Homeomorphic subsurfaces and the omnipresent arcs

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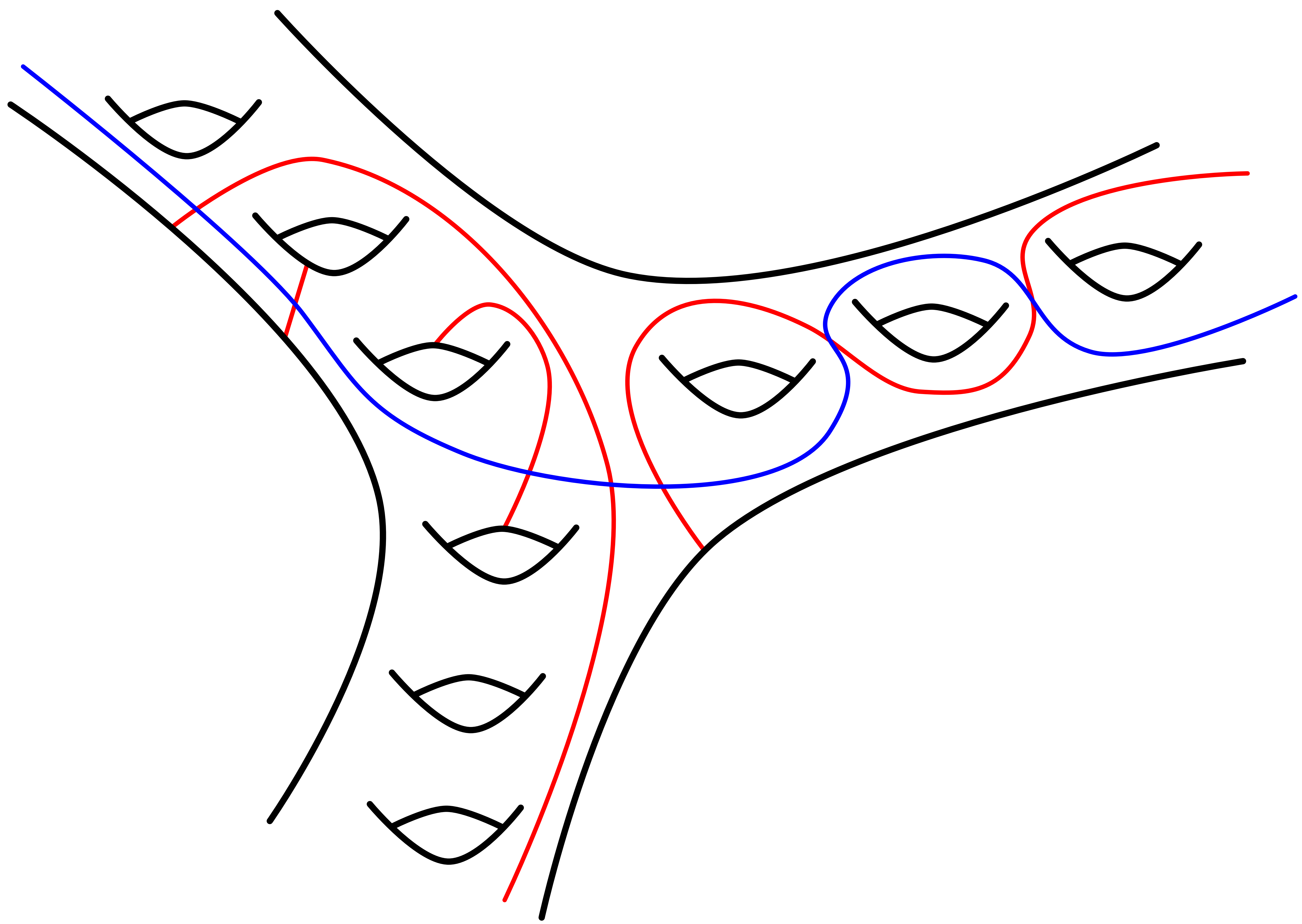
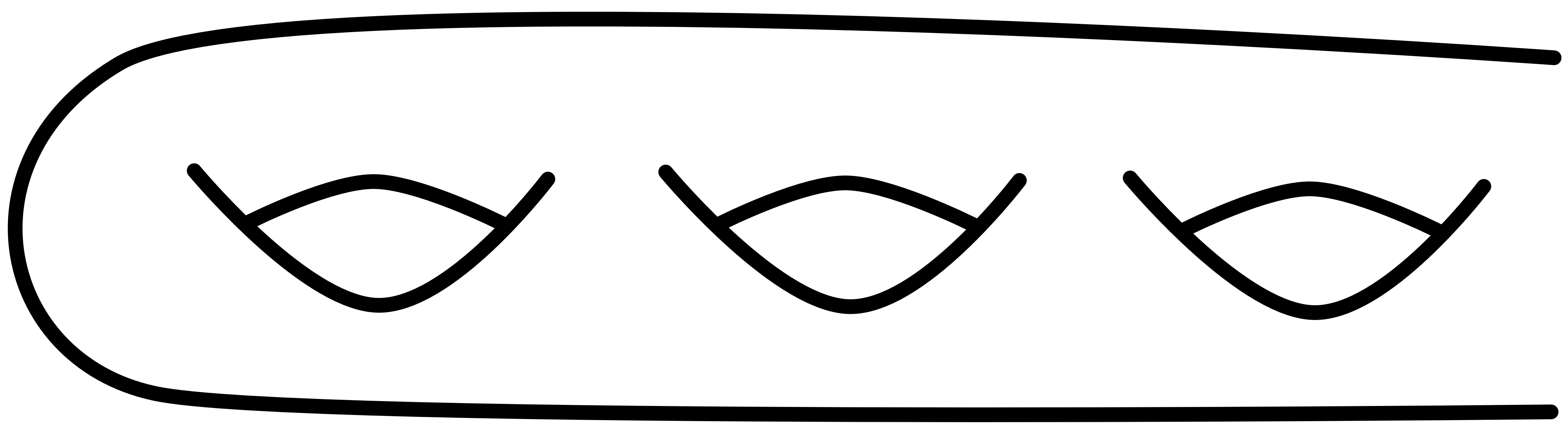
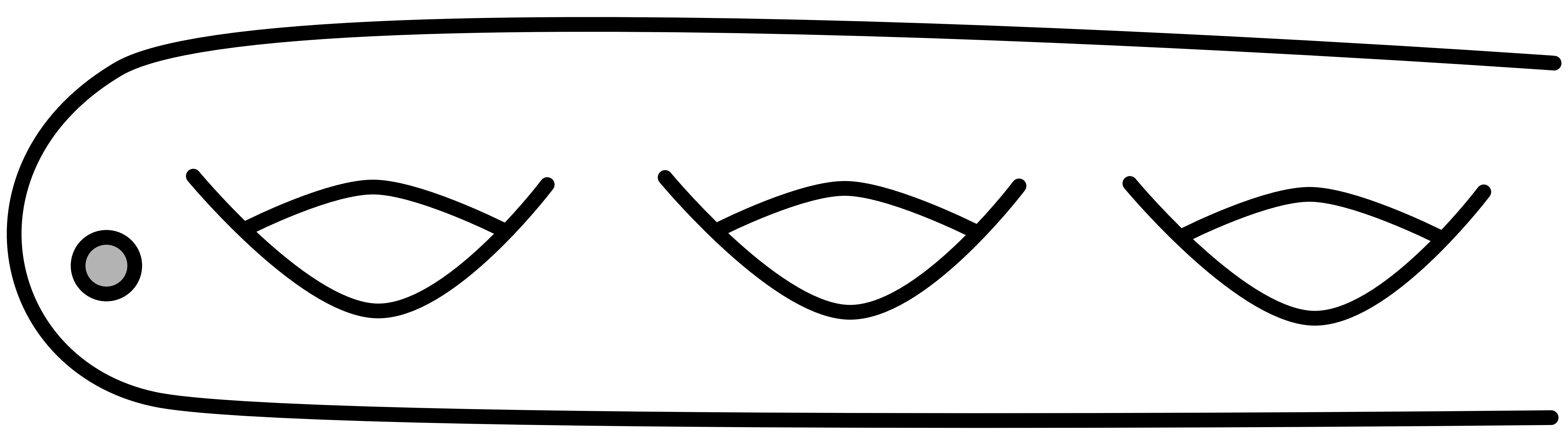
joint with Federica Fanoni and Tyrone Ghaswala



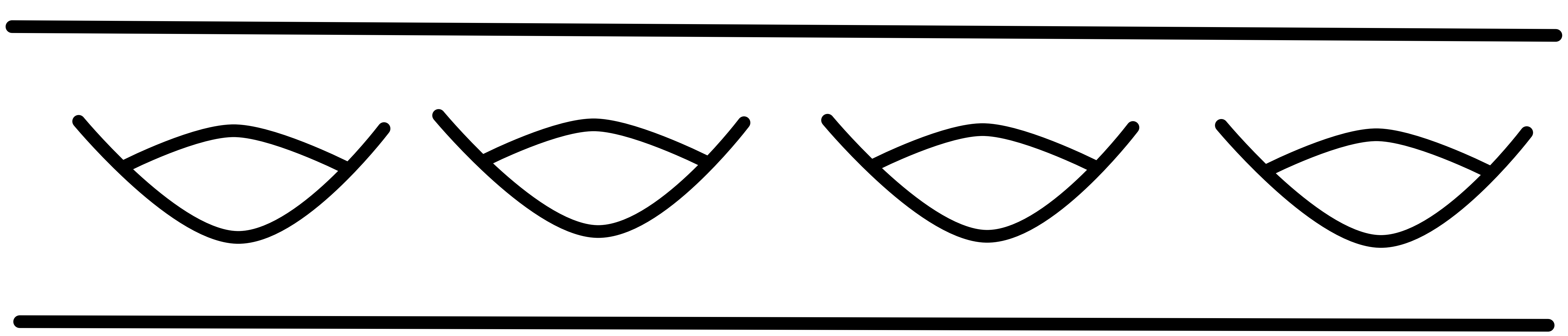
A surface Σ is of infinite-type if $\pi_1(\Sigma)$ is not finitely generated.



The Loch Ness Monster L_1

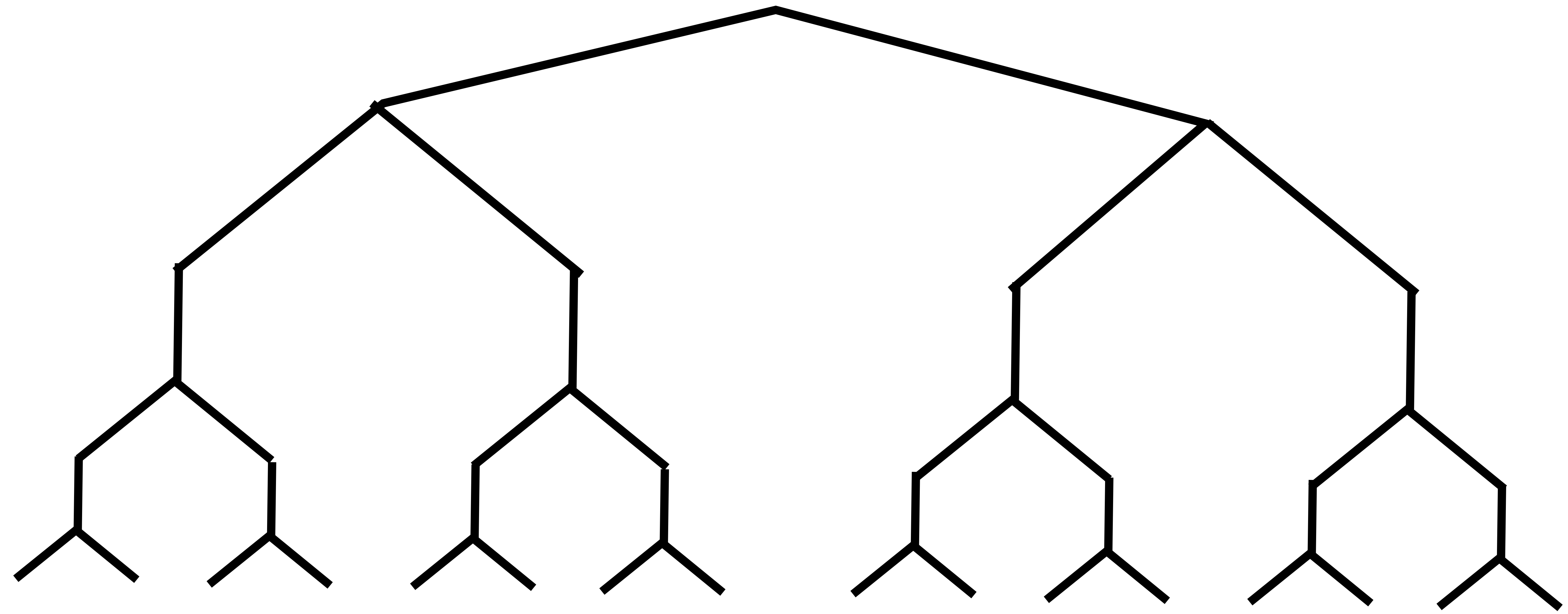


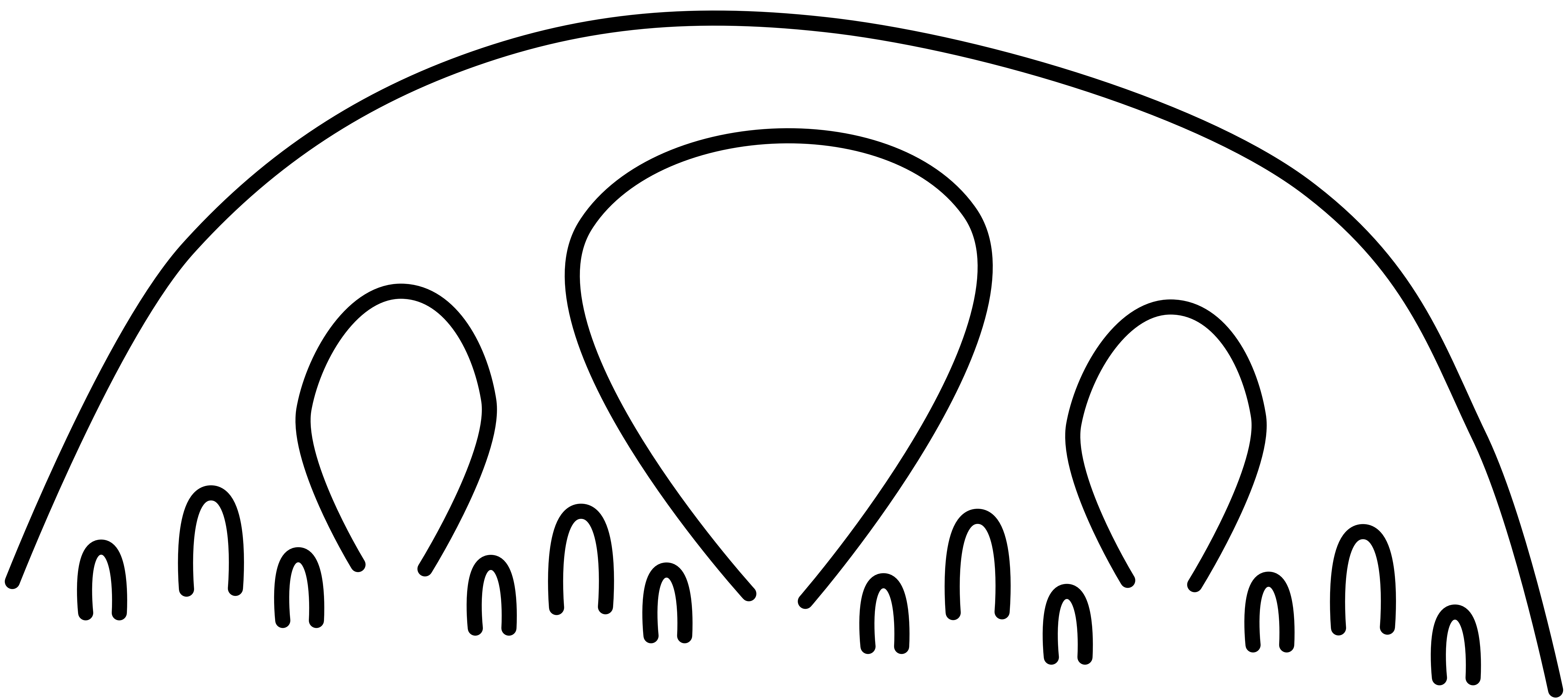
The punctured Monster L_1°



The Ladder L_2

The Cantor Tree



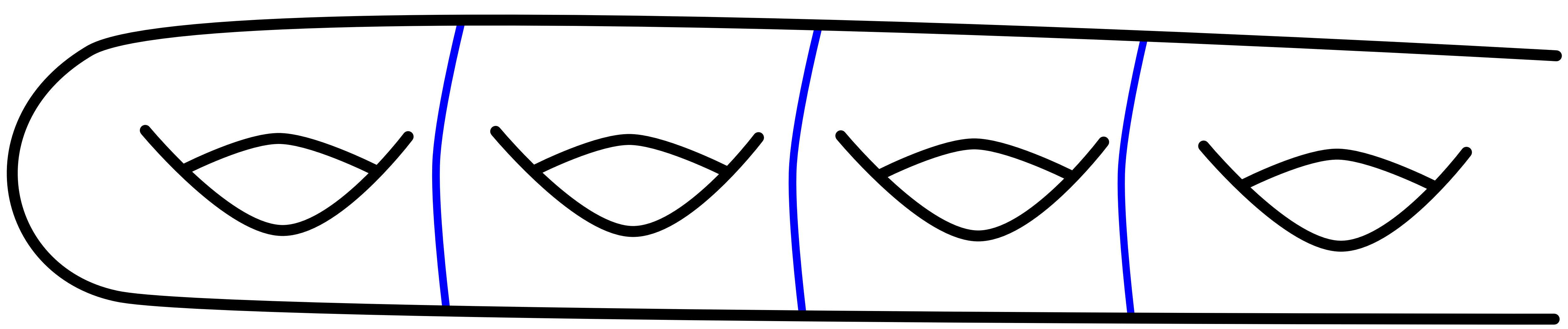


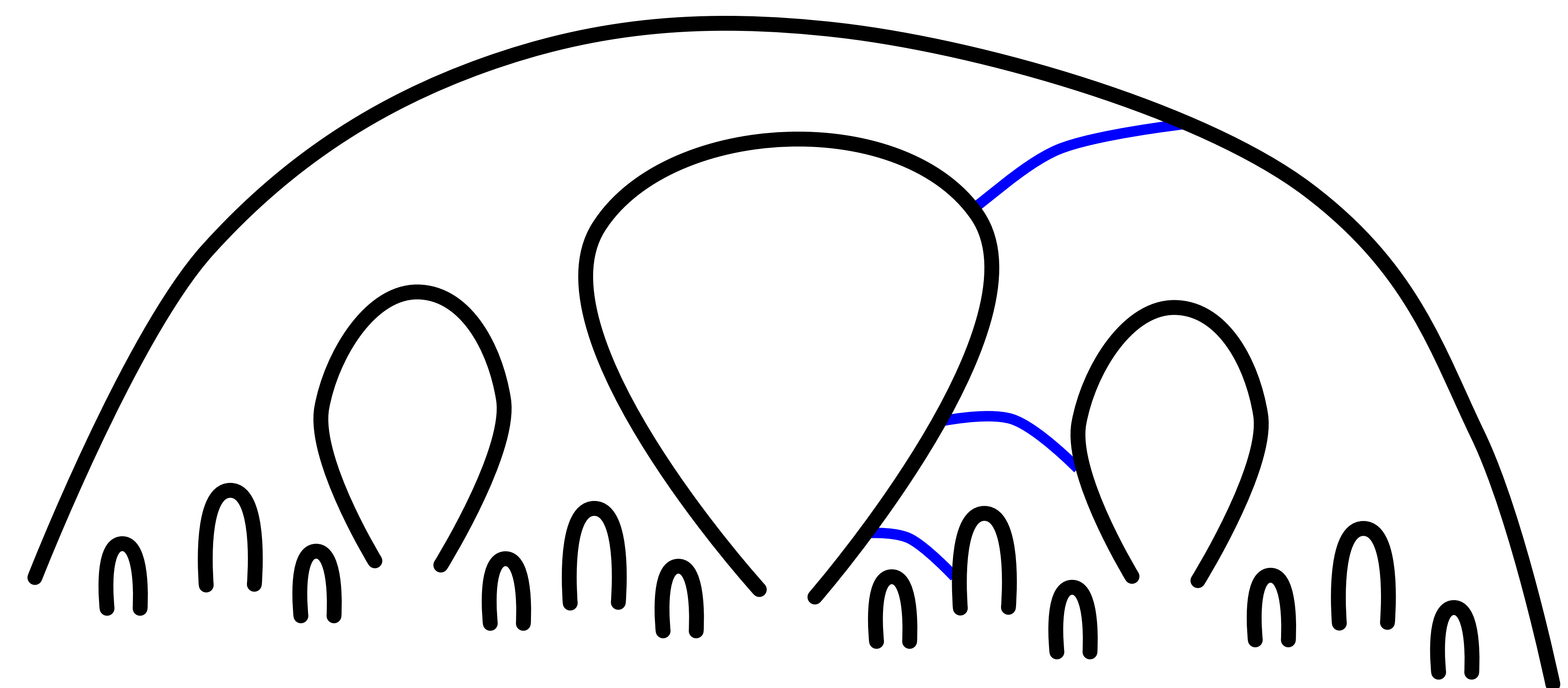
An end is where the surface "goes off to infinity".

Admissable chain: $U_1 \supset U_2 \supset \ldots$

$U_i \subset \Sigma$ noncompact, and ∂U_i a separating curve

for every compact $K \subset \Sigma$, $U_n \cap K = \emptyset$, for large n.





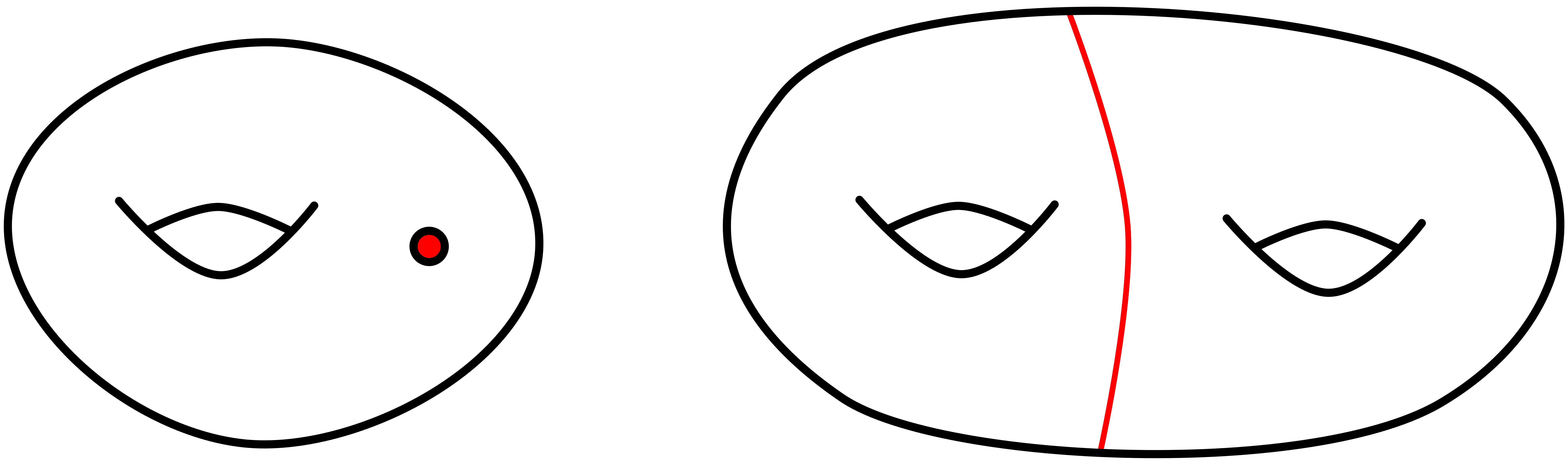
$U_1 \supset U_2 \supset \ldots \sim V_1 \supset V_2 \supset \ldots$

if $\forall n \exists N$ such that $U_n \subset V_N$ and vice versa.

An *end* is $e = [U_1 \supset U_2 \supset \ldots]$

If every U_i has genus then e is *nonplanar*, and *planar* otherwise.

Does it make sense to say one surface is bigger than another?

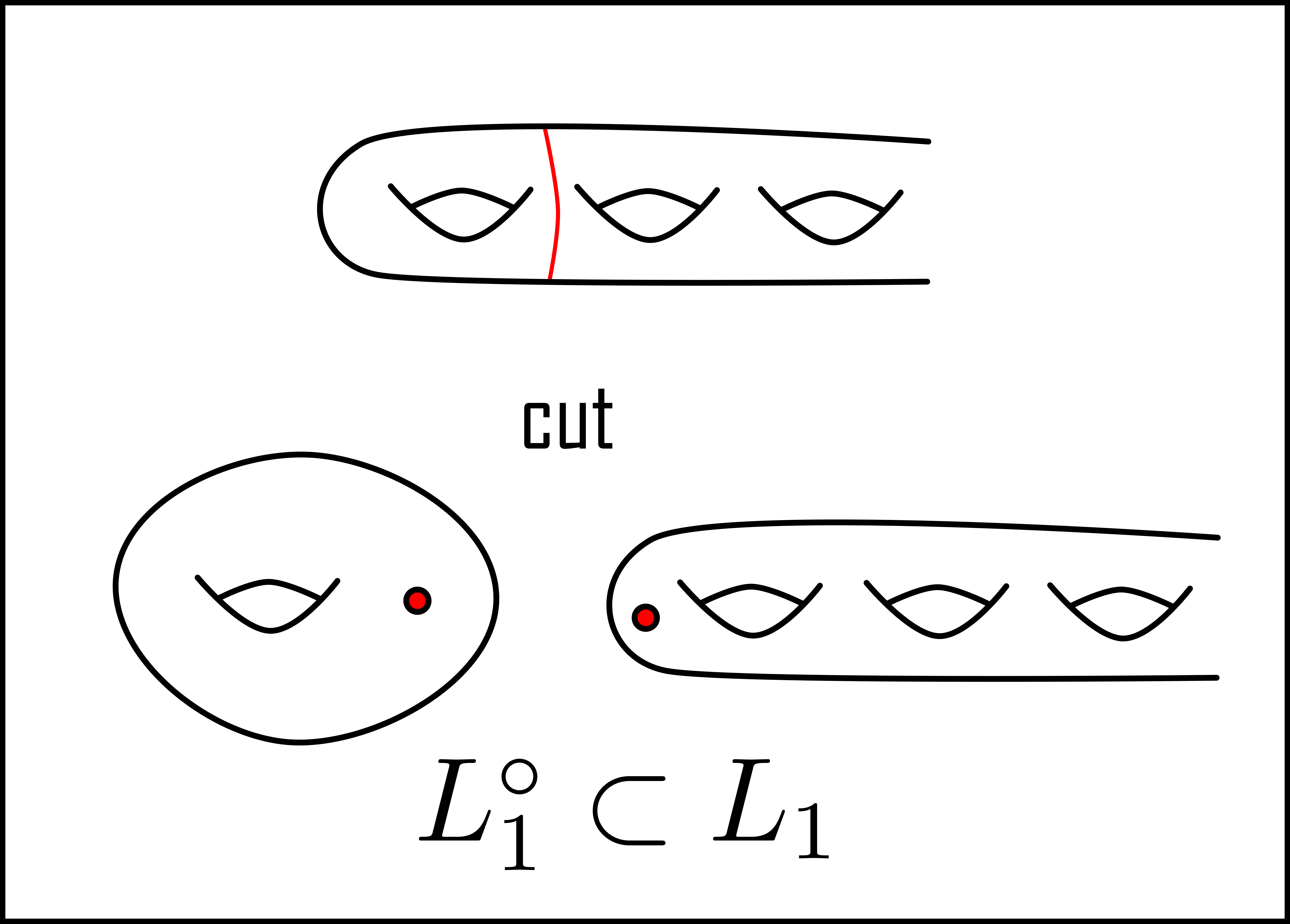


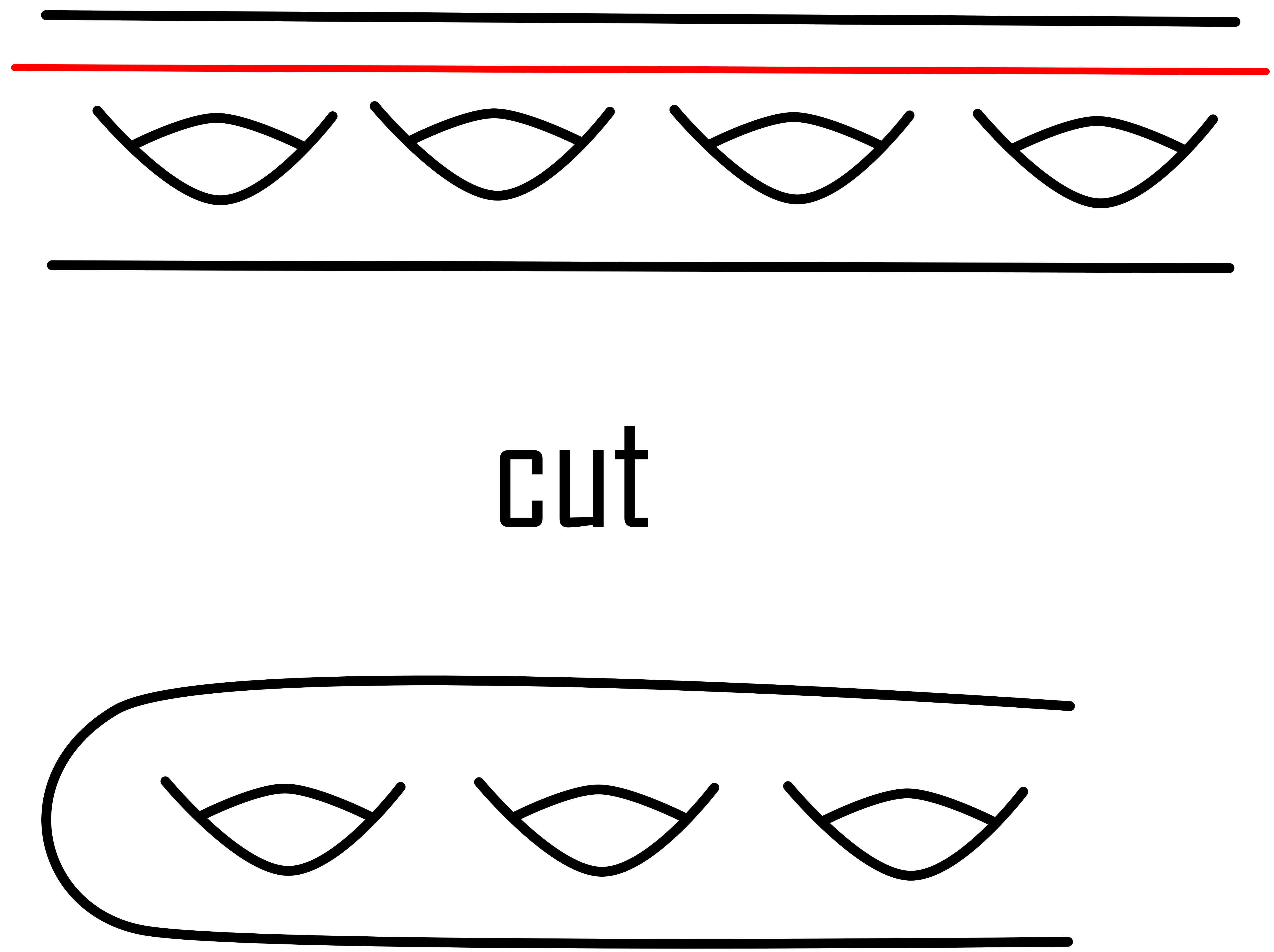
$$\Sigma_1^\circ \subset \Sigma_2 \text{ and } \Sigma_2 \not\subset \Sigma_1^\circ$$

Given two infinite-type surfaces,

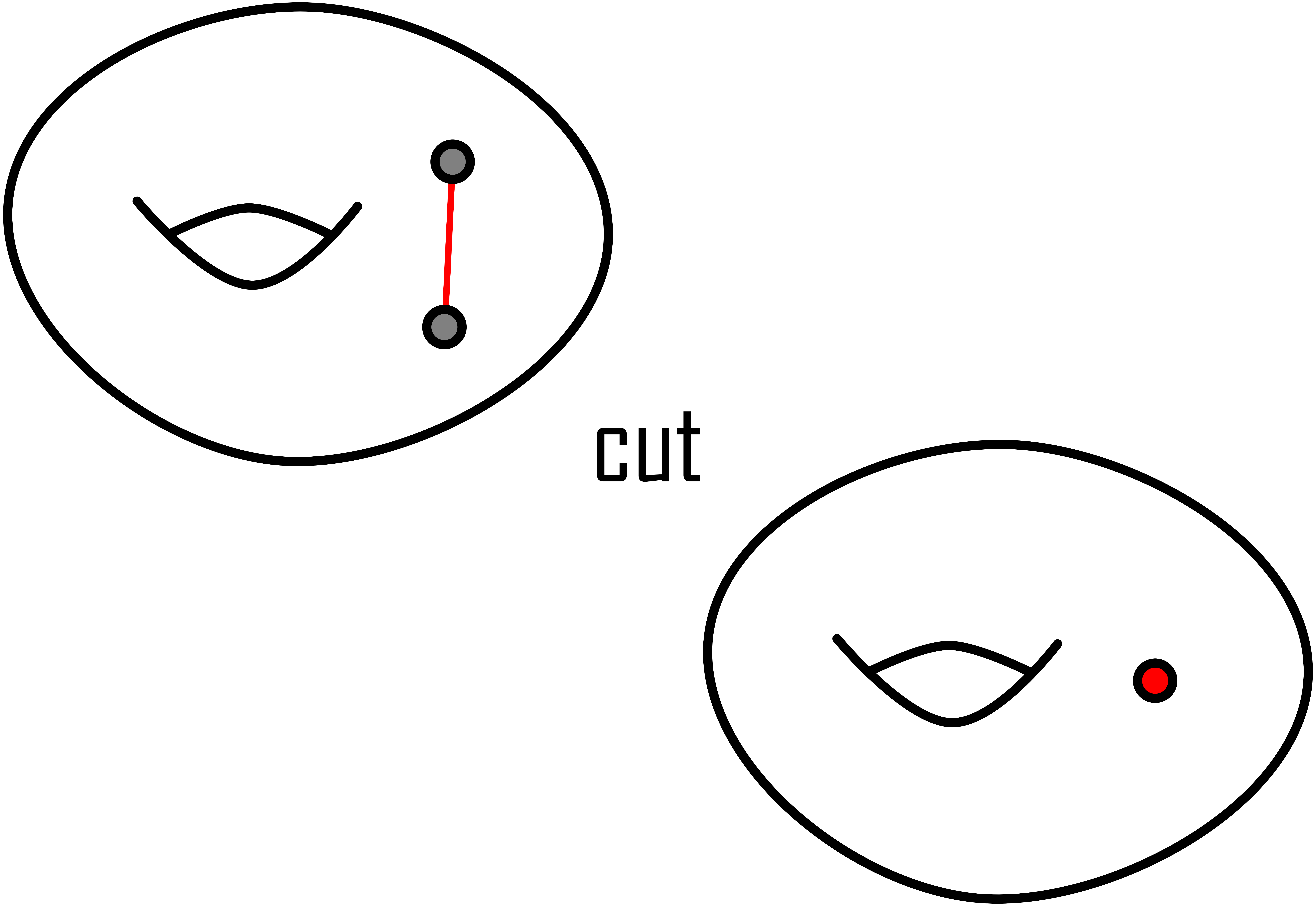
when can one be realized as a subsurface of the other?

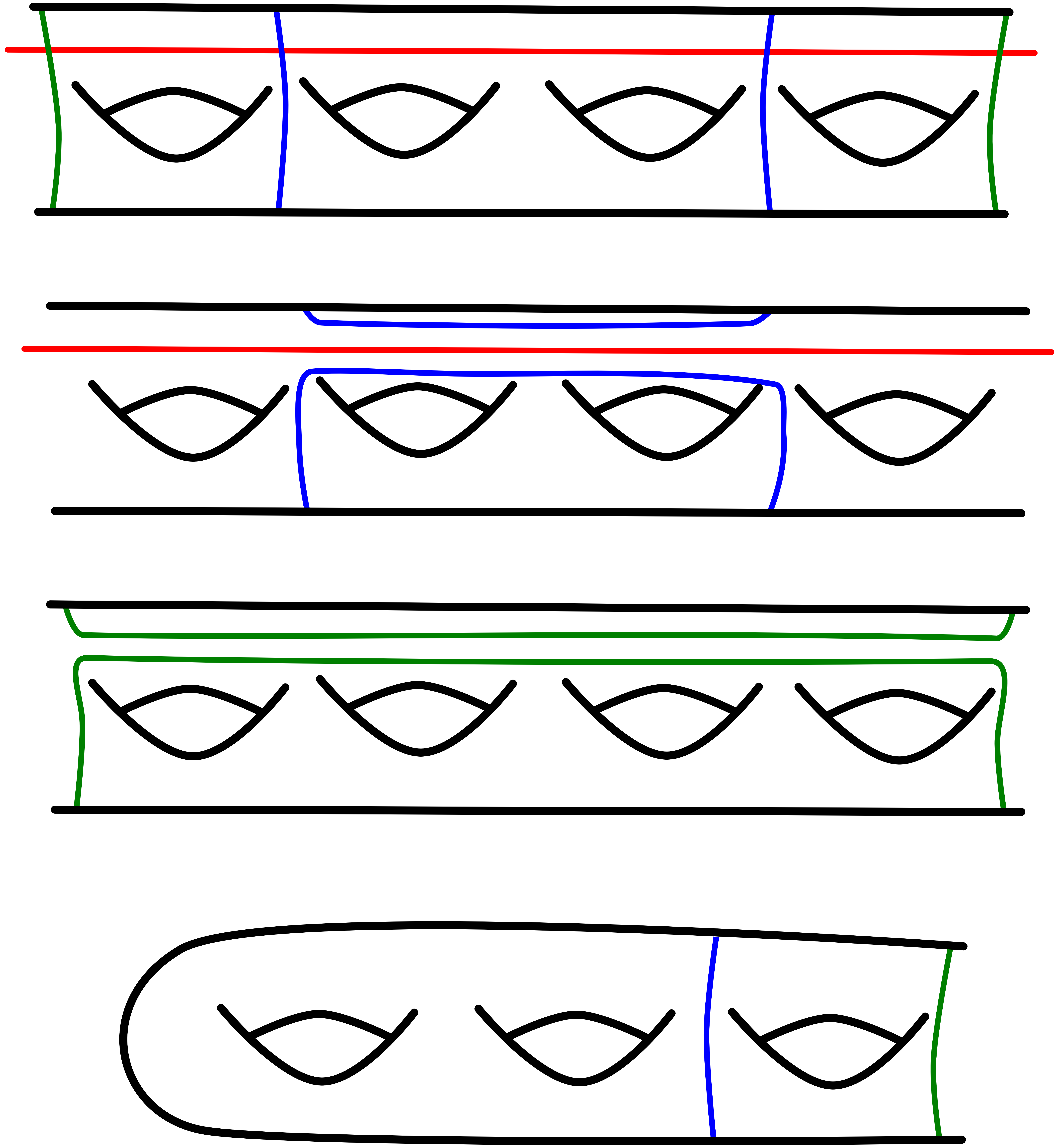
Restrict focus to cutting along arcs or curves.

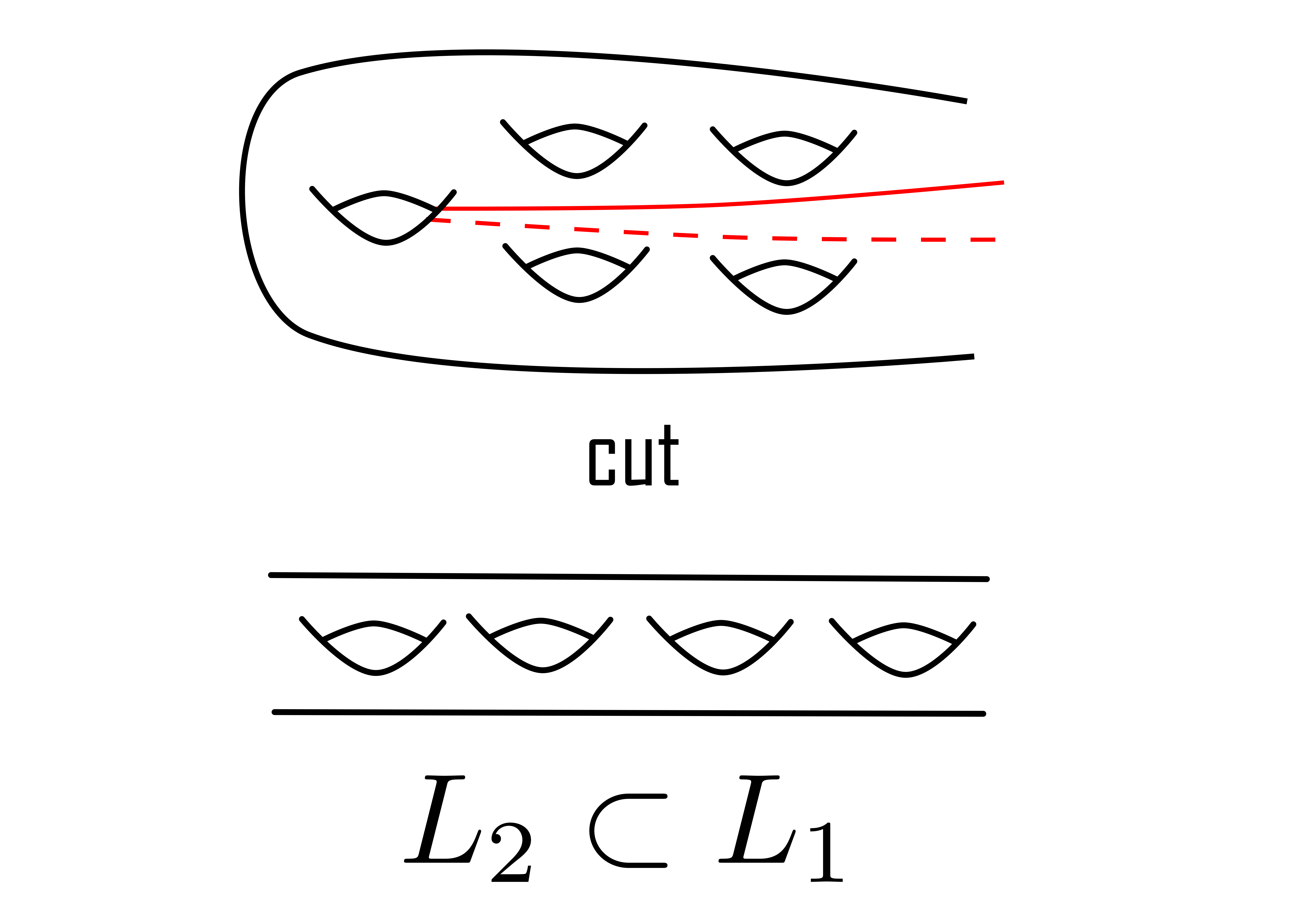

cut
$L_1^\circ \subset L_1$

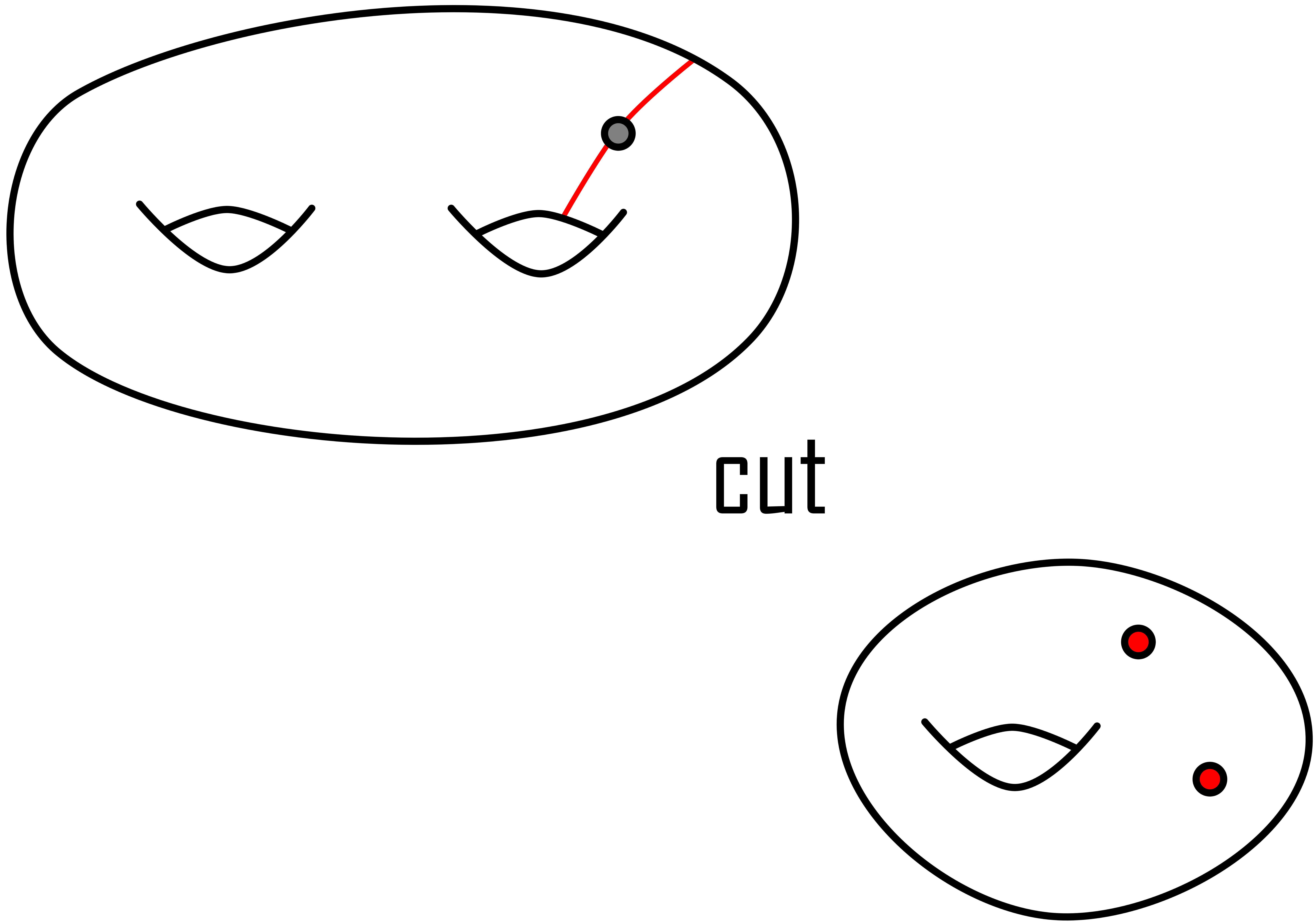


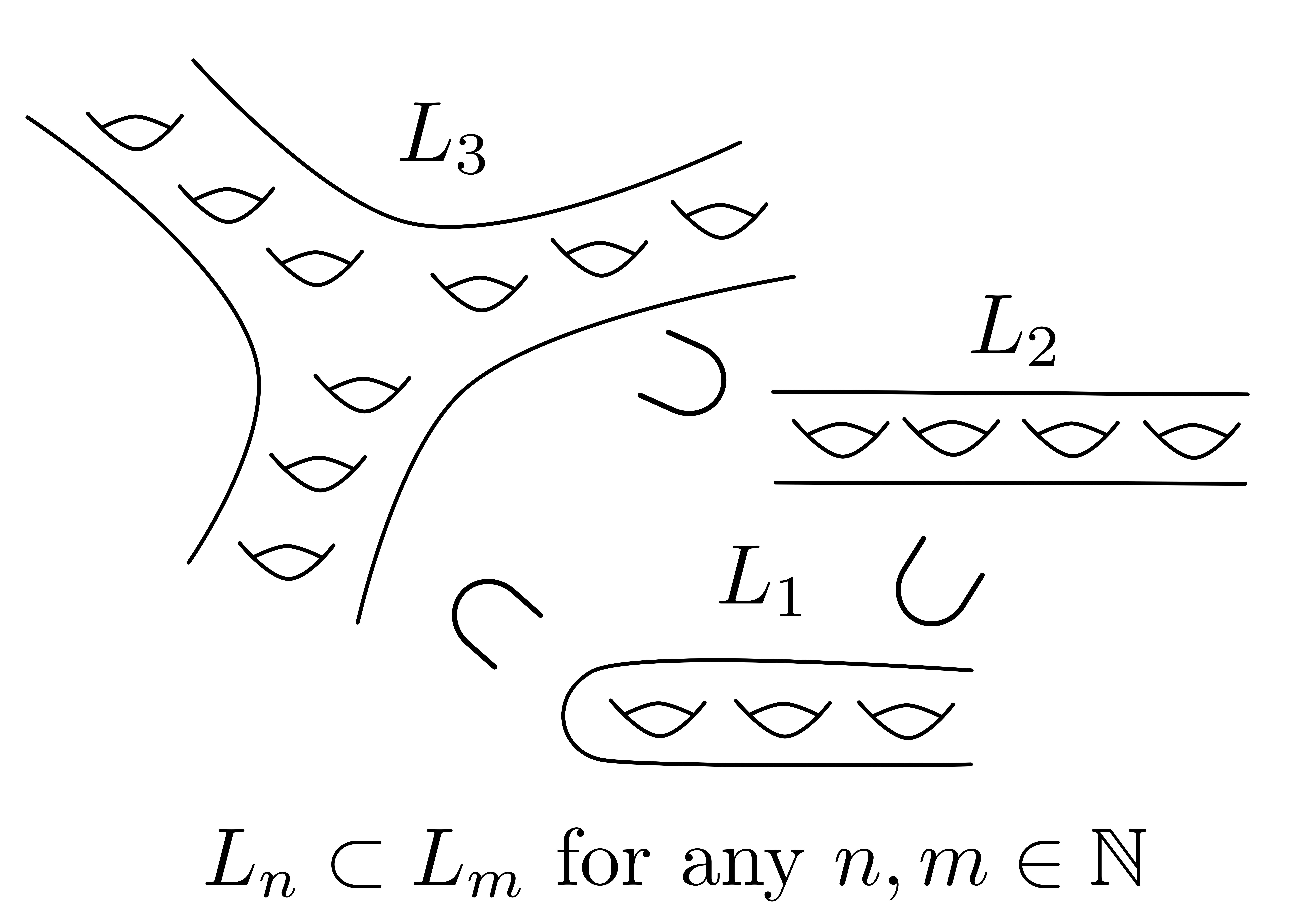
$$L_1 \subset L_2$$


cut




cut
$L_2 \subset L_1$

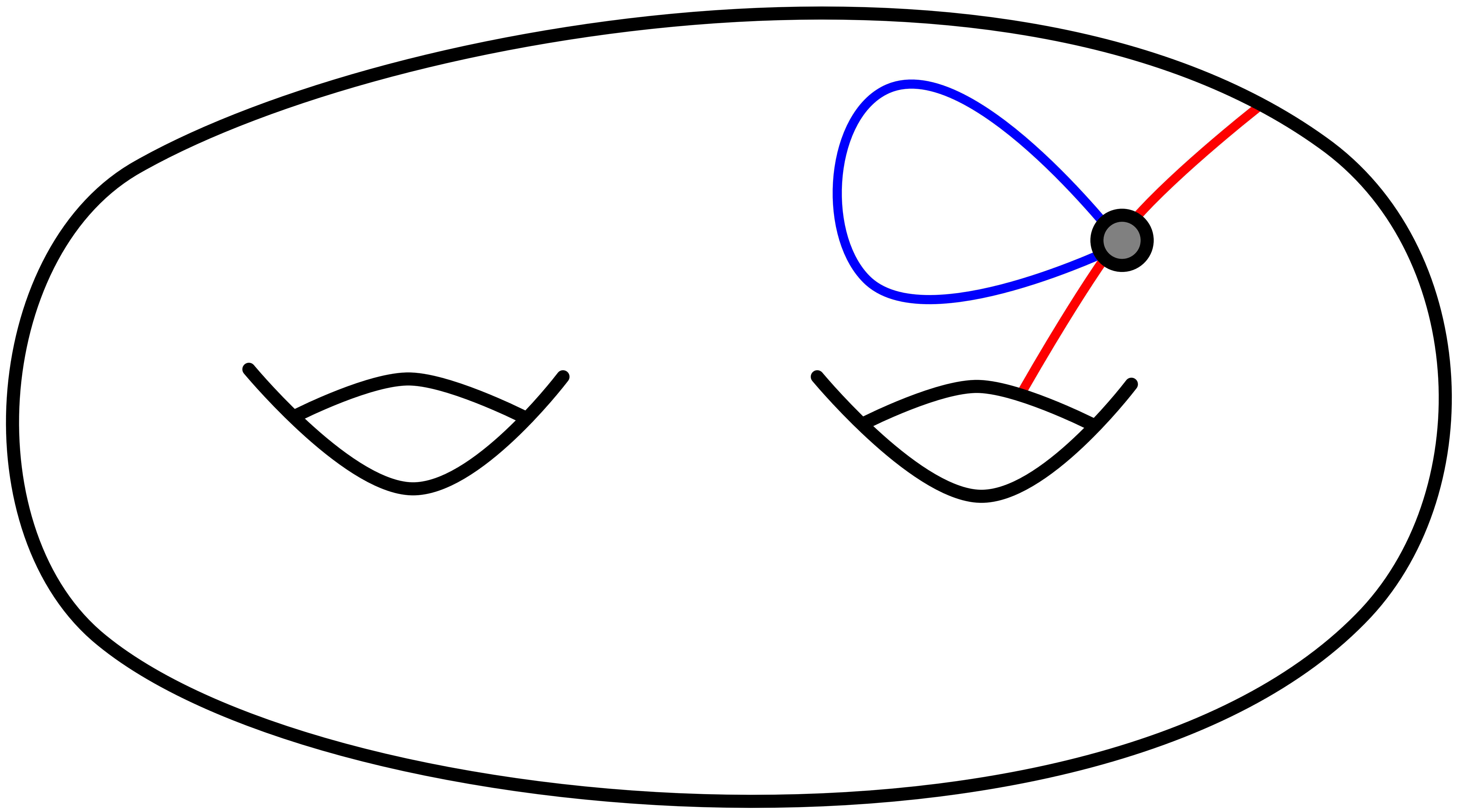

cut


L_3
L_2
L_1
$L_n \subset L_m$ for any $n, m \in \mathbb{N}$

If α is a separating arc,

call a component S of $\Sigma \setminus \alpha$ a *one-cut* subsurface.

If Σ is finite-type then $S \ncong \Sigma \Leftrightarrow \alpha$ is essential.



If $S \cong \Sigma$, call it a homeomorphic one-cut subsurface.

Are there infinite-type surfaces

containing (essential) homeomorphic one-cut subsurfaces?

Theorem (Fanoni-Ghaswala-M)

A surface is infinite-type $\Leftrightarrow$ it contains

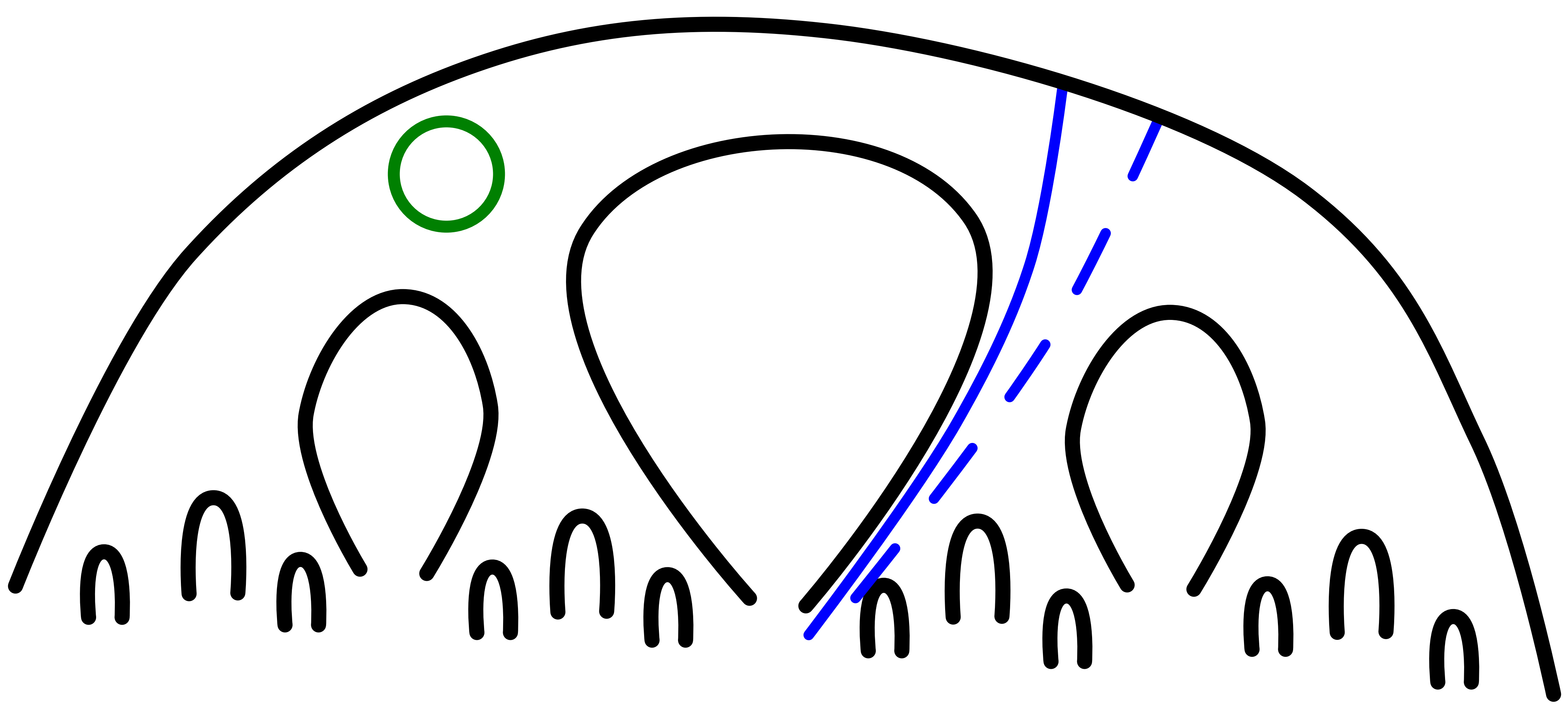
essential one-cut homeomorphic subsurfaces.

Proof by pictures

Case 1- There is an isolated nonplanar end.

Case 2- Infinitely many isolated planar ends.

Case 3- Finite genus, all planar ends nonisolated.



Case 4- Infinite genus, all nonplanar ends nonisolated.

Finite-type: α is essential $\Leftrightarrow$ it intersects every homeomorphic one-cut subsurface.

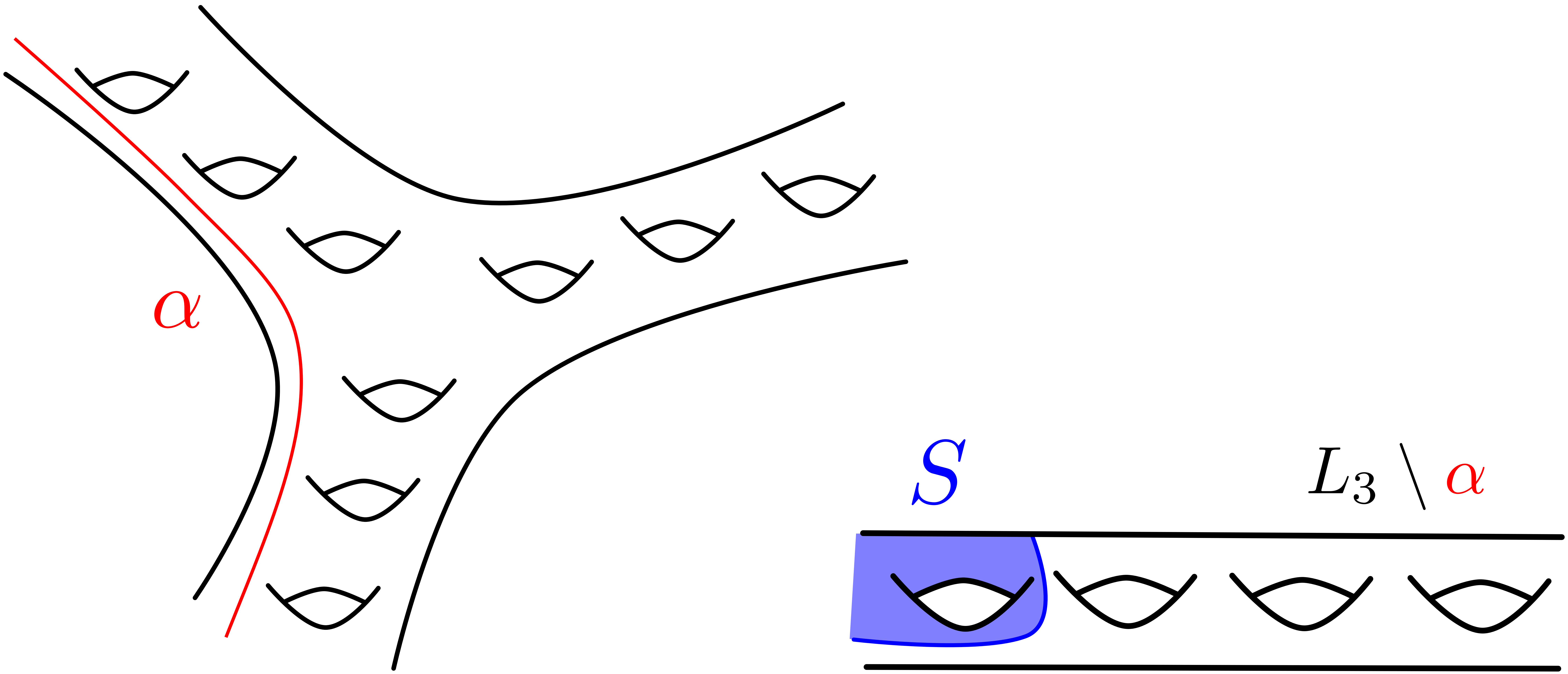
Infinite-type: Not true!

α is *omnipresent* if it is 2-ended and it intersects every homeomorphic one-cut subsurface.

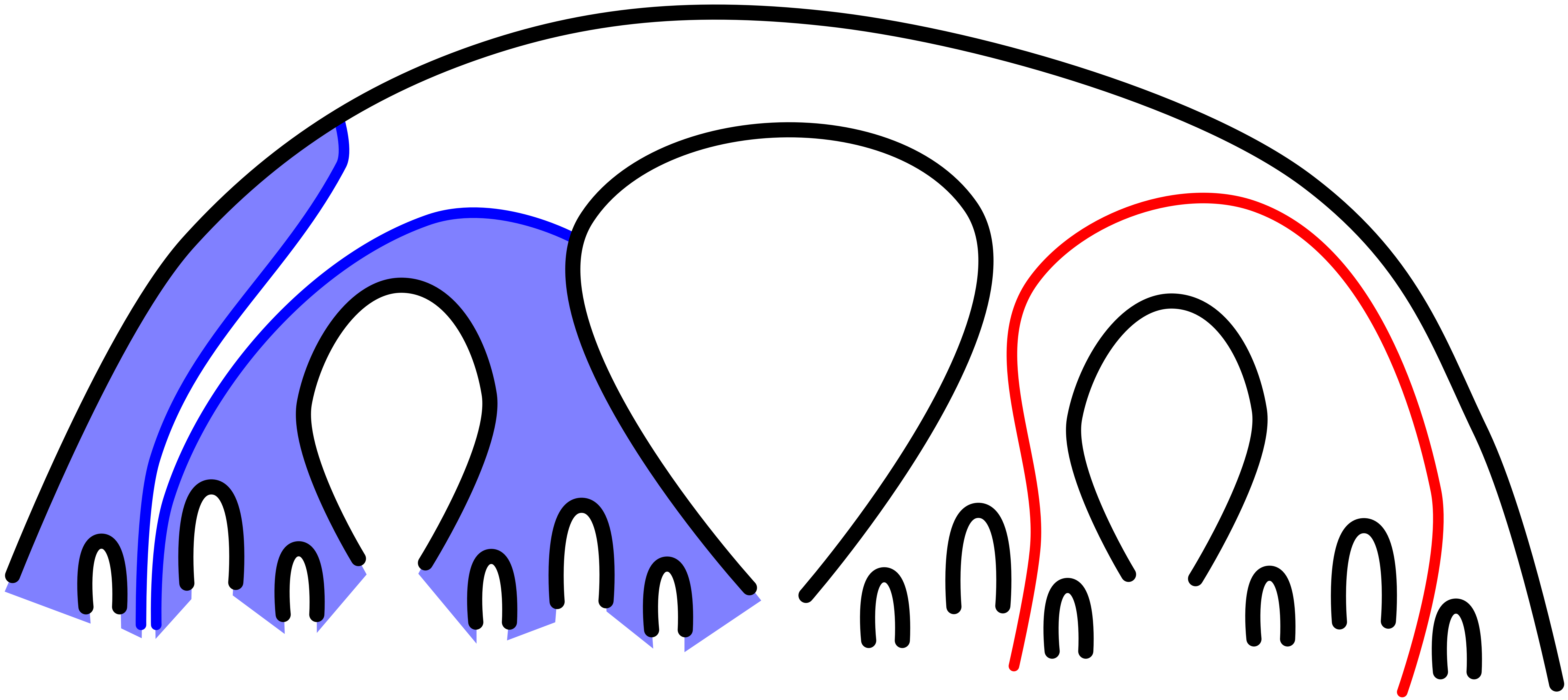
Every 2-ended arc in L_3 is omnipresent.

α not omnipresent $\Rightarrow$

There exists a one-cut subsurface S of $L_3 \setminus \alpha$ such that $S \cong L_3$.



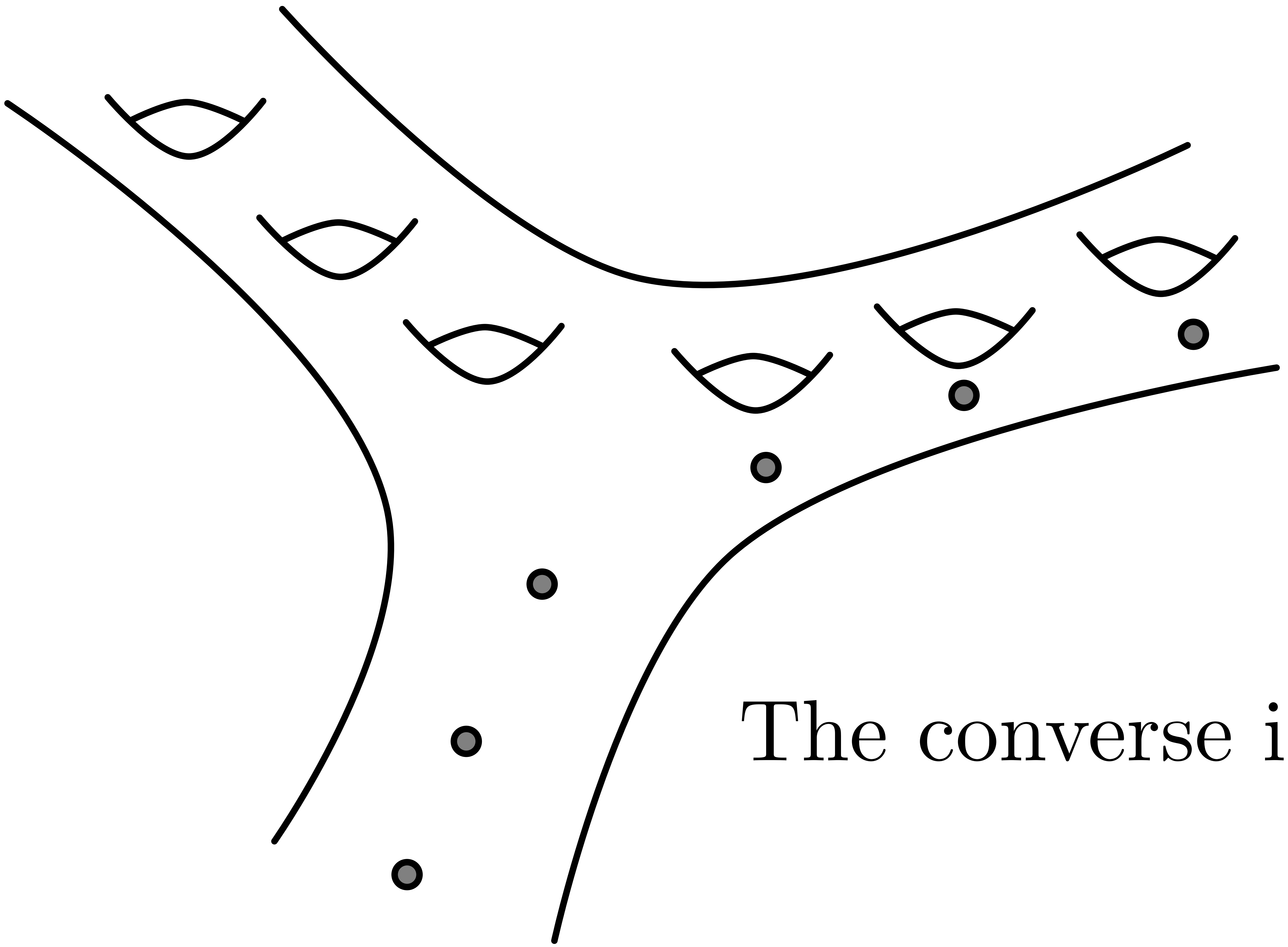
The Cantor Tree has no omnipresent arcs.



$$\mathrm{Map}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{isotopy}$$

A *finite orbit* end is an end with finite $\mathrm{Map}(\Sigma)$-orbit.

If an arc connects two finite-orbit ends, then it is omnipresent.



The converse is not always true!

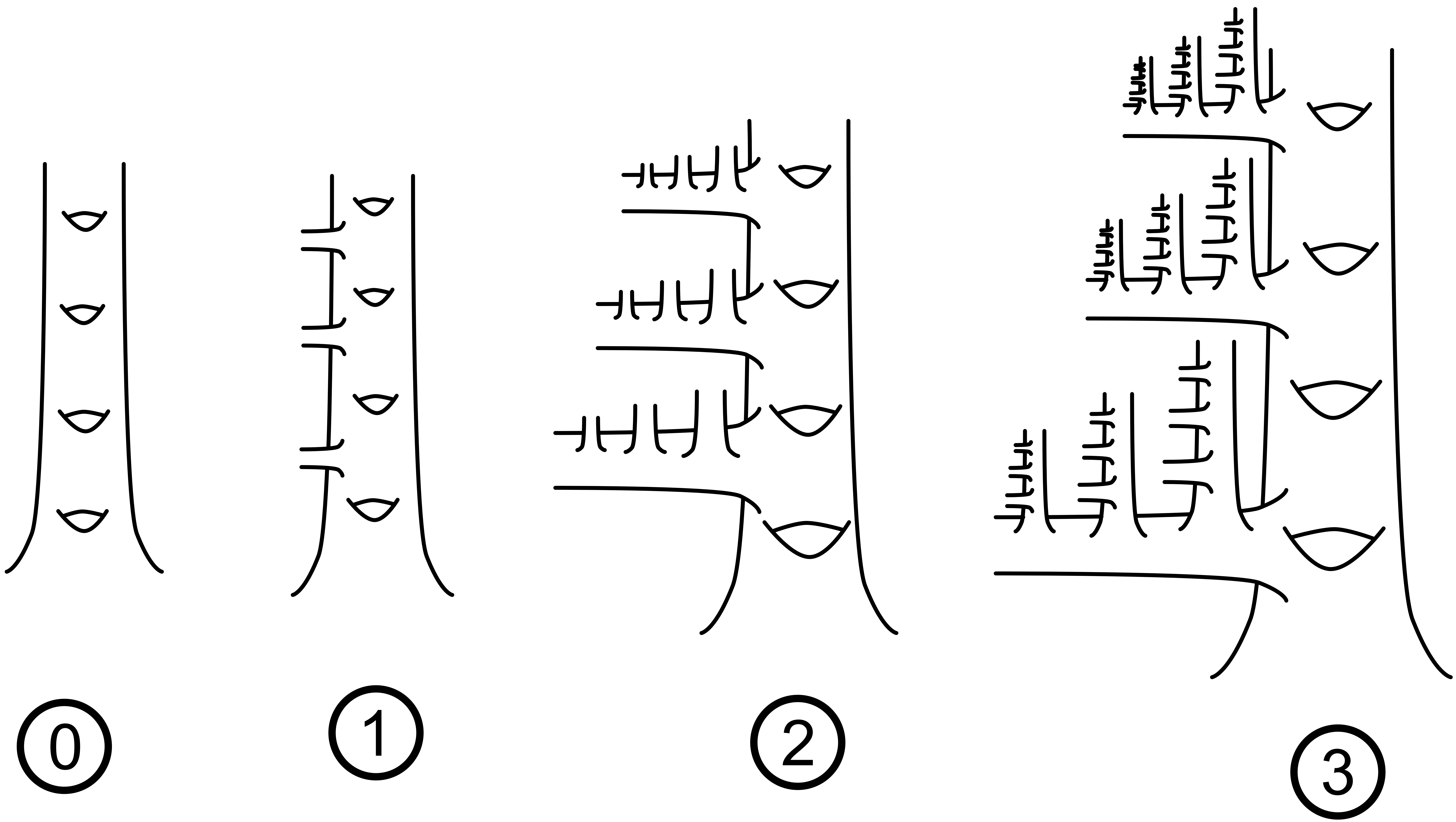
An end e is *stable* if $e = [U_1 \supset U_2 \supset \ldots]$,

where $U_i \cong U_{i+1}$ for all $i \in \mathbb{N}$.

A surface is *stable* if every end is stable.

Theorem(Fanoni-Ghaswala-M)

For stable surfaces, an arc is omnipresent if and only if it connects two finite orbit ends.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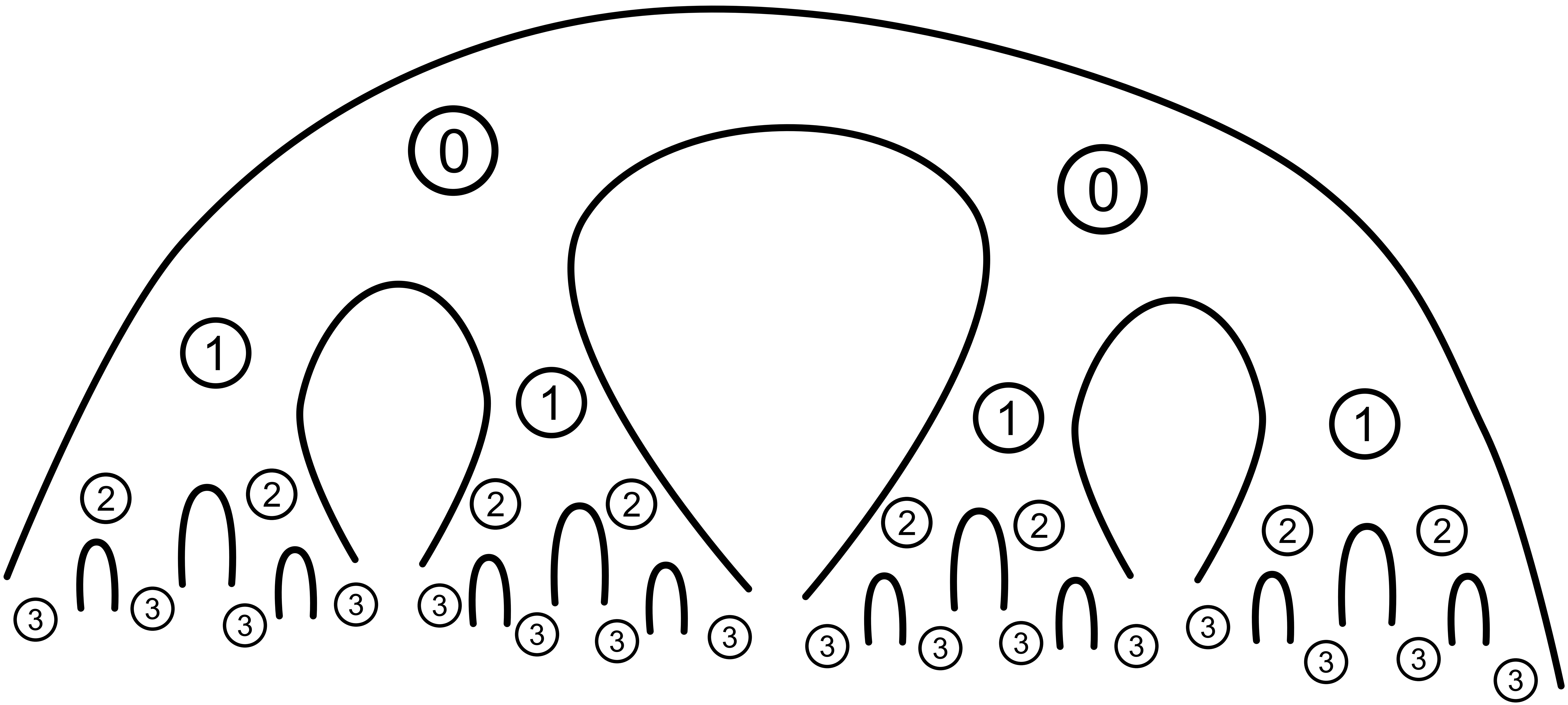
2
3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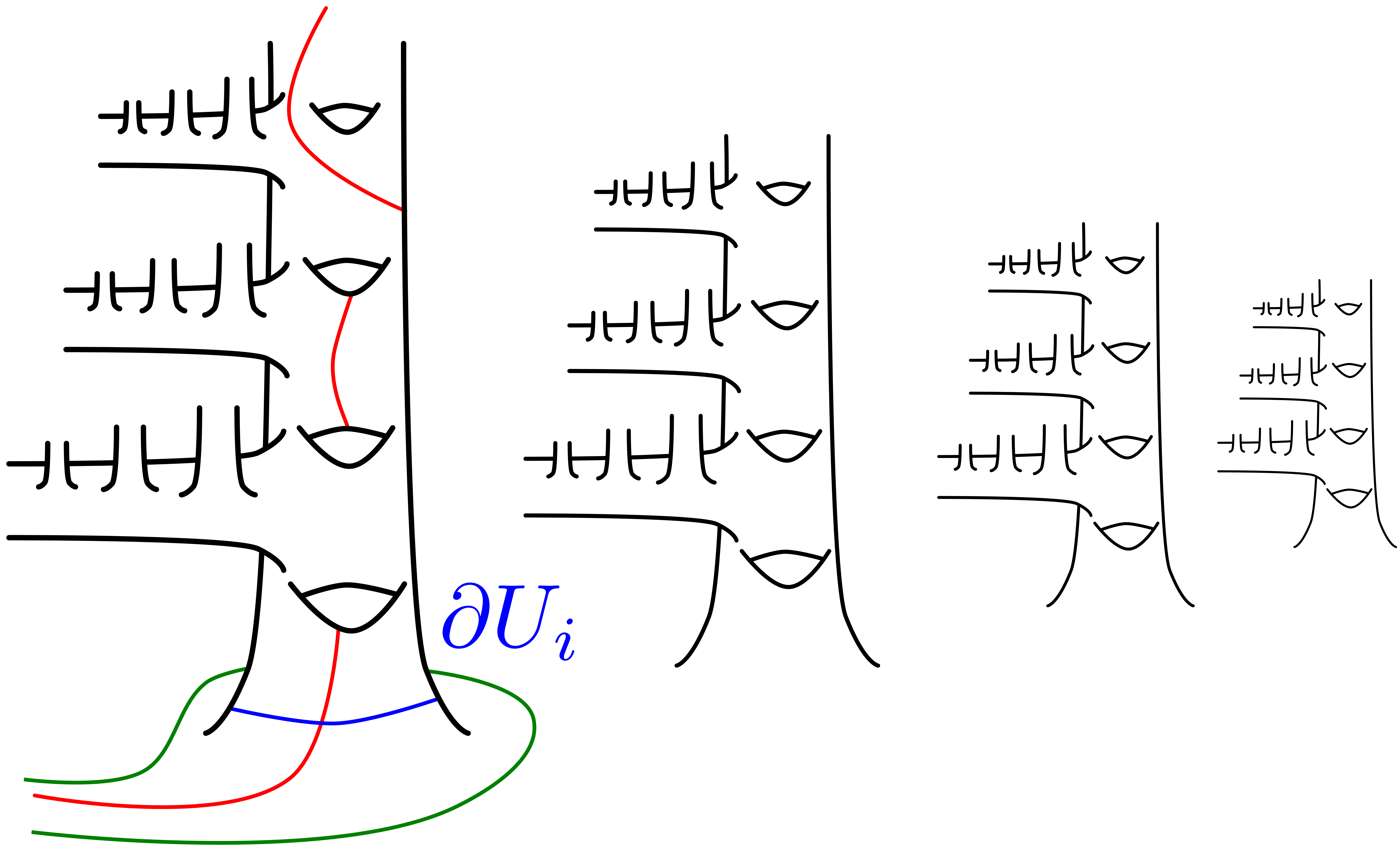
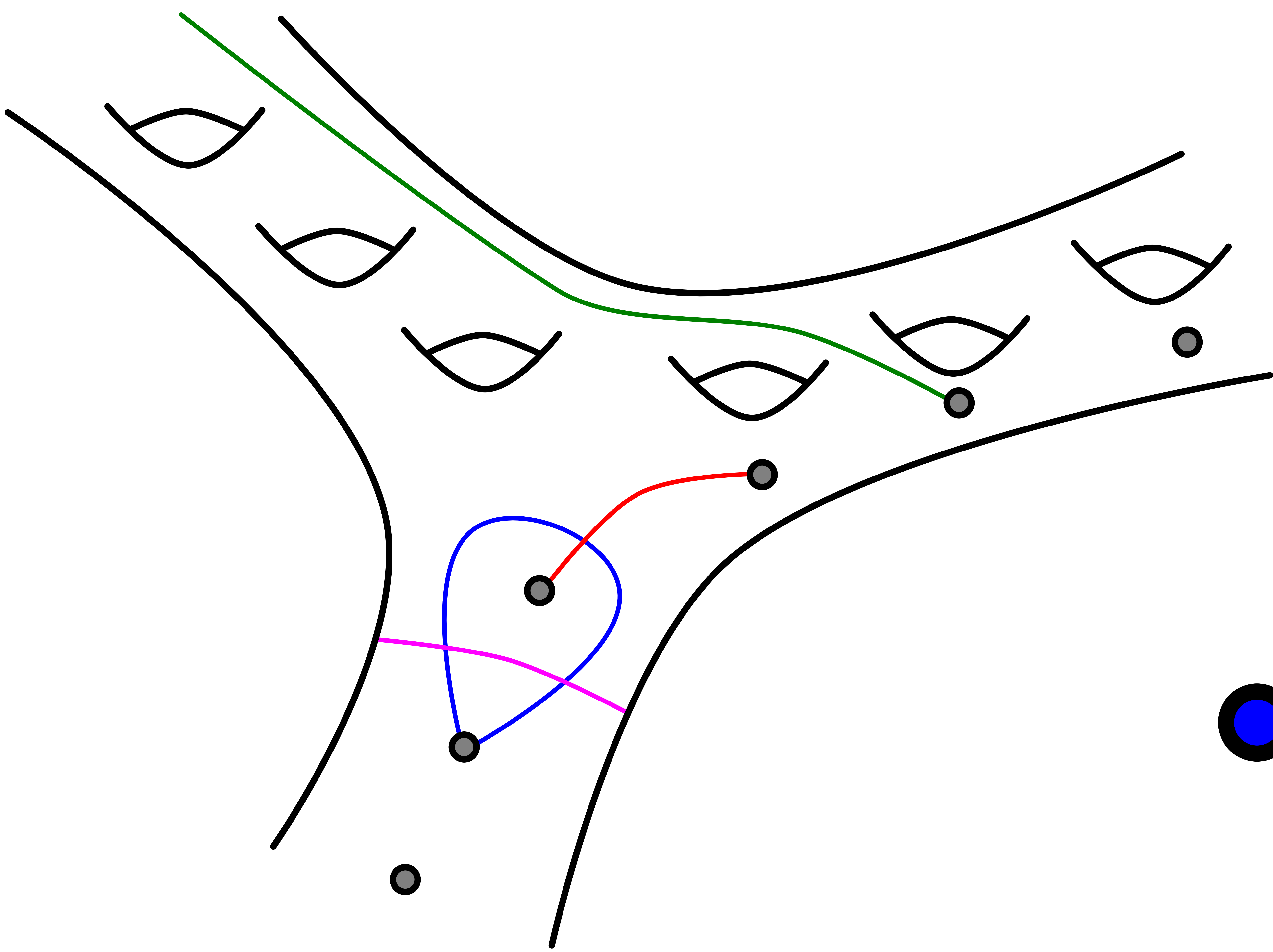

∂U_i

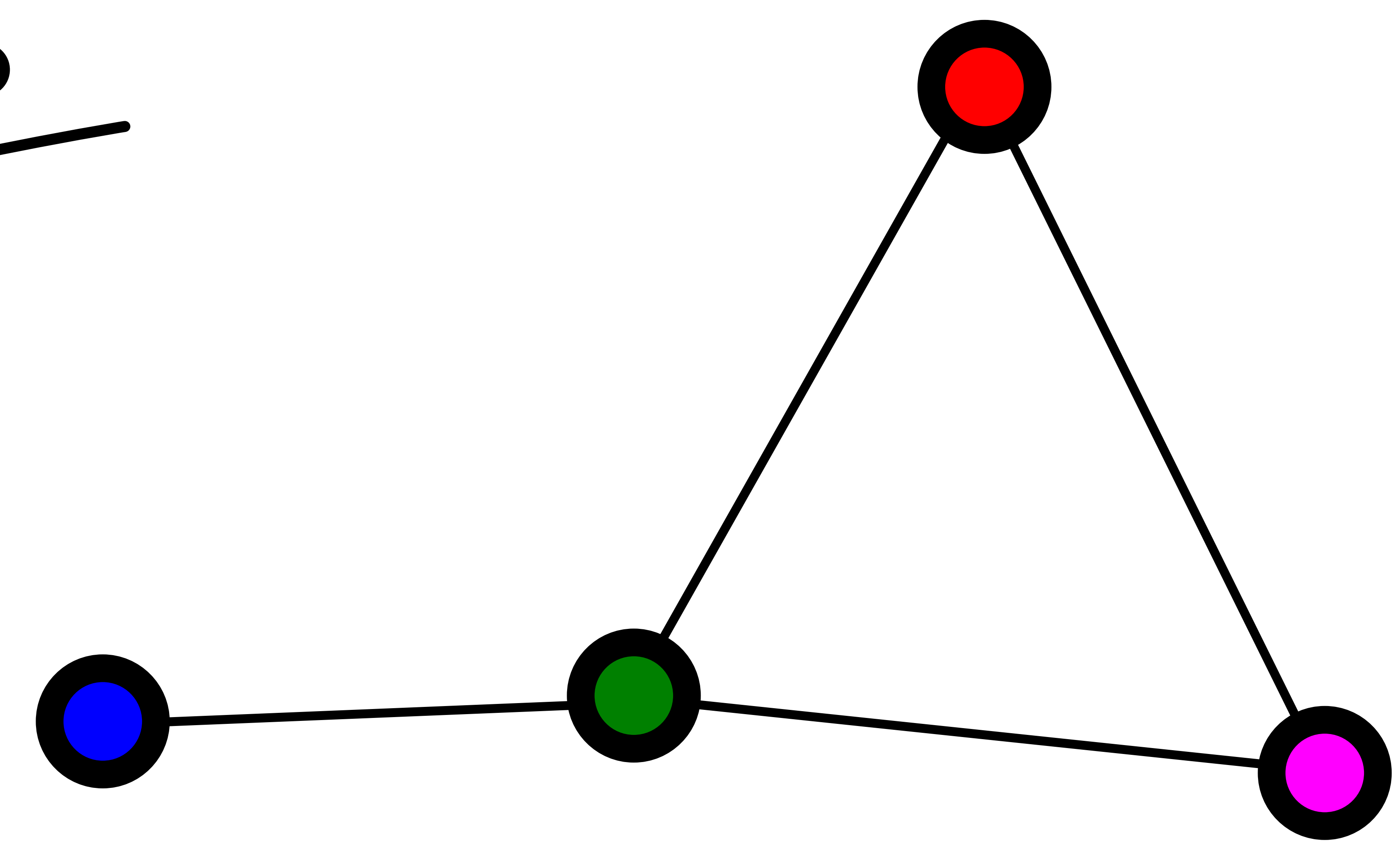
Graphs for infinite-type surfaces

The arc and curve graph $\mathcal{AC}(\Sigma)$

Vertices : isotopy classes of arcs and curves

Edges : pairs of disjoint arcs and curves





Bavard, Aramayona-Fossas-Parlier

Let P be the set of isolated planar ends.

$\mathcal{A}(\Sigma, P)$ is the full subgraph whose vertices are arcs with endpoints in P.

If $|P| < \infty$ then $\mathcal{A}(\Sigma, P)$ is connected, infinite diameter, and δ-hyperbolic.

Rasmussen

Let g be the genus of Σ.

$\mathcal{N}(\Sigma)$ is the full subgraph of nonseparating curves.

If $0 < g < \infty$ then $\mathcal{N}(\Sigma)$ is connected, infinite diameter, and δ-hyperbolic.

Durham-Fanoni-Vlamis

Let Σ have at least four finite orbit ends,

There exists a graph of curves; connected, infinite diameter, and δ-hyperbolic.

Theorem (Fanoni-Ghaswala-M)

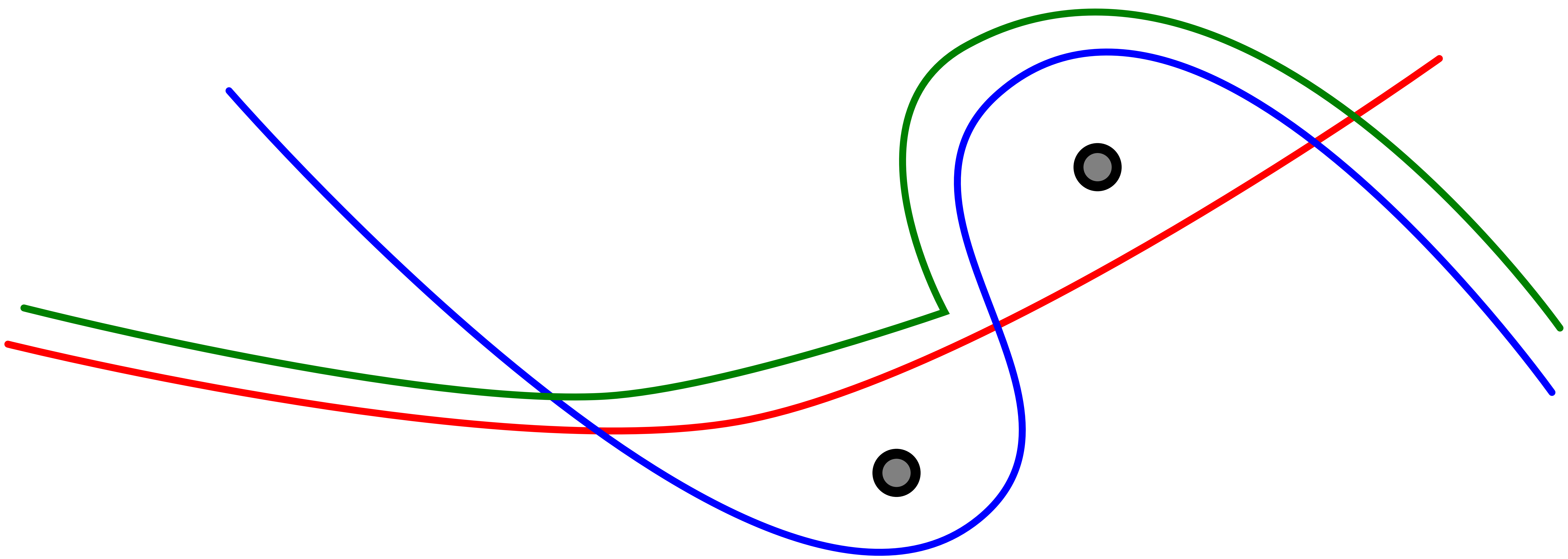
If Σ is stable with at least three finite orbit ends, then the omnipresent arc graph $\Omega(\Sigma)$ is connected, δ-hyperbolic, and infinite diameter.

Infinitely intersecting unicorns

A *unicorn* of α, β is an arc given by $a \cup b$,

where $a \subset \alpha$, $b \subset \beta$,

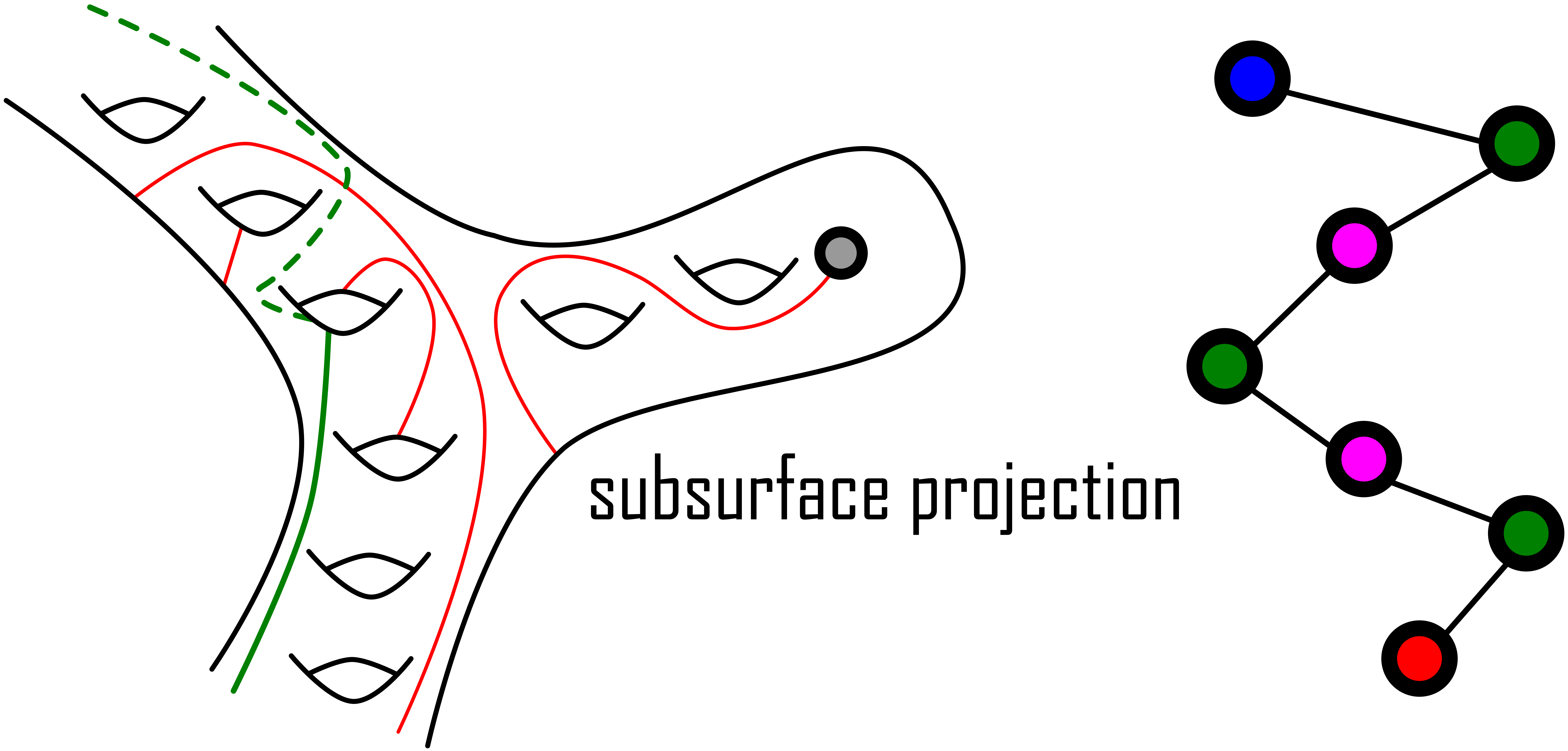
and $p = a \cap b$ is the unique corner.



If α, β are 2-ended, a 2-ended unicorn always exists.

Let $A_0(\alpha, \beta)$ be the set of 2-ended unicorns.

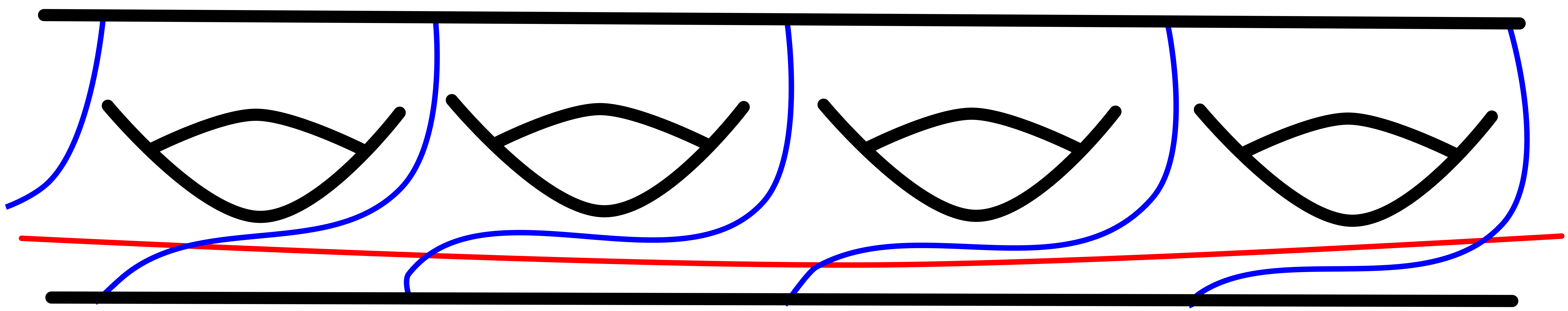
If $|\alpha \cap \beta| = \infty$ then $A_0(\alpha, \beta)$ is *almost* connected.



Let $A_1(\alpha, \beta)$ be the 1-nbhd of $A_0(\alpha, \beta)$.

Then $A_1(\alpha, \beta)$ is connected.

Is $\Omega(L_2)$ connected?



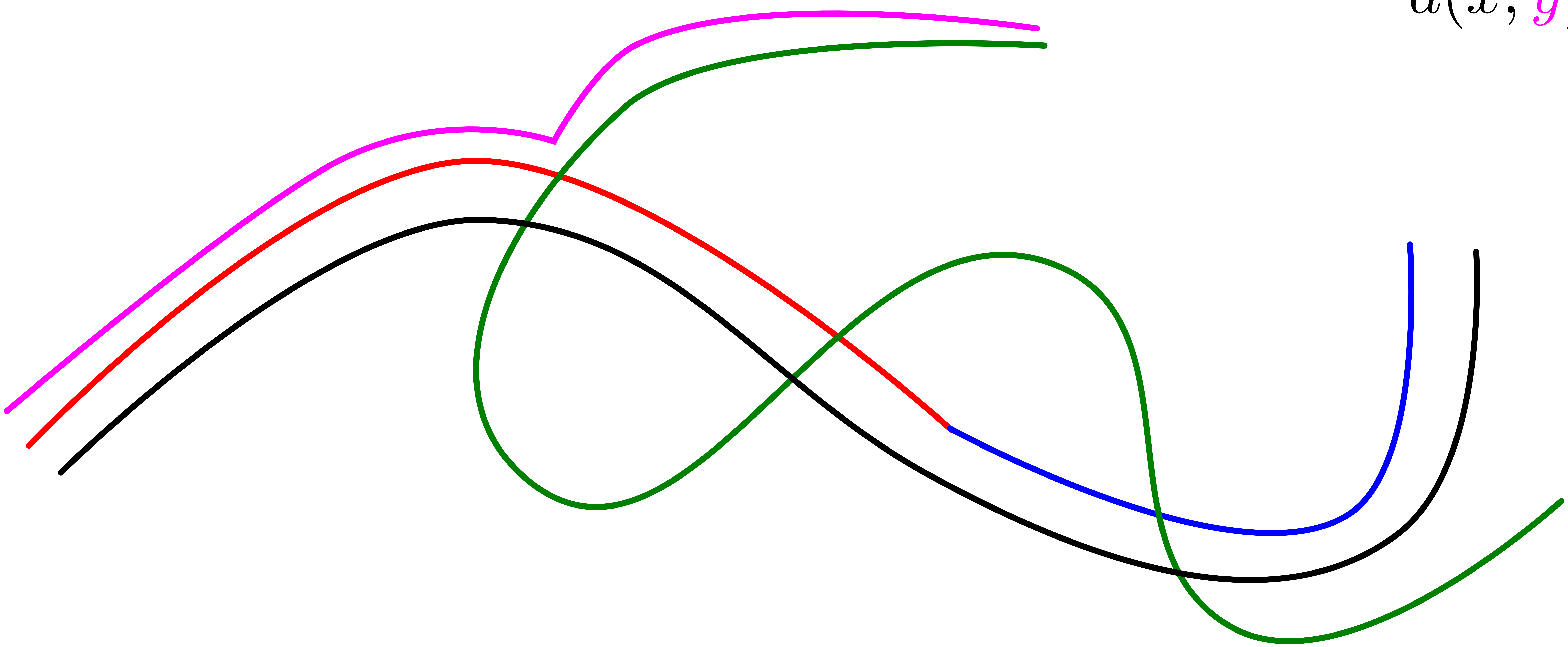
Guessing geodesics lemma

For all $x \in [\alpha, \beta]$ there exists $y \in [\alpha, \gamma] \cup [\gamma, \beta]$ such that

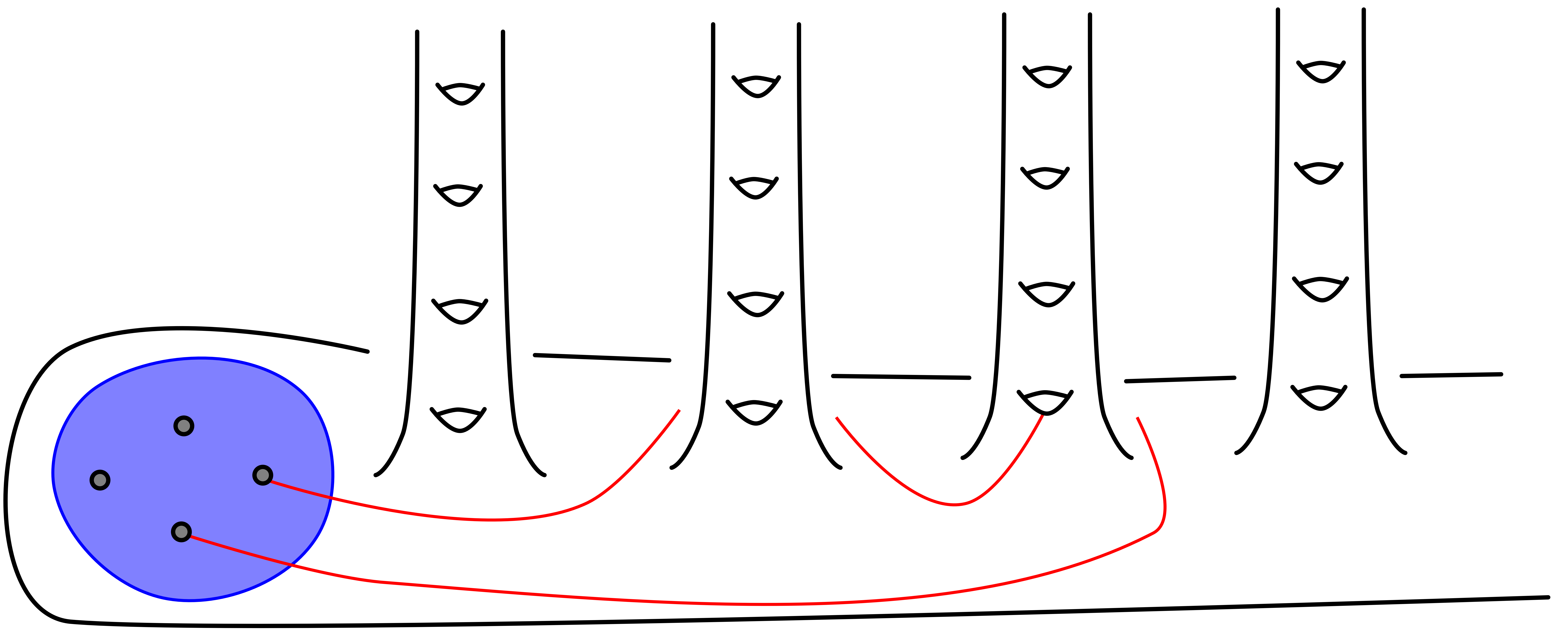
$$d(x, y) < \delta.$$

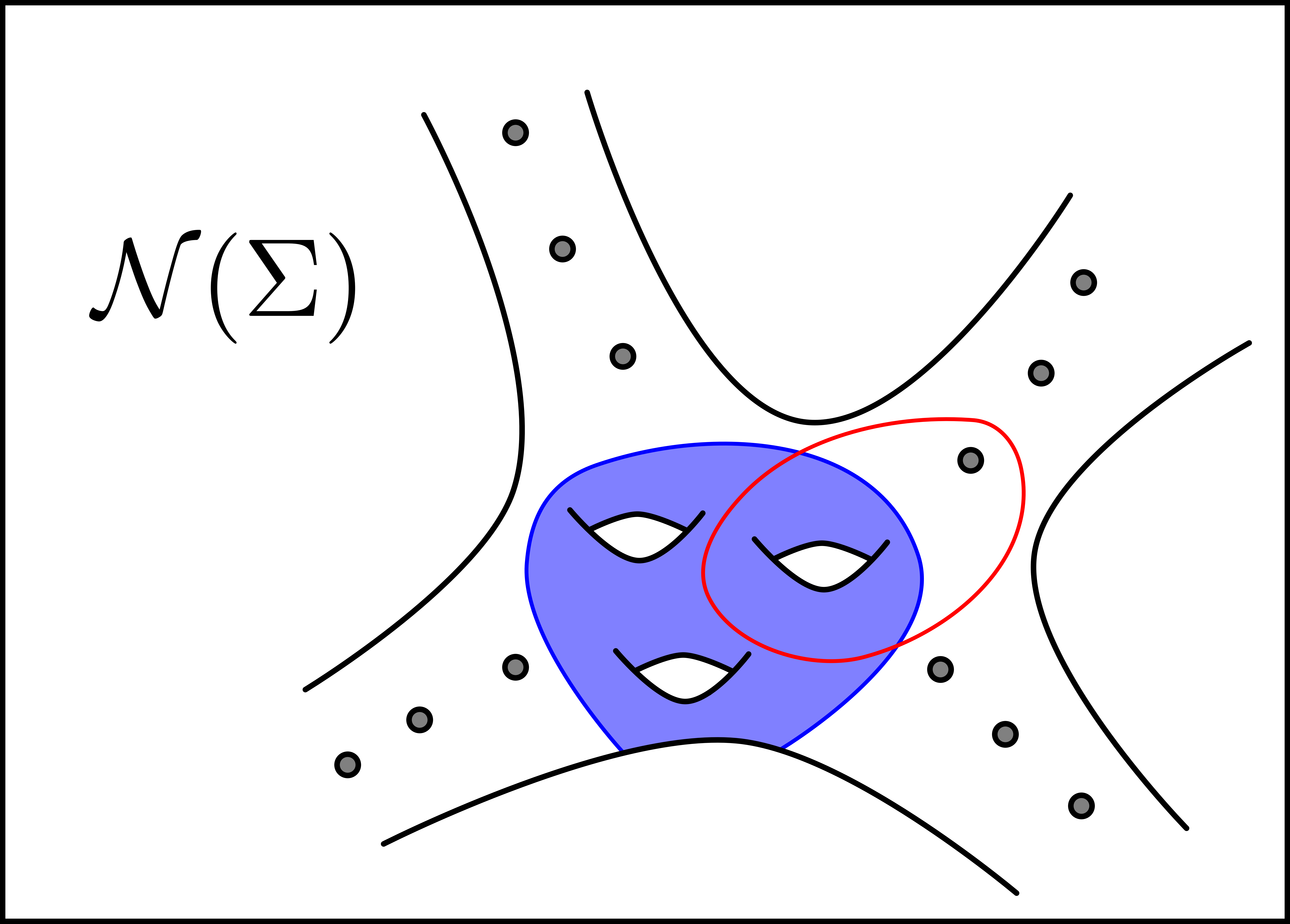
For all $x \in A_1(\alpha, \beta)$ there exists $y \in A_1(\alpha, \gamma) \cup A_1(\gamma, \beta)$ such that

$$d(x, y) < M.$$



$\mathcal{A}(\Sigma, P)$




$\mathcal{N}(\Sigma)$

$\Omega(\Sigma)$