Genericity of pseudo-Anosovs

joint w/ V. Erlandsson

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Classification of of elements in $SL_2\mathbb{Z}$

Trace	Jordan Form	$SL_2\mathbb{Z} = MCG(\ )$
$\lvert Tr(A)\rvert < 2$	$A \sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ finite order	Finit order
$\lvert Tr(A)\rvert = 2$	$A \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	Reducible
$\lvert Tr(A)\rvert > 2$	$A \sim \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \lambda > 1$	Anosov.

$S = S_g \quad g \geq 2, \quad \Gamma(S) = MCG(S) = \mathrm{Homeo}^+(S) / \text{isotopy}$

Thm [Nielsen-Thurston Classification]

$\forall\, \phi \in \Gamma(S)$ $\exists$ a representative $f \in \phi$ s.t f is

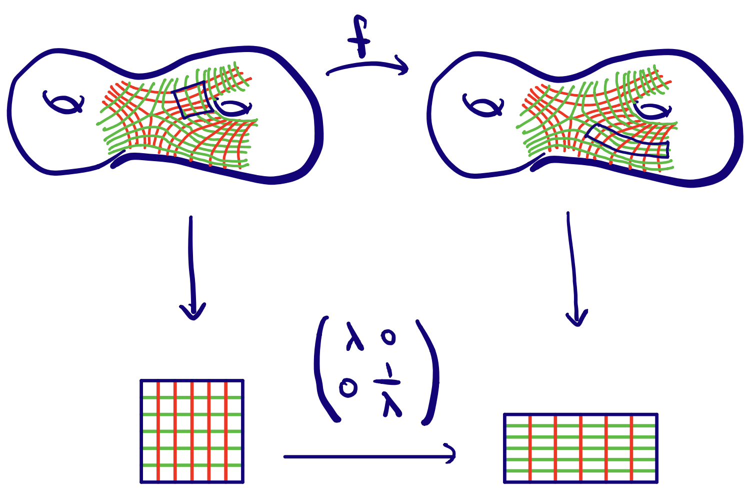
1) finite order

2) reducible

3) OR <u>pseudo-Anosov</u>

$\Updownarrow$

$\exists$ a pair of transverse (singular) measured foliations F_+ & F_- on S & $\lambda > 1$ s.t $f(F_\pm) = \lambda^{\pm 1} F_\pm$



Philosophy: pA are the "generic" elements of $\Gamma(S)$

Thurston Construction: pA are abundant

Thm: Suppose α & β are multi-curves on S s.t

$\alpha \cup \beta$ fill S. Then $\exists$ a representation

$$\Gamma(S) \geq \langle T_\alpha, T_\beta \rangle \xrightarrow{\rho} SL_2\mathbb{R}$$

$$T_\alpha \longmapsto \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \qquad \mu = \mu(\alpha,\beta) > 0$$

$$T_\beta \longmapsto \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}$$

s.t $\forall \ \phi \in \langle T_\alpha, T_\beta \rangle$

$$\phi \text{ is pA} \iff |\operatorname{Tr} \rho(\phi)| > 2$$

Notion of Genericity:

$\rho : \Gamma(S) \longrightarrow \mathbb{R}_{\geq 0}$, proper function

L-ball: $B_L^\rho = \{ \phi \in \Gamma(S) \mid \rho(\phi) \leq L \}$

Defn. Say $Y \subseteq \Gamma(S)$ is ρ-generic if

$$\lim_{L\to\infty} \frac{|Y \cap B_L^\rho|}{|B_L^\rho|} \longrightarrow 1$$

The complement of ρ-generic is ρ-neglible

Thm [EST]: For any filling curve γ on S, the set of pseudo-Anosov element of $\Gamma(S)$ is generic with respect to $\rho_\gamma(\phi) = i(\gamma, \phi(\gamma))$

Related Results

Recall: $\Gamma(S) \curvearrowright$ Teichmüller Space $(T(S), d)$

as isometries w.r.t $d =$ Teichmüller or Thurston metric.
d_{Teich} d_{Th}

Fix basepoint $x \in T(S)$

$$p_d^x : \Gamma(S) \longrightarrow \mathbb{R}_{\geq 0}$$

$$p_d^x(\phi) = e^{d(x, \phi(x))}$$

Corollary: pAs are also generic w.r.t p_d^x, $\forall x \in T(S)$
and $d = d_{Teich}$ or d_{Th}.

Proof: $P_\gamma \asymp P^x_{d_{Th}}$ & $P^2_\gamma \asymp P^x_{d_{Teich}}$

Analogies w/ Norms on $SL_2\mathbb{Z}$

- Operator $\quad \|A\| = \sup\limits_{\|v\|=1} \|Av\| \longleftrightarrow P_{d_{Th}}(A)$

- Spectral^2 $\quad \|A\| = \lambda_{max} A^T A \longleftrightarrow P_{d_{Teich}}(A)$

- Entries $\quad \|A\| = \sum\limits_{i,j} |a_{ij}| \longleftrightarrow P_\gamma(A) \quad \gamma = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Remark: [Maher 2010] pA are generic w.r.t. Teichmüller metric

Ingredients :

Our Proof	Maher's Proof
• Counting using geod currents	• Lattice counting in $(\Gamma(S), d_{Teich})$
• Ergodicity of $\Gamma(S) \curvearrowright$ Thurston measure	• Mixing of Teich geod flow

Geodesic Currents

$$\tilde{S} \cong \mathbb{H}^2 \qquad \partial\tilde{S} \cong S^1$$

$$\mathcal{G}(S) = \left(S^1 \times S^1 - \Delta\right) \Big/ \text{flip}$$

$$\mathcal{C}(S) = \left\{ \pi_1(S)\text{-invariant Radon-measures on } \mathcal{G}(S) \right\}$$

Properties [Bonahon]:

1) $\mathcal{C}(S)$ is locally-compact metrizable space (weak*), w/ a natural cone structure

$$\mathcal{C}(S) \times \mathcal{C}(S) \longrightarrow \mathcal{C}(S), \qquad \mathbb{R}_{\geq 0} \times \mathcal{C}(S) \longrightarrow \mathcal{C}(S)$$

$$(\lambda, \mu) \longmapsto \lambda + \mu \qquad\qquad (t, \mu) \longmapsto t\mu$$

2) $\mathbb{R}_+ \cdot \{\text{closed geodesics in } S\} \hookrightarrow \mathcal{C}(S)$ with dense image

3) $\mathcal{ML}(S) \cong \mathbb{R}^{6g-6} \hookrightarrow \mathcal{C}(S)$

$P\mathcal{ML}(S) \hookrightarrow P\mathcal{C}(S)$

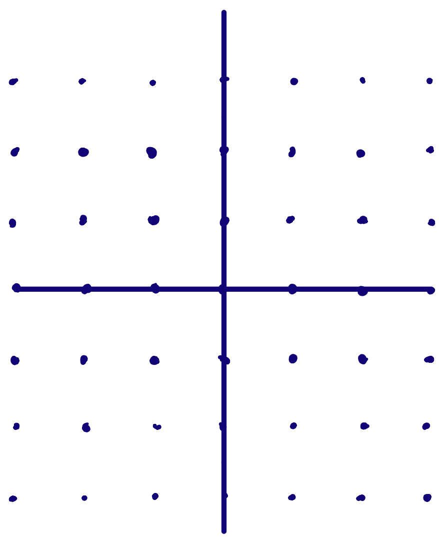
4) $i(\ ,\ )$ of curves extend to a continuous, symmetric, bilinear

$$\mathcal{C}(S) \times \mathcal{C}(S) \longrightarrow \mathbb{R}_{\geq 0}$$

$$\mathcal{ML}(S) = \{ \xi \mid i(\xi, \xi) = 0 \}$$

5) $\Gamma(S) \curvearrowright \mathcal{C}(S)$ continuously by linear automorphisms

Thurston Measure on $\mathcal{ML}(S)$



$\mathbb{Z}^2 \subseteq \mathbb{R}^2$

$L>0$

$$\frac{1}{L^2}\,\chi_{\frac{1}{L}\mathbb{Z}^2} \xrightarrow{L\to\infty} \lambda_{leb} \text{ on } \mathbb{R}^2$$

$$\underbrace{\mathbb{Z}_+\{\text{simple multicurves}\}}_{\mathcal{ML}_{\mathbb{Z}}} \hookrightarrow \underset{\mathbb{R}^{6g-6}}{\underset{\cong}{\mathcal{ML}(S)}} \subseteq \mathcal{C}(S)$$

Thm [Mirzakhani]

$$\frac{1}{L^{6g-6}}\,\chi_{\frac{1}{L}\mathcal{ML}_{\mathbb{Z}}} \xrightarrow{L\to\infty} m_{Th}$$

measure on $\mathcal{C}(S)$
support on $\mathcal{ML}(S)$
in the lebsgue class.

Thm [Masur]

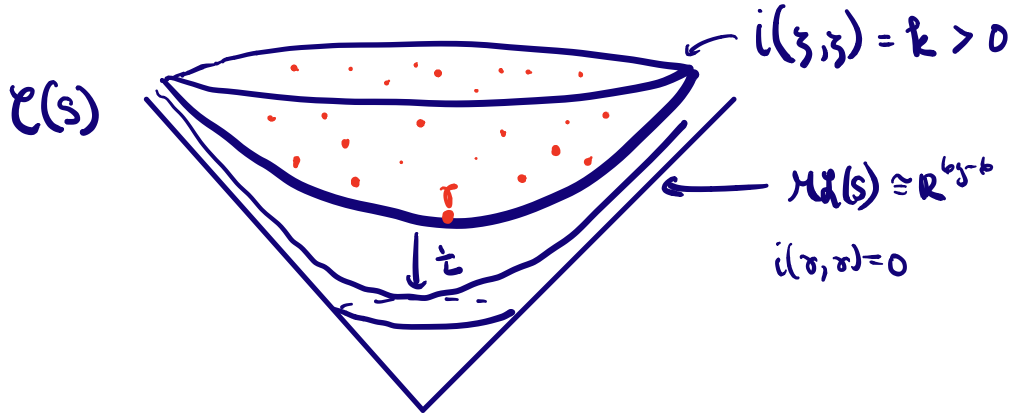
$$\Gamma(S) \curvearrowright (\mathcal{ML}(S), m_{Th}) \text{ ergodically.}$$

$\mathbb{R}^2 \cong \mathcal{ML}($$)$, $\mathbb{Z}^2 \cong \mathcal{ML}_{\mathbb{Z}}($$)$, $SL_2\mathbb{Z} \curvearrowright (\mathbb{R}^2, \lambda_{Leb})$ ergodically

Toward P_γ - genericity

γ filling curve, $\Upsilon \leq \Gamma(S)$, $L > 0$

$$\rightsquigarrow m_\gamma^\Upsilon(L) = \frac{1}{L^{6g-6}} \chi_{\frac{1}{L}\Upsilon\gamma} \quad \text{measure on } \mathcal{C}(S)$$



Relationship to Counting

$F : \mathcal{T}(S) \longrightarrow \mathbb{R}_{\geq 0}$ Proper homogeneous function

$$\rightsquigarrow \quad F_\gamma : \Gamma(S) \longrightarrow \mathbb{R}_{\geq 0} \qquad \text{proper.}$$
$$\phi \longmapsto F(\phi(\gamma))$$

$$\left[\text{e.g.} \quad F = i(\gamma, -) \quad , \quad F_\gamma = P_\gamma\right]$$

$$\frac{|B_L^{F_\gamma} \cap Y|}{L^{6g-6}} = \frac{|B_L^F \cap Y_\gamma|}{L^{6g-6}} = \frac{|B_1^F \cap \frac{1}{L} Y_\gamma|}{L^{6g-6}}$$

$$= \underbrace{\frac{1}{L^{6g-6}} \mathcal{X}_{\frac{1}{L}Y_\gamma}}_{m_\gamma^Y(L)} \underbrace{\left(B_1^F\right)}_{\text{compact ball}}$$

Gauss' Circle Problem: $\dfrac{|B_L \cap \mathbb{Z}^2|}{L^2} \longrightarrow \lambda_{leb}(B_1) = \pi$

Main Results

Thm 1 [Erlandsson-Souto]

$$m_\gamma^{\Gamma(S)}(L) \xrightarrow{L\to\infty} C_\gamma m_{Th}, \quad C_\gamma > 0$$

Thm2 [EST] $R = \{\text{non pAs}\}$

$$m_\gamma^R(L) \xrightarrow{L\to\infty} 0$$

Corollary: $R = \{\text{non-pAs}\}$ is P_γ - neglible.

Proof: $P = i(\gamma, -)$ $\mathcal{C}(S) \longrightarrow \mathbb{R}_{\geq 0}$

1) $$\frac{|B_L^{P_\gamma}|}{L^{6g-6}} = m_\gamma^{\Gamma(S)}(L)\left(B_1^P\right) \to C_\gamma m_{Th}(B_1^P) > 0$$

2) $$\frac{|B_L^{P_\gamma} \cap R|}{L^{6g-6}} = m_\gamma^R(L) \; (B_1^P) \longrightarrow 0$$

Proof of Thm 2: $\lim_{L\to\infty} m_\gamma^R(L) = 0$

Step 1: Precompactness of $\{m_\gamma^R(L)\}_{L\geq 1}$ as measures on $\mathcal{C}(S)$

& every limit point is absolutely cont w.r.t m_{Th}

proof: $R \subseteq \Gamma(S) \Rightarrow \mathcal{X}_{\frac{1}{L}R\gamma} \leq \mathcal{X}_{\frac{1}{L}\Gamma(S)\gamma}$

$$\limsup_{L\to\infty} m_\gamma^R(L) \leq \limsup_{L\to\infty} m_\gamma^{\Gamma(S)}(L) \overset{\text{Thm 1}}{=\!=} C_\gamma m_{Th}$$

$\Rightarrow \{m_\gamma^R(L)\}_{L\geq 1}$ is bounded by $C_\gamma m_{Th}$

Step 2: For $m_\gamma^R = \lim_{L_n\to\infty} m_\gamma^R(L_n)$ any limit point

$\phi\in\Gamma(S)$ $\quad \phi_* m_\gamma^R << m_{Th}$

$$\sum_{\phi\in\Gamma(S)} \phi_* m_\gamma^R \leq C_\gamma m_{Th}$$

[Uses $R = \{\text{non-pAs}\}$]

proof of Thm 2: $\lim_{L\to\infty} m_\gamma^0(L) = 0$

Any limit pt $m_\gamma^R = \lim_{n\to\infty} m_\gamma^R(L_n) \overset{①}{\ll} m_{Th}$

$\Rightarrow dm_\gamma^R = K\, dm_{Th}$, $K : \mathcal{C}(S) \to \mathbb{R}_{\geq 0}$ m_{Th}- measurable

$\phi \in \Gamma(S)$ $\phi_* m_\gamma^R \ll m_{Th}$

$\phi_* K = K\phi^{-1}$ m_{Th} a.e.

$$\sum_{\phi\in\Gamma(S)} \phi_* m_\gamma^R \overset{②}{\leq} C_\gamma m_{Th}$$

$$\Rightarrow \sum_{\phi\in\Gamma(S)} K(\phi(\xi)) \leq C_\gamma \quad m_{Th} \text{ a.e } \xi.$$

Ergodicity of $\Gamma(S) \curvearrowright (\mathcal{ML}, m_{Th}) \Rightarrow K = 0$ m_{Th} a.e.

Proof of Step 2 $\sum_{\phi \in \Gamma(S)} \phi_* m_\gamma^R \leq C_\gamma m_{Th}$

Enough to show $\forall$ finite set $Z \subset \Gamma(S)$

$$\sum_{\phi \in Z} \phi_* m_\gamma^R \leq C_\gamma m_{Th}$$

Choose $K > 2 \cdot \max_{\phi \in Z} d_{\Gamma(S)}(1, \phi)$

Decomposition of $R = I_K \cup D_K$

$I_K = \{ \phi \in R \mid d_{\Gamma(S)}(\phi, \phi') \geq K, \forall \phi' \in R - \{\phi\} \}$

$\uparrow$ K-isolated

$D_K = R - I_K$ K-dense

$$m_\gamma^R = m_\gamma^{I_K} + m_\gamma^{D_K}$$

Upshot: I_K has low density

D_K is negligible

<u>Low density</u> of I_K: K is chosen so that

$$I_K \phi \cap I_K \phi' = \emptyset \quad \forall\, \phi \neq \phi' \in Z$$

$\bigcup_{\phi \in Z} I_K \phi$ disjoint union of distinct subsets in $\Gamma(S)$

$$\sum_{\phi \in Z} \phi_* m^R_\gamma(L_n) = \sum_{\phi \in Z} m^{\phi R}_\gamma(L_n)$$

$[\phi R = R \phi]$

$$= \sum_{\phi \in Z} m^{R\phi}_\gamma(L_n)$$

$[(R\phi)\gamma = R(\phi\gamma)]$

$$= \sum_{\phi \in Z} m^{R}_{\phi\gamma}(L_n)$$

$$= \sum_{\phi \in Z} m^{J_k}_{\phi\gamma}(L_n) + \sum_{\phi \in Z} m^{D_k}_{\phi\gamma}(L_n)$$

$$= \sum_{\phi \in Z} m^{J_k\phi}_{\gamma}(L_n) + \quad " \quad\quad "$$

$$= m^{\cup J_k \phi}_{\gamma}(L_n) + \quad " \quad\quad "$$

$$\leq m^{\Gamma(S)}_{\gamma}(L_n) + \quad " \quad\quad "$$

$$\Big\downarrow \text{E-S} \qquad\qquad \Big\downarrow \; D_k \text{ is negligible}$$

$$\sum_{\phi \in Z} \phi_x m^R_\gamma \leq C_\gamma m_{Th} + 0$$

Negligibility of D_K : A Technical Result of Maher.

$CG(S)$ = curve graph of S

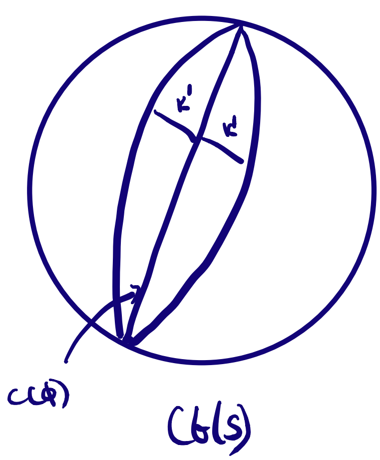
$\phi \in R \iff \phi \curvearrowright CG(S)$ elliptically

Thm [Maher] $\forall K$, $\exists$ finite set $F \subset \Gamma(S)$; $K' > 0$

$$D_K \subseteq \bigcup_{\phi \in F} N_{rel}(C(\phi), K')$$

K'-relative nghd of $C(\phi)$ centralizer of ϕ

$$\mathrm{RelDist}(\phi, \psi) = d_{CG(S)}(\phi x, \psi x), \quad x \in CG(S)$$



Key Point of Proof:

$\phi \in R \iff \phi \sim \phi' \in R$

$d_{CG(S)}(x, \phi' x) \leq B$

Lemma: $m_\gamma^{D_K} = 0 \quad \forall K \; ; \; \forall \gamma.$

proof: $m_\gamma^{D_K} \ll m_{Th} \Rightarrow \mathrm{supp}(m_\gamma^{D_K}) \subseteq \mathcal{ML}(S)$

$\mathcal{UE} = \{\text{uniquely ergodic meas lam}\} \subseteq \mathcal{ML}(S)$

↳ has full m_{Th} - measure

$\mathrm{supp}(m_\gamma^{D_K}) \cap \mathcal{UE} \qquad \left[D_K \subseteq \bigcup_{\phi \in F} N_{rel}(C(\phi), K') \right]$

$\subseteq \bigcup_{\phi \in F} \{ \lambda \in \mathcal{UE} \mid \phi([\lambda]) = [\lambda] \}$

$\Rightarrow \mathrm{Supp}(m_\gamma^{D_K})$ has m_{Th}- measure 0

$\Rightarrow m_\gamma^{D_K} = 0$