Actions of Homeo and Diff groups on manifolds

Lei Chen @Caltech

Joint with Kathryn Mann

Ghys in 1990s has studied "*extension problem*" and proved the following:

$\mathrm{Diff}(D^2) \xrightarrow{\pi} \mathrm{Diff}(S^1)$ has no sections.

A section of π can be considered as extensions of elements in $\mathrm{Diff}(S^1)$ that preserves group structure.

His argument:

If $\exists$ a section $\Rightarrow$ ① $\exists$ a global fixed point (Smith theory)

② Take a derivative at 0

$\mathrm{Diff}(S^1) \xrightarrow{D} GL_2\mathbb{R}$

not hard to show D is trivial

③ Thurston Stability (Reeb St)

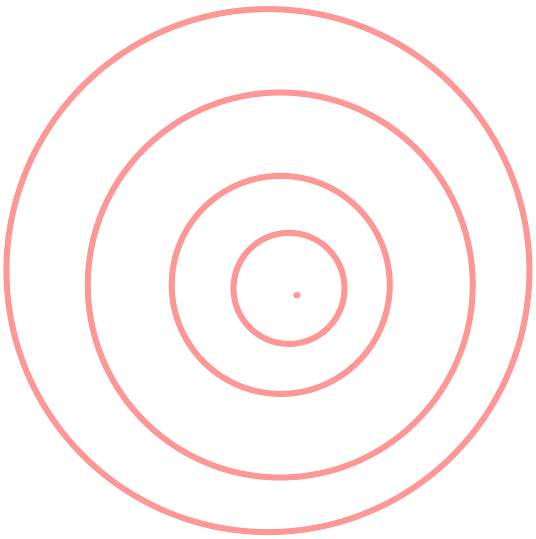
trivial derivative $\Rightarrow$ torsion free

$\Rightarrow$ contradiction

"A flat S^1-bundle" is not always the boundary of a flat D^2-bundle.

Topologically, the extension exists.

$\mathrm{Homeo}(\mathbb{D}^2) \longrightarrow \mathrm{Homeo}(S^1)$ has a section.



"Coning-off"

$f \in \mathrm{Homeo}(S^1)$

$$\rho(f)(r,\theta) = (r, f(\theta))$$

Naturally, we ask a question

Is this "coning-off" the only extension?

I asked Katie about it and she gave me a paper of Militon

$\boxed{\text{Militon's example}}$ "Militon's action"

$\mathrm{Homeo}(S^1) \longrightarrow \mathrm{Homeo}(A^2)$

$\rotatebox{90}{$\in$}$
f

$A^2 = S^1 \times [0,1]$

$$\rho(f)(\theta, r) = \left(f(\theta), \tilde{f}(\tilde{\theta}+r) - \tilde{f}(\tilde{\theta})\right)$$

$\tilde{f}, \tilde{\theta}$ are lifts to universal cover $\mathbb{R} \to S^1$

Now the t-coordinate also change!

Militon's Theorem

fully classified actions of $\mathrm{Homeo}(S^1) \curvearrowright \mathbb{A}^2$

$K \subseteq [0,1]$ closed subset

$\mathrm{Homeo}(S^1) \curvearrowright S^1 \times K$ as coning

complement of $S^1 \times K \subseteq S^1 \times [0,1]$

by Militon action

This construction apparently doesn't extend to higher dimension.

Bigger goal

Can we classify all $\mathrm{Homeo}(M)$, $\mathrm{Diff}(M)$ action on N?

for M, N two manifolds.

In this talk, $\mathrm{Homeo}(M)$, $\mathrm{Diff}(M)$ denote id component. We will not talk about mapping class group.

Basic Properties of Homeo(M) and Diff(M)

1. Locally generated and locally contractible

Diff $\rightsquigarrow$ Banach manifold structure

Homeo $\longrightarrow$ Kirby-Edward's result

2. Fragmentation property (Kirby—Edward)

$\forall g \in G \quad \forall$ cover $\{\mathcal{U}_\alpha\}$ of M

$\Longrightarrow g = \prod g_i$ s.t. $\operatorname{supp}(g_i) \subseteq \mathcal{U}_\alpha$ some α

Diff $\longrightarrow$ Partition of unity

Homeo $\longrightarrow$ Kirby—Edward

3. Simplicity (Thurston, Epstein, Herman, Yoccoz, Mather)

$r \neq \dim(M)+1 \Longrightarrow \mathrm{Diff}^r(M)$ is simple

r is regularity.

4. Automatic continuity (Rosendal-Solecki, Rosendal, Mann, Hurtado)

(RS, R, M) $\mathrm{Homeo}(M) \xrightarrow{\rho} G$ G separable

$\Longrightarrow$ ρ is continuous

(H) $\mathrm{Diff}^{\infty}(M) \xrightarrow{\rho} \mathrm{Diff}^{\infty}(N)$ always continuous

5. Uniqueness (Whittaker, Filipkewicz)

Any isomorphism of $\mathrm{Diff}^r(M)$ and $\mathrm{Diff}^s(N)$ is induced by a diffeomorphism of r

$M \xrightarrow{f} N$.

1-1 correspondence

$(M, r) \longrightarrow \mathrm{Diff}^r(M)$

It's possible to study (M, r) property by studying its transformation gp $\mathrm{Diff}^r(M)$

Example Zoo

$\mathrm{Diff}^r(M) \curvearrowright N$

Homeo(M)

① coning-off

$\mathrm{Homeo}(S^n) \curvearrowright D^{n+1}$

② suspension

$\mathrm{Homeo}(S^n) \curvearrowright S^{n+1}$

③ double suspension

$\mathrm{Homeo}(H^n) \curvearrowright S^{n+2}$

Cannon's thm: $\Sigma^2 H^n$ for H^n homology sphere is $\cong S^{n+2}$.

④ M, $M \times M$,

$M \times M \times K$

$M \times M \times K/\sim$

$(x, x, t) \sim (x, x, t')$

Diff(M)

M, TM, PM

FM, $G_k(M)$

$\mathrm{Jet}^r(M)$

Observation:

$\dim N \geq \dim M$

(Conj of Ghys)

Hurtado 2015

$\mathrm{Diff}^\infty(M) \xrightarrow{\rho} \mathrm{Diff}^\infty(N)$ closed

ρ nontrivial

$\Longrightarrow \dim N \geq \dim M$

Orbit Classification Theorem (C—Mann)

X finite dim CW complex and M closed manifold

Any continuous action of $\mathrm{Homeo}(M) \curvearrowright X$,

every orbit is homeomorphic to a cover of $\mathrm{Conf}_n(M)$

$\mathrm{Conf}_n(M)$ unoriented configuration space

$$\overset{\shortparallel}{\{}\{x_1, \dots, x_n\} \mid x_i \neq x_j, x_i \in M\} \subseteq \mathrm{Sym}^n(M)$$

For $\mathrm{Diff}^r(M)$, every orbit is $\mathrm{Jet}_n^r(M)/A$

$A \subseteq \mathrm{Jet}^r$ a Lie subgroup.

You may wonder which cover can appear as orbits:

Eg: $\mathrm{Homeo}(S^1)$ has no action on $\mathbb{R}$

but $\mathrm{Homeo}(S_g)$ acts on $\tilde{S}_g$ universal cover.

$g \geq 2$

This question is closely related to $\pi_1(\mathrm{Homeo}(M))$

Applications

1. Ghys' dimension growth conjecture

pf: OCT + $\underline{\text{invariance of domain}}$

no injective cont map from $\mathbb{R}^n \to \mathbb{R}^m$ if $m < n$.

2. When M,N have the same dimension.

$\Rightarrow$ $\nexists$ an embedding of $\tilde{n} \hookrightarrow N$.

3. When dim(M)+1=dim(N) and "extension problem"

① $\nexists$ extension for Diff case

② When $\pi_1(M) = 1$,

$\exists$ extension for Homeo case

if only if $M \cong S^n$.

Open problem: Does $\mathrm{Homeo}(H_g) \longrightarrow \mathrm{Homeo}(S_g)$ have a section?

handlebody

Outline the proof of Orbit Classification Theorem

$G = \mathrm{Homeo}(M)$

<u>Def:</u> $A \subseteq G$ a closed subgp

we say $A \subseteq G$ has finite codim if

G/A can be embedded in finite dim CW complex.

Example: $G \curvearrowright N \leftarrow$ continuous

$n \in N$

$\mathrm{Stab}(n) \subseteq G$ satifies that

$$G/\mathrm{Stab}(n) \cong O(n) \subseteq N$$

$\Longrightarrow$ $\mathrm{Stab}(n)$ is a finite codim subgp.

Thm (C.–Mann) OCT2

If $A \leq G$ is finite codim $\Longrightarrow$

$\exists\ x_1, \ldots, x_n \in M$ st

$$\underbrace{\mathrm{Stab}(x_1, \ldots, x_n)_0}_{\text{identity component}} \leq A \leq \mathrm{Stab}(x_1, \ldots, x_n)$$

in general $\mathrm{Stab}(x_1, \ldots, x_n)$ is not connected.

Now OCT2 $\Longrightarrow$ OCT1

$$\begin{array}{ccc} G/\mathrm{Stab}(x_1, \ldots, x_n) & \cong & \mathrm{Conf}_n(M) \\ \uparrow & & \\ G/\mathrm{Stab}(x_1 \cdots x_n)_0 & & \text{cover} \end{array}$$

G/A is an intermediate cover.

Recall Militon's example

$$\mathrm{Homeo}(S^1) \curvearrowright A^2 = S^1 \times [0,1]$$

$$PConf_2(S^1) = S^1 \times (0,1)$$

Militon example is the compactified action

$$\mathrm{Homeo}(S^1) \curvearrowright PConf_2(S^1) \cup S^1 \times \{0,1\}$$

How to prove OCT2 $\quad G = \mathrm{Homeo}_c(M)$.

Pf: ① $B \subseteq M$ a ball

$$G_B = \{ f \in G \mid \mathrm{supp}(f) \subseteq B \}$$

Claim: If $A \subseteq G$ finite codim $\Rightarrow$ $A \supseteq G_B$ some B.

Pf: If not $\quad B_1 \quad B_2 \quad B_3 \quad \cdots$

$$A \not\supseteq G_{B_i} \Rightarrow \underbrace{G_{B_i} / A \cap G_{B_i}}_{\text{contributes a copy of } \mathbb{R} \text{ in } G/A.} \subseteq G/A$$

Some argument with simplicity of G_{B_i}

$\Rightarrow$ $G_{B_1} \times \cdots G_{B_n} / A \cap (G_{B_1} \times \cdots \times G_{B_n})$ has a copy of $\mathbb{R}^n$

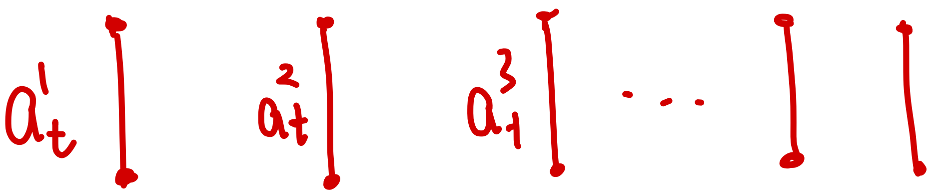
$\Rightarrow$ blow up the dim of G/A. $\square$

② Consider the action of $A \curvearrowright M$

$$\mathrm{Orbit}(A) = Ax \qquad \forall x \in M$$

Claim: Ax is either finite or cofinite $\forall x \in M$.
(finite complement)

Pf: If not, $\exists$ two infinite orbits



$$O(x) = Ax \neq O(y) = Ay.$$

$\Rightarrow \quad \mathbb{R}^\infty \longrightarrow G/A$ is almost injective blowup dim.

③ ① + ② $\Rightarrow$ $\exists$ finite set $S \subseteq M$

s.t. $f \in \mathrm{Homeo}(M-S)_c \subseteq A$

(fragmentation)

$A \supset G_B \quad B \subset M-S$

$\{B_i\}$ cover $M-S$

$\Rightarrow \quad * \; \mathrm{Stab}(S)_0 = \mathrm{Homeo}(M-S)_c \subseteq A \subseteq \mathrm{Stab}(S)$

In low dimensional case $\quad \mathrm{Homeo}(S^n) \curvearrowright S^{n+2} = \Sigma^2 S^n$

$\dim N < 2 \dim M \qquad \rho : \mathrm{Homeo}(M) \curvearrowright N$

Every orbit is either a point or a cover of M.

$N - \mathrm{Fix}(\rho)$ ← global fixed point. $\quad n \in N - \mathrm{Fix}(\rho)$

$\downarrow \pi$ $\qquad$ Apply OCT2 to $\mathrm{Stab}(n)$

M $\qquad$ $m = \pi(n)$ s.t $\mathrm{Stab}(n) \subseteq \mathrm{Stab}(m)$

Thm (C.–Mann) π is a flat bundle.

Diff case.

Flat bundle Thm can be used to prove extension problem

① $\exists$ global fixed point by flat bundle Thm

② the derivative is trivial

③ Thurston type argument.

What's next?

① How orbits can be glued together?

- generalisation of flat bundle Thm

② Shape of global Fixed Points.

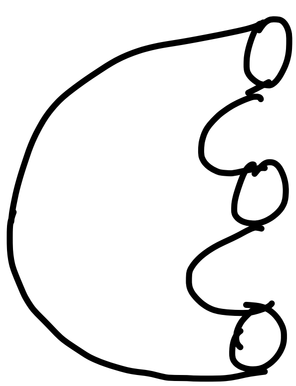
Thm(C.) If $\mathrm{Homeo}(S^n) \curvearrowright N^{n+k}$ $k<n$

$$\Rightarrow \quad N = S^n \times K_1 / \sim$$

K_1 is a $\mathbb{Z}$-homology manifold with boundary K_0

$(m, x) \sim (n, y) \quad \forall\, x = y \in K_0$

Ex: $S^n \times [0,1] / \sim \; = S^{n+1}$



$S^n \times K_1 / \sim$ is always a manifold.

Obstructing global fixed points in a special case

Thm (C.) If Homeo(M) contains a free p-torsion and M is not $\mathbb{Z}/p$-homology sphere

$\Rightarrow$ $N = M \times K$. $\curvearrowleft$ Homeo(M).

Cor: Homeo($S^1 \times M^n$) acts nontrivially on N^{k+n}, $k \leq n$

always without fixed points

$\Rightarrow$ N^{k+n} is a cover of $S^1 \times M^n$.

Future goal: Show the above Thm without conditions on M.

THANKS For listening!