The many facets of Basmajian's identity

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Hyperbolic Lunch
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Overview

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hyperbolic geometry

S

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hyperbolic geometry $\longrightarrow$ Teichmüller space

S $\qquad$ $\rho : \pi_1 S \to \mathrm{PSL}(2, \mathbb{R})$

Kleinian groups

$\rho : \pi_1 S \to \mathrm{PSL}(2, \mathbb{C})$

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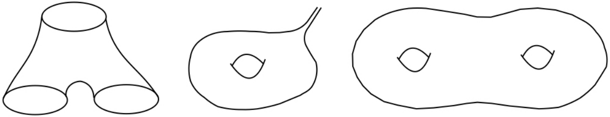

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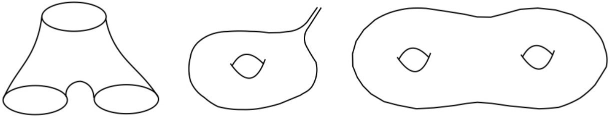
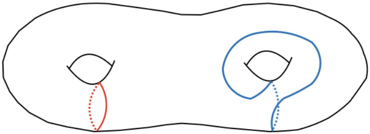


Geodesics on hyperbolic surfaces:



Basmajian's identity

If S is a compact hyperbolic surface with geodesic boundary, an orthogeodesic γ on S is a properly immersed geodesic arc perpendicular to the boundary at both ends.

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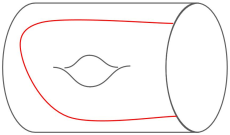

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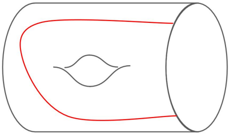

Theorem (Basmajian, 1991)

$$length(\partial S) = \sum_{\gamma} 2 \log \coth \left(\frac{length(\gamma)}{2} \right)$$

where the sum is taken over all orthogeodesics γ in S.

II. Teichmüller spaces

Let S be a compact surface with negative Euler characteristic. The *Teichmüller space* of S

$$\begin{aligned}\mathrm{Teich}(S) &= \{\text{hyperbolic structures on } S\}/\text{homotopy} \\ &= \{\text{discrete faithful } \rho : \pi_1 S \to \mathrm{PSL}(2,\mathbb{R})\}/\mathrm{PSL}(2,\mathbb{R})\end{aligned}$$

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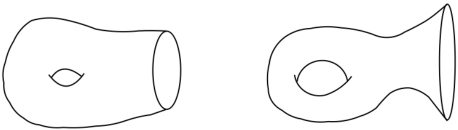

Complexifying Basmajian's identity

Thinking of the hyperbolic structure on S as a discrete faithful representation $\rho : \pi_1 S \to \mathrm{PSL}(2, \mathbb{R})$, our goal is to complexify the identity as we deform the representation into a Schottky representation $\rho : \pi_1 S \to \mathrm{PSL}(2, \mathbb{C})$.

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We need to address the convergence issue of the right hand side series.

Complexified Basmajian's identity: convergence theorem

Theorem (H., 2018)

Given a marked Schottky representation $\rho : F_n \to PSL(2,\mathbb{C})$, the series converges absolutely if and only if the Hausdorff dimension of the limit set Λ_Γ of the Schottky group $\Gamma = \rho(F_n)$ is strictly less than one.

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Proof: conformal dynamics.

Complexified Basmajian's identity

Denote by $\mathcal{S}_{<1}$ the space of Schottky groups whose limit set has Haudorff dimension less than one.

Theorem (H., 2018)

Suppose $\rho_0 : F_n \to PSL(2, \mathbb{C})$ *is a Fuchsian marking corresponding to a hyperbolic surface* S *with geodesic boundary* ∂S*. Let* $\alpha \in \pi_1 S$ *represent the free homotopy class of* ∂S*. If* ρ *is in the same path component as* ρ_0 *in* $\mathcal{S}_{<1}$*, then*

$$l(\rho(\alpha)) = \sum_{w \in \mathcal{L}} \log[\infty, 0; \rho(w) \cdot \infty, \rho(w) \cdot 0] \quad mod \quad 2\pi i \qquad (1)$$

Moreover, the series converges absolutely.

III. Higher Teichmüller theory

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- Let $K = \mathbb{R}$ or $\mathbb{C}$. Following Pozzetti-Sambarino-Wienhard, we study $(1,1,2)$-hyperconvex Anosov representations $\rho : \pi_1 S \to \mathrm{PGL}(n, K)$ and establish identities for such representations.

Identities for real $(1,1,2)$-hc Anosov representations

Theorem (H., 2019)

Let S be a connected compact oriented hyperbolic surface with geodesic boundary ∂S whose double $\hat{S}$ has genus at least 2. Let $\alpha \in \pi_1 S$ represent the free homotopy classes of ∂S. If $\rho : \pi_1 S \to PGL(n, \mathbb{R})$ is the restriction to $\pi_1 S$ of a $(1,1,2)$-hyperconvex representation $\hat{\rho} : \pi_1 \hat{S} \to PGL(n, \mathbb{R})$, then

$$\ell_\rho(\rho(\alpha)) = \sum_{w \in \mathcal{L}} \log C_\rho(\alpha_j^+, \alpha_j^-; w \cdot \alpha_j^+, w \cdot \alpha_j^-) \qquad (2)$$

where ℓ_ρ is a notion of length with respect to ρ, C_ρ is a cross ratio defined for four points on the boundary at infinity $\partial \pi_1 S$ and α_j^+, α_j^- are the attracting and repelling fixed points of α_j, respectively. Furthermore, if ρ is Hitchin, it is Vlamis-Yarmola's identity.

Identities for complex $(1,1,2)$-hc Anosov representations

Let $\mathcal{S}_{<1}$ be the space of $(1,1,2)$-hyperconvex Anosov representations $\rho : \pi_1 S \to \mathrm{PGL}(n,\mathbb{C})$ whose limit set $\zeta^1(\partial\pi_1 S) \subset \mathbb{P}(\mathbb{C}^n)$ has Hausdorff dimension strictly smaller than one.

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$$\ell_\rho(\rho(\alpha)) = \sum_{w\in\mathcal{L}} \log C_\rho(\alpha_j^+, \alpha_j^-; w\cdot\alpha_j^+, w\cdot\alpha_j^-) \mod 2\pi i,$$

where α_j^+, α_j^- are the attracting and repelling fixed points of α_j, respectively. Moreover, the series converges absolutely.

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Proof: projective dynamics.

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Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f_c(z) = z^2 + c$ where $c \in \mathbb{C}$.

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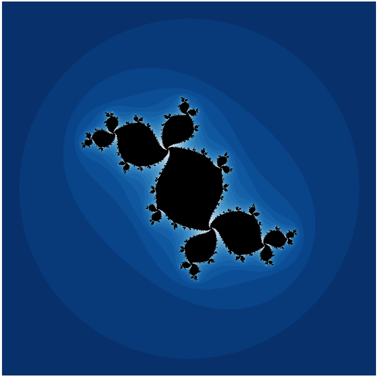
Example: $f(z) = z^2$. K_0 is the closed unit disk and $J_0 = S^1$.

Julia sets: Douady rabbit and Basilica



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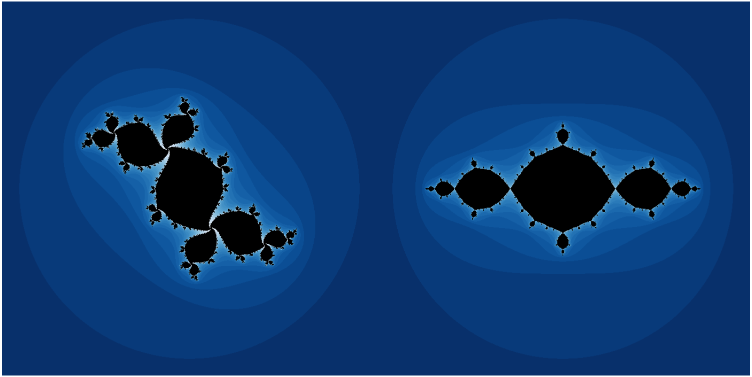
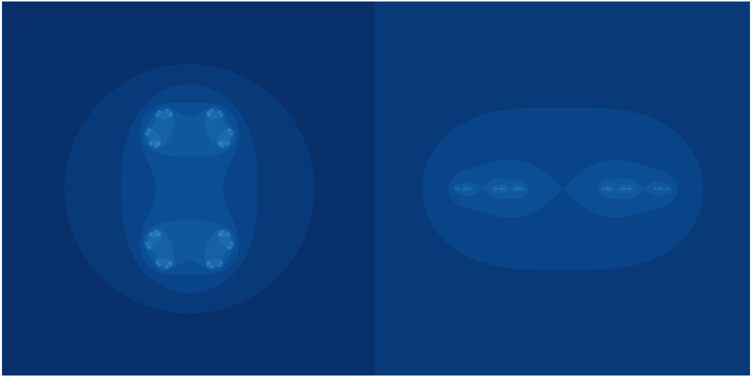
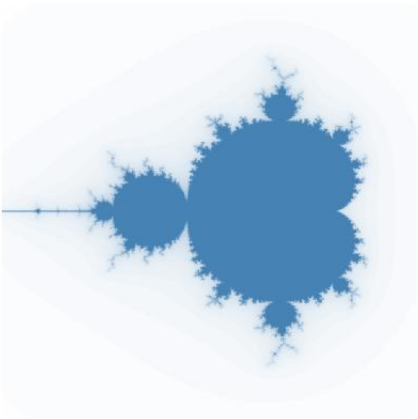


Cantor Julia sets



Quadratic polynomials: parameter space

Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} \mid f_c^n(0) \not\to \infty\}$.



Basmajian-type identities for $z^2 + c$

Denote by $(\mathbb{C} \setminus \mathcal{M})_{<1}$ the set of $c \in \mathbb{C} \setminus \mathcal{M}$ such that $\dim_{\mathrm{H}} J_c < 1$.

Theorem (H., 2018)

For complex parameter $c \in (\mathbb{C} \setminus \mathcal{M})_{<1}$, let T_1 and T_2 be the two branches of f_c^{-1} and z_1 be the fixed point of T_1, then the following identity holds

$$z_1 - (-z_1) = \sum_{w \in \{T_1, T_2\}^*} (-1)^\eta \Big(w(T_1(-z_1)) - w(T_2(-z_1)) \Big)$$

where η is the number of T_2's in the word w.

V. Geometry and topology of parameter spaces

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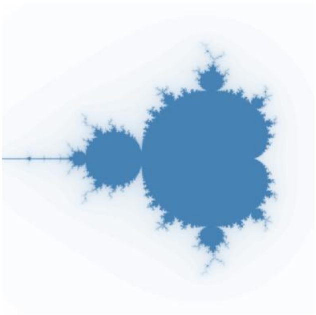

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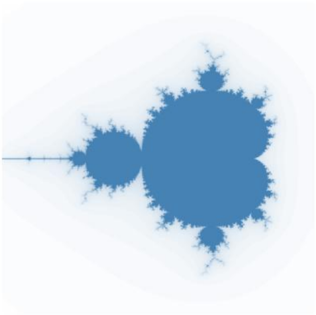

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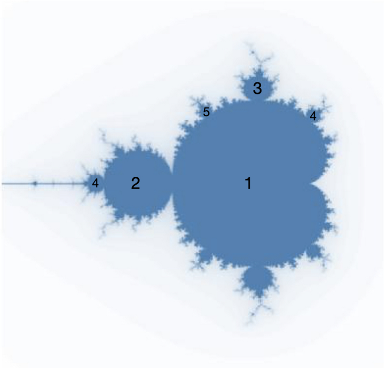

- Topology of the complement of the Mandelbrot set.
- Geometry of a hyperbolic component of $\mathcal{M}$.

Topology of the shift locus

We consider the space X_d of monic and centered complex polynomials of degree $d \geq 2$, i.e. the space of polynomials of the form

$$f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1 z + a_0$$

Hence, X_d is naturally homeomorphic to $\mathbb{C}^{d-1}$.

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Example: 0 is the only critical point of $f_c(z) = z^2 + c$.
Recall: Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} \mid f_c^n(0) \not\to \infty\}$. Therefore S_2 is the complement of the Mandelbrot set.

Topology of the shift locus (Cont'd)

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Theorem (Bavard-Calegari-H.-Koch-Walker, 2019)

For $d \geq 2$, we develop a combinatorial model for the shift locus S_d. Using this model, we compute the fundamental group of S_3 and study the monodromy map $\pi_1 S_d \to MCG(\mathbb{R}^2 - \text{Cantor set})$.

Geometry of hyperbolic components of rational maps

For each $d \geq 2$, let Rat_d be the space of degree d rational maps. Denote $\mathrm{rat}_d := \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$ the moduli space of degree d rational maps.

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Theorem (H.-Nie, 2020)

Let $\mathcal{H}$ be a hyperbolic component in rat_d such that $dim_H(\mathcal{H}) \subset (1,2)$. Then we construct a Riemannian metric on $\mathcal{H}$ which is conformal equivalent to the standard pressure metric.

VI. Number theory: relations to L-functions

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Basmajian's identity expresses in a similar way a (co)volume as a series over topological terms.

The Riemann zeta function

Recall the *Riemann zeta function* $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

As a complex function of s,

1. $\zeta(s)$ is analytic in the half-plane $Re(s) > 1$.
2. $\zeta(s)$ has an analytic continuation to the whole s-plane except for a simple pole at $s = 1$ with residue 1.
3. $\zeta(s)$ has no zeros in the half-plane $Re(s) > 1$. Zeros of $\zeta(s)$ are mysterious.

The Prime Number Theorem

Let $\pi(n) = \#\{\text{primes} \leq n\}$.

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Theorem (Hadamard, de la Vallée-Poussin, 1899)

$$\pi(n) = Li(n) + O(ne^{-a\sqrt{\log n}}) \text{ as } n \to \infty$$

for some positive constant a.

$Li(x) = \int_2^x \frac{dt}{\log t}$.

The Prime Ideal Theorem

Theorem (Hecke's Prime Ideal Theorem, 1918)

1. *The number of prime ideals in Gaussian integers* $\mathbb{Z}[i]$ *with norm less than* n *grows like* $Li(n)$*;*
2. *The angular components of Gaussian primes are equidistributed over the circle.*

L-functions from the identities

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$$F(s, m) = \sum_{w \in \mathcal{L}} \chi\left(\frac{\log c_w}{|\log c_w|}\right) |\log c_w|^s$$

where $\chi : S^1 \to S^1$ is a unitary character given by $\chi(z) = z^m$ for some $m \in \mathbb{Z}$ and $s \in \mathbb{C}$. Moreover, $F(1, 1)$ gives the RHS series in the identity.

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For quadratic polynomials:

$$G(s, m) = \sum_{w \in \{T_1, T_2\}^*} \left(\frac{w(I)}{|w(I)|}\right)^m |w(I)|^s$$

where $s \in \mathbb{C}$, $m \in \mathbb{Z}$.

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where $s \in \mathbb{C}$, $m \in \mathbb{Z}$. Note that $F(s)$ and $G(s)$ converge absolutely at least when $Re(s)$ is large.

Analytic properties of $F(s, m)$ (resp. $G(s, m)$)

Theorem (H., 2018)

For any $m \in \mathbb{Z}$, $F(s, m)$ (resp. $G(s, m)$) converges absolutely if and only if $Re(s) > \delta$, where δ is the Hausdorff dimension of the limit set (resp. Julia set).

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1. *If $m = 0$, $F(s,m)$ (resp. $G(s,m)$) is analytic on the half-plane $Re(s) > \delta - \varepsilon$ for some $\varepsilon > 0$ except a simple pole at $s = \delta$.*
2. *If $m \neq 0$, $F(s,m)$ (resp. $G(s,m)$) is analytic on the half-plane $Re(s) > \delta - \varepsilon$ for some $\varepsilon > 0$.*

Counting complex orthospectrum

Theorem (H. 2018)

There exist constants $C_1 > 0$ *and* $d_1 \in (0, \delta)$ *such that*

$$\#\{w \in \mathcal{L} \mid \Re(d(\ell, w \cdot \ell)) < x\} = C_1 e^{\delta x} + O(e^{d_1 x}) \text{ as } x \to \infty.$$

where ℓ *is the axis of the boundary element.*

This result also appeared in Parkkonen-Paulin, Pollicott.

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Equidistribution of holonomy:

Theorem (H. 2018)

For $m \neq 0$ and any non-Fuchsian Schottky group, there exist $C > 0$ and $0 < d_1 < \delta$ such that for any $f \in C^2(S^1)$, we have

$$\sum_{|\log c_w|^{-1} \leq x} f\left(\frac{\log c_w}{|\log c_w|}\right) = C x^{\delta} \int_0^1 f(e^{2\pi i t}) dt + O(x^{d_1})$$

where the implied constant depends on the C^2-norm of f.

Orbit counting for quadratic polynomials

Parallel counting results for quadratic polynomials:

Theorem (H., 2018)

There exist constants $C_2 > 0$ *and* $d_2 \in (0, \delta)$ *such that*

$$\#\{w \in \{T_1, T_2\}^* \mid |w(I)| > 1/x\} = C_2 x^{\delta} + O(x^{d_2}) \text{ as } x \to \infty.$$

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Thank you!