Big mapping class groups and rigidity of the simple circle

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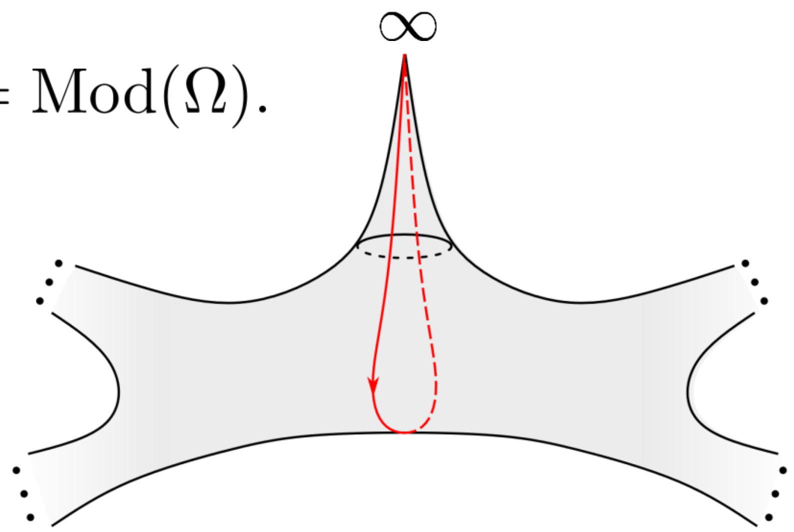
Big MCG and actions on S^1

$\Omega = \mathbb{R}^2 - K$, K Cantor set, $\Gamma = \mathrm{Mod}(\Omega)$.

Fact: Γ acts faithfully on S^1.

$\implies \Gamma \hookrightarrow \mathrm{Homeo}^+(S^1)$.



Question 1: Does Γ act on S^1 in different ways?

Any nontrivial action with a global fixed point?

Question 2: Is Γ generated by torsion?

$\mathrm{Homeo}^+([0,1])$ is torsion-free.

Question 3: Any further obstruction for $G \hookrightarrow \Gamma$?

Main results

Question 1: Does Γ act on S^1 in different ways? No.

Question 2: Is Γ generated by torsion? Yes.

Question 3: Any further obstruction for $G \leq \Gamma$? Yes.

Theorem 3: Any countable subgroup of $\mathrm{Homeo}^+(S^1)$ embeds into Γ. Not true if uncountable, e.g. $\mathrm{PSL}_2\mathbb{R}$.

Theorem 2: Γ is normally generated by a single 2-torsion.

Theorem 1: Γ acts faithfully and minimally on the simple circle S^1_S. Any nontrivial action of Γ on S^1 is semi-conjugate to this one.

$$\begin{array}{ccc} S^1 & \xrightarrow{\rho(g)} & S^1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ S^1_S & \xrightarrow{g} & S^1_S \end{array}$$

Similar results by Mann–Wolff

Rays

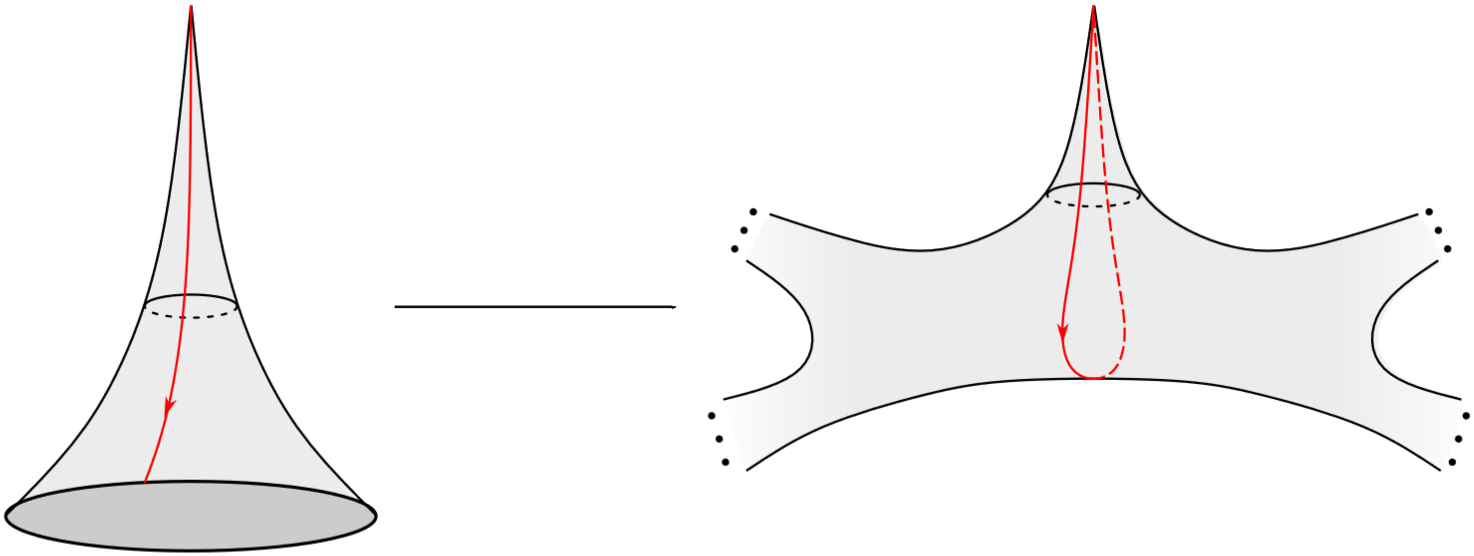
S finite type
curve graph $\mathcal{C}(S)$
$\mathcal{C}(S)$ is hyperbolic
(Masur–Minsky)

$S = \Omega$
ray graph $\mathcal{R}$
$\mathcal{R}$ is hyperbolic
(Bavard,
Aramayona–Fossas–Parlier)

Rays

Fix a hyperbolic structure on Ω.

conical cover Ω_C and conical circle S^1_C.



Rays

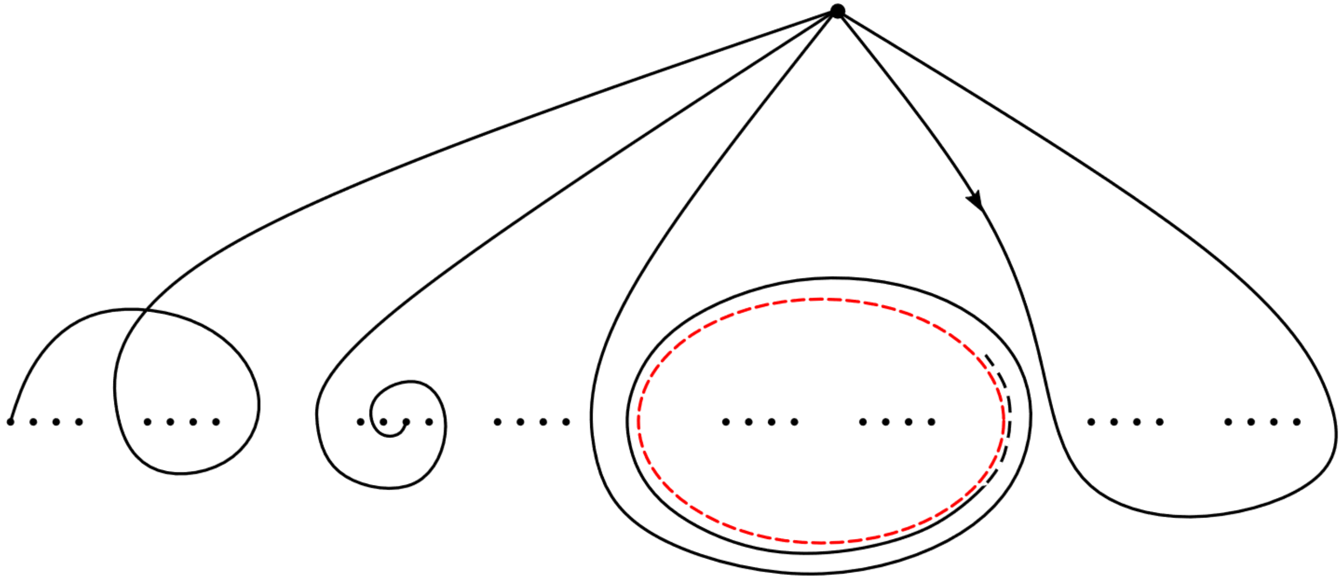
Fix a hyperbolic structure on Ω.

conical cover Ω_C and conical circle S^1_C.

Simple rays $= R \sqcup L \sqcup X$, Γ-invariant and closed.

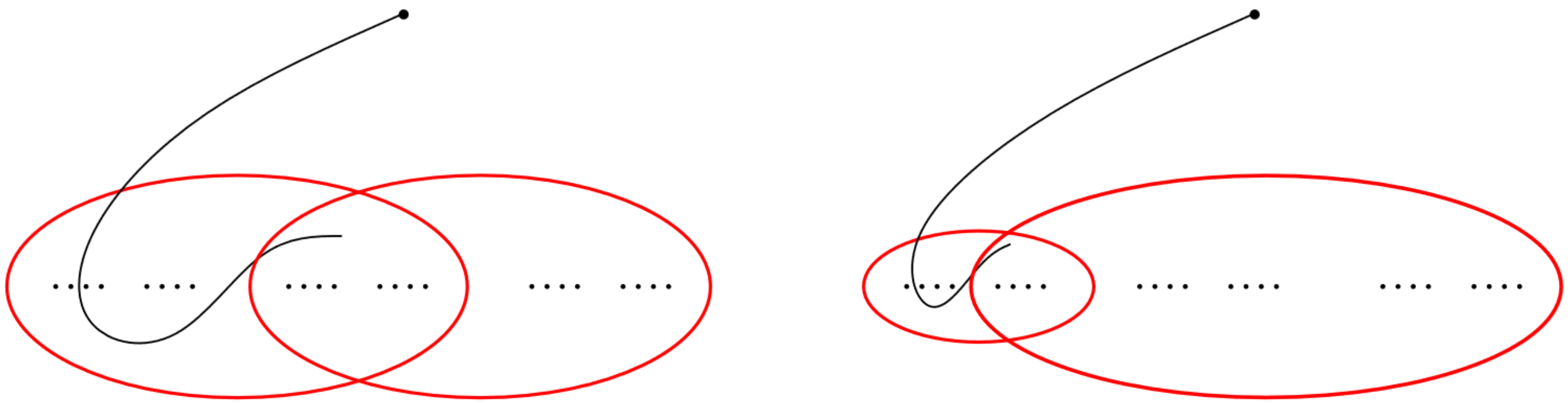
$R =$ short rays, $L =$ lassos, $X =$ long rays.

one orbit | one orbit | uncountably many orbits, related to $\partial\mathcal{R}$



Unique minimal set

Lemma: For any $x \in S^1_C$, the orbit closure $\overline{\Gamma x} \supset R$.

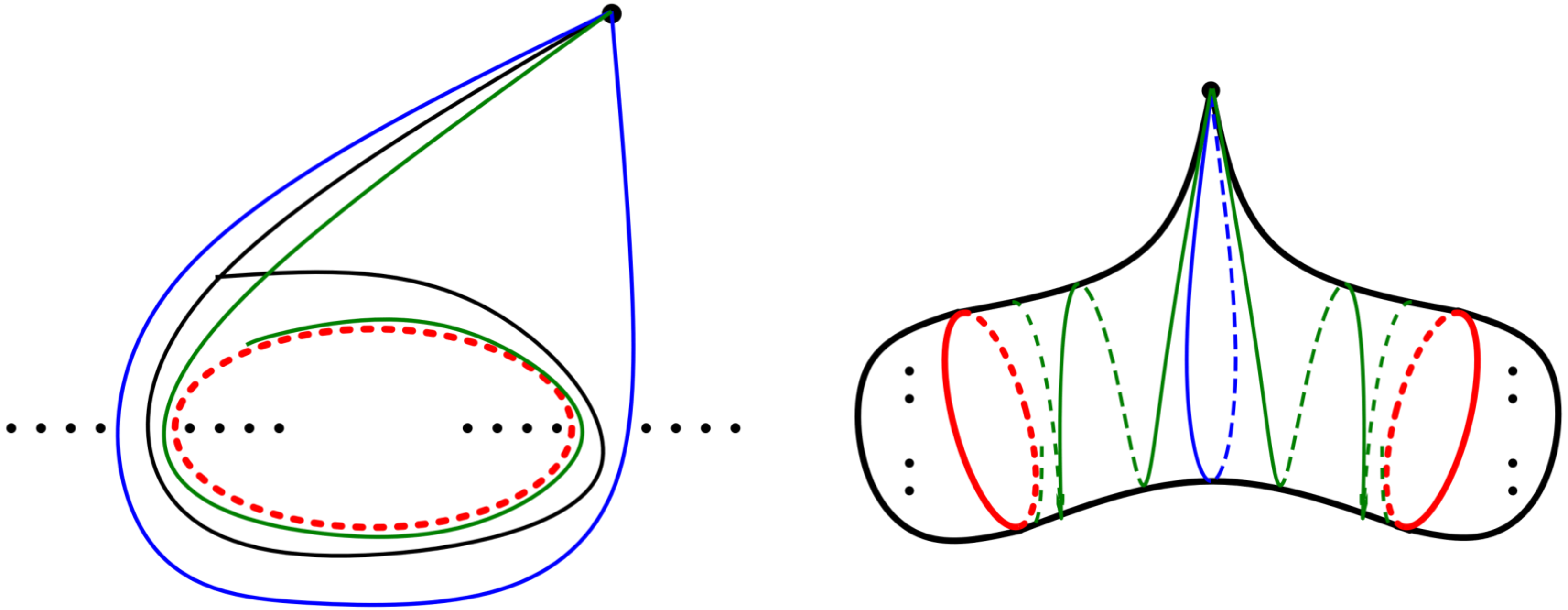


Unique minimal set

Lemma: The unique minimal set $\bar{R} = R \sqcup X$ is a Cantor set on S^1_C, complementary intervals $\leftrightarrow L$.

Key points:

- each $\ell \in L$ is isolated in the simple set.



Unique minimal set

Lemma: The unique minimal set $\bar{R} = R \sqcup X$ is a Cantor set on S^1_C, complementary intervals $\leftrightarrow L$.

Key points:

- each $\ell \in L$ is isolated in the simple set. $\implies R \cup X$ closed.
- the simple set is nowhere dense. $\implies R \cup X$ nowhere dense.
- can approximate each $x \in X$ by short rays.

$\implies \bar{R} = R \cup X$, perfect.

The simple circle

Collapse complementary intervals to points $S^1_C \rightsquigarrow S^1_S$

simple circle

- Γ acts faithfully and minimally on S^1_S
- has an uncountable orbit R.
- has a countable orbit L.

$\implies$ every subgroup of Γ has a faithful action on S^1 with a countable orbit.

$\mathrm{PSL}_2\mathbb{R}$ does not embed

Theorem 3: Any countable subgroup of $\mathrm{Homeo}^+(S^1)$ embeds into Γ. Not true if uncountable, e.g. $\mathrm{PSL}_2\mathbb{R}$.

Prop.: Every faithful action of $\mathrm{PSL}_2\mathbb{R}$ on S^1 is standard.

Tool: The bounded Euler class $\mathrm{eu}_b^\rho \in H_b^2(G)$.

$$\rho : G \to \mathrm{Homeo}^+(S^1), \rightsquigarrow \mathrm{eu}_b^\rho = \rho^* \mathrm{eu}_b.$$

- (Ghys) eu_b^ρ determines the action up to semi-conjugacy.
- $\mathrm{eu}_b^\rho = 0$ iff the action has a global fixed point.
- $H_b^2(G) \hookrightarrow H^2(G)$ if G is uniformly perfect.

$$\mathrm{eu}_b^\rho \mapsto \mathrm{eu}^\rho,\ \mathrm{eu}^\rho = 0 \text{ iff action lifts to } \mathbb{R}.$$

$\mathrm{PSL}_2\mathbb{R}$ does not embed

Theorem 3: Any countable subgroup of $\mathrm{Homeo}^+(S^1)$ embeds into Γ. Not true if uncountable, e.g. $\mathrm{PSL}_2\mathbb{R}$.

Prop.: Every faithful action of $\mathrm{PSL}_2\mathbb{R}$ on S^1 is standard.

Proof sketch:

- any $g \in \mathrm{PSL}_2\mathbb{R}$ is a commutator, so uniformly perfect.
- $H^2(\mathrm{PSL}_2\mathbb{R}; \mathbb{Z}) \cong \mathbb{Z}$, generated by eu^{std}.
- $\mathrm{eu}^\rho = \lambda \cdot \mathrm{eu}^{std}$, $\lambda = \pm 1$ (torsion+rigid subgroup).

Then show the action ρ is transitive, so no countable orbit.

Countable subgroups

Theorem 3: Any countable subgroup of $\mathrm{Homeo}^+(S^1)$ embeds into Γ. Not true if uncountable, e.g. $\mathrm{PSL}_2\mathbb{R}$.

Proof: Denjoy's blow-up construction + suspension.

Rigidity

Theorem 1: Γ acts faithfully and minimally on the simple circle S^1_S. Any nontrivial action of Γ on S^1 is semi-conjugate to this one (up to a change of orientation).

Proof sketch: Fix an action ρ without fixed points.

r short ray, $\Gamma_r := \mathrm{Stab}(r)$.

Step 1: Each Γ_r acts with a global fixed point.

Fix r_0, pick $P(r_0) \in \mathrm{Fix}(\Gamma_{r_0})$. Let $P(r) = \rho(g).P(r_0)$ if $r = g.r_0$.

$$\begin{array}{ccc} R & \longrightarrow & S^1_S \\ {\scriptstyle P}\downarrow & {\scriptstyle \Gamma\text{-equivariant}} & \\ S^1 & & \end{array} \qquad \text{fixed by } \Gamma_r = g\Gamma_{r_0}g^{-1}$$

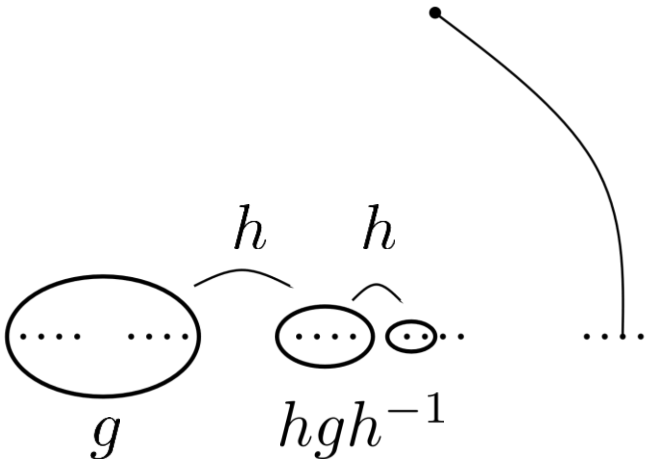
Step 2: P preserves the circular order and is injective.

Step 1 details

r short ray, $\Gamma_r := Stab(r)$, $\Gamma_{(r)} \trianglelefteq \Gamma_r$ (id on a nbhd of r).

Step 1: Each Γ_r acts with a global fixed point.

- Any circle action of $\Gamma_{(r)}$ has a fixed point ($H_b^2(\Gamma_{(r)}) = 0$).
  uniformly perfect (suspension trick) $g = a_g(h a_g^{-1} h^{-1})$
  $H^k(\Gamma_{(r)}) = 0$ (Mather's suspension argument)



$a_g = g \cdot (hgh^{-1}) \cdot (h^2 g h^{-2}) \cdots$

Step 1 details/structural results

r short ray, $\Gamma_r := Stab(r)$, $\Gamma_{(r)} \trianglelefteq \Gamma_r$ (id on a nbhd of r).

Step 1: Each Γ_r acts with a global fixed point.

- Any circle action of $\Gamma_{(r)}$ has a fixed point ($H_b^2(\Gamma_{(r)}) = 0$).
- $\Gamma_r = \langle \Gamma_{(r)}, \Gamma_{(s)} \cap \Gamma_r \rangle$ if $\mathrm{end}(r) \neq \mathrm{end}(s)$.

  $\Gamma_{(s)} \cap \Gamma_r$ preserves $\mathrm{Fix}(\Gamma_{(r)})$ and has fixed points.

- $\Gamma = \langle \Gamma_r, \Gamma_s \rangle$ if $\mathrm{end}(r) \neq \mathrm{end}(s)$.

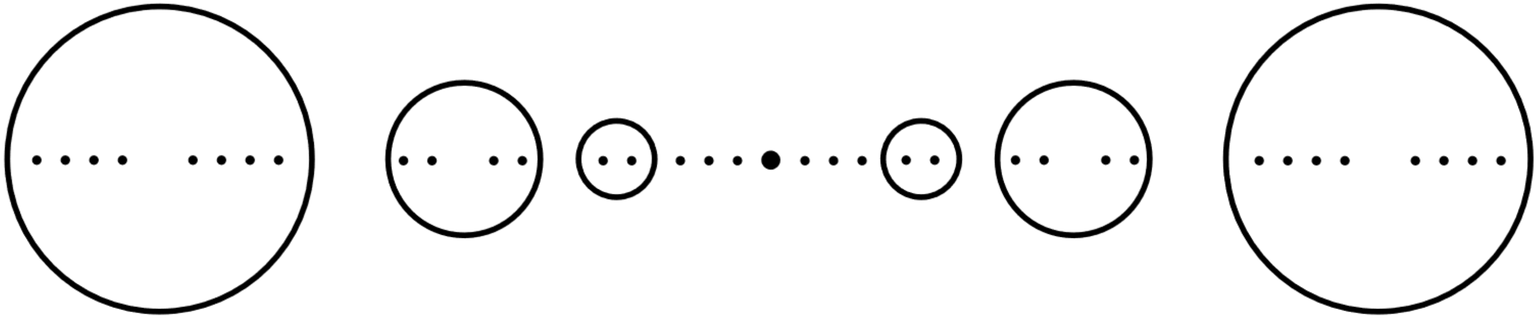
Cor: Γ is generated by elements supported in disks.

use this + suspension to prove Theorem 2.

Generated by torsion

Theorem 2: Γ is normally generated by a single 2-torsion.

Proof: The suspension trick.



Step 2 details

Step 2: P preserves the circular order and is injective.

• $P(r) \neq P(s)$ if $\mathrm{end}(r) \neq \mathrm{end}(s)$ since $\Gamma = \langle \Gamma_r, \Gamma_s \rangle$.

• $\mathrm{Or}(P(r_1), P(r_2), P(r_3)) = 1$ if $\mathrm{Or}(r_1, r_2, r_3) = 1$ and r_1, r_2, r_3 are disjoint with distinct endpoints.

goal: remove "disjoint".

The filtration

Induction using a filtration associated to an "equator" γ.

equator: embedded circle containing K.

$R_0(\gamma) \subset R_1(\gamma) \subset \cdots \subset R$, each is a Cantor set

each step adds Cantor to each complementary interval

The induction