Big mapping class groups acting on homology

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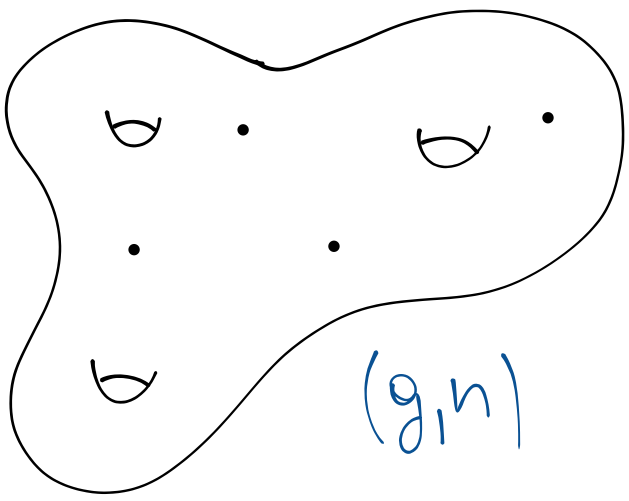
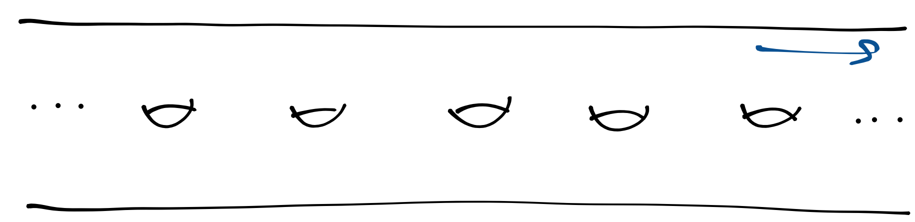
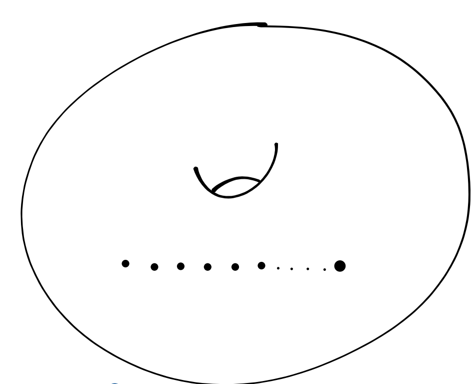
Joint with Sebastian Hensel and Nicholas Vlamis

S surface (conn, oriented, $\partial S = \emptyset$)







finite-type

$\pi_1 S$ f.g.

infinite-type

Ends(S) = space of ends

Mapping class group: $\mathrm{MCG}(S) :=$ Homeo$^+(S)/$homotopy

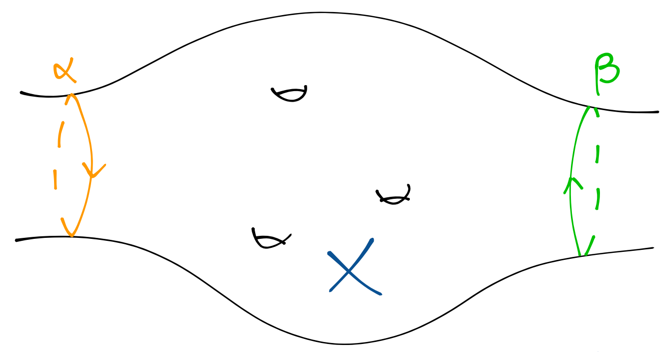
f.g.
discrete

not f.g.
not discrete

Homology

$\mathrm{H}_1(S;\mathbb{R})$: vector space generated by $\{[\alpha] \mid \alpha \text{ curve}\}$ ← simple, closed, oriented

$\mathrm{H}_1(S;\mathbb{Z})$ $\mathbb{Z}$-span of $\{[\alpha] \mid \alpha \text{ curve}\}$

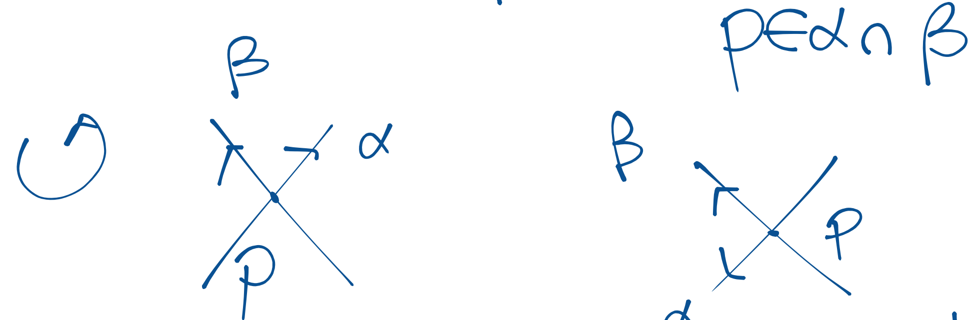


X cpt $\Rightarrow [\alpha]+[\beta]=0$

Rmk: α sep. cutting off X cpt $\Leftrightarrow [\alpha]=0$

Algebraic intersection form $\hat{\imath}$: $\mathrm{H}_1(S;\mathbb{R}) \times \mathrm{H}_1(S;\mathbb{R}) \to \mathbb{R}$
bilinear, antisymm.

$$\alpha,\beta \to \hat{\imath}([\alpha],[\beta]) = \sum_{p\in\alpha\cap\beta} \varepsilon_p$$



$\varepsilon_p = 1$ $\varepsilon_p = -1$

Rmk: α sep. $\Rightarrow \hat{\imath}([\alpha],\cdot) \equiv 0$ $[\alpha]$ isotropic

Remark: $\hat{\imath}$ is symplectic if and only if S has at most one end.

v isotropic $\Rightarrow v=0$ $\Leftrightarrow$ $\forall \alpha$ sep, $[\alpha]=0$ $\Leftarrow$

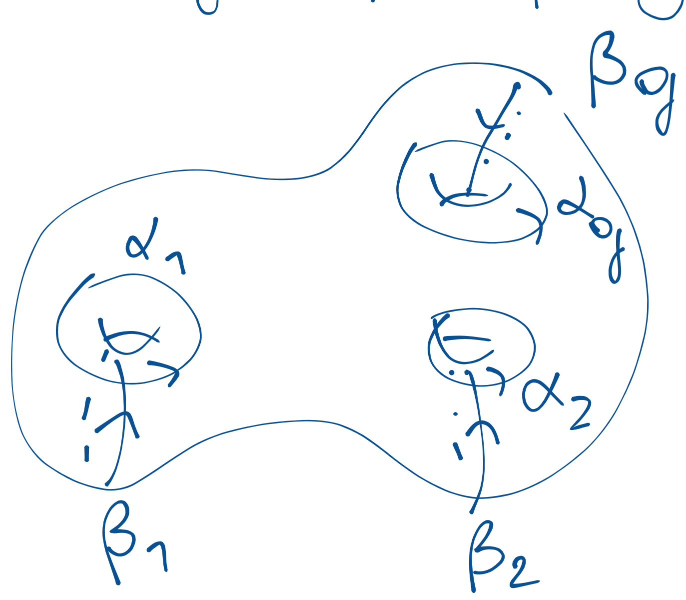
Action of $\mathrm{MCG}(S)$ on homology $\rightsquigarrow \rho_S : \mathrm{MCG}(S) \to \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z});\hat{\imath})$ $\subseteq GL(H_1(S;\mathbb{R}))$

Question: what is the image of ρ_S? I.e. which $\varphi \in GL(H_1)$ come from a mapping class?

A classical result

Theorem (Burkhardt)
If S is closed of genus g, ρ_S is surjective onto $\mathrm{Aut}(\underbrace{\mathrm{H}_1(S;\mathbb{Z})}_{\mathrm{rk}\ 2g};\hat{\imath}) \simeq \mathrm{Sp}(2g;\mathbb{Z})$.

"Proof:" $\varphi \in \mathrm{Sp}(2g;\mathbb{Z})$



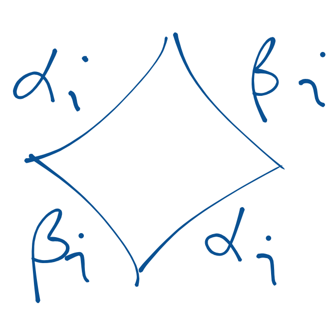
$[\alpha_i], [\beta_i]$ symplectic basis

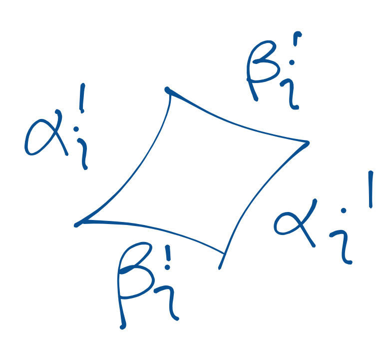
Only α_i & β_i intersect, in 1! pt

$$\begin{array}{ccc} \varphi([\alpha_i]) & , & \varphi([\beta_i]) \\ \parallel & & \parallel \\ [\alpha_i'] & & [\beta_i'] \end{array}$$

α_i', β_i' curves w/ same int. pattern

$S \setminus \bigcup_i \alpha_i \cup \beta_i \simeq S^2$ w/ g disks removed $\simeq S \setminus \bigcup \alpha_i' \cup \beta_i'$





f homeo
$\alpha_i \longmapsto \alpha_i'$
$\beta_i \longmapsto \beta_i'$

$\rightsquigarrow F: S \longrightarrow S$ homeo
$\alpha_i \to \alpha_i'$, $\beta_i \to \beta_i'$

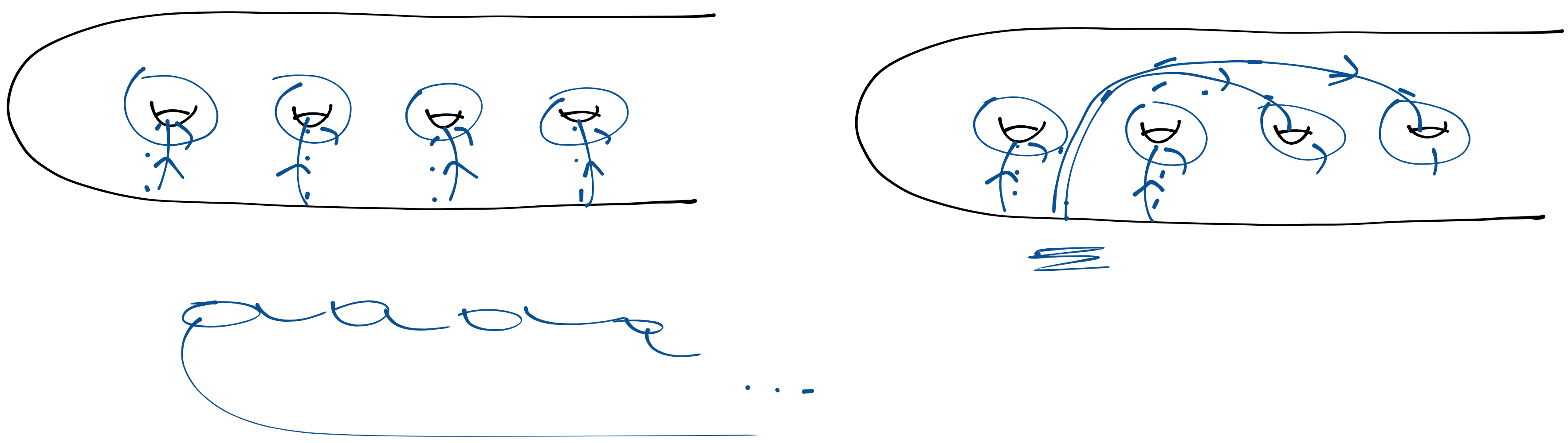
$F_* = \varphi$

$\square$

The Loch Ness monster

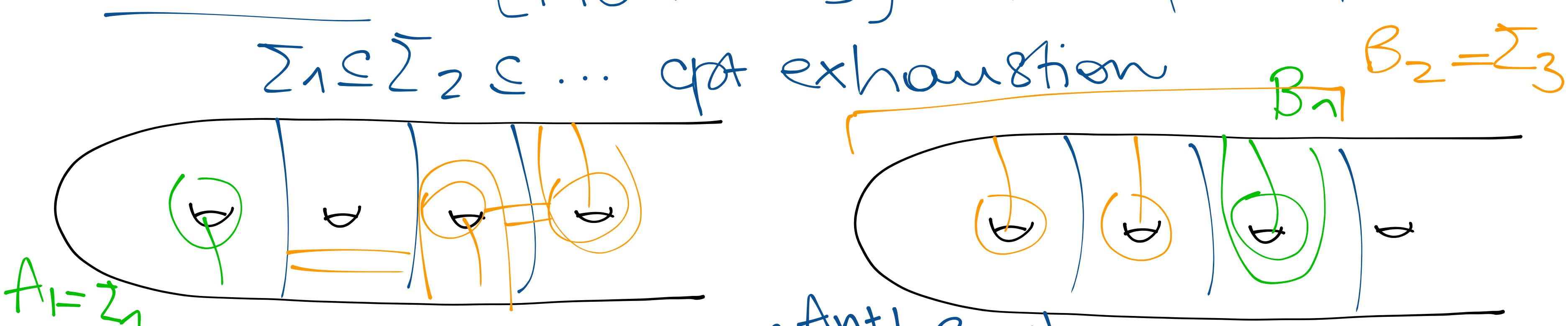
Theorem (F.–Hensel–Vlamis)
If S is the Loch Ness monster, ρ_S is surjective onto $\mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z});\hat{\imath}) \simeq \mathrm{Sp}(\mathbb{N};\mathbb{Z})$.

Problem:



Solution: [Richards] $\varphi \in \mathrm{Sp}(\mathbb{N}, \mathbb{Z})$

$\Sigma_1 \subseteq \Sigma_2 \subseteq \dots$ cpt exhaustion



Construct subs. $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$, $f_n : A_n \xrightarrow[\text{homeo}]{} B_n$

- n odd $A_n \supseteq \Sigma_n$
- n even $B_n \supseteq \Sigma_n$

$\Rightarrow A_n, B_n$ exh.

- $f_n|_{A_{n-1}} = f_{n-1} \Rightarrow f = \varinjlim f_n$
- $(f_n)_* = \varphi|_{H_1(A_n; \mathbb{Z})} \Rightarrow f_* = \varphi$

The general case

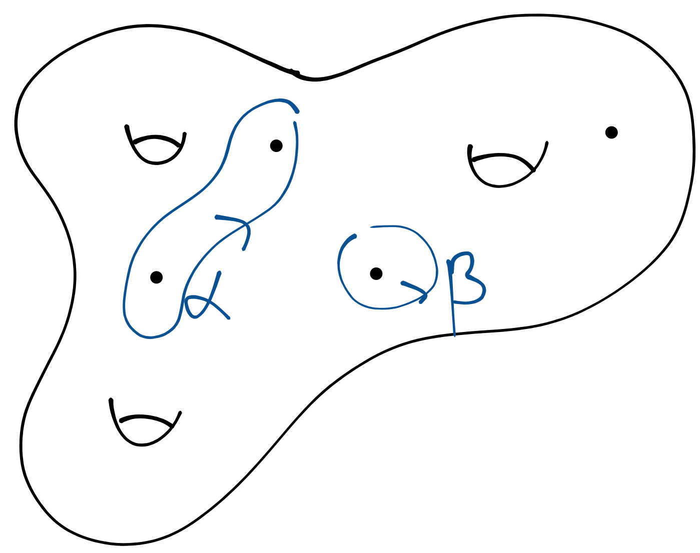
What if $\hat{\imath}$ is not symplectic?

$\exists\, \varphi \in \underline{\mathrm{Aut}(H_1(S,\mathbb{Z}), \hat{\imath})}$

$\varphi([\alpha]) = [\beta]$

but $\nexists\, F \in \mathrm{MCG}$:

$F_* = \varphi$



$\longrightarrow$ need to:

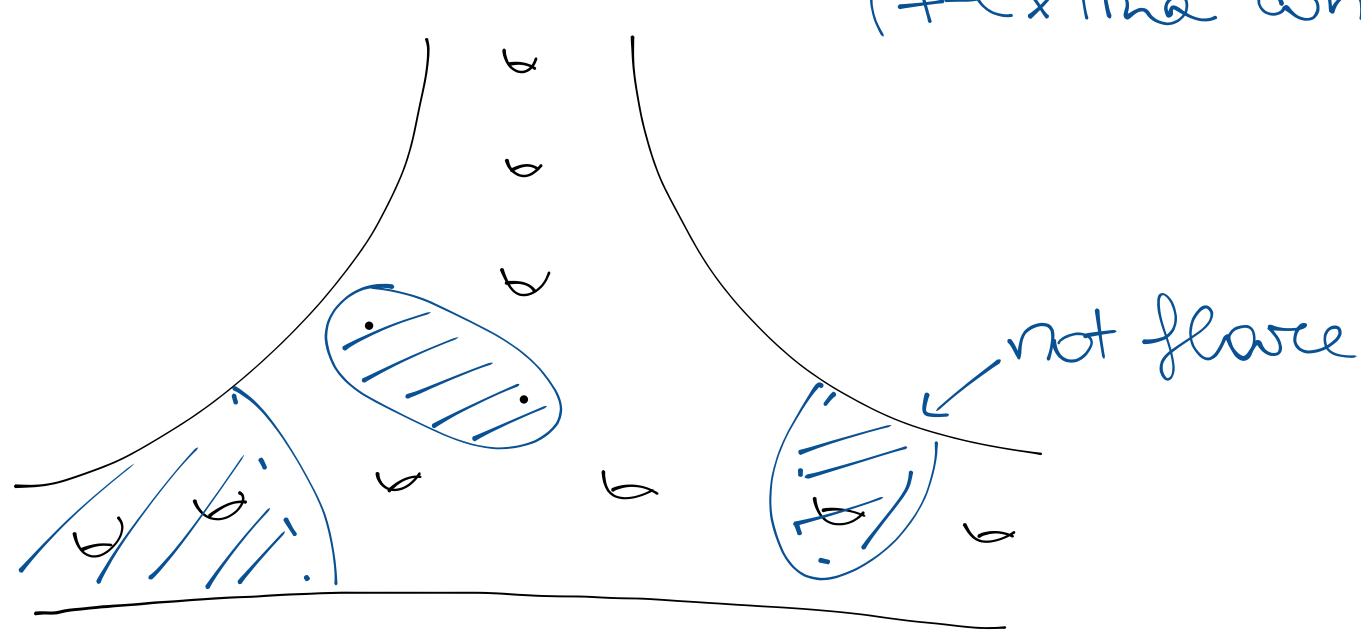
(1) detect ends

(2) deal w/ sep. curves

Ends

X **flare surface** if ∂X is a single separating curve and X is unbounded.

$\subseteq$ S (+ extra cond)



$\mathcal{F} := \{\mathrm{H}_1(X;\mathbb{Z}) \mid X \text{ flare surface}\}$ w/ partial order $\leq$ given by inclusion

$= \langle [\alpha] \mid \alpha \text{ curves}, \alpha \subseteq X \rangle$

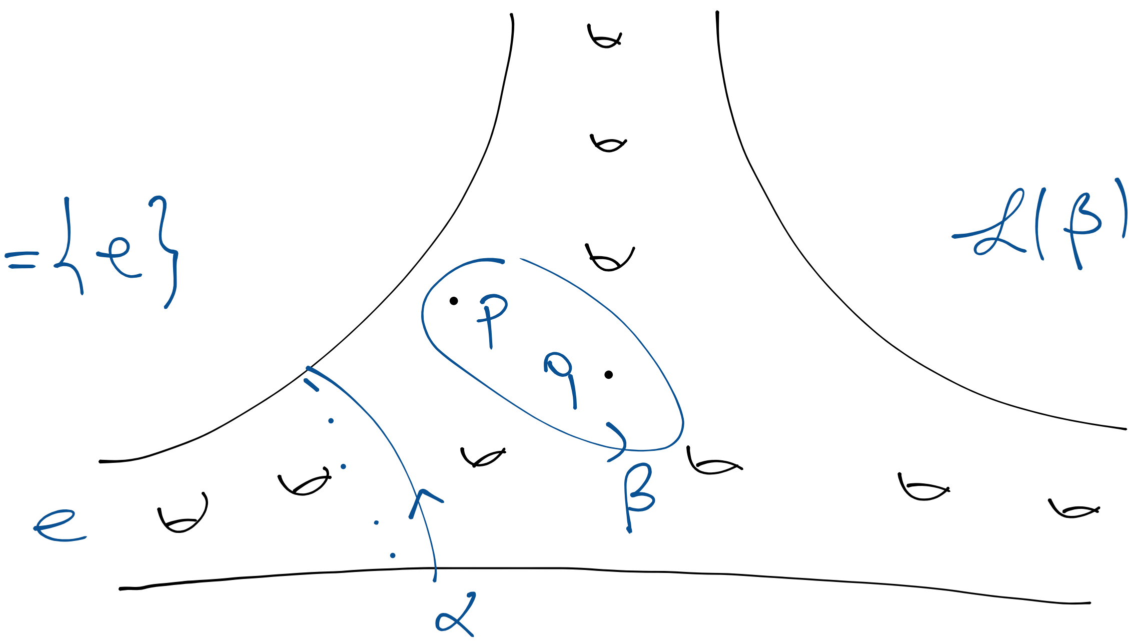
<u>Fact</u>: ends $\longleftrightarrow$ ultrafilters for $(\mathcal{F}, \leq)$

$e \longleftrightarrow \mathcal{F}_e = \{ V \in \mathcal{F} \mid \exists X \text{ flare}: V = H_1(X;\mathbb{Z}) \ \&\ e \in \mathrm{Ends}(X) \}$

Separating curves

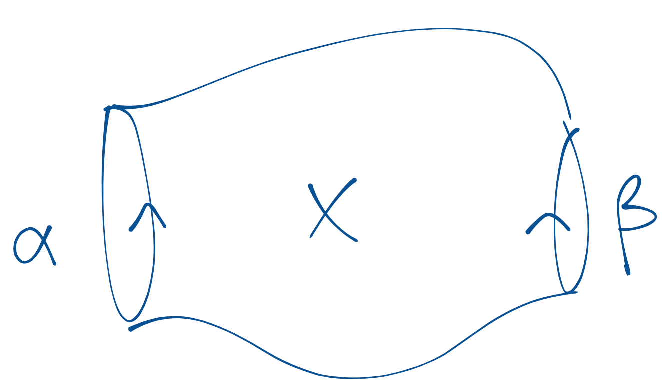
γ separating curve $\rightsquigarrow \mathcal{L}(\gamma) := \{e \in \mathrm{Ends}(S) \mid e$ is to the left of $\gamma\}$

$\mathcal{L}(\alpha) = \{e\}$



$\mathcal{L}(\beta) = \{p, q\}$

Lemma
α, β separating. Then $[\alpha] = [\beta]$ if and only if $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$.



X cpt $\iff [\alpha] = [\beta]$

$\Updownarrow$

no end in X $\iff \mathcal{L}(\alpha) = \mathcal{L}(\beta)$

$c = [\gamma]$, γ sep. $\Rightarrow \mathcal{L}(c) := \mathcal{L}(\gamma)$

The general case

$\mathcal{F} = \{ H_1(X;\mathbb{Z}) \mid X \text{ plane} \}$

Theorem (F.–Hensel–Vlamis)
Let S be either:

- a finite-type surface with at least 4 punctures, or
- an infinite-type surface different from the Loch Ness monster or the once-punctured Loch Ness monster.

Then: $\quad \forall V \in \mathcal{F}, \varphi(V) \in \mathcal{F}$

- $\varphi \in \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z}); \hat{\imath})$ preserving $\mathcal{F} \Rightarrow$ it preserves

  $$\{[\alpha] \mid \alpha \text{ separating}\}$$

  and it induces a homeomorphism

  $e \mapsto f$

  $$f_\varphi : \mathrm{Ends}(S) \to \mathrm{Ends}(S);$$

  $\varphi \nearrow \varphi(\mathcal{F}_e) = \mathcal{F}_f$

  $e \leftrightarrow \mathcal{F}_e = \{ H_1(X;\mathbb{Z}) \mid X \ni e \}$.

- the image of ρ_S is

$$\left\{ \varphi \in \mathrm{Aut}(\mathrm{H}_1(S;\mathbb{Z}); \hat{\imath}) \;\middle|\; \begin{array}{c} \varphi \text{ preserves } \mathcal{F} \text{ and} \\ \exists \alpha \text{ separating}, [\alpha] \neq 0 : \\ \mathcal{L}(\underbrace{\varphi([\alpha])}_{[\beta]}) = f_\varphi(\mathcal{L}([\alpha])) \end{array} \right\}.$$