# McMullen's approach to minimal volume entropy of graphs

Matt Clay

McMullen, C. "Entropy & the clique polynomial." Journal of Topology 8 (2015), 184-212.

$(X,g)$ - Riemannian manifold

$(\tilde{X},\tilde{g})$ - universal cover of $X$ equipped w/ pullback metric

<u>Volume entropy</u>:

$$\mathrm{ent}(X,g) = \lim_{T\to\infty} \frac{1}{T} \log \mathrm{vol}\, \underbrace{B_{\tilde{g}}(x,T)}_{\text{ball of radius } T \text{ centered at } x\in\tilde{X}}$$

Introduced by: Efremovich, Švarc, Milnor

Related to growth of fundamental group & dynamics of geodesic flow

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Ex: (closed surface, $g_{hyp}$), universal cover is $\mathbb{H}^2$

$$\mathrm{vol}\, B_{\mathbb{H}^2}(x,T) = 4\pi \sinh^2(T/2) \approx e^T$$

$$\Rightarrow \mathrm{ent}(\text{closed surface}, g_{hyp}) = 1$$

Question: Given $X$, what metric minimizes volume entropy?

not an interesting question as volume entropy scales:

$$\mathrm{ent}(X, \alpha\cdot g) = \frac{1}{\alpha}\,\mathrm{ent}(X, g) \quad \Big| \quad B_{\widetilde{\alpha\cdot g}}(x, T) = B_{\tilde{g}}(x, \tfrac{1}{\alpha}T)$$

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Counteract scaling by multiplying by volume:

$$\omega(X, g) = \mathrm{ent}(X, g)\,\mathrm{vol}(X, g)^{1/m} \quad \Big| \quad m = \dim X$$

<u>minimal volume entropy</u>

$$\omega(X) = \inf_g \omega(X, g)$$

Introduced by Gromov

$$\omega(X) = \inf_{g} \underbrace{\mathrm{ent}(X,g)\,\mathrm{vol}(X,g)^{1/m}}_{\omega(X,g)}$$

Facts:

① $\omega(\text{closed surface}) = \omega(\text{closed surface}, g_{hyp}) = \sqrt{2\pi|\chi|}$ (Katok)

② $\omega(\text{closed hyperbolic } m\text{-manifold}) = \omega(\text{closed hyperbolic } m\text{-manifold}, g_{hyp})$

(Besson - Courtois - Gallot)

③ $\omega(X)^m \geq c_m \|X\|_\Delta$ ← simplicial volume (Gromov)

proportional in dim $\leq 3$

Open question: $\exists\, d_m > 0$ s.t. $\|X\|_\Delta \geq d_m \omega(X)^m$? In particular does $\omega(X) > 0$ imply $\|X\|_\Delta > 0$?

Apply definition to $(X,g)$ where $X$ is a simplicial complex and $g$ is a piecewise Riemannian metric.

$$\omega(X) = \inf_{g} \underbrace{\mathrm{ent}(X,g)\,\mathrm{vol}(X,g)^{1/m}}_{\omega(X,g)} \qquad \Big|\ m = \dim X$$

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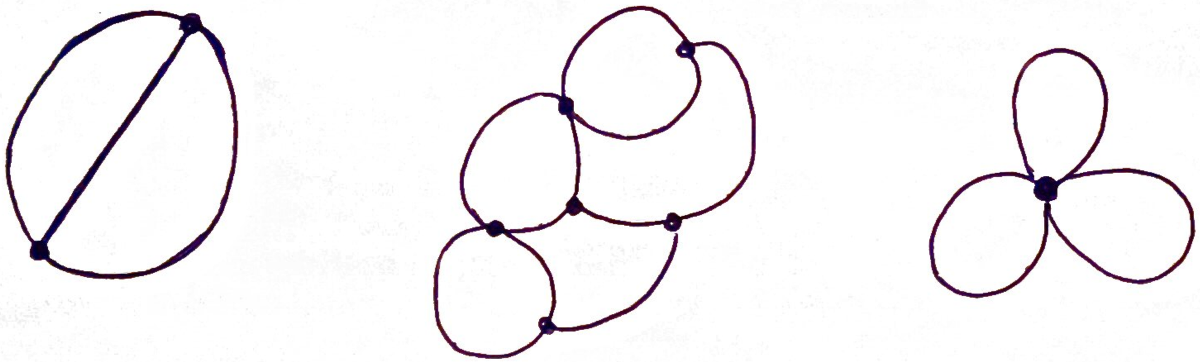
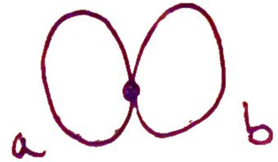
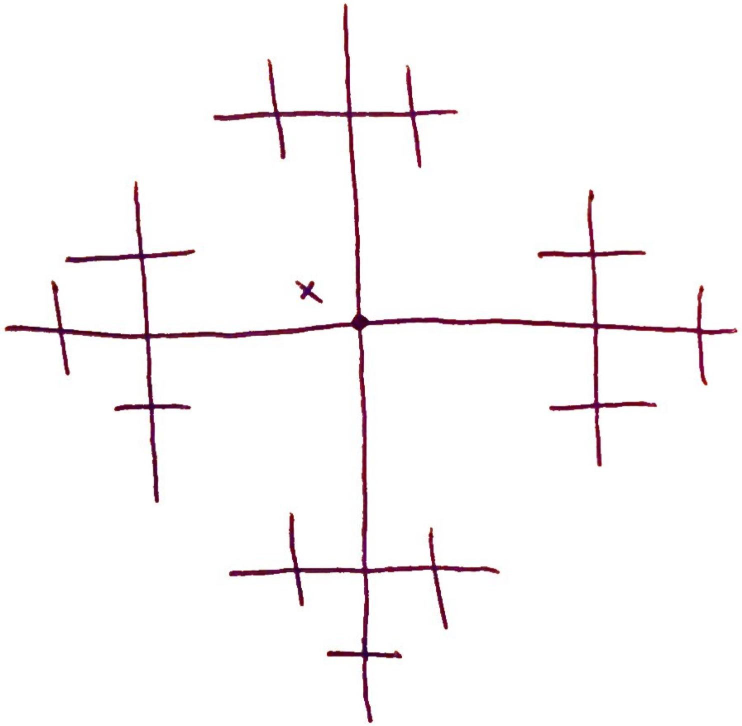
We will consider the case when $X$ is a graph:



Ex: $G =$ 


$g(a) = g(b) = 1 \Rightarrow \mathrm{vol}(G,g) = 2$

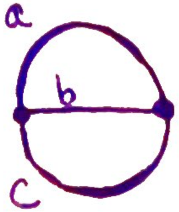
$\tilde{G}$: 


$$\mathrm{vol}\, B_{\tilde{g}}(x,T) = \sum_{i=0}^{T-1} 4 \cdot 3^i = 2(3^T - 1)$$

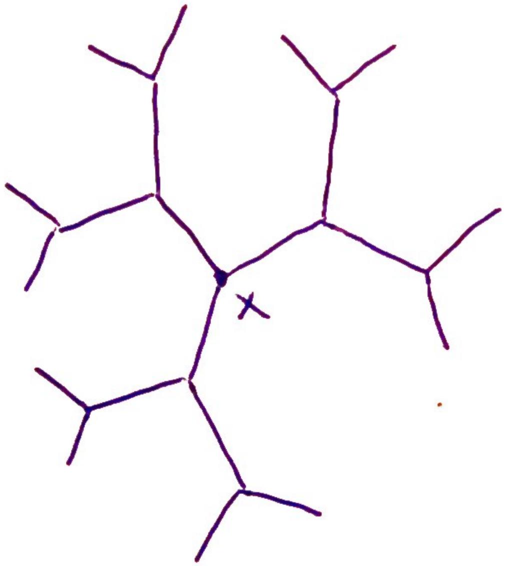
$\Rightarrow \mathrm{ent}(G,g) = \log 3$

$\Rightarrow \omega(G,g) = \mathrm{ent}(G,g)\,\mathrm{vol}(G,g) = 2\log 3$

In fact, $\omega(G) = 2\log 3$ say $g$ is an <u>optimal</u> <u>metric</u>.

Ex: $G =$ 


$g(a) = g(b) = g(c) = 1 \Rightarrow \mathrm{vol}(G,g) = 3$

$\tilde{G}$: 


$$\mathrm{vol}\, B_{\tilde{g}}(x,T) = \sum_{i=0}^{T-1} 3 \cdot 2^i = 3(2^T - 1)$$

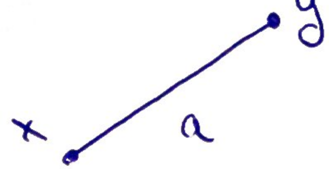
$$\Rightarrow \mathrm{ent}(G,g) = \log 2$$

$$\Rightarrow \omega(G,g) = \mathrm{ent}(G,g)\,\mathrm{vol}(G,g) = 3\log 2$$

In fact, $\omega(G) = 3\log 2$.

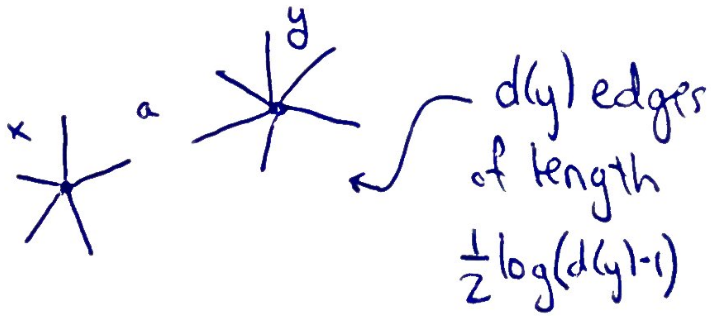
Thm (Lim '08, McMullen '15) Let $G$ be a finite connected graph where all vertices $x \in VG$ have degree $d(x) \geq 3$. Then $G$ carries a unique optimal metric up to scale given by:

$$g(a) = \log \sqrt{(d(x)-1)(d(y)-1)}$$




Further, $\mathrm{ent}(G, g) = 1$ and thus:

$$w(G) = \frac{1}{2} \sum_{x \in VG}' d(x) \log(d(x)-1)$$



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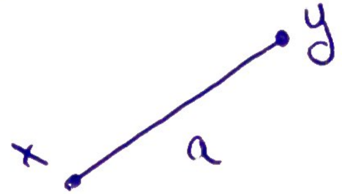


Further, $\mathrm{ent}(G,g) = 1$ and thus:

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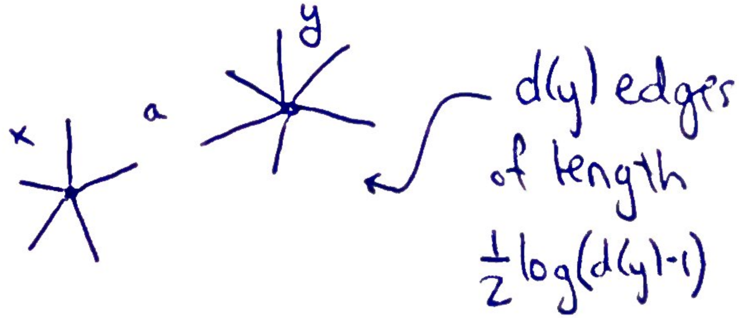


When $G$ is $d$-regular (i.e. $d(x) = d$ $\forall x \in VG$) this result was shown by I. Kapovich-Nagnibeda '07.

Identify metrics on $G$ with $MG = (\mathbb{R}_{>0})^{EG}$ & consider functions:

$$\mathrm{ent}: MG \to \mathbb{R} \qquad \mathrm{ent}(g) = \mathrm{ent}(G,g)$$
$$\mathrm{vol}: MG \to \mathbb{R} \qquad \mathrm{vol}(g) = \mathrm{vol}(G,g) = \sum_{a \in EG}{}' g(a)$$

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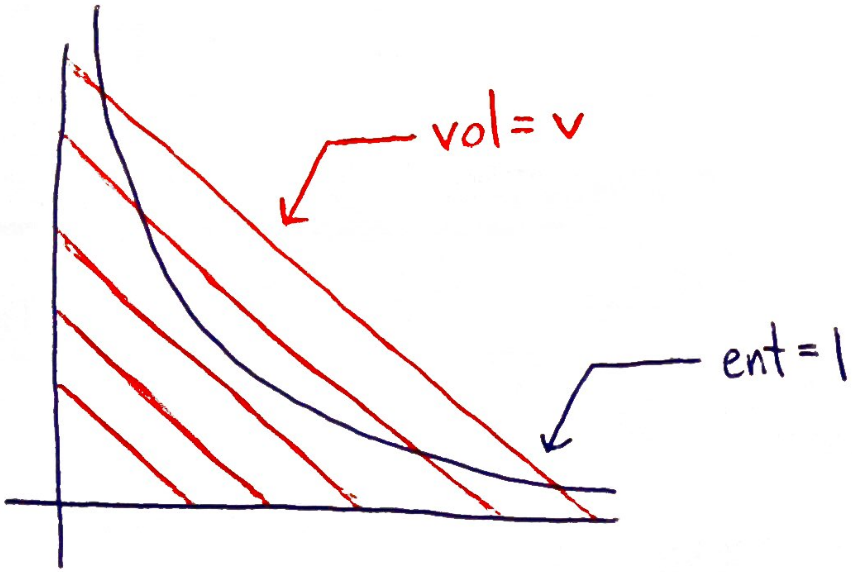
Thm: $\mathrm{ent} : MG \to \mathbb{R}$ is real-analytic & strictly convex.

MG




- at most one minimum to $\mathrm{vol}(\cdot)$ on $\mathrm{ent} = 1$.
- occurs at $\nabla \mathrm{ent}(g) \,//\, \nabla \mathrm{vol}(g)$.

$\Rightarrow$ want to compute $\nabla \mathrm{ent}(\cdot)$ and find metric where all components are equal

$E^{\pm}G$ = oriented edges in $G$, i.e., an edge $a \in EG$ and a direction

For $a, b \in E^{\pm}G$ we write <u>$a \triangleleft b$</u> if $\tau(a) = o(b)$ & $\overline{a} \neq b$:


A <u>closed geodesic</u> is a sequence $(a_1, a_2, \dots, a_n)$ with $a_i \triangleleft a_{i+1}$ (mod $n$).

For a metric $g \in MG$ and a geodesic $\gamma = (a_1, \dots, a_n)$ we set:

$$g(\gamma) = \sum_{i=1}^{n} g(a_i)$$

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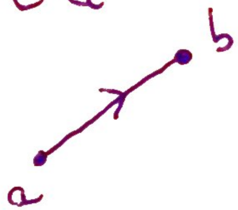
Fact: $\operatorname{ent}(g) = \lim_{T \to \infty} \frac{1}{T} \log \#\{\text{closed geodesics } \gamma \mid g(\gamma) \leq T\}$.

Count geodesics in $(G,g)$:

$\Gamma$ = directed graph

$V\Gamma = E^{\pm}G$

$E\Gamma$ =




if $a \triangleleft b$

$$A \in \mathrm{Mat}_{E^{\pm}G}(\mathbb{R})$$

$$A(a,b) = \begin{cases} 1 & \text{if } a \triangleleft b \\ 0 & \text{else} \end{cases}$$

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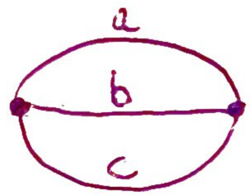
$G =$ 


$+$ : to right

$-$ : to left

$\Gamma$ = directed graph

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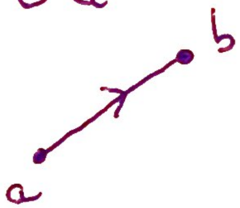
$E\Gamma =$ 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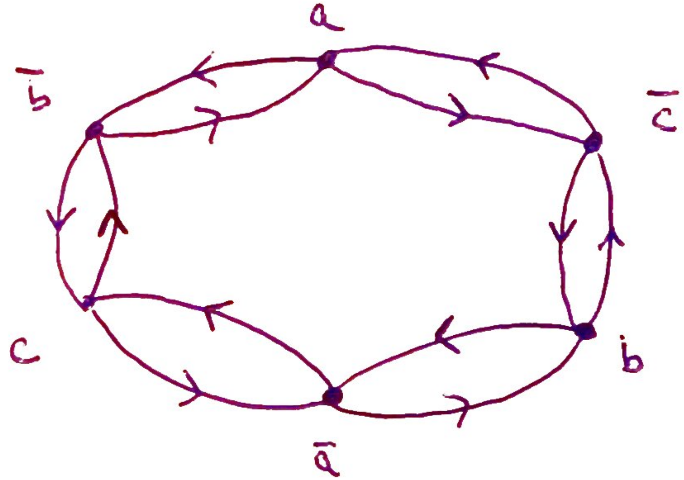
$\Gamma =$ 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$$A = \begin{array}{c} \begin{array}{cccccc} a & b & c & \bar{a} & \bar{b} & \bar{c} \end{array} \\ \left[\begin{array}{ccc|ccc} & & & 0 & 1 & 1 \\ & 0 & & 1 & 0 & 1 \\ & & & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & & & \\ 1 & 0 & 1 & & 0 & \\ 1 & 1 & 0 & & & \end{array}\right] \end{array}$$

closed geodesic in $G$ $\longleftrightarrow$ directed closed path in $\Gamma$

$$\Rightarrow \#\{\text{closed geodesic } \gamma \mid |\gamma| \leq T\} = \sum_{n=1}^{T} \operatorname{tr}(A^n)$$

combinatorial length

$$\Rightarrow \operatorname{ent}(G, |\cdot|) = \lim_{T\to\infty} \log \sum_{n=1}^{T} \operatorname{tr}(A^n) = \log \operatorname{spec}(A)$$

spectral radius

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spectral radius

How do we encorporate the metric $g \in MG$?

Use tools of the thermodynamic formalism as developed by

Bowen, Parry, Pollicott, Ruelle

For $s \in \mathbb{R}$ we define a new matrix $e^{-sg}A$ by:

$$e^{-sg}A(a,b) = e^{-sg(a)}A(a,b)$$, i.e., multiply row "a" by $e^{-sg(a)}$

$$\Rightarrow \sum_{n=1}^{T} \mathrm{tr}(e^{-sg}A)^n = \sum_{n=1}^{T} \sum_{|\gamma|=n} e^{-sg(\gamma)}$$

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Rearrange terms by g-length: $L_g : \mathbb{N} \to \mathbb{R}$ length spectrum of g

$$\sum_{n=1}^{\infty} \operatorname{tr}(e^{-sg}A)^n = \sum_{n=1}^{\infty} \#\{\text{closed geodesics } \gamma \mid g(\gamma) = L_g(n)\}\, e^{-L_g(n)s}$$

$$f(s) = \sum_{n=1}^{\infty} \mathrm{tr}\left(e^{-sg}A\right)^n = \sum_{n=1}^{\infty} \#\{\text{closed geodesics } \gamma \mid g(\gamma) = L_g(n)\} e^{-L_g(n)s}$$

$\uparrow$

generalized Dirichelt series $\sum_{n=1}^{\infty} c_n e^{-\lambda_n s}$

Fact: $\sum_{n=1}^{\infty} c_n e^{-\lambda_n s}$ converges for $s > \sigma$ and diverges for $s \leq \sigma$ where

$$\sigma = \limsup \frac{\log|c_1 + c_2 + \cdots + c_n|}{\lambda_n} \qquad \left(\text{if } \sum c_n \text{ diverges}\right)$$

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In our case $|c_1 + c_2 + \cdots + c_n| = \#\{\text{closed geodesics } \gamma \mid g(\gamma) \leq L_g(n)\}$

$$\Rightarrow \mathrm{ent}(g) = \lim_{T\to\infty} \frac{1}{T} \log \#\{\text{closed geodesics } \gamma \mid g(\gamma) \leq T\} = \sigma$$

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Fact: If $\operatorname{spec}(B) < 1$, then $\sum_{n=1}^{\infty} \operatorname{tr} B^n = \sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_i}$ where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues for $B$.

- if $\operatorname{spec}(e^{-sg}A) < 1$, then $f(s)$ converges so $s > \sigma = \operatorname{ent}(g)$
- if $\operatorname{spec}(e^{-sg}A) \geq 1$, then $f(s)$ diverges so $s \leq \sigma = \operatorname{ent}(g)$

$\Rightarrow s = \operatorname{ent}(g)$ is the unique number such that $\operatorname{spec}(e^{-sg}A) = 1$.

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Define <u>pressure</u> $P : MG \to \mathbb{R}$ by $P(g) = \log \mathrm{spec}(e^{-g}A)$

Thm: $\mathrm{ent}(g) = 1 \iff P(g) = 0$. Equivalently, $\mathrm{ent}(g)$ is characterized by $P(\mathrm{ent}(g)g) = 0$.

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$\Rightarrow \nabla \mathrm{ent}(g) \parallel \nabla P(g)$ when $\mathrm{ent}(g) = 1$

$\Rightarrow$ want to compute $\nabla P(\cdot)$ and find a metric where all components are equal.

Computing $\nabla P(\cdot)$ : (Parry measure)

Let $\rho = \operatorname{spec}(e^{-g}A)$, $\exists$ positive eigenvector $v \in \mathbb{R}^{E^{\pm}G}$ with $e^{-g}Av = \rho v$

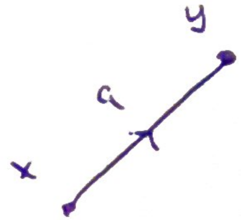
$$\Rightarrow \partial_a P(g) = \frac{e^{g(a)} v(a) v(\bar{a})}{\sum_{b \in EG} e^{g(b)} v(b) v(\bar{b})} = \frac{e^{g(a)} v(a) v(\bar{a})}{K_g}$$

Note: if $v \in \mathbb{R}^{E^{\pm}G}$ is positive and $e^{-g}Av = v$, then $\operatorname{spec}(e^{-g}A) = 1$ and thus

$P(g) = 0$.

Let's finish... set $f : V \to \mathbb{R}$, $f(x) = \sqrt{d(x)-1}$

Define $g \in MG$ and $v \in \mathbb{R}^{\vec{E}G}$ by:

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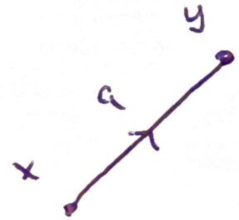


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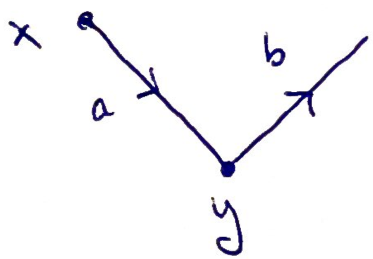
Check:

$$e^{-g}Av(a) = e^{-g(a)} \sum_{a \triangleleft b} v(b) = \frac{1}{f(x)f(y)} \frac{d(y)-1}{f(y)} = \frac{1}{f(x)} = v(a) \Rightarrow \boxed{e^{-g}Av = v}$$

↳ $\mathrm{ent}(g) = 1$



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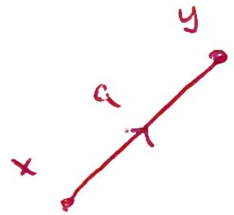


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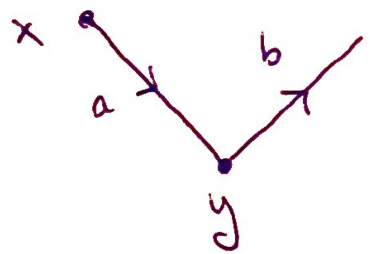


$$\partial_a P(g) = \frac{1}{K_g} e^{g(a)} v(a) v(\bar{a}) = \frac{1}{K_g} \, \frac{f(x)f(y)}{f(x)f(y)} = \frac{1}{K_g} \Rightarrow \boxed{\text{all components of } \nabla P(g) \text{ are equal.}}$$

↳ $g$ is optimal

Thank you