Counting conjugacy classes in groups

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Hyperbolic Lunch

Motivating examples: Cayley graph and word metric

Let G be a finitely generated group with a finite generating set S.

- The **Cayley graph of** G has the vertex set G so that two vertices $g_1 \Longleftrightarrow g_2$ are connected iff $g_2 = g_1 s$ for some $s \in S$
- The **word metric** is the combinatorial metric on the Cayley graph.

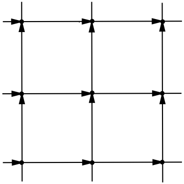


Figure: Standard Cayley graph of $\mathbb{Z}^2$

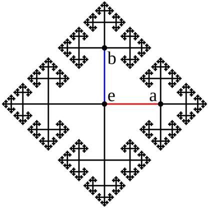


Figure: Standard Cayley graph of $\mathbb{F}_2$

Growth function and conjugacy growth

In this talk, we are interested in the following counting functions.

- The **growth function**

$$R \mapsto \sharp N(1, R)$$

counts the number of elements in a ball $N(1, R)$ of radius R at 1.

- The **conjugacy growth function**

$$R \mapsto \mathcal{C}(1, R)$$

counts the number of conjugacy classes in the ball $N(1, R)$.

Examples

1. In $\mathbb{Z}^n$, the growth function equals the conjugacy growth function.
2. [Coorneart; 2006] In $\mathbb{F}^n$, the conjugacy growth function is asymptotic to $C\frac{\exp(hR)}{R}$, where $h = \log(2n-1)$ and $C = (2n-1)/2(n-1)$.

Classification of groups by growth function

- **Exponential growth**: growth function is of order $C \cdot \exp(C \cdot R)$ for some $C > 1$.
- **Polynomial growth**: Gromov (1983) famously proved that polynomial growth function characterizes the class of virtually nilpotent group.
- **Immediate growth**: Grigorchuk (1983) constructed the first examples of groups which is neither polynomial nor exponential.

Remark

Many naturally occuring groups satisfy the **Tits alternative**: either it is virtually solvable or contains $\mathbb{F}_2$. As a consequence, the growth function is either polynomial or exponential.

Varieties of conjugacy growth

In 2010, Cuba and Sapir initiated a systematic study of conjugacy growth function in groups. It turns out that conjugacy growth functions could be very different with growth functions:

1. The solvable groups [Breuillard-Cornulier; 2010] and linear groups [Breuillard-Cornulier-Lubotzky-Meiri; 2013]: **The conjugacy growth function is either polynomially bounded or exponential**.
2. [Hull-Osin; 2011] **Conjugacy growth is not quasi-isometric invariant**: $\exists$ finitely generated group with exponential conjugacy growth but with a finite index subgroup with exactly two conjugacy classes.
3. [Hull-Osin; 2011] Any reasonable function (nondecreasing at most exponential) can be realized as conjugacy growth of a finitely generated group.

Setup: counting conjugacy classes in group actions

Suppose a group G acts properly on a geodesic metric space (X, d).

1. Fix a basepoint $o \in X$. Denote $N(o, n) := \{g \in G : d(o, go) \le n\} < \infty$. The function

$$\mathcal{G} : n \to \sharp N(o, n)$$

is called **growth function**.

2. Define the **algbraic length** of a conjugacy class $[g]$:

$$\ell_o([g]) = \min\{d(ho, o) : h \in [g]\}.$$

and consider the **conjugacy growth function**

$$\mathcal{C}(o, n) = \sharp\{[g] \in G : \ell_o([g]) \le n\}.$$

Problem: coarse asymptotic conjugacy growth

The growth function is called **purely exponential** if there exists a constant δ_G called **growth rate** such that

$$\sharp N(o,n) \asymp \exp(n\delta_G)$$

Here $\asymp$ means that both sides are equal, up to a **multiplicative** constant.

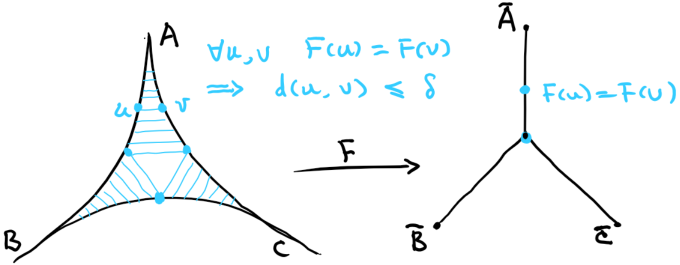
Question

Assume that the growth function is purely exponential, under which conditions, we have a coarse asymptotic conjugacy growth formula:

$$\mathcal{C}(o,n) \asymp \frac{\exp(n\delta_G)}{n} \quad ?$$

Hyperbolic groups in the sense of Gromov

► A geodesic metric space X is called δ-**hyperbolic** for $\delta \geq 0$ if any geodesic triangle is δ–thinner than the comparison triangle in a tree.



► A finitely generated group is called **hyperbolic**, if it acts properly and cocompactly on a δ-hyperbolic space for some $\delta > 0$.

Conjugacy growth for hyperbolic groups

Theorem (Coornaert-Knieper, 1997, torsion-free case; Antolin-Ciobanu, 2015, general case)

Let G be a group acting properly and cocompactly on a hyperbolic space (X, d). Fix a basepoint o.Then

$$\mathcal{C}(o, n) \asymp \frac{\exp(\delta_G n)}{n}$$

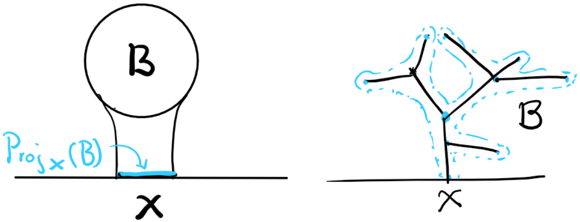
where

$$\delta_G = \lim_{n\to\infty} \frac{\log \sharp N(o, n)}{n}$$

Contracting subsets

Let (X, d) be a geodesic metric space.

- A subset S is called **(strongly) contracting** if any ball missing S has a uniform bounded projection to X: there exists $C > 0$ such that if a metric ball $B \cap S = \varnothing$ then $\text{Diam}(\text{Proj}_S(B)) \leq C$.



♣ Geodesics in trees are contracting: we can take $C = 0$.

Contracting elements

Definition

An element $g \in G$ is called **contracting** if for some basepoint $o \in S$, the map $n \in \mathbb{Z} \to g^n \cdot o$ is a quasi-isometric embedding map:

$$\exists \lambda \geq 1, c > 0: \quad \frac{1}{\lambda}|n-m| - c \leq d(g^n o, g^m o) \leq \lambda |n-m| + c$$

and **the orbit $\langle g \rangle \cdot o$ is a strongly contracting subset**.

Example

The prototype of a contracting element is the following.

1. In δ-hyperbolic spaces, (quasi-)geodesics are contracting.
2. An isometry g is called **loxodromic** if $\langle g \rangle$ preserves a quasi-geodesic.
3. Thus, a loxodromic element is contracting.

More examples of contracting elements

1. Rank-1 elements in CAT(0) spaces are contracting. [Fujiwara-Bestvina, 2008]
2. Rank-1 elements on a cubical CAT(0) space which is not a product of unbounded cube subcomplexes are contracting with respect to the combinatorial metric.
3. Every pseudo-Anosov element in Mapping class groups is contracting [Minsky, 1997].
4. Every hyperbolic element in a relatively hyperbolic group is contracting with respect to the Cayley graph. [Gerasimov - Potaygailo; 2010]

Main results: asymptotic growth of conjugacy classes

Theorem (Gekhtman - Y.; 2018)

Suppose a non-elementary group G admits a **properly discontinuous cocompact action** *on a geodesic metric space (X, d) with a* **contracting element**. *Then for a basepoint $o \in X$, we have*

$$\mathcal{C}(o, n) \asymp \frac{\exp(\delta_G n)}{n}.$$

Remark

Our theorem holds for a more general class of **statistically convex-cocompact actions** which is purely exponential [Y.2017]:

$$\sharp N(o, n) \asymp \exp(\delta_G n).$$

Applications to conjugacy growth series

Define the **conjugacy growth series**

$$\sum_{n\geq 1} \mathcal{C}(o,n) z^n$$

1. [Ciobanu-Hermiller-Holt-Rees] Virtually cyclic groups have rational conjugacy growth series.
2. Rivin conjectured that non-elementary hyperbolic groups always have transcendental conjugacy growth series. This was confirmed by Antolin-Ciobanu. We extend it to the following setting.

Corollary (Gekhtman - Y.)

Let G be a relatively hyperbolic group acting on its Cayley graph. The conjugacy growth series is transcendental iff G is not virtually cyclic.

The proof of main theorem is a study of **generic behaviours** of isometries when a contracting element is supplied.

Theorem (Y. 2017)

The set S of non-contracting elements in G is **exponentially negligible***:*

$$\exists \epsilon > 0, \forall n: \ \sharp S \cap N(o,n) \le \exp(-\epsilon n) \sharp N(o,n) \le \exp((\delta_G - \epsilon)n).$$

The goal of our theorem is to prove

$$\mathcal{C}(o,n) \asymp \frac{\exp(\delta_G n)}{n}$$

so the growth rate of $\mathcal{C}(o,n)$ is exactly δ_G. However, by Theorem, the growth rate of non-contracting elements is strictly less than δ_G.

Remark (Conclusion)

It suffices to consider the conjugacy classes of contracting elements in G.

Lower bound: an orbit closing lemma

1. We first prove **an orbit closing lemma**: for certain proportion of elements $g \in T \subset N(o, n)$, we purturb it by a universal element f to produce a contracting element $g \cdot f$.
2. Since G has purely exponential growth, this gives at least

$$\sharp\, T \cdot f \geq \theta_1 \exp(\delta_G n)$$

contracting elements for some uniform $\theta_1 > 0$.
3. We then show that each conjugacy class $[gf]$ in $T \cdot f$ contains at most $\theta_2 n$ elements. This gives the lower bound.

The idea in free groups

The general idea (as in free groups) is to notice that every conjugacy class $[g]$ produces n different elements as follows by cyclic permutations of a shortest representative $g = s_1 s_2 \cdots s_n$:

$$s_2 s_3 \cdots s_n s_1, \ \ \cdots, \ \ s_n s_1 \cdots s_{n-1}.$$

However, if the action $G \curvearrowright X$ is not cocompact, we are not able to produce n different elements for each $[g]$, when $[o, go]$ stays outside $N_M(Go)$ for large proportion of time.

Strongly primitive conjugacy classes

A contracting element g is contained in a maximal elementary group $E(g)$. It is a virtually cyclic group so $E(g)$ contain an index ≤ 2 subgroup $E^+(g)$ such that we have the following exact sequence

$$1 \to K \to E^+(g) \xrightarrow{\pi} \mathbb{Z} \to 1$$

where K is finite.

1. Every element h in $E^+(g)$ so that $\pi(h) = \pm 1 \in \mathbb{Z}$ is called **strongly primitive**.
2. A strongly primitive contracting element g must be **primitive**: it can not be written as a non-trivial power of some element.

Denote by $\mathcal{C}'(o, n)$ the set of strongly primitive conjugacy classes in $\mathcal{C}(o, n)$.

Obtaining upper bound on $\mathcal{C}'(o, n)$

Lemma (GY)

Fix $1 > \theta > 0$. There exists an exp. generic set of elements $g \in G$ such that the fraction of $[o, go]$ contained in $N_M(Go)$ is bigger than θ.

1. Let g be the minimal element so that $\ell_o[g] = d(o, go) = n$. Following the geodesic $[o, go]$, we can thus plot $N := (1-\theta) \cdot n$ orbital points in the M-nbhd of $[o, go]$.
2. Write the product form for $g = s_1 s_2 \cdots s_N$. If g is strongly primitive, then we show that all cyclic permutations give different elements.
3. Thus, each strongly primitive conjugacy classes $[g]$ of length n contain at least $(1-\theta)n$ elements.

We obtain the upper bound on $\mathcal{C}'(o, n) \prec \frac{\sharp N(o,n)}{(1-\theta)n} \prec \frac{\exp(\delta_G n)}{n}$.

Upper bound for all conjugacy classes

So far, we have established the lower bound for all conjugacy classes, and the upper bound for strongly primitive conjugacy classes:

$$\frac{\exp(\delta_G n)}{n} \prec \mathcal{C}(o,n) \prec ??$$

$$?? \prec \mathcal{C}'(o,n) \prec \frac{\exp(\delta_G n)}{n}$$

It remains therefore to prove the lower bound for strongly primitive ones, and the upper bound for all conjugacy classes.

▶ The solution is to prove that the set of strongly primitive contracting elements are exponentially generic:

$$\frac{\mathcal{C}'(o,n)}{\mathcal{C}(o,n)} \xrightarrow{exp.fast} 1$$

Non-Strongly Primitive elements are exp. negligible:

For each Non-Strongly Primitive element $g \in \mathcal{NSP}$ so that $\pi(g) \neq \pm 1$, we define a map

$$\Pi : [g] \mapsto [g_0]$$

where $g = g_0^k f$ for some $f \in K$ and $|\pi(g_0)| = 1$.

1. Since $\tau[g] = k \cdot \tau[g_0]$ for $|k| \geq 2$, we have that $[g_0]$ lies in $\mathcal{C}(\tau[g]/k)$ which is at most $\exp(n \cdot \delta_G/k) \leq \exp(\omega n)$ for $\omega < \delta_G$. Hence the image $\Pi(\mathcal{NSP})$ is indeed exp. negligible.
2. Since $g = g_0^k f$ for $f \in K$, we need to make sure the map Π is uniformly finite to one: the kernel K is uniform bounded. However, this is generally not true (eg. Dunwoody's group [Abbott 2016])!

Non-Strongly Primitive elements are exp. negligible:

Using the recent work of Bestvina-Bromberg-Fujiwara-Sisto, we can prove that this is true for a generic set of contracting elements. :

Lemma (GY)

There exists an exponentially generic set of elements $g \in G$ such that for the corresponding exact sequence $1 \to K \to E^+(g) \to \langle t \rangle \to 1$, there exists a uniform bound on $\sharp K$ independent of g.

Consequently, the map Π is uniformly finite to one, completing the proof that $\mathcal{NSP}$ is exp. negligible, thus the primitive element is exp. generic.

$$\frac{\exp(\delta_G n)}{n} \prec \mathcal{C}'(o,n) \prec \mathcal{C}(o,n) \prec \frac{\exp(\delta_G n)}{n}$$

Thank you for your attention!