QUASI-POISSON MANIFOLDS

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ABSTRACT. A quasi-Poisson manifold is a G-manifold equipped with an invariant bivector field whose Schouten bracket is the trivector field generated by the invariant element in $\wedge^3\mathfrak{g}$ associated to an invariant inner product. We introduce the concept of the fusion of such manifolds, and we relate the quasi-Poisson manifolds to the previously introduced quasi-Hamiltonian manifolds with group-valued moment maps.

1. Introduction

This paper is a sequel to [1], in which the notion of a *quasi-Poisson manifold* was introduced. While the purpose of [1] was to obtain a unified picture of various notions of "generalized moment maps" and their properties, the current article is devoted to a closer examination of a particular type of quasi-Poisson manifolds, defined as follows.

Let G be a Lie group, whose Lie algebra $\mathfrak{g}$ is equipped with an invariant inner product. From the Lie bracket and the invariant inner product, one obtains an invariant element $\phi \in \wedge^3\mathfrak{g}$, the Cartan 3-tensor of the Lie algebra with an invariant inner product. The quasi-Poisson manifolds studied in the present paper are G-manifolds M together with an invariant bivector field P, such that the Schouten bracket $[P, P]$ equals the trivector field ϕ_M generated by ϕ. Of particular interest are the quasi-Poisson manifolds admitting a *moment map*, $\Phi : M \to G$ (see Definition 2.2 below). The triple (M, P, Φ) will then be called a *Hamiltonian quasi-Poisson manifold*.

The basic example of a Hamiltonian quasi-Poisson G-manifold is $M = G$, where G acts by conjugation, and the moment map is the identity map. As we will explain in this paper, there are many parallels between this example and the usual linear Poisson structure on the dual of a Lie algebra. In particular, in analogy with the coadjoint orbits, all the conjugacy classes in G are quasi-Poisson submanifolds with a "non-degenerate" bivector field. Other examples are obtained using the methods of "fusion" and "exponentiation" introduced in this paper. As an application, we will show how to construct the usual Poisson structure on the representation variety, $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, for any oriented surface with boundary, Σ, by reduction of a quasi-Poisson structure on a fusion product of several copies of G.

One of the main results of this paper states that every Hamiltonian quasi-Poisson manifold has a generalized foliation, the leaves of which are non-degenerate Hamiltonian quasi-Poisson manifolds. Furthermore, every non-degenerate Hamiltonian quasi-Poisson manifold (M, P, Φ) carries an invariant 2-form ω such that (M, ω, Φ) satisfies the axioms of a *quasi-Hamiltonian G-manifold* with group-valued moment map, as introduced in [2].

Conversely, every such manifold carries a non-degenerate quasi-Poisson structure with moment map Φ.

In a recent paper, T. Treloar [8] has applied the concept of a Hamiltonian quasi-Poisson manifold developed here to study the symplectic geometry of spaces of polygons on the 3-sphere.

The organization of the paper is as follows. In Section 2 we give the definition of quasi-Poisson manifolds and of moment maps for Hamiltonian quasi-Poisson manifolds. Section 3 describes the fundamental examples of $M = G$ with the action by conjugation, and of the conjugacy classes in G. Section 4 contains the definitions of the *quasi-Poisson cohomology* and of the *modular class* of a quasi-Poisson manifold. In Sections 5 and 6, we study the *fusion* and *reduction* of quasi-Poisson manifolds. In Section 7, we show how to construct Hamiltonian quasi-Poisson manifolds out of Hamiltonian Poisson manifolds using the process of *exponentiation*. In Section 8 we define a generalized dynamical r-matrix and we prove the *cross-section theorem*, which is our main technical tool. Using this result, we show, in Section 9, that every Hamiltonian quasi-Poisson manifold is foliated into non-degenerate leaves. In Section 10 we prove the equivalence of the notion of a non-degenerate Hamiltonian quasi-Poisson manifold with that of a quasi-Hamiltonian manifold in the sense of [2], and we apply this result to describe the Poisson structure on the representation variety, $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. In Appendix A we explain the relation between the quasi-Poisson bivector on the group G and the Poisson bivector on the dual of the Lie algebra of the central extension of the corresponding loop group LG. In Appendix B, we prove that the new r-matrix introduced in Section 8 is indeed a solution of a generalized classical dynamical Yang-Baxter equation.

Notations

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Although many of the results of this paper hold for non-compact Lie groups, we shall assume for simplicity that G is compact. For any G-manifold M and any $\xi \in \mathfrak{g}$, the generating vector field of the induced infinitesimal action is defined by $\xi_M(m) := \frac{d}{dt}|_{t=0} \exp(-t\xi).m$, for $m \in M$. The Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M; TM)$, $\xi \mapsto \xi_M$, extends to an equivariant map,

$$\wedge^\bullet \mathfrak{g} \to C^\infty(M; \wedge^\bullet TM), \tag{1}$$

preserving wedge products and Schouten brackets.[1] More generally, for any function $\alpha \in C^\infty(M, \wedge^\bullet \mathfrak{g})$, we denote by α_M the multi-vector field, $\alpha_M(m) = (\alpha(m))_M(m)$.

[1]We briefly recall the main properties of the Schouten bracket:

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha],$$

$$[\alpha, [\beta, \gamma]] = [[\alpha, \beta], \gamma] + (-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, [\alpha, \gamma]],$$

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \wedge [\alpha, \gamma].$$

For any vector field X, the bracket $[X, \alpha]$ is just the Lie derivative.

If P is a bivector on M, then $P^\sharp$ denotes the induced map from differential forms to vector fields, with the convention $P^\sharp(a)(b) = P(a,b)$, for 1-forms a and b.

We shall denote the left- and right-invariant multivector fields on G generated by an element $\beta \in \wedge^\bullet \mathfrak{g}$ by β^L and $\beta^R \in C^\infty(G; \wedge^\bullet TG)$ respectively. The vector fields ξ^L for $\xi \in \mathfrak{g}$ are the generating vector fields for the right action, $(g,m) \in G \times G \mapsto g.m = m\,g^{-1} \in G$ and $-\xi^R$ are the generating vector fields for the left action, $(g,m) \mapsto g.m = gm$.

Our definitions involve the choice of an invariant inner product (positive definite, non-degenerate symmetric bilinear form) on $\mathfrak{g}$, which we also use to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. We denote the inner product by a dot. Using the inner product, one can define the canonical invariant skew-symmetric 3-tensor on $\mathfrak{g}$, sometimes called the Cartan 3-tensor. In terms of an orthonormal basis (e_a) of $\mathfrak{g}$, this canonical element, $\phi \in \wedge^3\mathfrak{g}$, is given by

$$\phi = \frac{1}{12} f_{abc} e_a \wedge e_b \wedge e_c \ ,$$

where $f_{abc} = e_a \cdot [e_b, e_c]$ are the structure constants of $\mathfrak{g}$. Here and below, we take the sum over repeated indices. (Normalizations for the element ϕ vary in the literature.)

We denote by θ^L the left-invariant Maurer-Cartan form on G, and by θ^R the right-invariant Maurer-Cartan form. Let θ^L_a and θ^R_a be the components of the Maurer-Cartan forms in the basis (e_a). Then $\iota(e^L_a)\theta^L_b = \delta_{ab}$ and $\iota(e^R_a)\theta^R_b = \delta_{ab}$. If $A : \mathfrak{g} \to \mathfrak{g}$ is a linear map, we define its components by $Ae_b = A_{ab}e_a$. This gives $(\mathrm{ad}_\xi)_{ab} = -f_{abc}\xi_c$, for $\xi \in \mathfrak{g}$. Also, at a point $g \in G$, $e^R_a = (\mathrm{Ad}_g)_{ab} e^L_b$, and $\theta^R_a = (\mathrm{Ad}_g)_{ab}\theta^L_b$.

2. Quasi-Poisson manifolds

Recall that a Hamiltonian Poisson G-manifold is a triple, (M, P_0, Φ_0), consisting of a G-manifold M, an invariant Poisson structure P_0, and an equivariant *moment map* $\Phi_0 : M \to \mathfrak{g}^*$ satisfying the condition,

$$P_0^\sharp(\mathrm{d}\langle \Phi_0, \xi\rangle) = \xi_M \ , \tag{2}$$

for all $\xi \in \mathfrak{g}$. The simplest example of a Hamiltonian Poisson G-manifold is $M = \mathfrak{g}^*$ with its linear Poisson structure and the coadjoint action; the identity map is then a moment map.

In this paper we will study a notion of Hamiltonian *quasi*-Poisson G-manifold, for which the moment map takes values in the group G itself. The terminology “quasi” is motivated by the relation to quasi-Poisson Lie groups (see [1]), which are the classical limits of quasi-Hopf algebras. For any G-manifold M, the Cartan 3-tensor ϕ corresponding to an invariant inner product on $\mathfrak{g}$ gives rise to an invariant trivector field ϕ_M on M.

Definition 2.1. A *quasi-Poisson manifold* is a G-manifold M, equipped with an invariant bivector field $P \in C^\infty(M; \wedge^2 TM)$ such that

$$[P, P] = \phi_M. \tag{3}$$

We note that if the group G is Abelian, P is a Poisson structure in the usual sense. We denote by $\{f,g\} = P(\mathrm{d}f, \mathrm{d}g)$ the bracket defined by P. It does not in general satisfy the Jacobi identity; instead

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 2\phi_M(\mathrm{d}f_1, \mathrm{d}f_2, \mathrm{d}f_3).$$

An equivariant map $F: M \to N$ between quasi-Poisson G-manifolds (M,P) and (N,Q) will be called a *quasi-Poisson map* if $\{F^*f_1, F^*f_2\} = F^*\{f_1, f_2\}$ for all $f_1, f_2 \in C^\infty(N, \mathbb{R})$. Equivalently, $F_*(P_m) = Q_{F(m)}$ for all $m \in M$.

To motivate the moment map condition for quasi-Poisson manifolds, we observe that the moment map condition for Poisson manifolds (2) is equivalent to,

$$P_0^\sharp(\mathrm{d}(\Phi_0^* f)) = (\Phi_0^*(\mathcal{D}_0 f))_M \ , \tag{4}$$

for all functions $f \in C^\infty(\mathfrak{g}^*, \mathbb{R})$. Here $\mathcal{D}_0 f$ is the exterior differential of f, viewed as a $\mathfrak{g}$-valued function on $\mathfrak{g}^*$. Equivalently, $\mathcal{D}_0$ is the $\mathfrak{g}$-valued differential operator on $\mathfrak{g}^*$ such that $(\mathcal{D}_0 f)_a = \frac{\partial f}{\partial \xi_a}$ in a basis (e_a) of $\mathfrak{g}$, defining coordinates ξ_a on $\mathfrak{g}^*$. Equation (4) becomes

$$P_0^\sharp(\mathrm{d}(\Phi_0^* f)) = \Phi_0^*(\frac{\partial f}{\partial \xi_a})(e_a)_M. \tag{5}$$

In the non-linear case, we assume that the basis (e_a) is orthonormal and we replace $\mathcal{D}_0$ by the $\mathfrak{g}$-valued differential operator $\mathcal{D}$ on G, with components

$$(\mathcal{D}f)_a = \frac{1}{2}(e_a^L + e_a^R) f \ .$$

We note that $\mathcal{D}$, unlike $\mathcal{D}_0$, depends upon the choice of the inner product on $\mathfrak{g}$.

Definition 2.2. An Ad-equivariant map $\Phi: M \to G$ is called a *moment map* for the quasi-Poisson manifold (M,P) if

$$P^\sharp(\mathrm{d}(\Phi^* f)) = (\Phi^*(\mathcal{D}f))_M \ , \tag{6}$$

for all functions $f \in C^\infty(G, \mathbb{R})$. The triple (M, P, Φ) is then called a *Hamiltonian quasi-Poisson manifold.*

In the basis (e_a), the moment map condition (6) reads

$$P^\sharp(\mathrm{d}(\Phi^* f)) = \frac{1}{2}\Phi^*((e_a^L + e_a^R) f)\,(e_a)_M \ . \tag{7}$$

The following Lemma gives an equivalent version of the moment map condition.

Lemma 2.3. *Let (M,P) be a quasi-Poisson G-manifold. An* Ad*-equivariant map $\Phi: M \to G$ is a moment map if and only if*

$$P^\sharp(\Phi^* \theta_a^R) = \frac{1}{2}(1 + \mathrm{Ad}_\Phi)_{ab}(e_b)_M. \tag{8}$$

Proof. First suppose that Φ satisfies (8). Using $\mathrm{d}f = (e_a^R f)\theta_a^R$, we find that

$$P^\sharp(\mathrm{d}(\Phi^* f)) = \frac{1}{2}\Phi^*(e_a^R f)(1 + \mathrm{Ad}_\Phi)_{ab}\,(e_b)_M \ .$$

Equation (7) follows since $e_a^R = (\mathrm{Ad}_g)_{ab} e_b^L$. The converse is proved similarly, since, for all $g \in G$, one can always find $f \in C^\infty(G, \mathbb{R})$ such that $\mathrm{d}f = \theta_a^R$ at g. □

The definition of a moment map for a quasi-Poisson manifold can also be cast in the more invariant form,

$$P^\sharp(\Phi^*(\theta^R \cdot \xi)) = \frac{1}{2}((1 + \mathrm{Ad}_{\Phi^{-1}})\xi)_M \ , \tag{9}$$

for all $\xi \in \mathfrak{g}$, from which it is clear that Definition 2.2 coincides with the definition given in [1].

3. Examples: the Lie group and its conjugacy classes

The basic example of a Hamiltonian quasi-Poisson manifold is the group G itself. Let $\psi \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ be the element

$$\psi = \frac{1}{2} e_a^1 \wedge e_a^2, \tag{10}$$

where the superscripts refer to the respective $\mathfrak{g}$-summand. A straightforward calculation shows that the Schouten bracket of ψ is the following element in $\wedge^3(\mathfrak{g} \oplus \mathfrak{g})$,

$$[\psi, \psi] = \frac{1}{4} f_{abc}(e_a^1 \wedge e_b^1 \wedge e_c^2 + e_a^1 \wedge e_b^2 \wedge e_c^2).$$

In terms of the map diag : $\wedge^\bullet \mathfrak{g} \to \wedge^\bullet(\mathfrak{g} \oplus \mathfrak{g})$, induced by the diagonal embedding $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, this can be written

$$[\psi, \psi] = \mathrm{diag}(\phi) - \phi^1 - \phi^2 \ , \tag{11}$$

where ϕ is the Cartan 3-tensor of $\mathfrak{g}$. Consider the map $\mathfrak{g} \oplus \mathfrak{g} \to C^\infty(G; TG), (\xi^1, \xi^2) \mapsto (\xi^2)^L - (\xi^1)^R$ given by the generators of the $G \times G$-action on G, where the first G-factor acts by the left action and the second G-factor by the right action. The image of $-\psi$ under the extended map $\wedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}) \to C^\infty(G; \wedge^\bullet TG)$ is the bivector field on G,

$$P_G = \frac{1}{2} e_a^R \wedge e_a^L \ .$$

When the action map is composed with the diagonal embedding of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}$, the elements of $\mathfrak{g}$ are mapped to the vector fields generating the conjugation action. We denote this infinitesimal action, and its extension to $\wedge^\bullet \mathfrak{g}$, by $\xi \mapsto \xi_G$. Therefore, Equation (11) yields

$$[P_G, P_G] = \phi_G - \phi^L + \phi^R.$$

Since ϕ is Ad-invariant, ϕ^L equals ϕ^R and therefore

$$[P_G, P_G] = \phi_G \ .$$

This proves the first part of

Proposition 3.1. *Let $M = G$, with G acting on itself by conjugation, and let P_G be the bivector field*

$$P_G = \frac{1}{2} e^R_a \wedge e^L_a. \tag{12}$$

Then (M, P_G) is a Hamiltonian quasi-Poisson G-manifold with moment map $\Phi:\ M \to G$ the identity map.

Proof. It remains to verify the moment map condition (7), which in this case is

$$P_G^\sharp(\mathrm{d}f) = \frac{1}{2}((e^L_a + e^R_a)f)(e_a)_G. \tag{13}$$

From the definition of P_G we obtain,

$$P_G^\sharp(\mathrm{d}f) = \frac{1}{2}((e^R_a f)e^L_a - (e^L_a f)e^R_a).$$

These two expressions differ by $(e^L_a f)e^L_a - (e^R_a f)e^R_a$, which vanishes by Ad-invariance of the inner product. □

Remark 3.2. In Appendix A, we shall give a heuristic derivation of the bivector field P_G as a quotient of a formal Poisson bivector $P_{0,L\mathfrak{g}^*}$ on the dual of the loop algebra $L\mathfrak{g}$ of $\mathfrak{g}$.

It is well-known that for any Hamiltonian Poisson manifold, the moment map is a Poisson map with respect to the linear Poisson structure on $\mathfrak{g}^*$. Similarly

Proposition 3.3. *Let (M, P) be a Hamiltonian quasi-Poisson G-manifold with moment map $\Phi:\ M \to G$. Then Φ is a quasi-Poisson map.*

Proof. Using (6) and (12) we find that

$$\Phi_* P^\sharp(\Phi^*\mathrm{d}f) = \Phi_*(\Phi^*(\mathcal{D}f))_M = (\mathcal{D}f)_G = P_G^\sharp(\mathrm{d}f),$$

for all $f \in C^\infty(G, \mathbb{R})$. This shows that $\Phi_* P = P_G$. □

To compare the bivector field P_G to the linear Poisson structure $P_{0,\mathfrak{g}}$ on $\mathfrak{g} \cong \mathfrak{g}^*$, consider the Taylor expansion of P_G near the origin, using the coordinates provided by the exponential map, $\exp:\ \mathfrak{g} \to G$. Let $\nu(s) = \frac{s}{1-e^{-s}} = 1 + \frac{s}{2} + \mathrm{O}(s^2)$. The operator $\nu(\mathrm{ad}_\xi):\ \mathfrak{g} \to \mathfrak{g}$ is well-defined for ξ small, and

$$e^L_a = (\nu(\mathrm{ad}_\xi))_{ab}\,\frac{\partial}{\partial \xi_b}, \quad e^R_a = (\nu(\mathrm{ad}_\xi))_{ca}\,\frac{\partial}{\partial \xi_c}\ , \tag{14}$$

where $\frac{\partial}{\partial \xi_a}$ are the coordinate vector fields. Therefore,

$$P_G = -\frac{1}{2}\big((\nu(\mathrm{ad}_\xi))^2\big)_{ab}\,\frac{\partial}{\partial \xi_a} \wedge \frac{\partial}{\partial \xi_b} = -\frac{1}{2}(\mathrm{ad}_\xi)_{ab}\,\frac{\partial}{\partial \xi_a} \wedge \frac{\partial}{\partial \xi_b} + \mathrm{O}(||\xi||^2),$$

showing that the linearization of P_G is the linear Poisson structure,

$$P_{0,\mathfrak{g}} = \frac{1}{2} f_{abc}\xi_c \frac{\partial}{\partial \xi_a} \wedge \frac{\partial}{\partial \xi_b} = -\frac{1}{2}(\mathrm{ad}_\xi)_{ab} \frac{\partial}{\partial \xi_a} \wedge \frac{\partial}{\partial \xi_b}. \tag{15}$$

Proposition 3.4. *For every conjugacy class $\mathcal{C} \subset G$, the bivector field P_G on G is tangent to $\mathcal{C}$. Thus $\mathcal{C}$ is a quasi-Poisson manifold, with moment map the embedding $\Phi : \mathcal{C} \hookrightarrow G$.*

Proof. By Equation (13), $P_G^\sharp : T^*G \to TG$ takes values in the image of the action map $\mathfrak{g} \to TG,\ \xi \to \xi_G$. Equivalently, P_G is tangent to the orbits for the conjugation action. □

Example 3.5. Let $\sigma : G \to G$ be a non-trivial involutive automorphism of G, with fixed point set G^σ, and $M = G/G^\sigma$ the corresponding symmetric space. Let ϵ be the generator of $\mathbb{Z}_2$, and define the semi-direct product $\hat{G} = \mathbb{Z}_2 \ltimes G$ such that $\epsilon g = \sigma(g)\,\epsilon$. The action of G on M extends to an action of $\hat{G}$, by $\epsilon.(gG^\sigma) = \sigma(g)G^\sigma$, and the map

$$\Phi :\ M \to \mathbb{Z}_2 \ltimes G,\ \ gG^\sigma \mapsto (\epsilon, \sigma(g)g^{-1})$$

is a $\hat{G}$-equivariant diffeomorphism from M onto the conjugacy class of the element $(\epsilon, e) \in \hat{G}$. Thus every symmetric space carries the structure of a quasi-Poisson manifold. Remark 3.6 below shows that $P_M = 0$, since each element of the form $(\epsilon, \sigma(g)g^{-1})$ squares to the identity element.

To obtain a more explicit description of the bivector field on a conjugacy class $\mathcal{C}$, identify the tangent space $T_g\mathcal{C}$ at $g \in \mathcal{C}$ with $\mathfrak{g}_g^\perp$, where $\mathfrak{g}_g = \{\xi \in \mathfrak{g} |\ \mathrm{Ad}_g \xi = \xi\}$ is the Lie algebra of the stabilizer of g. The operator $\mathrm{Ad}_g - 1$ is invertible on $\mathfrak{g}_g^\perp$. Using this inverse, we consider

$$\Big(\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}|\mathfrak{g}_g^\perp\Big) :\ \mathfrak{g}_g^\perp \to \mathfrak{g}_g^\perp\ .$$

We will view this as a linear operator on $\mathfrak{g}$ acting trivially on $\mathfrak{g}_g$. We claim that the bivector P_G at the point $g \in G$ can be written

$$P_G = \frac{1}{4}\Big(\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}|\mathfrak{g}_g^\perp\Big)_{ab} (e_a)_G \wedge (e_b)_G\ , \tag{16}$$

showing explicitly that P is tangent to the orbits. Indeed,

$$\begin{aligned} P_G &= \frac{1}{4}(\mathrm{Ad}_{g^{-1}} - \mathrm{Ad}_g)_{ab} e_a^L \wedge e_b^L = \frac{1}{4}(\mathrm{Ad}_g + 1)_{cb}(\mathrm{Ad}_{g^{-1}} - 1)_{ac} e_a^L \wedge e_b^L \\ &= \frac{1}{4}(\mathrm{Ad}_g + 1)_{cb}(\mathrm{Ad}_g - 1)_{ca} e_a^L \wedge e_b^L = -\frac{1}{4}(\mathrm{Ad}_g + 1)_{ab}(e_a)_G \wedge e_b^L\ . \end{aligned}$$

which yields (16), since $e_b^L = (((1 - \mathrm{Ad}_g)|\mathfrak{g}_g^\perp)^{-1})_{ab}(e_b)_G$.

Notice that, in analogy to (16), the linear Poisson bivector $P_{0,\mathfrak{g}}$ at the point $\xi \in \mathfrak{g}$ can be written

$$P_{0,\mathfrak{g}} = \frac{1}{2}((\mathrm{ad}_\xi |\mathfrak{g}_\xi^\perp)^{-1})_{ab}(e_a)_\mathfrak{g} \wedge (e_b)_\mathfrak{g}\ , \tag{17}$$

where G acts on $\mathfrak{g}$ by the adjoint action.

Remark 3.6. Equation (16) also show that P_G vanishes at those points $g \in G$ for which all eigenvalues of Ad_g are $+1$ or -1. That is, P_G vanishes at g if g^2 is contained in the center of G.

4. Quasi-Poisson cohomology

On a Poisson manifold (M, P), the graded algebra of multivectors, $C^\infty(M; \wedge^\bullet TM)$, with the differential $\mathrm{d}_P = [P, \cdot\,]$, is a complex. In fact, since
$d_P^2 = \frac{1}{2}[[P,P], \cdot\,]$, the property $[P,P] = 0$ ensures that d_P squares to zero. The cohomology of d_P is called the Poisson cohomology of (M, P).

Let (M, P) be a quasi-Poisson G-manifold. Then, $\mathrm{d}_P := [P, \cdot\,]$ defines an operator on the space of multivectors. Its square is in general non-vanishing, $\mathrm{d}_P^2 = \frac{1}{2}[\phi_M, \cdot\,]$. However, when restricted to the subspace of G-invariant multivectors, $C^\infty(M; \wedge^\bullet TM)^G$, d_P becomes a differential.

Definition 4.1. Let (M, P) be a quasi-Poisson G-manifold. The *quasi-Poisson cohomology* of (M, P) is the cohomology of the differential $\mathrm{d}_P = [P, \cdot\,]$ on the space of G-invariant multivectors, $C^\infty(M; \wedge^\bullet TM)^G$.

Let M be an orientable manifold and let $*_\mu$ be the isomorphism from multivectors to differential forms on M defined by a volume form, μ. The de Rham differential on the space of differential forms, d , translates into the operator $\partial_\mu := - *_\mu^{-1} \circ \mathrm{d} \circ *_\mu$ on multivectors, called the *Batalin-Vilkovisky* or *BV-operator*. The BV-operator, ∂_μ, is a generator of the Schouten bracket on $C^\infty(M; \wedge^\bullet TM)$, that is,

$$[\alpha, \beta] = (-1)^{|\alpha|}(\partial_\mu(\alpha \wedge \beta) - (\partial_\mu \alpha) \wedge \beta - (-1)^{|\alpha|}\alpha \wedge (\partial_\mu \beta)) \ ,$$

where $|\alpha|$ is the multivector degree of α. Since, moreover, the BV-operator is of square 0, it is a (super-) derivation of the Schouten bracket, $\partial_\mu[\alpha, \beta] = [\partial_\mu \alpha, \beta] + (-1)^{|\alpha|-1}[\alpha, \partial_\mu \beta]$.

Lemma 4.2. *Assume that μ is a G-invariant volume form on a G-manifold M. Then, the map $\wedge^\bullet \mathfrak{g} \to C^\infty(M; \wedge^\bullet TM)$ induced by the G-action on M, $\xi \mapsto \xi_M$, is a homomorphism of complexes with respect to the Lie algebra homology operator, $\partial : \wedge^\bullet \mathfrak{g} \to \wedge^{\bullet-1}\mathfrak{g}$ and the BV-operator, ∂_μ.*

Proof. The map $\xi \mapsto \xi_M$ is a homomorphism with respect to both the exterior product and the Schouten bracket. Moreover, the operator ∂ is a generator of the Schouten bracket on $\wedge^\bullet \mathfrak{g}$. Hence, it is sufficient to prove the property $\partial_\mu \xi_M = (\partial \xi)_M$ on the elements of degree 1. For all $\xi \in \mathfrak{g}$, $\partial \xi = 0$. To compute $\partial_\mu \xi_M = - *_\mu^{-1} \circ \mathrm{d} \circ *_\mu \xi_M$, we consider

$$\mathrm{d}(*_\mu \xi_M) = \mathrm{d}\iota(\xi_M)\mu = \mathcal{L}(\xi_M)\mu \ ,$$

where ι denotes an interior product and $\mathcal{L}$ a Lie derivation. Since μ is G-invariant, $\mathcal{L}(\xi_M)\mu = 0$, whence $\partial_\mu(\xi_M) = 0$, which shows that $\partial_\mu(\xi_M) = (\partial \xi)_M$, for all $\xi \in \mathfrak{g}$. ☐

We define the *modular vector field* on a quasi-Poisson G-manifold (M,P) with given G-invariant volume form μ by the formula, $X_\mu := \partial_\mu P$.

Proposition 4.3. *For any G-invariant volume form μ on the quasi-Poisson G-manifold (M,P), the modular vector field X_μ is G-invariant and a cocycle with respect to d_P. The quasi-Poisson cohomology class of X_μ is independent of the choice of μ.*

Proof. Since the BV-operator is a derivation of the Schouten bracket, for each $\xi \in \mathfrak{g}$, $[\xi_M, X_\mu] = [\xi_M, \partial_\mu P] = \partial_\mu[\xi_M, P] - [\partial_\mu \xi_M, P] = 0$. Moreover, $\mathrm{d}_P X_\mu = [P, \partial_\mu P] = \frac{1}{2}\partial_\mu \phi_M$. The element ϕ defines a cycle in Lie algebra homology. Hence, we obtain $\partial_\mu \phi_M = (\partial\phi)_M = 0$ and $\mathrm{d}_P X_\mu = 0$.

Choosing $\tilde{\mu} = f\mu$ with f a positive G-invariant function on M, one obtains the new BV-operator, $\partial_{\tilde{\mu}} = \partial_\mu - \iota(\mathrm{d}(\ln f))$. The modular vector field also changes,

$$X_{\tilde{\mu}} = X_\mu - \iota(\mathrm{d}(\ln f))P = X_\mu + [P, \ln(f)] = X_\mu + \mathrm{d}_P(\ln f).$$

We conclude that the class of X_μ in the quasi-Poisson cohomology is independent of the choice of μ. □

We refer to the class of X_μ in the quasi-Poisson cohomology as the *modular class* of the quasi-Poisson G-manifold (M,P). This definition extends the standard definition of the modular class of a Poisson manifold [10].

Proposition 4.4. *The modular class of the quasi-Poisson G-manifold (G, P_G), where P_G is defined by Formula (12), vanishes.*

Proof. Let μ be the bi-invariant volume form on G defined by the basis (e_a) of the Lie algebra $\mathfrak{g}$. Then, $X_\mu = \partial_\mu P_G = - *_\mu^{-1} \mathrm{d} *_\mu P_G$, where $*_\mu P_G = \frac{1}{2}\iota(e_a^R)\iota(e_a^L)\mu$. Applying the de Rham differential yields

$$\begin{aligned} \mathrm{d}\iota(e_a^R)\iota(e_a^L)\mu &= \mathcal{L}(e_a^R)\iota(e_a^L)\mu - \iota(e_a^R)\mathrm{d}\iota(e_a^L)\mu \\ &= \iota(e_a^L)\mathcal{L}(e_a^R)\mu - \iota(e_a^R)\mathcal{L}(e_a^L)\mu = 0\ , \end{aligned}$$

since e_a^R and e_a^L commute, and since μ is both left-and right-invariant and closed. This implies $X_\mu = 0$. □

5. Fusion of quasi-Poisson manifolds

Any Hamiltonian Poisson $G \times G$-manifold becomes a Hamiltonian G-manifold for the diagonal G-action, with moment map the sum of the moment map components. For Hamiltonian quasi-Poisson manifolds a similar statement is true using the *product* of the moment map components. However, it is necessary to change the bivector field. In this section H is a compact Lie group with an invariant inner product, possibly $H = \{e\}$.

Proposition 5.1. *Let (M,P) be a quasi-Poisson $G \times G \times H$-manifold. Then*

$$P_{fus} := P - \psi_M\ , \tag{18}$$

where ψ_M is the image of (10) *under the $G \times G$-action map, defines a quasi-Poisson structure on M for the diagonal $G \times H$-action. Moreover, if $(\Phi^1, \Phi^2, \Psi) : M \to G \times G \times H$ is a moment map for the $G \times G \times H$-action, the pointwise product*

$$(\Phi^1 \Phi^2, \Psi)$$

is a moment map for the diagonal action.

Proof. The trivector field for the $G \times G$-action is the sum of the trivector fields ϕ^1_M and ϕ^2_M for the two G-factors. By assumption, P is $G \times G$-invariant and $[P, P] = \phi^1_M + \phi^2_M$. Therefore $[P, \psi_M] = 0$, and using Formula (11) we obtain

$$[P - \psi_M, P - \psi_M] = \phi^1_M + \phi^2_M + [\psi_M, \psi_M] = (\mathrm{diag}(\phi))_M = \phi^{\mathrm{diag}}_M ,$$

where ϕ^{diag}_M is the image of ϕ under the map (1) extending the infinitesimal diagonal action of G on M.

Now suppose that the action is Hamiltonian. For any maps Φ^1 and Φ^2 from M to G,

$$(\Phi^1 \Phi^2)^* \theta^R_a = (\mathrm{Ad}_{\Phi^1})_{ab} (\Phi^2)^* \theta^R_b + (\Phi^1)^* \theta^R_a .$$

This relation together with the moment map property,

$$P^\sharp (\Phi^i)^* \theta^R_a = \frac{1}{2} (1 + \mathrm{Ad}_{\Phi^i})_{ab} (e_b)^i_M ,$$

for $i = 1, 2$, implies,

$$P^\sharp ((\Phi^1 \Phi^2)^* \theta^R_a) = \frac{1}{2} (1 + \mathrm{Ad}_{\Phi^1})_{ac} (e_c)^1_M + \frac{1}{2} (\mathrm{Ad}_{\Phi^1})_{ab} (1 + \mathrm{Ad}_{\Phi^2})_{bc} (e_c)^2_M .$$

By equivariance of Φ^i , $\langle (\Phi^i)^* \theta^R_a, (e_b)^i_M \rangle = (\mathrm{Ad}_{\Phi^i} - 1)_{ab}$, and therefore,

$$\psi^\sharp_M ((\Phi^1 \Phi^2)^* \theta^R_a) = \frac{1}{2} ((\mathrm{Ad}_{\Phi^1})_{ab} (1 - \mathrm{Ad}_{\Phi^2})_{bc} (e_c)^1_M + \frac{1}{2} (-1 + \mathrm{Ad}_{\Phi^1})_{ac} (e_c)^2_M .$$

We obtain

$$P^\sharp_{fus} ((\Phi^1 \Phi^2)^* \theta^R_a) = \frac{1}{2} (1 + \mathrm{Ad}_{\Phi^1 \Phi^2})_{ab} ((e_b)^1_M + (e_b)^2_M) ,$$

which is the moment map condition for the diagonal action. □

Example 5.2. Let $M = G$, viewed as a $G \times G$-space for the action defined by $(g_1, g_2).a = g_1 a g_2^{-1}$. The trivector field for this action is $\phi_M = \phi^L - \phi^R = 0$, hence $P = 0$ defines a quasi-Poisson structure. The diagonal action is the conjugation, and the quasi-Poisson structure P_{fus} is defined by the bivector field, $P_{fus} = \frac{1}{2} e^R_a \wedge e^L_a$, introduced above. We remark that the $G \times G$-action on $M = G$ does *not* admit a moment map. (However, one can view G as a symmetric space $G \times G/G$ for the involution $\sigma(a, b) = (b, a)$ of $G \times G$, and as such it is a Hamiltonian quasi-Poisson $\mathbb{Z}_2 \ltimes (G \times G)$-manifold by Example 3.5.)

Example 5.3. Let $M = D(G) := G \times G$ with $G \times G$-action

$$(g_1, g_2).(a_1, a_2) = (g_1 a_1 g_2^{-1}, g_2 a_2 g_1^{-1}) ,$$

and bivector field

$$P = \frac{1}{2}(e_a^{1,L} \wedge e_a^{2,R} + e_a^{1,R} \wedge e_a^{2,L}). \tag{19}$$

Then $(D(G), P)$ is a quasi-Poisson $G \times G$-manifold obtained from $G \times G$, viewed as a G^4-space, by fusing two pairs of G-factors. A simple calculation shows that $(a_1, a_2) \mapsto (a_1 a_2, a_1^{-1} a_2^{-1})$ is a moment map.

Example 5.4. By fusing the two G-factors acting on the Hamiltonian quasi-Poisson manifold $D(G)$ of Example 5.3, we obtain a Hamiltonian quasi-Poisson G-manifold, where G acts by conjugation on each factor of $G \times G$, and the moment map is the Lie group commutator, $(a_1, a_2) \mapsto [a_1, a_2] = a_1 a_2 a_1^{-1} a_2^{-1}$. We denote this Hamiltonian quasi-Poisson G-manifold by $\mathbf{D}(G)$.

If M_1 and M_2 are quasi-Poisson G-manifolds, we denote their direct product with diagonal G-action and bivector field $(P_1 + P_2)_{fus}$ by $M_1 \circledast M_2$. We remark that the two projection mappings, $M_1 \circledast M_2 \to M_j$ $(j = 1, 2)$, are quasi-Poisson maps.

Propositions 5.1 and 3.3 have the following consequence:

Corollary 5.5. *The group multiplication,* $\mathrm{Mult}_G : G \circledast G \to G,\ (g_1, g_2) \mapsto g_1 g_2$ *, is a quasi-Poisson map.*

Proof. Since the identity map is a moment map for (G, P), Proposition 5.1 shows that Mult_G is a moment map for $G \circledast G$. Hence it is a quasi-Poisson map by Proposition 3.3. □

Proposition 5.6. *For any quasi-Poisson manifold (M, P), the action map*

$$\mathcal{A} : G \circledast M \to M, \quad (g, m) \mapsto g.m$$

is a quasi-Poisson map.

Proof. Since P is G-invariant, $\mathcal{A}_* P = P$. To compute $\mathcal{A}_* P_G$ we observe that $\mathcal{A}_* e_a^R = -(e_a)_M$, since $-e_a^R$ is the vector field generating the left action on G, and $\mathcal{A}$ is equivariant with respect to this action. Thus

$$\mathcal{A}_* P_G = -\frac{1}{2}(\mathrm{Ad}_g)_{ab}(e_a)_M \wedge (e_b)_M.$$

Finally, since $\mathcal{A}_*(e_a)_M = (\mathrm{Ad}_{g^{-1}})_{ab}(e_b)_M$, we obtain

$$\mathcal{A}_*(-\frac{1}{2}(e_a^L - e_a^R) \wedge (e_a)_M) = \frac{1}{2}(\mathrm{Ad}_{g^{-1}} - 1)_{ab}(\mathrm{Ad}_{g^{-1}})_{ac}(e_b)_M \wedge (e_c)_M = \frac{1}{2}(\mathrm{Ad}_g)_{ab}(e_a)_M \wedge (e_b)_M\ ,$$

which cancels the term $\mathcal{A}_* P_G$. □

It is evident that the fusion operation is associative. Given a Hamiltonian quasi-Poisson $G \times G \times G \times H$-manifold M, we can begin by fusing the last two G-factors, or the first two G-factors. The $G \times H$-equivariant bivector fields on M thus obtained are identical. In the following sense, fusion is also commutative.

Proposition 5.7. *Let $(M, P, (\Phi^1, \Phi^2, \Psi))$ be a Hamiltonian quasi-Poisson $G \times G \times H$-manifold. Let*

$$R: M \to M, \quad m \mapsto (e, \Phi^1(m), e).m$$

be the action of the second G-factor by the value of the first moment map component. Then R is a diffeomorphism, with properties

$$R^*(\Phi^2\Phi^1) = \Phi^1\Phi^2 \tag{20}$$

and

$$R_*(P - \frac{1}{2}(e_a)^1_M \wedge (e_a)^2_M) = P - \frac{1}{2}(e_a)^2_M \wedge (e_a)^1_M \ . \tag{21}$$

Proof. Equation (20) follows by equivariance of the moment map,

$$R^*(\Phi^2\Phi^1) = R^*(\Phi^2)R^*(\Phi^1) = \mathrm{Ad}_{\Phi^1}(\Phi^2)\ \Phi^1 = \Phi^1\Phi^2 \ .$$

To prove (21), write R as the composition of two maps, $R = R_2 \circ R_1$, where $R_1 : M \to G \times M, m \mapsto (\Phi^1(m), m)$, and $R_2 : G \times M \to M$ is the action map $\mathcal{A}^2$ of the second G-factor. The tangent map to R_1 is defined by

$$(R_1)_*(X) = \langle (\Phi^1)^*\theta^R_a, X\rangle e^R_a + X \ .$$

In particular, $(R_1)_*(e_a)^1_M = (e_a)^1_M + (\mathrm{Ad}_{(\Phi^1)^{-1}} - 1)_{ab} e^R_b$ and $(R_1)_*(e_a)^2_M = (e_a)^2_M$, so that

$$\frac{1}{2}(R_1)_*(e_a)^1_M \wedge (e_a)^2_M = \frac{1}{2}(e_a)^1_M \wedge (e_a)^2_M - \frac{1}{2}(\mathrm{Ad}_{(\Phi^1)^{-1}} - 1)_{ab}(e_a)^2_M \wedge e^R_b \ .$$

We now use the fact that $(R_2)_*(e_a)^1_M = (e_a)^1_M$ and $(R_2)_*(e^R_a) = -(e_a)^2_M$, and the relation $(R_2)_*(e_a)^2_M = (\mathrm{Ad}_{g^{-1}})_{ab}(e_b)^2_M$. We find that

$$\begin{aligned}\frac{1}{2}(R_2R_1)_*(e_a)^1_M \wedge (e_a)^2_M &= \frac{1}{2}(\mathrm{Ad}_{(\Phi^1)^{-1}})_{ab}(e_a)^1_M \wedge (e_b)^2_M + \frac{1}{2}(\mathrm{Ad}_{(\Phi^1)^{-1}} - 1)_{ab}(e_a)^2_M \wedge (e_b)^2_M \\ &= -\frac{1}{2}(\mathrm{Ad}_{\Phi^1})_{ab}(e_a)^2_M \wedge (e_b)^1_M - \frac{1}{2}(\mathrm{Ad}_{\Phi^1})_{ab}(e_a)^2_M \wedge (e_b)^2_M.\end{aligned}$$

On the other hand, $(R_1)_*P$ is the sum of P, of $(\Phi^1)_*P = P_G$, and of cross-terms which can be found by pairing this bivector with θ^R_a and using the moment map condition. The result is

$$(R_1)_*P = P_G + P + \frac{1}{2}(1 + \mathrm{Ad}_{\Phi^1})_{ab}\ e^R_a \wedge (e_b)^1_M.$$

We next apply $(R_2)_*$ to this result. The push-forward of P and of P_G are obtained as in the proof of Proposition 5.6, and we obtain

$$(R_2R_1)_*P = P - \frac{1}{2}(\mathrm{Ad}_{\Phi^1})_{ab}(e_a)^2_M \wedge (e_b)^2_M - \frac{1}{2}(1 + \mathrm{Ad}_{\Phi^1})_{ab}(e_a)^2_M \wedge (e_b)^1_M.$$

Equation (21) is now obtained by subtracting the expression for $\frac{1}{2}R_*(e_a)^1_M \wedge (e_a)^2_M$ from that for R_*P. $\square$

6. Reduction of Hamiltonian quasi-Poisson manifolds

If (M, P_0) is a Poisson G-manifold, and M_* the open subset of M on which G acts freely, the quotient M_*/G carries a unique Poisson structure such that the quotient map is Poisson. If the action is Hamiltonian with moment map $\Phi_0 : M \to \mathfrak{g}^*$, then Φ_0 has maximal rank on M_*, and for every coadjoint orbit, $\mathcal{O} \subset \mathfrak{g}^*$, $(\Phi_0^{-1}(\mathcal{O}) \cap M_*)/G$ is a smooth Poisson submanifold of M_*/G. More generally, if M_{**} denotes the subset where the action is locally free, *i.e.*, the stabilizer groups are finite, M_{**}/G is a Poisson orbifold and $(\Phi_0^{-1}(\mathcal{O}) \cap M_{**})/G$ is a Poisson suborbifold of M_{**}/G. The space $M_{\mathcal{O}} := \Phi_0^{-1}(\mathcal{O})/G$ is called the *reduced space*. For the orbit of $0 \in \mathfrak{g}^*$, we also use the notation $M/\!\!/G := \Phi_0^{-1}(0)/G$.

Similar assertions hold for a quasi-Poisson manifold (M, P). Since ϕ_M vanishes on invariant forms, the space $C^\infty(M, \mathbb{R})^G$ of G-invariant functions is a Poisson algebra under $\{\cdot, \cdot\}$. Therefore M_*/G is a Poisson manifold in the usual sense.

Theorem 6.1 (quasi-Poisson reduction)**.** *Let (M, P, Φ) be a Hamiltonian quasi-Poisson manifold. Then Φ has maximal rank on the subset M_{**} where the action is locally free. For each conjugacy class $\mathcal{C} \subset G$, the intersection of $M_{\mathcal{C}} = \Phi^{-1}(\mathcal{C})/G$ with M_*/G (resp., M_{**}/G) is a Poisson submanifold (resp., suborbifold).*

Proof. Suppose that the action is locally free at the point $m \in M$. To prove that Φ has maximal rank at m, we need to show that the equation $\Phi^*(\theta^R \cdot \xi)(m) = 0$ does not admit a non-trivial solution $\xi \in \mathfrak{g}$. Let $\xi \in \mathfrak{g}$ be any solution of this equation. By equivariance of the moment map,

$$0 = \langle \Phi^*(\theta^R \cdot \xi), \eta_M \rangle(m) = (\mathrm{Ad}_{\Phi(m)^{-1}} - 1)\xi \cdot \eta$$

for all $\eta \in \mathfrak{g}$. Hence, $\mathrm{Ad}_{\Phi(m)^{-1}} \xi = \xi$, and the moment map condition (9) shows that

$$0 = P_m^\sharp(\Phi^*(\theta^R \cdot \xi)) = \xi_M(m).$$

Since the action is locally free at m, this implies $\xi = 0$.

Because Φ is of maximal rank on M_{**}, $\Phi^{-1}(\mathcal{C}) \cap M_{**}$ is a smooth submanifold of M_{**}, and $(\Phi^{-1}(\mathcal{C}) \cap M_{**})/G$ is a smooth suborbifold of M_{**}/G. To show that $(\Phi^{-1}(\mathcal{C}) \cap M_{**})/G$ is a Poisson suborbifold, we have to show that, for all invariant $F \in C^\infty(M, \mathbb{R})^G$, the Hamiltonian vector field $P^\sharp(\mathrm{d}F)$ is tangent to $\Phi^{-1}(\mathcal{C}) \cap M_{**}$. In fact, $P^\sharp(\mathrm{d}F)$ is tangent to all level sets of Φ, because

$$(P^\sharp(\mathrm{d}F))\Phi^* f = -(P^\sharp(\mathrm{d}\Phi^* f))\, F = 0$$

for all functions $f \in C^\infty(G, \mathbb{R})$. Here we have used the moment map condition (7) and $(e_a)_M F = 0$. □

For $\mathcal{C} = \{e\}$ we also write $M/\!\!/G := \Phi^{-1}(e)/G$.

Example 6.2. Let Σ be a compact oriented surface of genus h with $r \geq 1$ boundary components. It is known that the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ can be identified with the moduli space of flat G-bundles over Σ. Its smooth part carries a natural Poisson structure [6], the leaves of which correspond to flat bundles whose holonomies around the boundary circles belong to fixed conjugacy classes in G.

Using the notion of fusion, this Poisson structure can be described as follows. First we observe that

$$\mathrm{Hom}(\pi_1(\Sigma), G) = \{(a_1, \ldots, a_{2h}, b_1, \ldots, b_r) \in G^{2h+r} | \prod_j [a_j, a_{j+1}] \prod_k b_k = e\} .$$

Viewing this relation as a moment map condition, we can write

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G = \big(\underbrace{\mathbf{D}(G) \circledast \ldots \circledast \mathbf{D}(G)}_{h} \circledast \underbrace{G \circledast \ldots \circledast G}_{r}\big)/\!\!/G.$$

For any Hamiltonian quasi-Poisson manifold (M, P, Φ), the embedding $M \to M \circledast G$ defined by $m \mapsto (m, (\Phi(m))^{-1})$ is a G-equivariant diffeomorphism from M onto the identity level set of the moment map for $M \circledast G$, $(m, g) \in M \times G \mapsto \Phi(m)g \in G$. Thus it induces a bijection from M/G to $(M \circledast G)/\!\!/G$. It is easily shown that on the smooth part, this bijection is a Poisson diffeomorphism. In particular, the Poisson structure on the representation variety can also be written,

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G = \big(\underbrace{\mathbf{D}(G) \circledast \ldots \circledast \mathbf{D}(G)}_{h} \circledast \underbrace{G \circledast \ldots \circledast G}_{r-1}\big)/G.$$

In Section 10, Example 10.8 we shall explain that this Poisson structure coincides with the canonical Poisson structure on the moduli space of flat connections on Σ. A similar construction of the Poisson structure on the representation variety using the properties of the solutions of the classical Yang-Baxter equation was developed in [5].

7. Exponentials of Hamiltonian Poisson-manifolds

Let (M, P_0, Φ_0) be a Hamiltonian Poisson G-manifold. We will show in this section that if Φ_0 takes values in the open subset $\mathfrak{g}^\natural \subset \mathfrak{g}$ on which the exponential map has maximal rank, then P_0 can be modified into a quasi-Poisson structure with moment map $\exp \circ \Phi_0$. This construction involves a bivector field on $\mathfrak{g}^\natural$,

$$T = \frac{1}{2} T_{ab} e_a \wedge e_b \in C^\infty(\mathfrak{g}^\natural; \wedge^2 \mathfrak{g}),$$

where $T_{ab}(\xi) = (\varphi(\mathrm{ad}_\xi))_{ab}$, for $\xi \in \mathfrak{g}^\natural$, and φ is the meromorphic function of $s \in \mathbb{C}$,

$$\varphi(s) = \frac{1}{s} - \frac{1}{2} \coth\big(\frac{s}{2}\big) = -\frac{s}{12} + \mathrm{O}(s^3).$$

We observe that T is well-defined and smooth on $\mathfrak{g}^\natural$, but develops poles on the complement of this set. By a result of Etingof and Varchenko [4], T is a solution of the classical dynamical Yang-Baxter equation,

$$\mathrm{Cycl}_{abc}\big(\frac{\partial T_{ab}}{\partial \xi_c} + T_{ak} f_{kbl} T_{lc}\big) = \frac{1}{4} f_{abc}, \tag{22}$$

where Cycl_{abc} denotes the sum over cyclic permutations. It follows from (14) and from the relation $\frac{1}{2}(\nu(s)+\nu(-s)) = 1 - s\varphi(s)$ that T also satisfies the equation (see [3])

$$T_{ab}(e_b)_{\mathfrak{g}} = \frac{\partial}{\partial \xi_a} - \frac{1}{2}\exp^*(e_a^L + e_a^R), \tag{23}$$

where for a local diffeomorphism $F : M \to N$ and any multivector field u on N we denote by F^*u the unique multivector field on M such that $F_*(F^*u) = u$.

Theorem 7.1. *Let (M, P_0) be a Hamiltonian Poisson G-manifold with moment map Φ_0. Suppose that $\Phi_0(M) \subset \mathfrak{g}^\natural$. Then,* $\exp : \mathfrak{g} \to G$ *has maximal rank on the image of Φ_0 and $P = P_0 - (\Phi_0^* T)_M$ defines a Hamiltonian quasi-Poisson structure on M, with moment map $\Phi = \exp \circ \Phi_0$.*

Proof. Let $T_M = (\Phi_0^* T)_M$. We want to compute

$$[P, P] = [P_0, P_0] - 2[P_0, T_M] + [T_M, T_M].$$

The first term vanishes since P_0 is a Poisson structure. For any function $f \in C^\infty(\mathfrak{g}, \mathbb{R})$, $[P_0, \Phi_0^* f] = -P_0^\sharp(\Phi_0^* \mathrm{d}f)$. Using the moment map condition (5) of Φ_0 and the invariance property $[P_0, (e_a)_M] = 0$ of P_0, we obtain

$$[P_0, T_M] = -\frac{1}{2}\Phi_0^* \frac{\partial T_{ab}}{\partial \xi_c}(e_a)_M \wedge (e_b)_M \wedge (e_c)_M\,. \tag{24}$$

To calculate $[T_M, T_M]$ we use the relation

$$[(e_a)_M, \Phi_0^* T_{kl}] = \Phi_0^*(f_{akm}T_{ml} + f_{alm}T_{km}).$$

Taking symmetries into account, we obtain

$$\begin{aligned}
[T_M, T_M] = {} & \frac{1}{4}\Phi_0^*(T_{ab}T_{kl})\ [(e_a)_M \wedge (e_b)_M, (e_k)_M \wedge (e_l)_M] \\
& \qquad\qquad\qquad + \Phi_0^* T_{ab}[(e_b)_M, \Phi_0^* T_{kl}](e_a)_M \wedge (e_k)_M \wedge (e_l)_M \\
= {} & \Phi_0^*(T_{ab}T_{kl}) f_{bkm}(e_a)_M \wedge (e_m)_M \wedge (e_l)_M + 2\Phi_0^*(T_{ab} f_{bkm} T_{ml})(e_a)_M \wedge (e_k)_M \wedge (e_l)_M \\
= {} & \Phi_0^*(T_{ak} f_{kbl} T_{lc})(e_a)_M \wedge (e_b)_M \wedge (e_c)_M\ .
\end{aligned}$$

Together with (24), and using the classical dynamical Yang-Baxter equation (22), this yields

$$[P, P] = \Phi_0^*\Big(\frac{\partial T_{ab}}{\partial \xi_c} + T_{ak} f_{kbl} T_{lc}\Big)(e_a)_M \wedge (e_b)_M \wedge (e_c)_M = \phi_M.$$

To prove that $\Phi = \exp \circ \Phi_0$ is a moment map, we use Equation (23). For all functions $f \in C^\infty(G)$,

$$\begin{aligned} T_M^\sharp(\mathrm{d}\,\Phi^* f) &= \Phi_0^* T_{ab}\,(e_a)_M(\Phi_0^* \exp^* f)\ (e_b)_M \\ &= \Phi_0^*(T_{ab}(e_a)_{\mathfrak{g}}(\exp^* f))\ (e_b)_M \\ &= \Phi_0^*\big(\frac{\partial}{\partial \xi_a}(\exp^* f) - \frac{1}{2}\exp^*((e_a^L + e_a^R)f)\big)\ (e_a)_M \\ &= \Phi_0^*(\frac{\partial}{\partial \xi_a}(\exp^* f))\ (e_a)_M - \frac{1}{2}\Phi^*((e_a^L + e_a^R)f)(e_a)_M. \end{aligned}$$

The moment map property (7) of Φ follows from this relation together with the moment map map property (5) of Φ_0. □

Example 7.2. Let P_G be the quasi-Poisson structure on G defined by (12). Under the exponential map, it pulls back to a bivector field, $\exp^* P_G$, on $\mathfrak{g}^\natural$. On the other hand, let $P_{0,\mathfrak{g}}$ be the linear Poisson structure on $\mathfrak{g}^* \cong \mathfrak{g}$. Then $\exp^* P_G = P_{0,\mathfrak{g}} - T_{\mathfrak{g}}$. This follows from Equation (23), together with the expressions (16) for P_G and (17) for $P_{0,\mathfrak{g}}$.

The process described in Theorem 7.1 can also be reversed.

Corollary 7.3. *Let (M, P, Φ) be a Hamiltonian quasi-Poisson G-manifold. Suppose that the image of Φ is contained in an open subset $U \subset G$ on which the exponential map admits a smooth inverse,* $\log : U \to \mathfrak{g}$. *Set $\Phi_0 = \log \circ \Phi$, and $P_0 = P + T_M$. Then (M, P_0, Φ_0) is a Hamiltonian Poisson G-manifold in the usual sense.*

8. Cross-section theorem

The Guillemin-Sternberg symplectic cross-section theorem states that for any symplectic G-manifold with moment map $\Phi_0 : M \to \mathfrak{g}^*$, the inverse image $Y = \Phi_0^{-1}(U)$ of a slice $U \subset \mathfrak{g}^*$ at a given point $\xi \in \mathfrak{g}^*$ is a G_ξ-invariant symplectic submanifold, with moment map the restriction of Φ_0. Thus, Y is a Hamiltonian G_ξ-manifold, called a *symplectic cross-section* of M. In this section we will prove a similar result for the quasi-Poisson case.

Given $g \in G$, we denote the stabilizer of g with respect to the conjugation action of G on itself by $H = G_g$. Any sufficiently small connected open neighborhood U of g in H is a slice for this action. That is, the natural map $G \times_H U \to G.U$, $(g, h) \mapsto ghg^{-1}$, is a diffeomorphism onto its image. There is an orthogonal decomposition

$$TG|_U = TU \oplus (U \times \mathfrak{h}^\perp), \tag{25}$$

where $\mathfrak{h} = \mathrm{Lie}(H)$ and the second summand is embedded by means of the vector fields generating the adjoint action.

Let (M, P, Φ) be a Hamiltonian quasi-Poisson manifold. By equivariance of Φ, the inverse image $Y = \Phi^{-1}(U)$ is a smooth H-invariant submanifold of M, and there is an

H-equivariant splitting of the tangent bundle,

$$TM|_Y = TY \oplus (Y \times \mathfrak{h}^\perp). \tag{26}$$

Dually,

$$T^*M|_Y = T^*Y \oplus (Y \times (\mathfrak{h}^\perp)^*) \ . \tag{27}$$

Lemma 8.1. *The splitting* (27) *is $P|_Y$-orthogonal, that is, there is a decomposition, $P|_Y = P_Y + P_Y^\perp$, where P_Y is a bivector field on Y and $P_Y^\perp \in C^\infty(Y; \wedge^2\mathfrak{h}^\perp)$.*

Proof. At $m \in Y \subset M$, the fiber $\{m\} \times (\mathfrak{h}^\perp)^*$ is the space of covectors $\alpha = \Phi^*(\theta^R \cdot \xi)(m)$ with $\xi \in \mathfrak{h}^\perp$. Let $\xi' = \frac{1}{2}(1 + \mathrm{Ad}_{\Phi(m)^{-1}})\xi \in \mathfrak{h}^\perp$. By the moment map condition (9), it follows that,

$$P_m(\alpha, \beta) = \langle \xi'_M(m), \beta \rangle = 0 \ ,$$

for all $\beta \in T_m^*Y$. ☐

An explicit description of the bivector $P_Y^\perp$ can be given in terms of the moment map. Define $r \in C^\infty(U; \wedge^2\mathfrak{h}^\perp)$ as

$$r(h) = \frac{1}{2} r_{ab}(h) e_a \wedge e_b \ , \tag{28}$$

where

$$r_{ab}(h) = -\frac{1}{2}\Big(\frac{\mathrm{Ad}_h + 1}{\mathrm{Ad}_h - 1}\Big|\mathfrak{h}^\perp\Big)_{ab} \ . \tag{29}$$

Since Φ is a quasi-Poisson map, the description (16) of the quasi-Poisson structure on G shows that $P_Y^\perp = -\Phi_Y^* r$, where $\Phi_Y = \Phi|_Y$.

We will need the following property of the tensor field r. For every vector field $X \in C^\infty(U; TG)$ on U, let $X_\mathfrak{h}$ be the orthogonal projection of X to a vector field tangent to U, using the splitting (25). Let $f^\mathfrak{h}_{abc}$ be the structure constants of $\mathfrak{h}$, viewed as the components of a skew-symmetric tensor of $\mathfrak{g}$ under the inclusion $\wedge^3\mathfrak{h} \to \wedge^3\mathfrak{g}$.

Lemma 8.2. *The tensor field r defined by Equation* (29) *satisfies*

$$\mathrm{Cycl}_{abc}\Big(\frac{1}{2}(e_a^L + e_a^R)_\mathfrak{h} r_{bc} + r_{ak} f_{kbl} r_{lc}\Big) = \frac{1}{4}(f_{abc} - f^\mathfrak{h}_{abc}). \tag{30}$$

This lemma, which is proved in Appendix B, generalizes the result proved by Etingof and Varchenko in [4] (see also [3]) in the special case where H is Abelian, where this equation reduces to the classical dynamical Yang-Baxter equation.

Theorem 8.3. *For any Hamiltonian quasi-Poisson G-space (M, P, Φ), and any slice U at a given $g \in G$ with stabilizer $G_g = H$, the cross-section $Y = \Phi^{-1}(U)$ is a Hamiltonian quasi-Poisson H-manifold, with bivector field $P_Y = P|_Y - P_Y^\perp$ and moment map $\Phi_Y = \Phi|_Y$. Conversely, given a Hamiltonian quasi-Poisson H-manifold (Y, P_Y, Φ_Y), let the associated bundle $G \times_H Y$ be equipped with the unique equivariant map $\Phi : G \times_H Y \to G$ and the unique invariant bivector field P, such that Φ restricts to Φ_Y and P to $P_Y - \Phi_Y^* r$. Then $(G \times_H Y, P, \Phi)$ is a Hamiltonian quasi-Poisson G-manifold.*

Proof. Replacing M by $G.Y$, we may assume that $M = G\times_H Y$. It is clear that the moment map condition for (Y, P_Y, Φ_Y) is equivalent to that for (M, P, Φ). We have to show that $[P, P] = \phi_M$ if and only if $[P_Y, P_Y] = \phi_Y^{\mathfrak{h}}$. Let $\widetilde{P}_Y$ denote the extension of P_Y to a G-invariant bivector field on M. Also, let $\widetilde{r} \in C^\infty(G.U, \wedge^2\mathfrak{g})$ be the G-invariant extension of r. By definition, $P = \widetilde{P}_Y - \Phi^*\widetilde{r}$, so what we need to show is that

$$([\Phi^*\widetilde{r}, \Phi^*\widetilde{r}] - 2[\widetilde{P}_Y, \Phi^*\widetilde{r}])|_Y = \phi_M|_Y - \phi_Y^{\mathfrak{h}}. \tag{31}$$

By a calculation similar to that of the term $[T_M, T_M]$ in the proof of Theorem 7.1,

$$[\Phi^*\widetilde{r}, \Phi^*\widetilde{r}] = \Phi^*(\widetilde{r}_{ak} f_{kbl}\widetilde{r}_{lc})(e_a)_M \wedge (e_b)_M \wedge (e_c)_M.$$

To compute the term $[\widetilde{P}_Y, \Phi^*\widetilde{r}]$, first observe that $[\widetilde{P}_Y, (e_a)_M] = 0$ by G-invariance of $\widetilde{P}_Y$. Using (7), we obtain

$$[P_Y, \Phi_Y^* r_{bc}] = -P_Y^\sharp(\mathrm{d}\Phi_Y^* r_{bc}) = -\Phi^*\Big(\frac{1}{2}(e_a^L + e_a^R)_{\mathfrak{h}} r_{bc}\Big)(e_a)_Y.$$

Therefore,

$$([\Phi^*\widetilde{r}, \Phi^*\widetilde{r}] - 2[\widetilde{P}_Y, \Phi^*\widetilde{r}])|_Y = \Phi^*\big(\frac{1}{2}(e_a^L + e_a^R)_{\mathfrak{h}} r_{bc} + r_{ak} f_{kbl} r_{lc}\Big)(e_a)_M \wedge (e_b)_M \wedge (e_c)_M\,.$$

Together with Lemma 8.2, this proves (31). ☐

Using the construction from Section 7, the cross-section Y can be equipped with an ordinary Poisson structure as follows. Since g is in the center of $H = G_g$, $\Phi'_Y = g^{-1}\Phi_Y$ is also a moment map for (Y, P_Y). For U small enough, the exponential map for $\mathfrak{h}$ admits an inverse on $g^{-1}U \subset H$, $\log:\ g^{-1}U \to \mathfrak{h}$. Let

$$\Phi_{0,Y} = \log(g^{-1}\Phi_Y),$$

and set $P_{0,Y} = P_Y + T_Y$. By Corollary 7.3, $(Y, P_{0,Y}, \Phi_{0,Y})$ is a Hamiltonian Poisson H-manifold.

9. The generalized foliation of a Hamiltonian quasi-Poisson manifold

It is a well-known result of Lie, for the constant rank case, and of Kirillov, for the general case, that for any Poisson-manifold (M, P_0), the generalized distribution[2] $\mathfrak{D}_0 = \mathrm{im}(P_0^\sharp)$ is integrable.

In this section we show that every Hamiltonian quasi-Poisson manifold is foliated by non-degenerate quasi-Poisson submanifolds. Given a Hamiltonian quasi-Poisson manifold (M, P, Φ), define a generalized distribution $\mathfrak{D}$ on M by

$$\mathfrak{D}_m := \mathrm{im}(P_m^\sharp) + T_m(G.m).$$

[2]A (differentiable) generalized distribution on a manifold M is a family of subspaces $\mathfrak{D}_m \subset T_mM$, such that for all $m \in M$, there exists a finite number of vector fields $X_1, \ldots, X_k \in C^\infty(M; TM)$ taking values in $\mathfrak{D} = \cup_{m\in M}\mathfrak{D}_m$ and spanning $\mathfrak{D}_m$ at m. An in-depth discussion of generalized distributions can be found in Vaisman's book [9].

Because G is compact, $\mathfrak{g} = \mathrm{im}(1 + \mathrm{Ad}_g) \oplus \ker(1 + \mathrm{Ad}_g)$, for any $g \in G$. By the moment map condition, the image of $P_m^\sharp$ always contains all $\eta_M(m)$ with η in the image of the operator $1 + \mathrm{Ad}_{\Phi(m)}$ on $\mathfrak{g}$, therefore $\mathfrak{D}_m$ can be re-written

$$\mathfrak{D}_m = \mathrm{im}(P_m^\sharp) \oplus \{\xi_M(m) | (1 + \mathrm{Ad}_{\Phi(m)})\xi = 0\}$$

In particular, if $m \in M$ is such that $1 + \mathrm{Ad}_{\Phi(m)}$ is invertible, then $\mathfrak{D}_m = \mathrm{im}(P_m^\sharp)$.

Definition 9.1. A Hamiltonian quasi-Poisson manifold (M, P, Φ) is non-degenerate if $\mathfrak{D}_m = T_m M$ for all $m \in M$.

For example, the conjugacy classes $\mathcal{C}$ of a group G are non-degenerate Hamiltonian quasi-Poisson manifolds. The decomposition of G into conjugacy classes is a special case of the following:

Theorem 9.2. *The distribution $\mathfrak{D}$ is integrable, that is, through every point $m \in M$ there passes a unique maximal connected submanifold N of M such that $TN = \mathfrak{D}|_N$. Each submanifold N is G-invariant, and the restrictions of P and Φ to N define a non-degenerate Hamiltonian quasi-Poisson structure on N.*

Proof. We show that $\mathfrak{D}$ is integrable near any given point $m \in M$. Let $g = \Phi(m)$, and $U \subset H = G_g$ be an H-invariant slice through g, and $Y = \Phi^{-1}(U)$ the corresponding cross-section. Since $G.Y = G \times_H Y$, it suffices to show that the distribution $\mathfrak{D}_Y$ induced by the quasi-Poisson structure P_Y is integrable. However, as explained in the previous section, $P_Y = P_{0,Y} - T_Y$, where $P_{0,Y}$ is an H-invariant Poisson structure in the usual sense, and T_Y is a bivector field taking v alues in the second exterior power of the H-orbit directions. Moreover, $P_{0,Y}$ admits a moment map $\Phi_{0,Y} = \log(g^{-1}\Phi_Y)$, which implies that the image of $P_{0,Y}^\sharp$ contains the H-orbit directions. Hence $\mathfrak{D}_Y$ is *equal* to the distribution defined by the Poisson-structure $P_{0,Y}$, and therefore integrable by the theorem of Lie and Kirillov. □

10. Non-degenerate quasi-Poisson manifolds

Any non-degenerate bivector field P_0 on a manifold M determines a non-degenerate 2-form ω_0 by the condition $\omega_0^\flat = (P_0^\sharp)^{-1}$. It is well-known that under this correspondence, the Poisson condition $[P_0, P_0] = 0$ is equivalent to the closure of the 2-form ω_0.

In this section, we extend this correspondence between non-degenerate Poisson manifolds and symplectic manifolds to the "quasi" case. While the role of the non-degenerate Poisson manifolds is played by the non-degenerate Hamiltonian quasi-Poisson G-manifolds, that of the symplectic manifolds is played by the "quasi-Hamiltonian G-spaces", introduced in [2]. First, we recall their definition, which includes the non-degeneracy assumption (c) below. Then, we show that every non-degenerate Hamiltonian quasi-Poisson manifold (M, P, Φ) carries a canonically determined 2-form ω such that (M, ω, Φ) is a quasi-Hamiltonian G-space.

Let $\eta \in \Omega^3(G)$ be the bi-invariant closed 3-form,

$$\eta = \frac{1}{12} f_{abc} \theta^R_a \wedge \theta^R_b \wedge \theta^R_c \ .$$

Definition 10.1. [2] A quasi-Hamiltonian G-space is a triple (M, ω, Φ) where M is a G-manifold, ω an invariant differential 2-form, and $\Phi : M \to G$ is an Ad-equivariant map, such that

(a) $\mathrm{d}\omega = \Phi^* \eta$,
(b) $\iota((e_a)_M)\omega = \frac{1}{2}\Phi^*(\theta^L_a + \theta^R_a)$,
(c) for all $m \in M$, the kernel of ω_m is the space of all $\xi_M(m)$ such that ξ is in the kernel of $1 + \mathrm{Ad}_{\Phi(m)}$.

We will need the following two results that were proved in [2]. First, there is an exponentiation construction. Given a Hamiltonian symplectic G-manifold (M, ω_0, Φ_0) such that $\Phi_0(M)$ is contained in the set $\mathfrak{g}^\natural \subset \mathfrak{g}$ of regular values of the exponential map, exp, one obtains a quasi-Hamiltonian G-space (M, ω, Φ) by setting $\Phi = \exp \circ \Phi_0$ and

$$\omega = \omega_0 + \Phi_0^* \varpi,$$

where $\varpi \in \Omega^2(\mathfrak{g})$ is the image of the closed 3-form $\exp^* \eta$ under the homotopy operator $\Omega^\star(\mathfrak{g}) \to \Omega^{\star-1}(\mathfrak{g})$. Secondly, there is a cross-section theorem. Suppose that (M, ω, Φ) is a quasi-Hamiltonian G-space, and that $U \subset G$ is a slice at $g \in G$. Then the cross-section $Y = \Phi^{-1}(U)$ with the 2-form ω_Y and the moment map Φ_Y, defined as the pull-backs of ω and Φ, is a quasi-Hamiltonian H-space, where $H = G_g$. The canonical decomposition $TM|_Y = TY \oplus (Y \times \mathfrak{h}^\perp)$ is ω-orthogonal. Conversely, given a quasi-Hamiltonian H-space (Y, ω_Y, Φ_Y), the associated bundle $M = G \times_H Y$ carries a unique structure of quasi-Hamiltonian G-space (M, ω, Φ) such that ω and Φ pull back to ω_Y and Φ_Y.

If (M, P, Φ) is a non-degenerate Hamiltonian quasi-Poisson manifold such that Φ admits a smooth logarithm $\Phi_0 : M \to \mathfrak{g}$, one can define a 2-form ω on M in the following way. The bivector $P_0 = P + (\Phi_0^* T)_M$ is invertible, we denote its inverse by ω_0 , and we set $\omega = \omega_0 + \Phi_0^* \varpi$. The following Lemma describes the relation between ω and P.

Lemma 10.2. *Let (M, P_0, Φ_0) be a non-degenerate Hamiltonian Poisson manifold, and let ω_0 be the symplectic form corresponding to P_0. Suppose that $\Phi_0(M)$ is contained in the set of regular values of* $\exp : \mathfrak{g} \to G$, *and let $\Phi = \exp \circ \Phi_0$, $\omega = \omega_0 + \Phi_0^* \varpi$, and $P = P_0 - (\Phi_0^* T)_M$. Then*

$$P^\sharp \circ \omega^\flat = \mathrm{Id}_{TM} - \frac{1}{4}(e_a)_M \otimes \Phi^*(\theta^L_a - \theta^R_a). \tag{32}$$

Proof. Given $m \in M$, let U be a slice through $\Phi_0(m)$, and let $Y = \Phi_0^{-1}(U)$ be the corresponding cross-section. We first evaluate both sides of (32) on elements of $T_m Y$, and then on orbit directions. The 2-form $\Phi_0^* \varpi$ vanishes on $T_m Y$, and the bivector $(\Phi_0^* T)_M$ vanishes on $T_m^* Y$. Hence $P^\sharp \circ \omega^\flat|_{T_m Y} = P_0^\sharp \circ \omega_0^\flat|_{T_m Y} = \mathrm{Id}_{T_m Y}$, which agrees with the right-hand side of (32) since $\Phi^*(\theta^L_a - \theta^R_a)$ also vanishes on $T_m Y$. To complete the proof we evaluate both sides of (32) on orbit directions. The moment map properties of P and ω yield

$$\begin{aligned} P^\sharp(\omega^\flat(e_a)_M) &= \frac{1}{2}(\mathrm{Ad}_{\Phi^{-1}}+1)_{ab}P^\sharp(\Phi^*\theta_b^R) \\ &= \frac{1}{4}(\mathrm{Ad}_{\Phi^{-1}}+1)_{ab}(\mathrm{Ad}_\Phi+1)_{bc}(e_c)_M \\ &= \frac{1}{4}(2+\mathrm{Ad}_{\Phi^{-1}}+\mathrm{Ad}_\Phi)_{ab}(e_b)_M\ , \end{aligned}$$

and the same result is obtained by applying the right-hand side of (32) to $(e_a)_M$. □

Generalizing the Lemma, we can state the main result of this section:

Theorem 10.3. *Every non-degenerate Hamiltonian quasi-Poisson manifold (M, P, Φ) carries a unique 2-form ω such that (M, ω, Φ) is a quasi-Hamiltonian G-space, and such that ω and P satisfy Equation* (32)*. Conversely, on every quasi-Hamiltonian G-space (M, ω, Φ) there is a unique bivector field P such that (M, P, Φ) is a non-degenerate Hamiltonian quasi-Poisson G-manifold, and Equation* (32) *is satisfied.*

Proof. Given (M, P, Φ), let Y be a cross-section at m, as in the proof of Lemma 10.2, and let $g = \Phi(m)$. Thus (Y, P_Y, Φ_Y) is a non-degenerate Hamiltonian quasi-Poisson H-manifold that corresponds to a quasi-Hamiltonian H-space, (Y, ω_Y, Φ_Y). Let ω be the unique 2-form on $G.Y \subset M$ such that $(G.Y, \omega, \Phi)$ is a quasi-Hamiltonian G-space, and ω_Y is the pull-back of ω. We have to show that ω satisfies Equation (32). For orbit directions, this follows from the moment map conditions (see the proof of Lemma 10.2), while, for directions tangent to Y, it follows by applying the Lemma to $(Y, P_Y, g^{-1}\Phi_Y)$. Uniqueness is clear since the equation,

$$\omega^\flat \circ P^\sharp = \mathrm{Id}_{T^*M} - \frac{1}{4}\Phi^*(\theta_a^L - \theta_a^R) \otimes (e_a)_M\ , \tag{33}$$

the transpose of (32), defines $\omega^\flat$ on the image of $P^\sharp$, while the moment map condition determines $\omega^\flat$ on orbit directions.

The converse is proved similarly, using the cross-section theorem for Hamiltonian quasi-Poisson manifolds. □

By Theorem 10.3, all the constructions and examples for quasi-Hamiltonian G-spaces given in [2] can be translated into the quasi-Poisson picture.

Example 10.4. Let $\mathcal{C} \subset G$ be the conjugacy class of a point g. By Proposition 3.4, there exists a unique bivector P on $\mathcal{C}$ such that $(\mathcal{C}, P, \Phi)$, with $\Phi : \mathcal{C} \to G$ the embedding, is a Hamiltonian quasi-Poisson space. Similarly, by Proposition 3.1 of [2], there exists a unique 2-form ω on $\mathcal{C}$ such that $(\mathcal{C}, \omega, \Phi)$ is a quasi-Hamiltonian space. Theorem 10.3 implies that the bivector P and the 2-form ω are related by Equation (32).

Example 10.5. Let $D(G) = G \times G$ with bivector P given by formula (19), and with the 2-form ω defined in Section 3.2 of [2],

$$\omega = -\frac{1}{2}(\theta_a^{1,L} \wedge \theta_a^{2,R} + \theta_a^{1,R} \wedge \theta_a^{2,L}).$$

Both the left- and the right-hand sides of (33) yield the same expression,

$$\begin{aligned} \mathrm{Id}_{TD(G)} \quad &- \quad \frac{1}{4}\Big((\mathrm{Ad}_{a_2})_{ab}\, e_a^{1,L} \otimes \theta_b^{1,R} - (\mathrm{Ad}_{a_2^{-1}})_{ab}\, e_a^{1,R} \otimes \theta_b^{1,L} \\ &- \quad (\mathrm{Ad}_{a_1})_{ab}\, e_a^{2,L} \otimes \theta_b^{2,R} - (\mathrm{Ad}_{a_1^{-1}})_{ab}\, e_a^{2,R} \otimes \theta_b^{2,L}\Big)\ . \end{aligned}$$

Thus, the quasi-Poisson and quasi-Hamiltonian definitions of $D(G)$ agree.

In both the quasi-Poisson and the quasi-Hamiltonian settings there is a notion of reduction. We show that these notions agree as well.

Proposition 10.6. *Let (M, P, Φ) be a non-degenerate quasi-Poisson manifold and let ω be the corresponding 2-form on M. Then, for any conjugacy class $\mathcal{C} \subset G$, the intersection of the reduced space $M_{\mathcal{C}}$ with M_*/G carries a Poisson bivector $P^{\mathcal{C}}$ induced by P and a symplectic form $\omega^{\mathcal{C}}$ induced by ω, such that $(P^{\mathcal{C}})^\sharp \circ (\omega^{\mathcal{C}})^\flat = \mathrm{Id}$.*

Proof. Choose $g \in \mathcal{C}$ and a slice U containing g and let $Y = \Phi^{-1}(U)$ be the cross-section. We observe that in both the quasi-Poisson and the quasi-Hamiltonian settings, the reduction of M at $\mathcal{C}$ is canonically isomorphic to the reduction of Y at the group unit of $H = G_g$. The cross-section Y carries the Poisson bivector $P_{0,Y}$ and the symplectic form $\omega_{0,Y}$. According to Lemma 10.2, $(P_{0,Y})^\sharp \circ (\omega_{0,Y})^\flat = \mathrm{Id}_{TY}$. Hence, the same relation holds for the reduced space. □

Next, we will show that the fusion operation for quasi-Hamiltonian G-spaces given in [2] coincides with the fusion operation for quasi-Poisson spaces. Let $(M, \omega, (\Phi_1, \Phi_2))$ be a quasi-Hamiltonian $G \times G$-space, and let $(M, P, (\Phi_1, \Phi_2))$ be the corresponding Hamiltonian quasi-Poisson $G \times G$-space. By [2, Theorem 6.1], the space M with the diagonal G-action, the pointwise product of the moment map components $\Phi = \Phi_1\Phi_2$, and the fusion 2-form

$$\omega_{fus} = \omega - \frac{1}{2}\Phi_1^*\theta_a^L \wedge \Phi_2^*\theta_a^R\ ,$$

is a quasi-Hamiltonian G-space. On the other hand, Proposition 5.1 yields a bivector field $P_{fus} = P - \psi_M$, which, together with the diagonal G-action and the moment map $\Phi_1\Phi_2$, defines the structure of a Hamiltonian quasi-Poisson G-manifold on M. The following proposition asserts that, as expected, ω_{fus} corresponds to P_{fus} under the equivalence established in Theorem 10.3.

Proposition 10.7. *The bundle maps $P^\sharp_{fus} : T^*M \to TM$ and $\omega^\flat_{fus} : TM \to T^*M$ are related by*

$$P^\sharp_{fus} \circ \omega^\flat_{fus} = \mathrm{Id}_{TM} - \frac{1}{4}(e_a)_M \otimes \Phi^*(\theta_a^L - \theta_a^R), \tag{34}$$

where $(e_a)_M = (e_a)^1_M + (e_a)^2_M$ *are the vector fields generating the diagonal* G*-action on* M*.*

Proof. We have to compute

$$P^\sharp_{fus} \circ \omega^\flat_{fus} = \Big(P - \frac{1}{2}(e_a)^1_M \wedge (e_a)^2_M\Big)^\sharp \circ \Big(\omega - \frac{1}{2}\Phi_1^*\theta^L_a \wedge \Phi_2^*\theta^R_a\Big)^\flat.$$

We compute the four terms in the expansion of the right-hand side. By Theorem 10.3, the first term is

$$P^\sharp \circ \omega^\flat = \mathrm{Id}_{TM} - \frac{1}{4}(e_a)^1_M \otimes \Phi_1^*(\theta^L_a - \theta^R_a) - \frac{1}{4}(e_a)^2_M \otimes \Phi_2^*(\theta^L_a - \theta^R_a).$$

Next, using $\iota((e_a)^i_M)\omega = \frac{1}{2}\Phi_i^*(\theta^L_a + \theta^R_a)$, $i = 1, 2$, we find that

$$\frac{1}{2}\big((e_a)^1_M \wedge (e_a)^2_M\big)^\sharp \circ \omega^\flat = \frac{1}{4}(e_a)^1_M \otimes \Phi_2^*(\theta^L_a + \theta^R_a) - \frac{1}{4}(e_a)^2_M \otimes \Phi_1^*(\theta^L_a + \theta^R_a).$$

From $P^\sharp(\Phi_i^*\theta^L_a) = \frac{1}{2}(1 + \mathrm{Ad}_{\Phi_i})_{ab}(e_b)^i_M$, $i = 1, 2$, we obtain

$$\frac{1}{2}P^\sharp \circ \big(\Phi_1^*\theta^L_a \wedge \Phi_2^*\theta^R_a\big)^\flat = \frac{1}{4}(1 + \mathrm{Ad}_{\Phi_2^{-1}})_{ab}(e_a)^2_M \otimes \Phi_1^*\theta^L_b - \frac{1}{4}(1 + \mathrm{Ad}_{\Phi_1})_{ab}(e_a)^1_M \otimes \Phi_2^*\theta^R_b\ .$$

Finally,

$$\begin{aligned} \frac{1}{4}((e_a)^1_M \wedge (e_a)^2_M)^\sharp \quad &\circ \quad (\Phi_1^*\theta^L_b \wedge \Phi_2^*\theta^R_b)^\flat \\ &= \quad \frac{1}{4}(e_a)^1_M \otimes (1 - \mathrm{Ad}_{\Phi_2^{-1}})_{ab}\Phi_1^*\theta^L_b + \frac{1}{4}(e_a)^2_M \otimes (\mathrm{Ad}_{\Phi_1} - 1)_{ab}\Phi_2^*\theta^R_b. \end{aligned}$$

Putting everything together, we get

$$\begin{aligned} P^\sharp_{fus} \circ \omega^\flat_{fus} &= \mathrm{Id}_{TM} - \frac{1}{4}(e_a)_M \otimes (\Phi_2^*\theta^L_a + (\mathrm{Ad}_{\Phi_2^{-1}})_{ab}\Phi_1^*\theta^L_b - \Phi_1^*\theta^R_a - (\mathrm{Ad}_{\Phi_1})_{ab}\Phi_2^*\theta^R_b) \\ &= \mathrm{Id}_{TM} - \frac{1}{4}(e_a)_M \otimes (\Phi_1\Phi_2)^*(\theta^L_a - \theta^R_a)\,, \end{aligned}$$

as required. □

Example 10.8. Consider

$$M = \underbrace{\mathbf{D}(G) \circledast \ldots \circledast \mathbf{D}(G)}_{h} \circledast \mathcal{C}_1 \circledast \ldots \circledast \mathcal{C}_r,$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are conjugacy classes in G. This space can be viewed either as a quasi-Hamiltonian space (see Section 9 of [2]), or as a quasi-Poisson manifold (see Section 6). Proposition 10.7 together with Examples 10.4 and 10.5 show that these two structures agree in the sense of Theorem 10.3. According to Theorem 9.3 of [2], the reductions of M are isomorphic to the moduli spaces of flat connections with the Atiyah-Bott symplectic form. Proposition 10.6 implies that the quasi-Poisson reduction yields the Poisson bivector inverse to the canonical symplectic form.

Appendix A. The formal Poisson structure of $L\mathfrak{g}^*$

In this appendix, we show that the quasi-Poisson structure P_G on G defined by Equation (12) can be viewed as a quotient of a formal Poisson structure on $L\mathfrak{g}^*$, the dual of the loop algebra $L\mathfrak{g}$ of $\mathfrak{g}$.

Let LG be the loop group of G, with Lie algebra $L\mathfrak{g}$. The inner product defines a central extension $\widehat{L\mathfrak{g}}$ of $L\mathfrak{g}$, with dual $\widehat{L\mathfrak{g}}^* \cong \Omega^1(S^1, \mathfrak{g}) \times \mathbb{R}$. The adjoint action of LG on $\widehat{L\mathfrak{g}}$ dualizes to the following action on $\widehat{L\mathfrak{g}}^*$,

$$g.(\mu, \lambda) = (\mathrm{Ad}_g(\mu) - \lambda g^*\theta^R, \lambda), \quad g \in LG, \mu \in \Omega^1(S^1, \mathfrak{g}),\ \lambda \in \mathbb{R}.$$

In particular, the affine action on $L\mathfrak{g}^* := \Omega^1(S^1, \mathfrak{g}) \times \{1\}$ is identified with the gauge action of LG on G-connections over S^1. Let $L\mathfrak{g}^*$ be equipped with the formal linear Poisson bivector obtained by restriction from that on $\widehat{L\mathfrak{g}}^*$. The action of the based loop group $\Omega G \subset LG$ on $L\mathfrak{g}^*$ is free, with quotient $L\mathfrak{g}^*/\Omega G = G$. We shall show that, under the quotient map, the formal Poisson structure on $L\mathfrak{g}^*$ projects to the bivector (12) on G.

In the finite-dimensional case, the restriction of the linear Poisson structure $P_{0,\mathfrak{g}^*}$ of $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subset G$ can be written in terms of the corresponding root space decomposition. Let $\mathfrak{R}$ be the set of roots of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and for any $\alpha \in \mathfrak{R}$, let e_α be a root vector of unit length, such that $e_{-\alpha}$ is the complex conjugate of e_α and $\langle e_\alpha, e_{-\alpha} \rangle = 1$, where $\langle\ ,\ \rangle$ is the Killing form. Then, if $\mu \in \mathfrak{t}$ with $\langle \alpha, \mu \rangle \neq 0$ for all roots α, the value of $P_{0,\mathfrak{g}^*}$ at μ is

$$P_{0,\mathfrak{g}^*} = \sum_{\alpha \in \mathfrak{R}_+} \frac{1}{2\pi i \langle \alpha, \mu \rangle} (e_\alpha)_{\mathfrak{g}^*} \wedge (e_{-\alpha})_{\mathfrak{g}^*}\ .$$

In the case of loop algebras, for any $\xi \in \mathfrak{g}^{\mathbb{C}}$ and $k \in \mathbb{Z}$, we denote by $\xi[k] \in L\mathfrak{g}^{\mathbb{C}}$ the loop defined by

$$\xi[k](e^{2\pi i s}) = e^{2\pi i k s}\xi.$$

Then the root vectors for $\widehat{L\mathfrak{g}}$ are all $e_\alpha[k]$, with corresponding affine roots (α, k), together with vectors $h_j[k]$ for $k \neq 0$, with affine roots $(0, k)$, where (h_j) is an orthonormal basis for $\mathfrak{t}$. From the identification of $L\mathfrak{g}^*$ with the hyperplane at level 1 in $\widehat{L\mathfrak{g}}^*$, we obtain the following formal expression for the Poisson structure on $L\mathfrak{g}^*$ at a constant loop $\mu \in \mathfrak{t} \subset L\mathfrak{g}^*$ with $\langle \alpha, \mu \rangle \notin \mathbb{Z}$, for all roots α,

$$\begin{aligned} P_{0,L\mathfrak{g}^*} &= \sum_{\alpha \in \mathfrak{R}_+} \sum_{k \in \mathbb{Z}} \frac{1}{2\pi i(\langle \alpha, \mu \rangle + k)} (e_\alpha[k])_{L\mathfrak{g}^*} \wedge (e_{-\alpha}[k])_{L\mathfrak{g}^*} \\ &+ \sum_j \sum_{k>0} \frac{1}{2\pi i k} (h_j[k])_{L\mathfrak{g}^*} \wedge (h_j[-k])_{L\mathfrak{g}^*}\ . \end{aligned}$$

In a finite-dimensional manifold, any Poisson structure which is invariant under a free group action reduces to a Poisson structure on the quotient. Formally, we can carry out this calculation for the free action of ΩG on $L\mathfrak{g}^*$. The quotient map takes a constant loop $\mu \in \mathfrak{t} \subset L\mathfrak{g}^*$ to the element $\exp\mu \in T \subset G$, and the corresponding tangent map takes the value of $(\xi[k])_{L\mathfrak{g}^*}$ at μ to the value of ξ_G at $\exp\mu$. Applying the tangent of the quotient map to the Poisson bivector of $L\mathfrak{g}^*$, we obtain

$$\sum_{\alpha\in\mathfrak{R}_+}\sum_{k\in\mathbb{Z}} \frac{1}{2\pi i(\langle\alpha,\mu\rangle + k)}(e_\alpha)_G \wedge (e_{-\alpha})_G\,.$$

Using the formula

$$\lim_{N\to\infty}\sum_{|k|\le N}\frac{1}{2\pi(x+k)} = \frac{1}{2}\cot(\pi x)$$

for $x \notin \mathbb{Z}$, we obtain

$$\frac{1}{2i}\sum_{\alpha\in\mathfrak{R}_+}\cot(\pi\langle\alpha,\mu\rangle)(e_\alpha)_G \wedge (e_{-\alpha})_G\ ,$$

which coincides with the bivector P_G defined by Equation (12) (see Equation (16)). An alternative procedure for projecting the bivector of $L\mathfrak{g}^*$ to G was considered in [7].

Appendix B. A generalized dynamical r-matrix

Let $r^{\mathfrak{g}/\mathfrak{h}} \in C^\infty(U, \wedge^2\mathfrak{h}^\perp)$ denote the r-matrix defined by Equation (29). We shall now prove Lemma 8.2. Let $T \subset H$ be a maximal torus, and on $T\cap U$ define

$$r^{\mathfrak{g}/\mathfrak{t}}(h) = -\frac{1}{2}\Big(\frac{\mathrm{Ad}_h+1}{\mathrm{Ad}_h-1}\Big|\mathfrak{t}^\perp\Big)_{ab} e_a\wedge e_b$$

and

$$r^{\mathfrak{h}/\mathfrak{t}}(h) = -\frac{1}{2}\Big(\frac{\mathrm{Ad}_h+1}{\mathrm{Ad}_h-1}\Big|\mathfrak{h}\cap\mathfrak{t}^\perp\Big)_{ab} e_a\wedge e_b\ .$$

Then $r^{\mathfrak{g}/\mathfrak{h}}|_{T\cap U} = r^{\mathfrak{g}/\mathfrak{t}} - r^{\mathfrak{h}/\mathfrak{t}}$. Our starting point will be the classical dynamical Yang-Baxter equations satisfied by $r^{\mathfrak{g}/\mathfrak{t}}$ and $r^{\mathfrak{h}/\mathfrak{t}}$ (see [3], Lemma A.5):

$$\mathrm{Cycl}_{abc}\Big(\frac{1}{2}(e^L_a+e^R_a)_{\mathfrak{t}} r^{\mathfrak{g}/\mathfrak{t}}_{bc} + r^{\mathfrak{g}/\mathfrak{t}}_{ak} f_{kbl} r^{\mathfrak{g}/\mathfrak{t}}_{lc}\Big) = \frac{1}{4} f_{abc}$$

and

$$\mathrm{Cycl}_{abc}\Big(\frac{1}{2}(e^L_a+e^R_a)_{\mathfrak{t}} r^{\mathfrak{h}/\mathfrak{t}}_{bc} + r^{\mathfrak{h}/\mathfrak{t}}_{ak} f^{\mathfrak{h}}_{kbl} r^{\mathfrak{h}/\mathfrak{t}}_{lc}\Big) = \frac{1}{4} f^{\mathfrak{h}}_{abc}.$$

Using the fact that $r^{\mathfrak{h}/\mathfrak{t}}_{ak} f^{\mathfrak{h}}_{kbl} r^{\mathfrak{h}/\mathfrak{t}}_{lc} = r^{\mathfrak{h}/\mathfrak{t}}_{ak} f_{kbl} r^{\mathfrak{h}/\mathfrak{t}}_{lc}$, upon subtracting the second equation from the first, we obtain

$$\mathrm{Cycl}_{abc}\Big(\frac{1}{2}(e^L_a+e^R_a)_{\mathfrak{t}} r^{\mathfrak{g}/\mathfrak{h}}_{bc} + r^{\mathfrak{g}/\mathfrak{h}}_{ak} f_{kbl} r^{\mathfrak{g}/\mathfrak{h}}_{lc} + 2 r^{\mathfrak{h}/\mathfrak{t}}_{ak} f_{kbl} r^{\mathfrak{g}/\mathfrak{h}}_{lc}\Big) = \frac{1}{4}(f_{abc} - f^{\mathfrak{h}}_{abc}). \tag{35}$$

To prove the Lemma, we need to evaluate $\mathrm{Cycl}_{abc}\big(\frac{1}{2}(e^L_a+e^R_a)_{\mathfrak{h}/\mathfrak{t}} r^{\mathfrak{g}/\mathfrak{h}}_{bc}\big)$, where the subscript $\mathfrak{h}/\mathfrak{t}$ denotes the projection of the vector field along $T\cap U$ onto the normal bundle of $T\cap U$

in U which is isomorphic to $(T \cap U) \times (\mathfrak{t}^\perp \cap \mathfrak{h})$. This projection can be expressed in terms of r itself and the vector fields generating the G-action as

$$\frac{1}{2}(e_a^L + e_a^R)_{\mathfrak{h}/\mathfrak{t}} = r_{ak}^{\mathfrak{h}/\mathfrak{t}}(e_k)_G.$$

By the H-invariance of $r^{\mathfrak{g}/\mathfrak{h}}$, this relation shows that

$$\mathrm{Cycl}_{abc}\left(\frac{1}{2}(e_a^L + e_a^R)_{\mathfrak{h}/\mathfrak{t}} r_{bc}^{\mathfrak{g}/\mathfrak{h}} - 2 r_{ak}^{\mathfrak{h}/\mathfrak{t}} f_{kbl} r_{lc}^{\mathfrak{g}/\mathfrak{h}}\right) = 0. \tag{36}$$

Adding Equation (35) to (36), we obtain Equation (30), proving the Lemma.

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