\ell(\Gamma, \rho_0)^2}{\mathrm{area}(\rho_0)}.$$

Pf First note that

$$\mathrm{area}(\rho_0) = \int_X \rho_0^2 = \int_{\Gamma_0} \left( \int_\gamma \rho_0 \right) d\mu(\gamma)$$

$$= \int_{\Gamma_0} \ell(\gamma, \rho_0)\, d\mu(\gamma)$$

$$= \int_{\Gamma_0} \ell(\Gamma, \rho_0)\, d\mu$$

$$= \ell(\Gamma, \rho_0) \cdot \mu(\Gamma_0)$$

"area = length · height"

Let $\rho$ be any competing metric with $0 < \mathrm{area}(\rho) < \infty$.

We need to show

$$\frac{\ell(\Gamma, \rho)^2}{\mathrm{area}(\rho)} \leq \frac{\ell(\Gamma, \rho_0)^2}{\mathrm{area}(\rho_0)}.$$

$$\frac{\ell(\Gamma',\rho)^2}{\text{area}(\rho)} \leq \frac{\ell(\Gamma',\rho_0)^2}{\text{area}(\rho_0)}.$$

We have

$$\ell(\Gamma,\rho) \leq \ell(\Gamma_0,\rho) \leq \int_\gamma \rho \quad \text{for every } \gamma \in \Gamma_0$$

integrate over $\Gamma_0$ $\Rightarrow$

$$\ell(\Gamma,\rho)\cdot\mu(\Gamma_0) \leq \int_{\Gamma_0} \left(\int_\gamma \rho\right) d\mu(\gamma)$$

$$= \int_{\Gamma_0} \int_\gamma \frac{\rho}{\rho_0}\cdot \overbrace{\rho_0 \; d\mu(\gamma)}^{\rho_0^2}$$

$$= \int_X \frac{\rho}{\rho_0}\cdot\rho_0^2 = \int_X \rho\cdot\rho_0$$

$$\underset{C-S}{\leq} \sqrt{\int_X \rho^2}\cdot\sqrt{\int_X \rho_0^2}$$

$$= \sqrt{\text{area}(\rho)}\cdot\sqrt{\text{area}(\rho_0)}$$

$$\Rightarrow \quad \frac{\ell(\Gamma,\rho)^2}{\text{area}(\rho)} \leq \frac{\text{area}(\rho_0)}{\mu(\Gamma_0)^2}$$

$$= \frac{\text{length}\cdot\text{height}}{\text{height}^2} = \frac{\text{length}}{\text{height}} = \frac{\text{length}^2}{\text{length}\cdot\text{height}}$$

$$= \ell(\Gamma,\rho_0)^2$$

$$= \frac{\ell(\Gamma, \rho_0)^2}{\mathrm{area}(\rho_0)}. \quad \square$$

More examples: 0) The rectangle example.

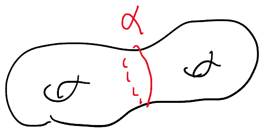
1) $X$ any Riemann surface, $\Gamma = [\,\alpha\,]$ where $\alpha \subset X$ is an essential simple closed curve.

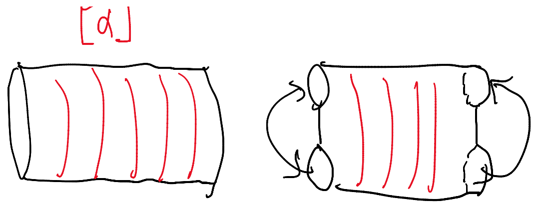
Then a theorem of Jenkins/Strebel says that there is a quadratic differential $q$ that makes $X$ isometric to a Euclidean cylinder modulo some gluings along the boundary.

The induced conformal metric is evenly covered by shortest curves in $[\,\alpha\,]$ so that

$$\mathrm{EL}([\alpha], X) = \frac{\text{circumference}^2}{\text{area}} = \frac{\text{circumference}}{\text{height}} = \frac{1}{\text{modulus}}$$

by Beurling's criterion.




A special case of this is when $X$ is a flat torus. Then $\mathrm{EL}([\alpha], X) =$ length^2 / area for the flat metric.

2) $X = \mathbb{R}P^2 = \mathbb{S}^2$ / antipodal map, $\Gamma =$ all non-contractible curves in $X$.

Then the spherical metric is evenly covered by quotients of great circles, which are shortest in $\Gamma$.

By Beurling's criterion,

$$\mathrm{EL}(\Gamma, X) = \frac{\pi^2}{2\pi} = \frac{\pi}{2}.$$

## 3. Systolic inequalities

$(M, g)$ — compact Riemannian $n$-manifold

Systole: $\mathrm{sys}(M, g) =$ length of shortest non-contractible curve in $(M, g)$

Systolic ratio: $\mathrm{SR}(M, g) = \frac{\mathrm{sys}(M,g)^n}{\mathrm{vol}(M,g)}$ (invariant under scaling)

Question (Berger,Gromov): Given a smooth manifold $M$, which Riemannian metrics on $M$ maximize the systolic ratio?

Answer known only if $M \in \{\, 2-\text{torus},\ \text{projective plane},\ \text{Klein bottle}\,\}$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

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Relationship with extremal length:

We can stratify the space of Riemannian metrics on $M$ by conformal classes:

$\sup\{\mathrm{SR}(M, \mathrm{g}) : \mathrm{g}\text{ is a Riemannian metric on } M\} = \sup\{\sup\{\mathrm{SR}(M, \mathrm{g}) : g \in c\} : c \text{ is a conformal class of metrics on } M\}$

so we can try to solve the problem for each conformal class, then maximize over moduli space.

For a fixed conformal structure X (or conformal class c) on M we have

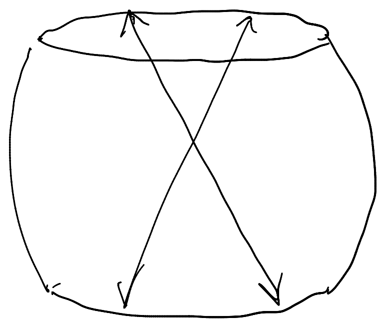
$$\sup\{\mathrm{SR}(M, \mathrm{g}) : g \in c\} = \sup_{g\in c} \frac{\mathrm{sys}(M,g)^n}{\mathrm{vol}(M,g)} = \sup_\rho \frac{\ell(\Gamma_{ess},\rho)^n}{\mathrm{vol}(\rho)} = \mathrm{EL}(\Gamma_{ess}, X).$$

where $\Gamma_{ess} =$ all non-contractible curves in $X$.

$$\ell(\Gamma_{ess}, \rho) = \inf\{\text{length}(\gamma, \rho) : \gamma \in \Gamma_{ess}\}$$

$$= \text{inf length of non-contr. curves}$$

$$= sys(X, \rho).$$



$$EL\ sys = \inf_{\alpha} EL(\alpha, X)$$

$$= \inf_{\alpha} \sup_{\rho} \frac{\ell^2}{\text{area}}$$

$$\max SR = \sup_{\rho} \inf_{\alpha} \frac{\ell^2}{\text{area}}$$