THE CUBIC DIRAC OPERATOR FOR INFINITE-DIMENSONAL LIE ALGEBRAS

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ABSTRACT. Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be an infinite-dimensional graded Lie algebra, with $\dim \mathfrak{g}_i < \infty$, equipped with a non-degenerate symmetric bilinear form B of degree 0. The quantum Weil algebra $\widehat{\mathcal{W}}\mathfrak{g}$ is a completion of the tensor product of the enveloping and Clifford algebras of $\mathfrak{g}$. Provided that the Kac-Peterson class of $\mathfrak{g}$ vanishes, one can construct a cubic Dirac operator $\mathcal{D} \in \widehat{\mathcal{W}}(\mathfrak{g})$, whose square is a quadratic Casimir element. We show that this condition holds for symmetrizable Kac-Moody algebras. Extending Kostant's arguments, one obtains generalized Weyl-Kac character formulas for suitable 'equal rank' Lie subalgebras of Kac-Moody algebras. These extend the formulas of G. Landweber for affine Lie algebras.

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0. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra, equipped with a non-degenerate invariant symmetric bilinear form B. For $\xi \in \mathfrak{g}$, the corresponding generators of the enveloping algebra $U(\mathfrak{g})$ are denoted $s(\xi)$, while those of the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ are denoted simply by ξ. The *quantum Weil algebra* [1] is the super algebra

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}),$$

with even generators $s(\xi)$ and odd generators ξ. Let $\mathcal{D} \in \mathcal{W}(\mathfrak{g})$ be the odd element, written in terms of a basis e_a of $\mathfrak{g}$ as

$$\mathcal{D} = \sum_a s(e_a)e^a - \frac{1}{12}\sum_{abc} f_{abc}e^a e^b e^c,$$

where e^a is the B-dual basis and f_{abc} are the structure constants. The key property of this element is that its square lies in the center of $\mathcal{W}(\mathfrak{g})$:

$$\mathcal{D}^2 = \mathrm{Cas}_\mathfrak{g} + \frac{1}{24}\,\mathrm{tr}_\mathfrak{g}(\mathrm{Cas}_\mathfrak{g}), \tag{1}$$

where $\mathrm{Cas}_\mathfrak{g} = \sum_a s(e_a)s(e^a) \in U(\mathfrak{g})$ is the quadratic Casimir element. The element $\mathcal{D}$ is called the *cubic Dirac operator*, following Kostant [10]. More generally, Kostant introduced cubic Dirac operators $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ for pairs of a quadratic Lie algebra $\mathfrak{g}$ and a quadratic Lie subalgebra $\mathfrak{u}$. For $\mathfrak{g}$ semi-simple and $\mathfrak{u}$ an equal rank subalgebra, he used this to prove, among other things, generalizations of the Bott-Borel-Weil theorem and of the Weyl character formula (see also [2, 11]).

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In this article, we will consider generalizations of this theory to infinite-dimensional Lie algebras. We assume that $\mathfrak{g}$ is $\mathbb{Z}$-graded, with finite dimensional graded pieces $\mathfrak{g}_i$, and equipped with a non-degenerate invariant symmetric bilinear form B of degree 0. A priori, the formal expressions defining $\mathcal{D}$, $\mathrm{Cas}_\mathfrak{g}$ are undefined since they involve infinite sums. It is possible to replace these expressions with 'normal-ordered' sums, leading to well-defined elements $\mathcal{D}', \mathrm{Cas}'_\mathfrak{g}$ in suitable completion of $\mathcal{W}(\mathfrak{g})$. However, it is no longer true in general that $(\mathcal{D}')^2 - \mathrm{Cas}'_\mathfrak{g}$ is a constant, and in any case $\mathrm{Cas}'_\mathfrak{g}$ is not a central element. One may attempt to define elements $\mathcal{D}$, $\mathrm{Cas}_\mathfrak{g}$ having these properties by adding lower order correction terms to $\mathcal{D}', \mathrm{Cas}'_\mathfrak{g}$. Our main observation is that this is possible if and only the *Kac-Peterson class* $[\psi_{KP}] \in H^2(\mathfrak{g})$ is zero. In fact, given $\rho \in \mathfrak{g}_0^*$ with $\psi_{KP} = \mathrm{d}\rho$, the elements $\mathcal{D} = \mathcal{D}' + \rho$ and $\mathrm{Cas}_\mathfrak{g} = \mathrm{Cas}'_\mathfrak{g} + 2\rho$ have the desired properties. These results are motivated by the work of Kostant-Sternberg [12], who had exhibited the Kac-Peterson class as an obstruction class in their BRST quantization scheme.

For symmetrizable Kac-Moody algebras, the existence of a corrected Casimir element $\mathrm{Cas}_\mathfrak{g}$ is a famous result of Kac [4]. In particular, $[\psi_{KP}] = 0$ in this case. As we will see, Kostant's theory carries over to the symmetrizable Kac-Moody case in a fairly straightforward manner. For suitable 'regular' Kac-Moody subalgebras $\mathfrak{u} \subset \mathfrak{g}$, we thus obtain generalized Weyl-Kac character formulas as sums over multiplets of $\mathfrak{u}$-representations.

For *affine* Lie algebras or loop algebras, similar Dirac operators were described in Kac-Todorov [7] and Kazama-Suzuki [8], and more explicitly in Landweber [14] and Wassermann [19]. In fact, Wassermann uses this Dirac operator to give a proof of the Weyl-Kac character formula for affine Lie algebras, while Landweber proves generalized Weyl character formulas for 'equal rank loop algebras'. The cubic Dirac operator $\mathcal{D}$ for general symmetrizable Kac-Moody algebras is very briefly discussed in Kitchloo [9].

1. Completions

In this Section we will define completions of the exterior and Clifford algebras of a graded quadratic vector space. We recall from [6] how the Kac-Peterson cocycle appears in this context.

1.1. **Kac-Peterson cocycle.** Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be a $\mathbb{Z}$-graded vector space over $\mathbb{C}$, with finite-dimensional graded components. The (graded) dual space is the direct sum over the duals of V_i, with grading $(V^*)_i = (V_{-i})^*$. Given another graded vector space V' with $\dim V'_i < \infty$, we let $\mathrm{Hom}(V, V')$ be the direct sum over the spaces $\mathrm{Hom}(V, V')_i = \bigoplus_r \mathrm{Hom}(V_r, V'_{r+i})$ of finite rank maps of degree i. We let

$$\widehat{\mathrm{Hom}}(V, V')_i = \prod_r \mathrm{Hom}(V_r, V'_{r+i})$$

be the space of *all* linear maps $V \to V'$ of degree i, and $\widehat{\mathrm{Hom}}(V, V')$ their direct sum. If $V = V'$ we write $\mathrm{End}(V) = \mathrm{Hom}(V, V)$ and $\widehat{\mathrm{End}}(V) = \widehat{\mathrm{Hom}}(V, V)$. Note that $\widehat{\mathrm{End}}(V)$ is an algebra with unit I.

Define a splitting $V = V_- \oplus V_+$ where $V_+ = \bigoplus_{i>0} V_i$, $V_- = \bigoplus_{i\le 0} V_i$. Denote by π_-, π_+ the projections to the two summands. The *Kac-Peterson cocycle* ([6]; see also [5, Exercise

7.28]) on $\widehat{\mathrm{End}}(V)$ is a Lie algebra cocycle given by the formula,

$$\psi_{KP}(A_1, A_2) = \tfrac{1}{2}\,\mathrm{tr}(A_1\pi_- A_2\pi_+) - \tfrac{1}{2}\,\mathrm{tr}(A_2\pi_- A_1\pi_+). \tag{2}$$

This is well-defined since the compositions $\pi_- A_i\pi_+ \colon V \to V$ have finite rank. Observe that ψ_{KP} has degree 0, that is, (2) vanishes unless the degrees of A_1, A_2 add to zero. On the Lie subalgebra $\mathrm{End}(V) \subset \widehat{\mathrm{End}}(V)$, the Kac-Peterson cocycle restricts to a coboundary:

$$\psi_{KP}(A_1, A_2) = \tfrac{1}{2}\,\mathrm{tr}(\pi_+[A_1, A_2]). \tag{3}$$

1.2. **Completion of symmetric and exterior algebras.** Let $S(V)$ be the symmetric algebra of V, with $\mathbb{Z}$-grading defined by assigning degree i to generators in V_i. Let V^* be the graded dual as above. The pairing between $S(V)$ and $S(V^*)$ identifies $S(V)_i$ as a subspace of the space of linear maps $S(V^*)_{-i} \to \mathbb{K}$. We define a completion $\widehat{S}(V)_i$ as the space of all linear maps $S(V^*)_{-i} \to \mathbb{K}$. Equivalently,

$$\widehat{S}(V)_i = \prod_{r\geq 0} S(V_-)_{i-r} \otimes S(V_+)_r.$$

We let $\widehat{S}(V)$ be the direct sum over the $\widehat{S}(V)_i$. The multiplication map of $S(V)$ extends to the completion, making $\widehat{S}(V)$ into a $\mathbb{Z}$-graded algebra. For each $k \geq 0$ one similarly has a completion $\widehat{S}^k(V) \subset \widehat{S}(V)$ of each component $S^k(V)$. Then $\widehat{S}(V)_i$ is the direct product over all $\widehat{S}^k(V)_i$. The space $\widehat{S}^2(V^*)_0$ may be identified with the space of symmetric bilinear maps $B \colon V \times V \to \mathbb{C}$ of degree 0, that is $B(V_i, V_j) = 0$ for $i + j \neq 0$.

In a similar fashion, one defines a completions $\widehat{\wedge}(V)_i$ as the spaces of all linear maps $\widehat{\wedge}(V^*)_{-i} \to \mathbb{K}$, or equivalently

$$\widehat{\wedge}(V)_i = \prod_{r\geq 0} \wedge(V_-)_{i-r} \otimes \wedge(V_+)_r.$$

We let $\widehat{\wedge}(V)$ be the $\mathbb{Z}$-graded super algebra given as the direct sum over all $\widehat{\wedge}(V)_i$. Again, one also has completions of the individual $\wedge^k(V)$. The space $\widehat{\wedge}^2(V^*)_0$ may be identified with the skew-symmetric bilinear maps $V \times V \to \mathbb{C}$ of degree 0. In particular:

$$\psi_{KP} \in \widehat{\wedge}^2(\widehat{\mathrm{End}}(V)^*)_0.$$

1.3. **Clifford algebras.** Suppose B is a (possibly degenerate) symmetric bilinear form on $V = \bigoplus_i V_i$ of degree 0. Let $\mathrm{Cl}(V)$ be the corresponding Clifford algebra, i.e. the super algebra with odd generators $v \in V$ and relations $vw + wv = 2\,B(v, w)$ for $v, w \in V$. The $\mathbb{Z}$-grading on V defines a $\mathbb{Z}$-grading on $\mathrm{Cl}(V)$, compatible with the algebra structure.

Using the restrictions of the bilinear form to $V_\pm$, we may similarly form the Clifford algebras $\mathrm{Cl}(V_\pm)$. These are $\mathbb{Z}$-graded subalgebras of $\mathrm{Cl}(V)$, and the multiplication map defines an isomorphism of super vector spaces, $\mathrm{Cl}(V) \cong \mathrm{Cl}(V_-) \otimes \mathrm{Cl}(V_+)$. Note that $\mathrm{Cl}(V_+) = \wedge(V_+)$ since B restricts to 0 on V_+.

We obtain a $\mathbb{Z}$-graded superalgebra $\widehat{\mathrm{Cl}}(V)$ as the direct sum over all

$$\widehat{\mathrm{Cl}}(V)_i = \prod_{r\geq 0} \mathrm{Cl}(V_-)_{i-r} \otimes \mathrm{Cl}(V_+)_r.$$

Let $q^0\colon \wedge(V) \to \mathrm{Cl}(V)$ denote the standard quantization map for the Clifford algebra, defined by super symmetrization:

$$q^0(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)},$$

where $\mathfrak{S}_k$ is the permutation group on k elements, and $\mathrm{sign}(\sigma) = \pm 1$ is the parity of the permutation σ. The map q^0 is an isomorphism of super spaces, preserving the $\mathbb{Z}$-gradings and taking $\wedge(V_\pm)$ to $\mathrm{Cl}(V_\pm)$. While q^0 itself does not extend to the completions, we obtain a well-defined *normal-ordered quantization map*

$$q\colon \widehat{\wedge}(V) \to \widehat{\mathrm{Cl}}(V)$$

by taking the direct sum over $i \in \mathbb{Z}$ and direct product over $r \geq 0$ of

$$q^0 \otimes q^0\colon \wedge(V_-)_{i-r} \otimes \wedge(V_+)_r \to \mathrm{Cl}(V_-)_{i-r} \otimes \mathrm{Cl}(V_+)_r.$$

The quantization map is an isomorphism of $\mathbb{Z}$-graded super vector spaces, with the property that for $\lambda \in \widehat{\wedge}^k(V),\ \mu \in \widehat{\wedge}^l(V)$,

$$q^{-1}(q(\lambda)q(\mu)) = \lambda \wedge \mu \mod \widehat{\wedge}^{k+l-2}(V).$$

Any element $v \in V$ defines an odd derivation ι_v, called *contraction*, of the super algebra $\wedge(V)$, given on generators by $\iota_v(w) = B(v, w)$. The same formula also defines a derivation of the Clifford algebra, again denoted ι_v. In both cases, the contractions extend to the completions. The map $q\colon \widehat{\wedge}(V) \to \widehat{\mathrm{Cl}}(V)$ intertwines contractions:

$$q \circ \iota_v = \iota_v \circ q,$$

since $q^0 \circ \iota_v = \iota_v \circ q^0$ and since contractions preserve $\wedge(V_\pm)$ and $\mathrm{Cl}(V_\pm)$.

Let $\mathfrak{o}(V) \subset \mathrm{End}(V)$ and $\widehat{\mathfrak{o}}(V) \subset \widehat{\mathrm{End}}(V)$ denote the B-skew-symmetric endomorphisms. Let

$$\widehat{\wedge}^2(V) \to \widehat{\mathfrak{o}}(V), \quad \lambda \mapsto A_\lambda \tag{4}$$

be the map defined by $A_\lambda(v) = -2\iota_v\lambda$. The map (4) is $\widehat{\mathfrak{o}}(V)$-equivariant, that is,

$$A_{L_X\lambda} = [X, A_\lambda]$$

for $X \in \widehat{\mathfrak{o}}(V)$.

Lemma 1.1. *For all $\lambda \in \wedge^2(V)$,*

$$q(\lambda) = q^0(\lambda) - \tfrac{1}{2}\,\mathrm{tr}(\pi_+ A_\lambda). \tag{5}$$

Proof. It suffices to check for elements of the form $\lambda = u \wedge v$ for $u, v \in V$. We have $A_{u\wedge v}(w) = 2(B(v, w)u - B(u, w)v)$, hence $\mathrm{tr}(\pi_+ A_{u\wedge v}) = 2(B(\pi_+ u, v) - B(\pi_+ v, u))$. On the other hand, by considering the special cases that u, v are both in V_-, both in V_+, or $u \in V_-, v \in V_+$ we find

$$q(u \wedge v) = q^0(u \wedge v) + B(\pi_+ v, u) - B(\pi_+ u, v). \tag{6}$$

□

The map q^0 is $\mathfrak{o}(V)$-equivariant. For the normal-ordered quantization map this is no longer the case.

Proposition 1.2 (Kac-Peterson). [6] *For all* $\lambda \in \widehat{\wedge}^2(V)$ *and* $X \in \widehat{\mathfrak{o}}(V)$, *one has*

$$L_X q(\lambda) = q(L_X \lambda) + \psi_{KP}(X, A_\lambda).$$

Proof. It is enough to prove this for $X \in \mathfrak{o}(V)$ and $\lambda \in \wedge^2(V)$. Since q^0 intertwines Lie derivatives, Lemma 1.1 together with (3) give

$$L_X q(\lambda) - q(L_X \lambda) = \tfrac{1}{2}\operatorname{tr}(\pi_+ A_{L_X\lambda}) = \tfrac{1}{2}\operatorname{tr}(\pi_+[X, A_\lambda]) = \psi_{KP}(X, A_\lambda). \qquad \square$$

If B is non-degenerate, the map $\lambda \mapsto A_\lambda$ defines an isomorphism $\wedge^2(V) \to \mathfrak{o}(V)$. Let

$$\lambda\colon \mathfrak{o}(V) \to \wedge^2(V),\ A \mapsto \lambda(A)$$

be the inverse map. It extends to a map $\widehat{\mathfrak{o}}(V) \to \widehat{\wedge}^2(V)$ of the completions. In a basis e_a of V, with B-dual basis e^a (i.e. $B(e_a, e^b) = \delta_a^b$), one has

$$\lambda(A) = \frac{1}{4}\sum_a A(e_a) \wedge e^a.$$

If $A \in \mathfrak{o}(V)$, the elements $\gamma^0(A) = q^0(\lambda(A))$ are defined. As is well-known, $[\gamma^0(A_1), \gamma^0(A_2)] = \gamma^0([A_1, A_2])$ for $A_i \in \mathfrak{o}(V)$, and

$$L_A = [\gamma^0(A), \cdot].$$

If $A \in \widehat{\mathfrak{o}}(V)$, one still has $L_A = [\gamma'(A), \cdot]$ with

$$\gamma'(A) = q(\lambda(A)),$$

but the map γ' is no longer a Lie algebra homomorphism. Instead, Proposition 1.2 shows [6]

$$[\gamma'(A_1), \gamma'(A_2)] = \gamma'([A_1, A_2]) + \psi_{KP}(A_1, A_2) \tag{7}$$

for $A_1, A_2 \in \widehat{\mathfrak{o}}(V)$.

2. Graded Lie algebras

We will now specialize to the case that $V = \mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra. We show that in the quadratic case, the obstruction to defining a reasonable 'Casimir operator' is precisely the Kac-Peterson class of $\mathfrak{g}$.

2.1. **Kac-Peterson cocycle of $\mathfrak{g}$.** Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a graded Lie algebra, with $\dim \mathfrak{g}_i < \infty$. That is, we assume that the grading is compatible with the bracket: $[\mathfrak{g}_i, \mathfrak{g}_j]_\mathfrak{g} \subset \mathfrak{g}_{i+j}$. The map $\operatorname{ad}_\xi\colon \mathfrak{g} \to \mathfrak{g}$ defines a homomorphism of graded Lie algebras

$$\operatorname{ad}\colon \mathfrak{g} \to \widehat{\operatorname{End}}(\mathfrak{g}).$$

Recall that $\mathfrak{g}^* = \bigoplus_i (\mathfrak{g}^*)_i$ denotes the restricted dual where $(\mathfrak{g}^*)_i = (\mathfrak{g}_{-i})^*$. The algebra $\wedge(\mathfrak{g}^*)$ carries contraction operators and Lie derivatives $\iota_\xi,\ L_\xi$ for $\xi \in \mathfrak{g}$, given on generators by $\iota_\xi \mu = \langle \mu, \xi\rangle$ and $L_\xi \mu = (-\operatorname{ad}_\xi)^* \mu$. If $\dim \mathfrak{g} < \infty$ it also carries a differential d, given on generators by

$$\mathrm{d}\mu = 2\lambda(\mu)$$

where $\lambda(\mu)$ is defined by $\iota_\xi \lambda(\mu) = \frac{1}{2} L_\xi \mu$. On generators,

$$(\mathrm{d}\mu)(\xi_1, \xi_2) = -\langle \mu, [\xi_1, \xi_2]_\mathfrak{g}\rangle.$$

In the infinite-dimensional case, $\lambda(\mu)$ and hence d are well-defined on the completion $\widehat{\wedge}(\mathfrak{g}^*)$. The operators $\iota_\xi, L_\xi, \mathrm{d}$ make $\widehat{\wedge}(\mathfrak{g}^*)$ into a $\mathfrak{g}$-differential algebra.

Define

$$\psi_{KP}(\xi_1, \xi_2) := \psi_{KP}(\mathrm{ad}_{\xi_1}, \mathrm{ad}_{\xi_2})$$

for $\xi_i \in \mathfrak{g}$. Thus $\psi_{KP} \in \widehat{\wedge}^2(\mathfrak{g}^*)_0$ is a degree 2 Lie algebra cocycle of $\mathfrak{g}$, called the *Kac-Peterson cocycle of* $\mathfrak{g}$. Its class $[\psi_{KP}] \in H^2(\mathfrak{g})$ will be called the Kac-Peterson class of the graded Lie algebra $\mathfrak{g}$. Note that d has $\mathbb{Z}$-degree 0, so that it restricts to a differential on each $\widehat{\wedge}(\mathfrak{g}^*)_i$. Hence, if ψ_{KP} admits a primitive in $\mathfrak{g}^*$, then it admits a primitive in $\mathfrak{g}_0^*$.

Example 2.1. [6] Suppose $\mathfrak{k}$ is a finite-dimensional Lie algebra, and let $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$ the loop algebra with its usual $\mathbb{Z}$-grading. Let $B^{\mathrm{Kil}}(x, y) = \mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}_x \, \mathrm{ad}_y)$ for $x, y \in \mathfrak{k}$ be the Killing form on $\mathfrak{k}$. One finds

$$\psi_{KP}(\xi, \zeta) = \mathrm{Res}\ B^{\mathrm{Kil}}\Big(\frac{\partial \xi}{\partial z}, \zeta\Big)$$

for $\xi, \zeta \in \mathfrak{k}[z, z^{-1}]$, where Res picks out the coefficient of z^{-1}. One may check that unless $B\mathrm{Kil} = 0$, the Kac-Peterson class $[\psi_{KP}]$ is non-zero.

Example 2.2 (Heisenberg algebra). Let $\mathfrak{g}$ be the Lie algebra with basis $K, e_1, f_1, e_2, f_2, \ldots$, where K is a central element and $[e_i, f_j]_{\mathfrak{g}} = \delta_{ij} K$. Define a grading on $\mathfrak{g}$ such that e_i has degree i and f_i has degree $-i$, while K has degree 0. One finds $\psi_{KP} = 0$.

Example 2.3. Suppose $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}$ and a system $\Delta^+ \subset \mathfrak{h}^*$ of positive roots. Let $\mathfrak{g}$ carry the principal grading, i.e. $\mathfrak{g}_0 = \mathfrak{h}$ while $\mathfrak{g}_i, i \neq 0$ is the direct sum of root spaces for roots of height i. Using (3) one finds that $\psi_{KP} = \mathrm{d}\rho$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

2.2. **Enveloping algebras.** The $\mathbb{Z}$-grading on $\mathfrak{g}$ defines a $\mathbb{Z}$-grading on the enveloping algebra $U(\mathfrak{g})$. Both $\mathfrak{g}_+ = \bigoplus_{i>0} \mathfrak{g}_i$ and $\mathfrak{g}_- = \bigoplus_{i \leq 0} \mathfrak{g}_i$ are graded Lie subalgebras, thus $U(\mathfrak{g}_\pm)$ are graded subalgebras of $U(\mathfrak{g})$. By the Poincaré-Birkhoff-Witt theorem, the multiplication map defines an isomorphism of vector spaces, $U(\mathfrak{g}) = U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+)$. We define a completion $\widehat{U}(\mathfrak{g})$ as a direct sum over

$$\widehat{U}(\mathfrak{g})_i = \prod_{r \geq 0} U(\mathfrak{g}_-)_{i-r} \otimes U(\mathfrak{g}_+)_r.$$

The multiplication map extends to the completion, making $\widehat{U}(\mathfrak{g})$ into a graded algebra. Let $q^0 \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ be the isomorphism given by the standard (PBW) symmetrization map,

$$q^0(\xi_1 \cdots \xi_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}.$$

This preserves $\mathbb{Z}$-degrees and takes $S(\mathfrak{g}_\pm)$ to $U(\mathfrak{g}_\pm)$. While the map itself does not extend to the completions, we define a normal-ordered symmetrization (quantization) map

$$q \colon \widehat{S}(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$$

by taking the direct sum over i and direct product over r of the maps

$$q^0 \otimes q^0 \colon S(\mathfrak{g}_-)_{i-r} \otimes S(\mathfrak{g}_+)_r \to U(\mathfrak{g}_-)_{i-r} \otimes U(\mathfrak{g}_+)_r.$$

Then q is an isomorphism of $\mathbb{Z}$-graded vector spaces. Let

$$S^2(\mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}),\ \ p \mapsto A_p$$

be the linear map given for $p = uv,\ \ u, v \in \mathfrak{g}$ by

$$A_p(\mu) = \langle \mu, u\rangle v + \langle \mu, v\rangle u.$$

It extends to a $\mathfrak{g}$-equivariant linear map $\widehat{S}^2(\mathfrak{g}) \to \widehat{\mathrm{Hom}}(\mathfrak{g}^*, \mathfrak{g})$. Let

$$\mathrm{br}\colon\ \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}) \to \mathfrak{g}$$

be the linear map, given by the identification $\mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \mathfrak{g}\otimes\mathfrak{g}$ followed by the Lie bracket. In a basis e_a of $\mathfrak{g}$ with dual basis $e^a \in \mathfrak{g}^*$, $\mathrm{br}(A) = \sum_a [A(e^a), e_a]_{\mathfrak{g}}$. The counterpart to Lemma 1.1 reads:

Lemma 2.4. *For* $p \in S^2(\mathfrak{g})$*,*

$$q(p) = q^0(p) - \tfrac{1}{2}\,\mathrm{br}(\pi_+ A_p).$$

Proof. It suffices to check for $p = uv$, where the formula reduces to (cf. (6))

$$q(uv) = q^0(uv) + \tfrac{1}{2}[u, \pi_+ v]_{\mathfrak{g}} + \tfrac{1}{2}[v, \pi_+ u]_{\mathfrak{g}}, \tag{8}$$

but this is straightforward in each of the cases that u, v are both in $\mathfrak{g}_+$, both in $\mathfrak{g}_-$, or $u \in \mathfrak{g}_+, v \in \mathfrak{g}_-$. □

In contrast to q^0, the map q is not $\mathfrak{g}$-equivariant. Similar to Proposition 1.2 we have:

Proposition 2.5. *On* $\widehat{S}^2(\mathfrak{g})$*,*

$$L_\xi(q(p)) - q(L_\xi(p)) = \tfrac{1}{2}\,\mathrm{br}\left((\pi_+ \,\mathrm{ad}_\xi\, \pi_- - \pi_- \,\mathrm{ad}_\xi\, \pi_+) A_p\right).$$

The right hand side is well-defined, since $\pi_- \,\mathrm{ad}_\xi\, \pi_+$ and $\pi_+ \,\mathrm{ad}_\xi\, \pi_-$ are in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$, hence $(\pi_+ \,\mathrm{ad}_\xi\, \pi_- - \pi_- \,\mathrm{ad}_\xi\, \pi_+) A_p \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$.

Proof. It suffices to verify this for $p \in S^2(\mathfrak{g})$, so that A_p has finite rank. Since $L_\xi q^0(p) - q^0(L_\xi p) = 0$, Lemma 2.4 gives

$$\begin{aligned}
L_\xi q(p) - q(L_\xi p) &= -\tfrac{1}{2}\left(L_\xi \,\mathrm{br}(\pi_+ A_p) - \mathrm{br}(\pi_+ A_{L_\xi p})\right)\\
&= -\tfrac{1}{2}\,\mathrm{br}\left([L_\xi, \pi_+ A_p] - \pi_+ [L_\xi, A_p]\right)\\
&= -\tfrac{1}{2}\,\mathrm{br}\left(L_\xi \pi_+ A_p - \pi_+ L_\xi A_p\right)\\
&= \tfrac{1}{2}\,\mathrm{br}\left((\pi_+ L_\xi \pi_- - \pi_- L_\xi \pi_+) A_p\right).
\end{aligned}$$

□

2.3. **Quadratic Lie algebras.** We assume that $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ comes equipped with a non-degenerate ad-invariant symmetric bilinear form B of degree 0. Thus, $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for $i + j \neq 0$, while B defines a non-degenerate pairing between $\mathfrak{g}_i, \mathfrak{g}_{-i}$. We will often use B to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. The examples we have in mind are the following:

(a) Let $\mathfrak{k}$ be a finite-dimensional Lie algebra, with an invariant symmetric bilinear form $B_{\mathfrak{k}}$. Then B extends to an inner product on the loop algebra $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$.

(b) Let $\mathfrak{l} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i$ be a graded Lie algebra, with finite-dimensional homogeneous components, and $\mathfrak{l}^* = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i^*$ its restricted dual, with grading $(\mathfrak{l}^*)_i = \mathfrak{l}^*_{-i}$. The semi-direct product $\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{l}^*$, with B given by the pairing, satisfies our assumptions. This case was studied by Kostant and Sternberg in [12].

(c) Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ be a symmetrizable Kac-Moody Lie algebra, with grading the principal grading (defined by the height of roots). Then $\mathfrak{g}$ carries a 'standard' non-degenerate invariant symmetric bilinear form, see [5]. We will return to the Kac-Moody case in Section 6.

Under the identification $\widehat{\wedge}^2(\mathfrak{g}) \cong \widehat{\mathfrak{o}}(\mathfrak{g})$, the Kac-Peterson cocycle ψ_{KP} corresponds to an element

$$\Psi_{KP} \in \widehat{\mathfrak{o}}(\mathfrak{g}), \quad \psi_{KP}(\xi,\zeta) = B(\Psi_{KP}(\xi),\zeta).$$

Since ψ_{KP} has $\mathbb{Z}$-degree 0, the transformation Ψ_{KP} preserves each $\mathfrak{g}_i$. Since ψ_{KP} is a cocycle, Ψ_{KP} is a derivation of the Lie bracket on $\mathfrak{g}$. Moreover, ψ_{KP} is a coboundary if and only if the derivation Ψ_{KP} is inner:

$$\psi_{KP} = \mathrm{d}\rho \quad \Leftrightarrow \quad \Psi_{KP} = [\rho^\sharp, \cdot]_{\mathfrak{g}}, \tag{9}$$

where $\rho^\sharp$ is the image of $\rho \in \mathfrak{g}_0^*$ under the isomorphism $B^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$.

Example 2.6. Let $\mathfrak{g} = \mathfrak{k}[z,z^{-1}]$, with $\mathfrak{k}$ semi-simple, and with bilinear form defined in terms of the Killing form on $\mathfrak{k}$ as $B(\xi,\zeta) = \mathrm{Res}(z^{-1}B^{\mathrm{Kil}}(\xi,\zeta))$, for $\xi,\zeta \in \mathfrak{k}[z,z^{-1}]$. Then Ψ_{KP} is the degree operator:

$$\Psi_{KP}(\xi) = z\frac{\partial \xi}{\partial z}.$$

2.4. **Casimir elements.** Let $p \in \widehat{S}^2(\mathfrak{g})$ be the element

$$p = \sum_a e_a e^a \in \widehat{S}^2(\mathfrak{g}),$$

where e_a is a homogeneous basis of $\mathfrak{g}$, with B-dual basis e^a. The corresponding transformation $A_p \in \mathrm{Hom}(\mathfrak{g}^*,\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$ is $2\,\mathrm{Id}_{\mathfrak{g}}$. We refer to

$$\mathrm{Cas}'_{\mathfrak{g}} = q(p) \in \widehat{U}(\mathfrak{g})$$

as the *normal-ordered Casimir element*. It is not an element of the center, in general:

Theorem 2.7. *The normal-ordered Casimir element satisfies*

$$L_\xi \,\mathrm{Cas}'_{\mathfrak{g}} = 2\Psi_{KP}(\xi),$$

for all $\xi \in \mathfrak{g}$.

Proof. From the definition of br, one finds

$$B(\mathrm{br}(A),\zeta) = \mathrm{tr}(\mathrm{ad}_\zeta A)$$

for all $A \in \mathrm{End}(\mathfrak{g})$ and $\zeta \in \mathfrak{g}$. Since $A_p = 2\,\mathrm{Id}_{\mathfrak{g}}$ and $L_\xi p = 0$, Proposition 2.5 therefore gives

$$\begin{aligned} B(L_\xi \,\mathrm{Cas}'_{\mathfrak{g}},\zeta) &= B(\mathrm{br}(\pi_+ \,\mathrm{ad}_\xi\, \pi_- - \pi_- \,\mathrm{ad}_\xi\, \pi_+),\zeta)\\ &= \mathrm{tr}(\mathrm{ad}_\zeta\, \pi_+ \,\mathrm{ad}_\xi\, \pi_- - \mathrm{ad}_\zeta\, \pi_- \,\mathrm{ad}_\xi\, \pi_+)\\ &= 2\psi_{\mathrm{KP}}(\xi,\zeta)\\ &= 2B(\Psi_{\mathrm{KP}}(\xi),\zeta). \end{aligned}$$

□

The normal-ordered Casimir element $\mathrm{Cas}'_{\mathfrak{g}}$ admits a linear correction to a central element if and only if the Kac-Peterson class is zero. More precisely:

Corollary 2.8. *For $\rho \in \mathfrak{g}_0^*$,*

$$\mathrm{Cas}_{\mathfrak{g}} := \mathrm{Cas}'_{\mathfrak{g}} + 2\rho^\sharp \tag{10}$$

lies in the center of $\widehat{U}(\mathfrak{g})$ if and only if $\psi_{KP} = d\rho$.

Proof. This is a direct consequence of Theorem 2.7, since $\psi_{KP} = \mathrm{d}\rho$ if and only if $L_\xi \rho^\sharp = -\Psi_{KP}(\xi)$, see Equation (9). □

Example 2.9. For a loop algebra $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$, with $\mathfrak{k}$ a semi-simple Lie algebra, the Kac-Peterson coycle of $\mathfrak{g}$ defines a non-trivial cohomology class. Hence it is impossible to make $\mathrm{Cas}'_{\mathfrak{g}}$ invariant by adding linear terms. On the other hand, for a symmetrizable Kac-Moody algebra $\mathfrak{g}$, a classical result of Kac shows that $\mathrm{Cas}'_{\mathfrak{g}}$ becomes invariant after a ρ-shift. Hence the Kac-Peterson class of such a $\mathfrak{g}$ is trivial. See Section 6 below.

2.5. **The structure constants tensor and its quantization.** Recall the definition of $\lambda\colon \widehat{\mathfrak{o}}(\mathfrak{g}) \to \widehat{\wedge}^2(\mathfrak{g})$. We will write

$$\lambda(\xi) = \lambda(\mathrm{ad}_\xi),$$

that is $\iota_\xi \lambda(\zeta) = \frac{1}{2}[\xi, \zeta]_{\mathfrak{g}}$. In a basis e_a of $\mathfrak{g}$, with B-dual basis e^a, we have $\lambda(\xi) = \frac{1}{4}\sum_a [\xi, e_a]_{\mathfrak{g}} \wedge e^a$.

Lemma 2.10. *There is a unique element $\phi \in \widehat{\wedge}^3(\mathfrak{g})_0$ with the property*

$$\iota_{\xi_1}\iota_{\xi_2}\iota_{\xi_3}\phi = \tfrac{1}{2}B([\xi_1, \xi_2]_{\mathfrak{g}}, \xi_3), \quad \xi_1, \xi_2, \xi_3 \in \mathfrak{g}. \tag{11}$$

Proof. The right-hand side is a skew-symmetric trilinear form of degree 0 on $\mathfrak{g}$. Hence it defines an element of $\widehat{\wedge}^3(\mathfrak{g})$. □

Equivalently, $\iota_\xi \phi = 2\lambda(\xi), \quad \xi \in \mathfrak{g}$. In a basis,

$$\phi = -\frac{1}{12}\sum_{abc} f_{abc} e^a \wedge e^b \wedge e^c, \tag{12}$$

where $f_{abc} = B([e_a, e_b]_{\mathfrak{g}}, e_c)$ are the structure constants. From the definition, it is clear that ϕ is $\mathfrak{g}$-invariant. This need no longer be true of its normal-ordered quantization. Write

$$\gamma'(\xi) = q(\lambda(\xi)), \quad \phi'_{\mathrm{Cl}} = q(\phi),$$

so that $L_\xi = [\gamma'(\xi), \cdot)]$. Denote by $\psi^\sharp_{KP} \in \widehat{\wedge}^2(\mathfrak{g})$ the image of $\psi_{KP} \in \widehat{\wedge}^2(\mathfrak{g}^*)$ under the isomorphism $B^\sharp\colon \widehat{\wedge}(\mathfrak{g}^*) \to \widehat{\wedge}(\mathfrak{g})$.

Proposition 2.11. *The element $\phi'_{\mathrm{Cl}} \in \widehat{\mathrm{Cl}}(\mathfrak{g})$ satisfies*

$$L_\xi \phi'_{\mathrm{Cl}} = \Psi_{KP}(\xi),$$

and its square is given by the formula

$$(\phi'_{\mathrm{Cl}})^2 = q(\psi^\sharp_{KP}) + \frac{1}{24}\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}).$$

Here $\mathrm{Cas}_{\mathfrak{g}_0} \in U(\mathfrak{g}_0)$ is the quadratic Casimir element for $\mathfrak{g}_0$, and $\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0})$ is its trace in the adjoint representation.

Proof. The first formula follows from the second, since

$$L_\xi \phi'_{\mathrm{Cl}} = [\gamma'(\xi), \phi'_{\mathrm{Cl}}] = \iota_\xi (\phi'_{\mathrm{Cl}})^2.$$

Since

$$\iota_\xi (\phi'_{\mathrm{Cl}})^2 = [\gamma'(\xi), \phi'_{\mathrm{Cl}}] = L_\xi \phi'_{\mathrm{Cl}} = \Psi_{KP}(\xi) = \iota_\xi q(\psi^\sharp_{KP}),$$

the difference $(\phi'_{\mathrm{Cl}})^2 - q(\psi^\sharp_{KP})$ is a constant. Let ϕ_r be the component of ϕ in $(\wedge \mathfrak{g}_-)_{-r} \otimes (\wedge \mathfrak{g}_+)_r$. The commutator of ϕ'_{Cl} with a term $q(\phi_r)$ for $r > 0$ is contained in the right ideal generated by $\mathfrak{g}_+$, and hence does not contribute to the constant. Hence the constant equals $q(\phi_0)^2$, where $\phi_0 \in \wedge^3 \mathfrak{g}_0$ is the structure constants tensor of $\mathfrak{g}_0 \subset \mathfrak{g}$. By [1, 10] this constant is given by $\frac{1}{24} \operatorname{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0})$. □

Corollary 2.12. *Suppose $\psi_{KP} = d\rho$ for some $\rho \in \mathfrak{g}_0^*$. Define elements of $\widehat{\mathrm{Cl}}(\mathfrak{g})$ by*

$$\phi_{\mathrm{Cl}} := \phi'_{\mathrm{Cl}} + \rho^\sharp, \quad \gamma(\xi) = \gamma'(\xi) + \langle \rho, \xi \rangle,$$

for $\xi \in \mathfrak{g}$. The following commutator relations hold in $\widehat{\mathrm{Cl}}(\mathfrak{g})$:

$$\begin{aligned}
[\xi, \zeta] &= 2B(\xi, \zeta), \\
[\gamma(\xi), \phi_{\mathrm{Cl}}] &= 0, \\
[\xi, \phi_{\mathrm{Cl}}] &= 2\gamma(\xi), \\
[\gamma(\xi), \gamma(\zeta)] &= \gamma([\xi, \zeta]_{\mathfrak{g}}), \\
[\gamma(\xi), \zeta] &= [\xi, \zeta]_{\mathfrak{g}}, \\
[\phi_{\mathrm{Cl}}, \phi_{\mathrm{Cl}}] &= 2B(\rho^\sharp, \rho^\sharp) + \frac{1}{12} \operatorname{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}).
\end{aligned}$$

Thus $\widehat{\mathrm{Cl}}(\mathfrak{g})$ becomes a $\mathfrak{g}$-differential algebra (see e.g. [16]) with differential $d = [\phi_{\mathrm{Cl}}, \cdot]$, contractions $\iota_\xi = \frac{1}{2}[\xi, \cdot]$, and Lie derivatives $L_\xi = [\gamma(\xi), \cdot]$.

Proof. Observe first that $\lambda(\rho^\sharp) = -\psi_{KP}$, since

$$\iota_\zeta \iota_\xi \lambda(\rho^\sharp) = \iota_\zeta [\xi, \rho^\sharp]_{\mathfrak{g}} = B(\zeta, [\xi, \rho^\sharp]_{\mathfrak{g}}) = -\langle \rho, [\xi, \zeta]_{\mathfrak{g}} \rangle.$$

Consequently $[\rho^\sharp, \phi'_{\mathrm{Cl}}] = -q(\psi_{KP})$, which implies the formula for $[\phi_{\mathrm{Cl}}, \phi_{\mathrm{Cl}}]$. The other assertions are verified similarly. □

Still assuming $\psi_{KP} = \mathrm{d}\rho$, consider the algebra morphism

$$\gamma \colon U(\mathfrak{g}) \to \widehat{\mathrm{Cl}}(\mathfrak{g}) \tag{13}$$

extending the Lie algebra homomorphism $\xi \mapsto \gamma(\xi)$.

Proposition 2.13. *The map* (13) *extends to an algebra morphism*

$$\gamma \colon \widehat{U}(\mathfrak{g}) \to \widehat{\mathrm{Cl}}(\mathfrak{g}).$$

Proof. We claim that for all $i > 0$, $\gamma(\mathfrak{g}_i)$ is contained in

$$\coprod_{r \geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{-r} \mathrm{Cl}(\mathfrak{g}_+)_{i+r} \subset \widehat{\mathrm{Cl}}(\mathfrak{g})_i \tag{14}$$

(i.e. the components in $\mathrm{Cl}(\mathfrak{g}_+)$ have degree $\geq i$). Indeed, suppose $\xi \in \mathfrak{g}_i$ with $i > 0$. In particular, $\langle \rho, \xi \rangle = 0$. Let $e_a \in \mathfrak{g}$ be a basis consisting of homogeneous elements, and e^a the dual basis. Since $\langle \rho, \xi \rangle = 0$, and since $[\xi, e_a]_\mathfrak{g}$ Clifford commutes with e^a, we have

$$\gamma(\xi) = \frac{1}{2}\sum\nolimits_+ ([\xi, e^a]e_a - e^a[\xi, e_a]) + \frac{1}{4}\sum\nolimits_0 [\xi, e_a]e^a$$

where $\sum_+$ is a summation over indices with $e_a \in \mathfrak{g}_+$, and $\sum_0$ is a summation over indices with $e_a \in \mathfrak{g}_0$. The second and third term in this expression are in (14), as are the summands $[\xi, e^a]e_a$ from the first sum for $e_a \in \mathfrak{g}_s$ with $s \geq i$. In the remaining case $s < i$ we have $[\xi, e^a] \in \mathfrak{g}_{i-s} \subset \mathfrak{g}_+$, and hence $[\xi, e^a]e_a \in \mathrm{Cl}(\mathfrak{g}_+)_i$. This proves the claim. By induction, one deduces that

$$\gamma(U(\mathfrak{g}_+)_i) \subset \prod_{r\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{-r}\mathrm{Cl}(\mathfrak{g}_+)_{i+r}.$$

Similarly, if $j \leq 0$,

$$\gamma(U(\mathfrak{g}_-)_j) \subset \prod_{r\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{j-r}\mathrm{Cl}(\mathfrak{g}_+)_r.$$

It follows that

$$\gamma(U(\mathfrak{g}_-)_{-r}U(\mathfrak{g}_+)_{i+r}) \subset \prod_{m\geq 0} \mathrm{Cl}(\mathfrak{g}_-)_{-r-m}\mathrm{Cl}(\mathfrak{g}_+)_{i+r+m}.$$

Summing over all $r \geq 0$, one obtains a well-defined map $\widehat{U}(\mathfrak{g})_i \to \widehat{\mathrm{Cl}}(\mathfrak{g})_i$. □

3. Double extension

For the loop algebra $\mathfrak{g} = \mathfrak{k}[z, z^{-1}]$ of a semisimple Lie algebra $\mathfrak{k}$, the Kac-Peterson class is non-trivial. On the other hand, the usual double extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra, hence its Kac-Peterson class is zero. In fact, one has a similar double extension in the general case, as we now explain.

We continue to work with the assumptions from the last sections; in particular $\mathfrak{g}$ carries an invariant non-degenerate symmetric bilinear form B of degree 0. As noted above, the Kac-Peterson cocycle ψ_{KP} gives rise to a skew-symmetric derivation $\Psi_{KP} \in \widehat{\mathfrak{o}}(\mathfrak{g})$. By a general construction of Medina-Revoy [15], such a derivation can be used to define a double extension

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}\delta \oplus \mathbb{C}K,$$

with the following bracket: For $\xi, \xi_1, \xi_2 \in \mathfrak{g}$,

$$\begin{aligned}
[\xi_1, \xi_2]_{\tilde{\mathfrak{g}}} &= [\xi_1, \xi_2]_\mathfrak{g} + \psi_{KP}(\xi_1, \xi_2)K,\\
[\delta, \xi]_{\tilde{\mathfrak{g}}} &= \Psi_{KP}(\xi),\\
[\delta, K]_{\tilde{\mathfrak{g}}} &= 0,\\
[\xi, K]_{\tilde{\mathfrak{g}}} &= 0
\end{aligned}$$

The bilinear form B on $\mathfrak{g}$ extends to a non-degenerate invariant bilinear form on $\tilde{\mathfrak{g}}$, in such a way that $\mathfrak{g}$ and $\mathbb{C}\delta \oplus \mathbb{C}K$ are orthogonal and

$$\tilde{B}(\delta, K) = 1, \quad \tilde{B}(\delta, \delta) = \tilde{B}(K, K) = 0.$$

Introduce the grading $\tilde{\mathfrak{g}}_i = \mathfrak{g}_i$ for $i \neq 0$ and $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus \mathbb{C}\delta \oplus \mathbb{C}K$. The resulting splitting is

$$\tilde{\mathfrak{g}}_- = \mathfrak{g}_- \oplus \mathbb{C}\delta \oplus \mathbb{C}K,\ \ \tilde{\mathfrak{g}}_+ = \mathfrak{g}_+.$$

Let $\tilde{\psi}_{KP}$ be the Kac-Peterson cocycle for this splitting, $\tilde{\Psi}_{KP}$ the associated derivation, and denote by $\tilde{\pi}_\pm \colon \tilde{\mathfrak{g}} \to \mathfrak{g}_\pm$ the projections along $\tilde{\mathfrak{g}}_\mp$. The adjoint representation for $\tilde{\mathfrak{g}}$ will be denoted $\widetilde{\mathrm{ad}}$.

Proposition 3.1. *The derivation $\tilde{\Psi}_{KP}$ is inner:*

$$\tilde{\Psi}_{KP} = [\delta, \cdot]_{\tilde{\mathfrak{g}}}.$$

Equivalently $\tilde{\psi}_{KP} = d\rho$ where $\rho = \tilde{B}(\delta, \cdot)$.

Proof. The desired equation $\tilde{\Psi}_{KP} = [\delta, \cdot]_{\tilde{\mathfrak{g}}}$ means that $\tilde{\Psi}_{KP}(\xi) = \Psi_{KP}(\xi),\ \tilde{\Psi}_{KP}(\delta) = 0,\ \tilde{\Psi}_{KP}(K) = 0$. Equivalently, we have to show that $\tilde{\psi}_{KP}(\xi_1, \xi_2) = \psi_{KP}(\xi_1, \xi_2)$ for $\xi_1, \xi_2 \in \mathfrak{g}$, while both K, δ are in the kernel of $\tilde{\psi}_{KP}$. The last claim follows from

$$\tilde{\pi}_- \widetilde{\mathrm{ad}}_\delta \tilde{\pi}_+ = 0 = \tilde{\pi}_+ \widetilde{\mathrm{ad}}_\delta \tilde{\pi}_-,$$

and similarly for ad_K, since ad_δ and ad_K preserve degrees. On the other hand, one checks that for $\xi_1, \xi_2 \in \mathfrak{g}$, the composition

$$\pi_+ \,\mathrm{ad}_{\xi_1}\, \pi_- \,\mathrm{ad}_{\xi_2}\, \pi_+ \colon \mathfrak{g}_+ \to \mathfrak{g}_+$$

of operators on $\mathfrak{g}$ coincides with the composition

$$\tilde{\pi}_+ \widetilde{\mathrm{ad}}_{\xi_1} \tilde{\pi}_- \widetilde{\mathrm{ad}}_{\xi_2} \tilde{\pi}_+ \colon \mathfrak{g}_+ \to \mathfrak{g}_+$$

of operators on $\tilde{\mathfrak{g}}$. Hence the Kac-Peterson coycles agree on elements of $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. ☐

4. The cubic Dirac operator

We will define the cubic Dirac operator as an element of a completion of the quantum Weil algebra $\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. Following [1], we take the viewpoint that the commutator with $\mathcal{D}$ defines a differential, making $\widehat{\mathcal{W}}(\mathfrak{g})$ into a $\mathfrak{g}$-differential algebra.

4.1. Weil algebra.

We begin with an arbitrary $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ with $\dim \mathfrak{g}_i < \infty$. As usual $\mathfrak{g}^*$ denotes the restricted dual. Consider the tensor product $W(\mathfrak{g}^*) = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$ with grading

$$W^k(\mathfrak{g}^*) = \bigoplus_{2r+s=k} S^r(\mathfrak{g}^*) \otimes \wedge^s(\mathfrak{g}^*).$$

For $\mu \in \mathfrak{g}^*$ we denote by $s(\mu) = \mu \otimes 1$ the degree 2 generators and by $\mu = 1 \otimes \mu$ the degree 1 generators. Any $\xi \in \mathfrak{g}$ defines contraction operators ι_ξ; these are derivations of degree -1 given on generators by $\iota_\xi \mu = \mu(\xi),\ \iota_\xi s(\mu) = 0$. The co-adjoint action on $\mathfrak{g}^*$ defines Lie derivatives $L_\xi = L^S_\xi \otimes 1 + 1 \otimes L^\wedge_\xi$. If $\dim(\mathfrak{g}) < \infty$, the algebra $W(\mathfrak{g})$ carries a *Weil differential* d^W, given on generators by[1]

$$\mathrm{d}^W \mu = 2(s(\mu) + \lambda(\mu)),\ \ \mathrm{d}^W s(\mu) = \sum\nolimits_a s(L_{e_a}\mu) e^a. \tag{15}$$

[1]The conventions for the differential follow [16, §6.11]. They are arranged to make the relation with the quantum Weil algebra appear most natural. One recovers the more standard conventions used in e.g. [3] and [1] by a simple rescaling of variables.

Here e_a is a basis of $\mathfrak{g}$ with dual basis $e^a \in \mathfrak{g}^*$.

In the general case, we need to pass to a completion in order for the differential to be defined. Define a second $\mathbb{Z}$-grading on $W(\mathfrak{g}^*)$, in such a way that the generators $s(\mu), \mu$ for $\mu \in (\mathfrak{g}^*)_i = (\mathfrak{g}_{-i})^*$ have degree i. Letting $\mathfrak{g}^*_+ = \bigoplus_{i>0}(\mathfrak{g}^*)_i$ and $\mathfrak{g}^*_- = \bigoplus_{i\le 0}(\mathfrak{g}^*)_i$ we define a completion $\widehat{W}(\mathfrak{g}^*)$ as the graded algebra with

$$\widehat{W}(\mathfrak{g}^*)_i = \prod_{r\ge 0} W(\mathfrak{g}^*_-)_{i-r} \otimes W(\mathfrak{g}^*_+)_r.$$

(Equivalently, $\widehat{W}(\mathfrak{g}^*)_i$ is the space of all linear maps $(S(\mathfrak{g}) \otimes \wedge(\mathfrak{g}))_{-i} \to \mathbb{K}$.) The Weil differential d^W is define on generators by the formulas (15). Together with the natural extensions of ι_ξ, L_ξ this makes $\widehat{W}(\mathfrak{g}^*)$ into a $\mathfrak{g}$-differential algebra.

4.2. **Quantum Weil algebra.** Suppose now that $\mathfrak{g}$ carries an invariant symmetric bilinear form B of degree 0. We use B to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and will thus write $W(\mathfrak{g})$, $\widehat{W}(\mathfrak{g})$ and so on. The non-commutative *quantum Weil algebra* is the tensor product

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

It is a super algebra, with even generators $s(\zeta) = \zeta \otimes 1$ and odd generators $\zeta = 1 \otimes \zeta$. Any $\xi \in \mathfrak{g}$ defines Lie derivatives $L_\xi = L^U_\xi \otimes 1 + 1 \otimes L^{Cl}_\xi$ and contraction operators ι_ξ, given as odd derivations with $\iota_\xi \zeta = B(\xi, \zeta)$, $\iota_\xi s(\zeta) = 0$. Super symmetrization defines an isomorphism

$$q^0 \colon W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g}), \tag{16}$$

given simply as the tensor product of $q^0 \colon S(\mathfrak{g}) \to U(\mathfrak{g})$ and $q^0 \colon \wedge(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$. Note that (16) intertwines the contractions and Lie derivatives. We define a completion $\widehat{\mathcal{W}}(\mathfrak{g})$ as the graded super algebra with

$$\widehat{\mathcal{W}}(\mathfrak{g})_i = \prod_{r\ge 0} \mathcal{W}(\mathfrak{g}_-)_{i-r} \otimes \mathcal{W}(\mathfrak{g}_+)_r.$$

The 'normal-ordered' quantization map $q \colon \widehat{W}(\mathfrak{g}) \to \widehat{\mathcal{W}}(\mathfrak{g})$ is defined by summing over all

$$q^0 \otimes q^0 \colon W(\mathfrak{g}_-)_{i-r} \otimes W(\mathfrak{g}_+)_r \to \mathcal{W}(\mathfrak{g}_-)_{i-r} \otimes \mathcal{W}(\mathfrak{g}_+)_r.$$

It extends the quantization maps $q \colon \widehat{S}(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ and $q \colon \widehat{\wedge}(\mathfrak{g}) \to \widehat{\mathrm{Cl}}(\mathfrak{g})$.

4.3. **The element** $q(D)$**.** If $\dim \mathfrak{g} < \infty$, one obtains a differential $\mathrm{d}^{\mathcal{W}}$ on $\mathcal{W}\mathfrak{g}$, as a derivation given on generators by formulas similar to (15),

$$\mathrm{d}^{\mathcal{W}}\zeta = 2(s(\zeta) + q_0(\lambda(\zeta))),\ \mathrm{d}^{\mathcal{W}} s(\zeta) = \sum\nolimits_a s(L_{e_a}\zeta)e^a,$$

see [1]. In fact, $\mathrm{d}^{\mathcal{W}} = [q^0(D), \cdot]$, where $D \in W^3(\mathfrak{g})$ is the element

$$D = \sum_a s(e_a)e^a + \phi,$$

with $\phi \in \wedge^3\mathfrak{g} \subset W^3(\mathfrak{g})$ the structure constants tensor. The fact that $\mathrm{d}^{\mathcal{W}}$ squares to zero means that $q^0(D)$ squares to a central element, and indeed one finds

$$q^0(D)^2 = \mathrm{Cas}_\mathfrak{g} + \frac{1}{24}\,\mathrm{tr}_\mathfrak{g}(\mathrm{Cas}_\mathfrak{g}).$$

If $\dim \mathfrak{g} = \infty$, the element D is well-defined as an element of the completion $\widehat{W}^3(\mathfrak{g})$, but $q^0(D)$ is ill-defined. On the other hand,

$$\mathcal{D}' = q(D) = \sum_a s(e_a)e^a + \phi'_{\mathrm{Cl}}$$

is defined but does not square to a central element.

Proposition 4.1. *The square of $\mathcal{D}' = q(D)$ is given by*

$$(\mathcal{D}')^2 = \mathrm{Cas}'_{\mathfrak{g}} + q(\psi^\sharp_{KP}) + \frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}).$$

Proof. We have

$$L_\xi \mathcal{D}' = L_\xi \phi'_{\mathrm{Cl}} = \Psi_{KP}(\xi) = \iota_\xi q(\psi^\sharp_{KP})$$

because $\sum_a s(e_a)e^a \in \widehat{\mathcal{W}}(\mathfrak{g})$ is $\mathfrak{g}$-invariant. Using that

$$\iota_\xi \mathcal{D}' = s(\xi) + \iota_\xi(q(\phi)) = s(\xi) + \gamma'(\xi)$$

are generators for the $\mathfrak{g}$-action on $\widehat{\mathcal{W}}(\mathfrak{g})$, we have

$$\iota_\xi((\mathcal{D}')^2 - q(\psi^\sharp_{KP})) = [\iota_\xi \mathcal{D}', \mathcal{D}'] - q(\psi^\sharp_{KP}) = 0.$$

This shows $(\mathcal{D}')^2 - q(\psi^\sharp_{KP}) \in \widehat{U}(\mathfrak{g}) \subset \widehat{\mathcal{W}}(\mathfrak{g})$. To find this element we calculate, denoting by $\ldots$ terms in the kernel of the projection $\widehat{\mathcal{W}}(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$,

$$\begin{aligned}
(\mathcal{D}')^2 &= \sum_{ab} s(e_a)s(e_b)e^a e^b + (\phi'_{\mathrm{Cl}})^2 + \ldots \\
&= \tfrac{1}{2}\sum_{ab} s(e_a)s(e_b)[e^a, e^b] + \frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + \ldots \\
&= \mathrm{Cas}'_{\mathfrak{g}} + \frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + \ldots
\end{aligned}$$

□

If the Kac-Peterson class is trivial, one obtains an element $\mathcal{D}$ with better properties.

Corollary 4.2. *Suppose that $\psi_{KP} = d\rho$ for some $\rho \in \mathfrak{g}_0^*$. Define*

$$\mathcal{D} = \mathcal{D}' + \rho^\sharp, \quad \gamma_{\mathcal{W}}(\xi) = s(\xi) + \gamma'_{\mathrm{Cl}}(\xi) + \langle \rho, \xi\rangle,$$

and put $\mathrm{Cas}_{\mathfrak{g}} = \mathrm{Cas}'_{\mathfrak{g}} + 2\rho^\sharp$ as before. Then

$$\mathcal{D}^2 = \mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + B(\rho^\sharp, \rho^\sharp).$$

One has the following commutator relations in $\widehat{\mathcal{W}}(\mathfrak{g})$,

$$\begin{aligned}
[\mathcal{D}, \mathcal{D}] &= 2\,\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \frac{1}{12}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0}) + 2B(\rho^\sharp, \rho^\sharp), \\
[\gamma_{\mathcal{W}}(\xi), \mathcal{D}] &= 0, \\
[\xi, \mathcal{D}] &= 2\gamma_{\mathcal{W}}(\xi), \\
[\gamma_{\mathcal{W}}(\xi), \gamma_{\mathcal{W}}(\zeta)] &= \gamma_{\mathcal{W}}([\xi, \zeta]_{\mathfrak{g}}), \\
[\gamma_{\mathcal{W}}(\xi), \zeta] &= [\xi, \zeta]_{\mathfrak{g}}, \\
[\xi, \zeta] &= 2B(\xi, \zeta).
\end{aligned}$$

Thus $\widehat{\mathcal{W}}(\mathfrak{g})$ becomes a $\mathfrak{g}$-differential algebra, with differential, Lie derivatives and contractions given by

$$d^{\mathcal{W}} = [\mathcal{D}, \cdot], \quad L_\xi^{\mathcal{W}} = [\gamma_{\mathcal{W}}(\xi), \cdot], \quad \iota_\xi^{\mathcal{W}} = \tfrac{1}{2}[\xi, \cdot].$$

We will refer to $\mathcal{D} \in \widehat{\mathcal{W}}(\mathfrak{g})$ as the *cubic Dirac operator*, following Kostant [10].

5. Relative Dirac operators

In his paper [10], Kostant introduced more generally Dirac operators for any pair of a quadratic Lie algebra $\mathfrak{g}$ and a quadratic Lie subalgebra $\mathfrak{u}$. We consider now an extension of his results to infinite-dimensional graded Lie algebras.

Let $\mathfrak{g}, B$ be as in the last Section, and suppose $\mathfrak{u} \subseteq \mathfrak{g}$ is a graded quadratic subalgebra. That is, $\mathfrak{u}_i \subseteq \mathfrak{g}_i$ for all i, and the non-degenerate symmetric bilinear form B on $\mathfrak{g}$ restricts to a non-degenerate bilinear form on $\mathfrak{u}$. We have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$$

where $\mathfrak{p} = \mathfrak{u}^\perp$. For any $\xi \in \mathfrak{u}$, the operator $\mathrm{ad}_\xi \in \widehat{\mathfrak{o}}(\mathfrak{g})$ breaks up as a sum

$$\mathrm{ad}_\xi = \mathrm{ad}_\xi^{\mathfrak{u}} + \mathrm{ad}_\xi^{\mathfrak{p}}, \quad \xi \in \mathfrak{u}$$

of operators $\mathrm{ad}_\xi^{\mathfrak{u}} \in \widehat{\mathfrak{o}}(\mathfrak{u})$ and $\mathrm{ad}_\xi^{\mathfrak{p}} \in \widehat{\mathfrak{o}}(\mathfrak{p})$. Accordingly,

$$\lambda(\xi) = \lambda_{\mathfrak{u}}(\xi) + \lambda_{\mathfrak{p}}(\xi), \quad \xi \in \mathfrak{u}$$

with $\lambda_{\mathfrak{u}}(\xi) \in \widehat{\wedge}^2(\mathfrak{u})$ and $\lambda_{\mathfrak{p}}(\xi) \in \widehat{\wedge}^2(\mathfrak{p})$. Denote by $\gamma'_{\mathfrak{u}}(\xi)$, $\gamma'_{\mathfrak{p}}(\xi)$ their images under $q\colon \widehat{W}(\mathfrak{g}) \to \widehat{\mathcal{W}}(\mathfrak{g})$. We have (cf. (7))

$$[\gamma'_{\mathfrak{p}}(\xi), \gamma'_{\mathfrak{p}}(\zeta)] = \gamma'_{\mathfrak{p}}([\xi, \zeta]) + \psi^{\mathfrak{p}}_{KP}(\xi, \zeta),$$

where $\psi^{\mathfrak{p}}_{KP}(\xi, \zeta) = \psi^{\mathfrak{p}}_{KP}(\mathrm{ad}^{\mathfrak{p}}_\xi, \mathrm{ad}^{\mathfrak{p}}_\zeta)$ defines a cocycle $\psi^{\mathfrak{p}}_{KP} \in \widehat{\wedge}^2(\mathfrak{u}^*)$. If $\psi^{\mathfrak{p}}_{KP} = \mathrm{d}\rho_{\mathfrak{p}}$ for some $\rho_{\mathfrak{p}} \in \mathfrak{u}_0^*$, then

$$\gamma_{\mathfrak{p}}(\xi) = \gamma'_{\mathfrak{p}}(\xi) + \langle \rho_{\mathfrak{p}}, \xi \rangle$$

gives a Lie algebra homomorphism $\mathfrak{u} \to \widehat{\mathrm{Cl}}(\mathfrak{p})$, generating the adjoint action of $\mathfrak{u}$. One obtains an algebra homomorphism $j\colon \mathcal{W}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g})$, given on generators by

$$j(\xi) = \xi, \quad j(s(\xi)) = s(\xi) + \gamma_{\mathfrak{p}}(\xi), \quad \xi \in \mathfrak{u}.$$

Proposition 5.1. *The homomorphism $\mathcal{W}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g})$ extends to an algebra homomorphism for the completion:*

$$j\colon \widehat{\mathcal{W}}(\mathfrak{u}) \to \widehat{\mathcal{W}}(\mathfrak{g}).$$

It intertwines Lie derivatives and contraction by elements $\xi \in \mathfrak{u}$.

Proof. The first part follows by an argument parallel to that for Proposition 2.13. The second part follows from

$$j \circ L_\xi = j \circ [s(\xi) + \gamma'_{\mathfrak{u}}(\xi), \cdot] = [s(\xi) + \gamma'_{\mathfrak{g}}(\xi), \cdot] \circ j = L_\xi \circ j$$

and similarly $j \circ \iota_\xi = \frac{1}{2} j \circ [\xi, \cdot] = \frac{1}{2}[\xi, \cdot] \circ j = \iota_\xi \circ j$. □

Let

$$\mathcal{W}(\mathfrak{g},\mathfrak{u})=(U(\mathfrak{g})\otimes \mathrm{Cl}(\mathfrak{p}))^{\mathfrak{u}}$$

be the $\mathfrak{u}$-basic part of $\mathcal{W}(\mathfrak{g})$, i.e. the subalgebra of elements annilated by all L_ξ and all ι_ξ for $\xi\in\mathfrak{u}$. Similarly let $\widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$ be the $\mathfrak{u}$-basic part of $\widehat{\mathcal{W}}(\mathfrak{g})$.

Proposition 5.2. *The subalgebra $\widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$ is the commutant of the range $j(\widehat{\mathcal{W}}(\mathfrak{u}))$.*

Proof. Since $\iota_\xi=\frac{1}{2}[\xi,\cdot]$, an element of $\widehat{\mathcal{W}}(\mathfrak{g})$ commutes with the generators $j(\xi)$ for $\xi\in\mathfrak{u}$ precisely if it lies in the $\mathfrak{u}$-horizontal subspace, given as the completion of $U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{p})$. The elements $j(s(\xi))=s(\xi)+\gamma'_{\mathfrak{p}}(\xi)$ generate the $\mathfrak{u}$-action on that subspace. Hence, an element of $\widehat{\mathcal{W}}(\mathfrak{g})$ commutes with all $j(\xi),\ j(s(\xi))$ if and only if it is $\mathfrak{u}$-basic. □

We will now make the stronger assumption that the Kac-Peterson classes of both $\mathfrak{g},\mathfrak{u}$ are zero. Let $\rho\in\mathfrak{g}_0^*,\ \rho_{\mathfrak{u}}\in\mathfrak{u}_0^*$ be elements such that

$$\psi_{KP}=\mathrm{d}\rho,\quad \psi_{KP}^{\mathfrak{u}}=\mathrm{d}\rho_{\mathfrak{u}},$$

and take $\rho_{\mathfrak{p}}:=\rho|_{\mathfrak{u}_0}-\rho_{\mathfrak{u}}\in\mathfrak{u}_0^*$ so that $\psi_{KP}^{\mathfrak{p}}=\mathrm{d}\rho_{\mathfrak{p}}$. Put

$$\gamma(\zeta)=\gamma'(\zeta)+\langle\rho,\zeta\rangle,\quad \gamma_{\mathfrak{u}}(\xi)=\gamma'_{\mathfrak{u}}(\xi)+\langle\rho_u,\xi\rangle$$

for all $\zeta\in\mathfrak{g},\ \xi\in\mathfrak{u}$, and let

$$\mathcal{D}=\mathcal{D}'+\rho^\sharp\in\widehat{\mathcal{W}}(\mathfrak{g}),\quad \mathcal{D}_{\mathfrak{u}}=\mathcal{D}'_u+\rho_{\mathfrak{u}}^\sharp\in\widehat{\mathcal{W}}(\mathfrak{u})$$

be the cubic Dirac operators for $\mathfrak{g},\mathfrak{u}$. The commutator with these elements defines differentials on the two Weil algebras.

Lemma 5.3. *The map $j\colon\widehat{\mathcal{W}}(\mathfrak{u})\to\widehat{\mathcal{W}}(\mathfrak{g})$ is a homomorphism of $\mathfrak{u}$-differential algebras.*

Proof. It remains to show that the map j intertwines differentials. It suffices to check on generators. For $\xi\in\mathfrak{u}$,

$$j(\mathrm{d}\xi)=j(s_{\mathfrak{u}}(\xi)+\gamma_{\mathfrak{u}}(\xi))=s(\xi)+\gamma_{\mathfrak{p}}(\xi)+\gamma_{\mathfrak{u}}(\xi)=s(\xi)+\gamma(\xi)=\mathrm{d}j(\xi),$$

and similarly $j(\mathrm{d}s_{\mathfrak{u}}(\xi))=\mathrm{d}j(s_{\mathfrak{u}}(\xi))$. □

We define the relative cubic Dirac operator $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ as a difference,

$$\mathcal{D}_{\mathfrak{g},\mathfrak{u}}=\mathcal{D}-j(\mathcal{D}_{\mathfrak{u}}). \tag{17}$$

Proposition 5.4. *The element $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ lies in $\widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$, and squares to an element of the center of $\widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$. Explicitly,*

$$\mathcal{D}_{\mathfrak{g},\mathfrak{u}}^2=\mathrm{Cas}_{\mathfrak{g}}-j(\mathrm{Cas}_{\mathfrak{u}})+\frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}_0}(\mathrm{Cas}_{\mathfrak{g}_0})-\frac{1}{24}\,\mathrm{tr}_{\mathfrak{u}_0}(\mathrm{Cas}_{\mathfrak{u}_0})+B(\rho^\sharp,\rho^\sharp)-B(\rho_{\mathfrak{u}}^\sharp,\rho_{\mathfrak{u}}^\sharp).$$

Proof. Using that j intertwines contractions $\iota_\xi,\ \xi\in\mathfrak{u}$, we find

$$\begin{aligned}\iota_\xi\mathcal{D}_{\mathfrak{g},\mathfrak{u}}&=\iota_\xi\mathcal{D}-j(\iota_\xi\mathcal{D}_{\mathfrak{u}})\\&=s(\xi)+\gamma(\xi)-j(s_{\mathfrak{u}}(\xi)+\gamma_{\mathfrak{u}}(\xi))\\&=\gamma(\xi)-\gamma_{\mathfrak{p}}(\xi)-\gamma_{\mathfrak{u}}(\xi)=0.\end{aligned}$$

Thus $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ is $\mathfrak{u}$-horizontal, and it is clearly $\mathfrak{u}$-invariant as well. Thus $\mathcal{D}_{\mathfrak{g},\mathfrak{u}} \in \widehat{\mathcal{W}}(\mathfrak{g},\mathfrak{u})$. In particular, $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ commutes with $j(\mathcal{D}_{\mathfrak{u}})$. Consequently, $[\mathcal{D},\ \mathcal{D}] = j([\mathcal{D}_{\mathfrak{u}}, \mathcal{D}_{\mathfrak{u}}]) + [\mathcal{D}_{\mathfrak{g},\mathfrak{u}}, \mathcal{D}_{\mathfrak{g},\mathfrak{u}}]$, that is

$$\mathcal{D}_{\mathfrak{g},\mathfrak{u}}^2 = \mathcal{D}^2 - j(\mathcal{D}_{\mathfrak{u}}^2).$$

Now use Corollary 4.2. □

6. Application to Kac-Moody algebras

In his paper [10], Kostant used the cubic Dirac operator $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ to prove generalized Weyl character formulas for any pair of a semi-simple Lie algebra $\mathfrak{g}$ and equal rank subalgebra $\mathfrak{u}$. In this Section, we show that much of this theory carries over to symmetrizable Kac-Moody algebras, with only minor adjustments.

6.1. **Notation and basic facts.** Let us recall some notation and basic facts; our main references are the books by Kac [5] and Kumar [13].

Let $A = (a_{ij})_{1\le i,j\le l}$ be a generalized Cartan matrix, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. Thus $\mathfrak{h}$ is a vector space of dimension $2l - \mathrm{rk}(A)$, and $\Pi = \{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{h}^*$ (the set of simple roots) and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \subset \mathfrak{h}$ (the corresponding co-roots) satisfy $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$. The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is the Lie algebra generated by elements $h \in \mathfrak{h}$ and elements $e_j,\ f_j$ for $j = 1, \ldots, l$, subject to relations

$$[h, e_i] = \langle \alpha_i, h \rangle e_i,\ [h, f_i] = -\langle \alpha_i, h \rangle f_i,\ [h, h'] = 0,\ [e_i, f_j] = \delta_{ij} \alpha_i^\vee,$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0,\ \ \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0,\ \ i \neq j.$$

The non-zero weights $\alpha \in \mathfrak{h}^*$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ are called the roots, the corresponding root spaces are denoted $\mathfrak{g}_\alpha$. The set Δ of roots is contained in the lattice $Q = \bigoplus_{j=1}^l \mathbb{Z}\alpha_j \subset \mathfrak{h}^*$. Let $Q^+ = \bigoplus_{j=1}^l \mathbb{Z}_{\ge 0}\alpha_j$, and put $\Delta^+ = \Delta \cap Q^+$ and $\Delta^- = -\Delta^+$. One has $\Delta = \Delta^+ \cup \Delta^-$.

Let W be the Weyl group of $\mathfrak{g}$, i.e. the group of transformations of $\mathfrak{h}$ generated by the simple reflections $\xi \mapsto \xi - \langle \alpha_j, \xi \rangle \alpha_j^\vee$. The dual action of W as a reflection group on $\mathfrak{h}^*$ preserves Δ. Let Δ^{re} be the set of real roots, i.e. roots that are W-conjugate to roots in Π, and let Δ^{im} be its complement, the imaginary roots. For $\alpha \in \Delta^{\mathrm{re}}$ one has $\dim \mathfrak{g}_\alpha = 1$.

The length $l(w)$ of a Weyl group element may be characterized as the cardinality of the set

$$\Delta_w^+ = \Delta^+ \cap w\Delta^-$$

of positive roots that become negative under w^{-1} [13, Lemma 1.3.14]. We remark that $\Delta_w^+ \subset \Delta^{\mathrm{re}}$ [5, §5.2].

Fix a real subspace $\mathfrak{h}_\mathbb{R} \subset \mathfrak{h}$ containing $\Pi^\vee$. Let $C \subset \mathfrak{h}_\mathbb{R}$ be the *dominant chamber* and X the *Tits cone* [5, §3.12]. Thus C is the set of all $\xi \in \mathfrak{h}_\mathbb{R}$ such that $\langle \alpha, \xi \rangle \ge 0$ for all $\alpha \in \Pi$, while X is characterized by the property that $\langle \alpha, \xi \rangle < 0$ for at most finitely many $\alpha \in \Delta$. The W-action preserves X, and C is a fundamental domain in the sense that every W-orbit in X intersects C in a unique point.

For any $\mu = \sum_{j=1}^l k_j \alpha_j \in Q$ one defines $\mathrm{ht}(\mu) = \sum_{j=1}^l k_j$. The *principal grading* on $\mathfrak{g}$ is defined by letting $\mathfrak{g}_i$ for $i \neq 0$ be the direct sum of root spaces $\mathfrak{g}_\alpha$ with $\mathrm{ht}(\alpha) = i$, and $\mathfrak{g}_0 = \mathfrak{h}$. Letting $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$, it follows that $\mathfrak{g}_+ = \mathfrak{n}_+$ and $\mathfrak{g}_- = \mathfrak{n}_- \oplus \mathfrak{h}$.

6.2. **The Kac-Peterson cocycle.** Suppose from now on that A is *symmetrizable*, that is, there exists a diagonal matrix $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_l)$ such that $D^{-1}A$ is symmetric. In this case, $\mathfrak{g}$ carries a non-degenerate symmetric invariant bilinear form B with the property $B(\alpha_j^\vee, \xi) = \epsilon_j \langle \alpha_j, \xi\rangle,\ \xi \in \mathfrak{h}$ [5, §2.2]. One refers to B as a *standard* bilinear form. Choose $\rho \in \mathfrak{h}^*$ with $\langle \rho, \alpha_j^\vee\rangle = 1$ for $j = 1, \ldots, l$.

Proposition 6.1. *The Kac-Peterson cocycle of the symmetrizable Kac-Moody algebra $\mathfrak{g}$ is exact. In fact,*

$$\psi_{KP} = d\rho.$$

Proof. Use B to define $\mathrm{Cas}'_{\mathfrak{g}}$. As shown by Kac [5, Theorem 2.6] the operator $\mathrm{Cas}_{\mathfrak{g}} := \mathrm{Cas}'_{\mathfrak{g}} + 2\rho^\sharp$ is $\mathfrak{g}$-invariant. By Corollary 2.8 above this is equivalent to $\psi_{KP} = \mathrm{d}\rho$. □

6.3. **Regular subalgebras.** We now introduce a suitable class of 'equal rank' subalgebras. Following Morita and Naito [17, 18], consider a linearly independent subset $\Pi_{\mathfrak{u}} \subset \Delta^{\mathrm{re},+}$ with the property that the difference of any two elements in Π_u is not a root. We denote by $\mathfrak{u} \subset \mathfrak{g}$ the Lie subalgebra generated by $\mathfrak{h}$ together with the root spaces $\mathfrak{g}_{\pm\beta}$ for $\beta \in \Pi_{\mathfrak{u}}$. Let $\mathfrak{p} = \mathfrak{u}^\perp$, so that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$.

Examples 6.2. (a) If $\Pi_{\mathfrak{u}} = \varnothing$ one obtains $\mathfrak{u} = \mathfrak{h}$. (b) Suppose $\mathfrak{g}$ is an affine Kac-Moody algebra, i.e. the double extension of a loop algebra $\mathfrak{k}[z, z^{-1}]$ of a semi-simple Lie algebra $\mathfrak{k}$. Let $\mathfrak{l} \subset \mathfrak{k}$ be an equal rank subalgebra of $\mathfrak{k}$. Let $\Pi_{\mathfrak{l}} \subset \Delta_{\mathfrak{k}}^+$ be the simple roots of $\mathfrak{l}$, and $\Pi_{\mathfrak{u}} \subset \Delta^+$ the corresponding affine roots. Then $\mathfrak{u} = \mathfrak{l}[z, z^{-1}]$. This is the setting considered in Landweber's paper [14].

It was shown in [17, 18] that $\mathfrak{u}$ is a direct sum (as Lie algebras) of a symmetrizable Kac-Moody algebra $\tilde{\mathfrak{u}}$ with a subalgebra of $\mathfrak{h}$.[2] Furthermore, the standard bilinear form B on $\mathfrak{g}$ restricts to a standard bilinear form on $\tilde{\mathfrak{u}}$.

For any root $\alpha \in \Delta$ put $n_{\mathfrak{u}}(\alpha) = \dim \mathfrak{u}_\alpha$ and $n_{\mathfrak{p}}(\alpha) = \dim(\mathfrak{p}_\alpha)$. Thus $n(\alpha) = n_{\mathfrak{u}}(\alpha) + n_{\mathfrak{p}}(\alpha)$ is the multiplicity of α in $\mathfrak{g}$. Let $\Delta_{\mathfrak{u}}$ (resp. $\Delta_{\mathfrak{p}}$) be the set of roots such that $n_{\mathfrak{u}}(\alpha) > 0$ (resp. $n_{\mathfrak{p}}(\alpha) > 0$). Thus $\Delta_{\mathfrak{u}}$ is the set of roots of $\mathfrak{u}$. Let $W_{\mathfrak{u}} \subset W$ be the Weyl group of $\mathfrak{u}$ (generated by reflections for elements of $\Pi_{\mathfrak{u}}$), and define a subset

$$W_{\mathfrak{p}} = \{w \in W |\ w^{-1}\Delta_{\mathfrak{u}}^+ \subset \Delta^+\}.$$

Lemma 6.3. *We have $w \in W_{\mathfrak{p}} \Leftrightarrow \Delta_w^+ \subset \Delta_{\mathfrak{p}}$. Every $w \in W$ can be uniquely written as a product $w = w_1 w_2$ with $w_1 \in W_{\mathfrak{u}}$ and $w_2 \in W_{\mathfrak{p}}$.*

Proof. By definition, $w \in W_{\mathfrak{p}}$ if and only if the intersection $\Delta_{\mathfrak{u}}^+ \cap w\Delta^- = \Delta_{\mathfrak{u}} \cap \Delta_w^+$ is empty. Since Δ_w^+ consists of real roots, this means $\Delta_w^+ \subset \Delta_{\mathfrak{p}}$. For the second claim, let $C_{\mathfrak{u}} \subset X_{\mathfrak{u}}$ be the chamber and Tits cone for $\mathfrak{u}$. One has $w \in W_{\mathfrak{p}}$ if and only if $w^{-1}\Delta_{\mathfrak{u}}^+ \subset \Delta^+$, if and only if $wC \subset C_{\mathfrak{u}}$. Let $w \in W$ be given. Then $wC \subset X \subset X_{\mathfrak{u}}$ is contained in a unique chamber of $\mathfrak{u}$. Hence there is a unique $w_1 \in W_{\mathfrak{u}}$ such that $wC \subset w_1 C_{\mathfrak{u}}$. Equivalently, $w_2 := w_1^{-1} w \in W_{\mathfrak{p}}$. □

[2] In fact, Naito [18] constructs an explicit subspace $\widetilde{\mathfrak{h}} \subset \mathfrak{h}$ such that the Lie algebra $\widetilde{\mathfrak{g}}$ generated by $\widetilde{\mathfrak{h}}$ and the $\mathfrak{g}_{\pm\beta}$, $\beta \in \Pi_{\mathfrak{u}}$ is a Kac-Moody algebra. He also considers subsets $\Pi_{\mathfrak{u}}$ that do not necessarily consist of real roots, and finds that the resulting $\tilde{u}$ is a symmetrizable *generalized* Kac-Moody algebra.

We have a decomposition $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, where $\mathfrak{p}_\pm = \mathfrak{p} \cap \mathfrak{n}_\pm$. The splitting defines a spinor module $\mathsf{S}_\mathfrak{p} = \wedge \mathfrak{p}_-$ over $\mathrm{Cl}(\mathfrak{p})$, where the elements of $\mathfrak{p}_+$ act by contraction and those of $\mathfrak{p}_-$ by exterior multiplication. The Clifford action on this module extends to the completion $\widehat{\mathrm{Cl}}(\mathsf{S}_\mathfrak{p})$.

Fix $\rho_\mathfrak{u} \in \mathfrak{h}^*$ with $\langle \rho_\mathfrak{u}, \beta^\vee \rangle = 1$ for all $\beta \in \Pi_\mathfrak{u}$. Let $\rho_\mathfrak{p} = \rho|_\mathfrak{u} - \rho_\mathfrak{u}$, defining a Lie algebra homomorphism $\gamma_\mathfrak{p} = \gamma'_\mathfrak{p} + \rho_\mathfrak{p} \colon \mathfrak{u} \to \widehat{\mathrm{Cl}}(\mathfrak{p})$. By composition with the spinor action one obtains an integrable $\mathfrak{u}$-representation

$$\pi_\mathsf{S} \colon \mathfrak{u} \to \mathrm{End}(\mathsf{S}_\mathfrak{p}).$$

Proposition 6.4. *The restriction of π_S to $\mathfrak{h} \subset \mathfrak{u}$ differs from the adjoint representation of $\mathfrak{h}$ by a $\rho_\mathfrak{p}$-shift:*

$$\pi_\mathsf{S}(\xi) = \langle \rho_\mathfrak{p}, \xi \rangle + \mathrm{ad}(\xi), \quad \xi \in \mathfrak{h}.$$

Hence, the weights for the action of $\mathfrak{h}$ on $\mathsf{S}_\mathfrak{p}$ are of the form

$$\rho_\mathfrak{p} - \sum_{\alpha \in \Delta^+_\mathfrak{p}} k_\alpha \alpha,$$

where $0 \le k_\alpha \le n_\mathfrak{p}(\alpha)$. The parity of the corresponding weight space is $\sum_\alpha k_\alpha \mod 2$. For all $w \in W_\mathfrak{p}$, the element

$$w\rho - \rho_\mathfrak{u}$$

is a weight of $\mathsf{S}_\mathfrak{p}$, of multiplicity 1. The parity of the weight space $\mathsf{S}_\mathfrak{p}$ equals $l(w) \mod 2$.

Proof. For each $\alpha \in \Delta^+_\mathfrak{p}$, fix a basis $e^{(s)}_\alpha$, $s = 1, \ldots, n_\mathfrak{p}(\alpha)$ of $\mathfrak{p}_\alpha$, and let $e^{(s)}_{-\alpha}$ be the B-dual basis of $\mathfrak{p}_{-\alpha}$. By definition, we have $\gamma_\mathfrak{p}(\xi) = \langle \rho_\mathfrak{p}, \xi \rangle + \gamma'_\mathfrak{p}(\xi)$ with

$$\gamma'_\mathfrak{p}(\xi) = -\tfrac{1}{2} \sum_{\alpha \in \Delta^+_\mathfrak{p}} \sum_{s=1}^{n_\mathfrak{p}(\alpha)} \langle \alpha, \xi \rangle \; e^{(s)}_{-\alpha} e^{(s)}_\alpha.$$

The action of $\gamma'_\mathfrak{p}(\xi)$ on the spinor module is just the adjoint action of ξ. This proves the first assertion. It is now straightforward to read off the weights of the action on $\mathsf{S}_\mathfrak{p}$. For all $w \in W$ one has $\rho - w\rho = \sum_{\alpha \in \Delta^+_w} \alpha$ (cf. [13, Corollary 1.3.22]). If $w \in W_\mathfrak{p}$, so that $\Delta^+_w \subset \Delta^+_\mathfrak{p}$, it follows that $w\rho - \rho_\mathfrak{u} = w\rho - \rho + \rho_\mathfrak{p} = \rho_\mathfrak{p} - \sum_{\alpha \in \Delta^+_w} \alpha$ is a weight of $\mathsf{S}_\mathfrak{p}$. We now use

$$\mathsf{S}_{\mathfrak{h}^\perp} = \mathsf{S}_\mathfrak{p} \otimes \mathsf{S}_{\mathfrak{u} \cap \mathfrak{h}^\perp}$$

as modules over $\mathrm{Cl}(\mathfrak{h}^\perp) = \mathrm{Cl}(\mathfrak{p}) \otimes \mathrm{Cl}(\mathfrak{u} \cap \mathfrak{h}^\perp)$. Hence, the tensor product with a generator of the line $(\mathsf{S}_{\mathfrak{u} \cap \mathfrak{h}^\perp})_{\rho_\mathfrak{u}}$ defines an isomorphism of the weight space $(\mathsf{S}_\mathfrak{p})_{w\rho - \rho_\mathfrak{u}}$ with $(\mathsf{S}_{\mathfrak{h}^\perp})_{w\rho}$; but the latter is 1-dimensional, and its parity is given by $l(w) \mod 2$ (cf. [13, Lemma 3.2.6]). □

6.4. **Action of the cubic Dirac operator.** The subalgebra $\mathfrak{u}$ inherits a $\mathbb{Z}$-grading from $\mathfrak{g}$, with $\mathfrak{u}_i$ the direct sum of root spaces $\mathfrak{u}_\alpha$ for $\alpha = \sum_r k_r \beta_r$ and $i = \sum_r k_r m_r$. It is thus the *grading of type* $m = (m_1, \ldots, m_r)$ [5, §1.5] with $m_r = \mathrm{ht}(\beta_r)$. Let $\widehat{\mathcal{W}}(\mathfrak{u})$ be the completion of the quantum Weil algebra for this grading. (It is just the same as the completion defined by the principal grading of $\mathfrak{u}$).

Let $P \subset \mathfrak{h}^*$ be the weight lattice of $\mathfrak{g}$, and $P^+ \subset P$ the dominant weights. Thus $\mu \in P$ if and only if $\langle \mu, \alpha_j^\vee \rangle \in \mathbb{Z}$ for $j = 1, \ldots, l$, and $\mu \in P^+$ if these pairings are all non-negative.

For any $\mu \in P^+$ let $L(\mu)$ be the irreducible integrable representation of $\mathfrak{g}$ of highest weight μ. By [5, §11.4], $L(\mu)$ carries a unique (up to scalar) Hermitian form for which the elements of the real form of $\mathfrak{g}$ are represented as skew-adjoint operators. The weights ν of $L(\mu)$ satisfy $\mu - \nu \in Q^+$, hence there is a $\mathbb{Z}$-grading on $L(\mu)$ such that elements of $L(\mu)_\nu$ have degree $j = -\operatorname{ht}(\mu - \nu)$. The $\mathfrak{g}$-action is compatible with the gradings, i.e. the action map $\mathfrak{g} \otimes L(\mu) \to L(\mu)$ preserves gradings. The spinor module $\mathsf{S}_\mathfrak{p} = \wedge \mathfrak{p}_-$ carries the $\mathbb{Z}$-grading defined by the $\mathbb{Z}$-grading on $\mathfrak{p}_-$, and the module action $\mathrm{Cl}(\mathfrak{p}) \otimes \mathsf{S}_\mathfrak{p} \to \mathsf{S}_\mathfrak{p}$ preserves gradings. The action of $\mathcal{W}(\mathfrak{g}, \mathfrak{u})$ on the graded vector space $L(\mu) \otimes \mathsf{S}_\mathfrak{p}$ extends to an action of the completion $\widehat{\mathcal{W}}(\mathfrak{g}, \mathfrak{u})$. We denote by

$$\mathcal{D}_{L(\mu)} \in \widehat{\mathrm{End}}(L(\mu) \otimes \mathsf{S}_\mathfrak{p})$$

the image of $\mathcal{D}_{\mathfrak{g},\mathfrak{u}}$ under this representation. Then $\mathcal{D}_{L(\mu)}$ is an odd, skew-adjoint operator.

Since $\mathcal{D}_{L(\mu)}$ commutes with the diagonal action of $\mathfrak{u}$ on $L(\mu) \otimes \mathsf{S}_\mathfrak{p}$, its kernel $\ker(\mathcal{D}_{L(\mu)})$ is a $\mathbb{Z}_2$-graded $\mathfrak{u}$-representation.

Let $P_\mathfrak{u}^+ \subset P_\mathfrak{u} \subset \mathfrak{h}^*$ be the set of dominant weights for $\mathfrak{u}$. For any $\nu \in P_\mathfrak{u}^+$, let $M(\nu)$ be the corresponding irreducible highest weight representation of $\mathfrak{u}$. Parallel to [10, Theorem 4.24] we have:

Theorem 6.5. *The kernel of the operator $\mathcal{D}_{L(\mu)}$ is a direct sum,*

$$\ker(D_{L(\mu)}) = \bigoplus_{w \in W_\mathfrak{p}} M(w(\mu + \rho) - \rho_\mathfrak{u}).$$

Here the even (resp. odd) part of the kernel is the sum over the $w \in W_\mathfrak{p}$ such that $l(w)$ is even (resp. odd).

Proof. Given an integrable $\mathfrak{u}$-representation, and any $\mathfrak{u}$-dominant weight $\nu \in P_\mathfrak{u}^+$, let the subscript $[\nu]$ denote the corresponding isotypical subspace. We are interested in $\ker(D_{L(\mu)})_{[\nu]}$. Since $\mathcal{D}_{L(\mu)}$ is skew-adjoint, its kernel coincides with that of its square:

$$\ker(\mathcal{D}_{L(\mu)}) = \ker(\mathcal{D}_{L(\mu)}^2).$$

The action of $\mathrm{Cas}_\mathfrak{g}$ on $L(\mu)$ is as a scalar $B(\mu + \rho, \mu + \rho) - B(\rho, \rho)$, and similarly for the action of $\mathrm{Cas}_\mathfrak{u}$ on $M(\nu)$. Hence

$$\mathcal{D}_{L(\mu)}^2 = B(\mu + \rho, \mu + \rho) - j(\mathrm{Cas}_\mathfrak{u}) - B(\rho_\mathfrak{u}, \rho_\mathfrak{u})$$

acts on $(L(\mu) \otimes \mathsf{S}_\mathfrak{p})_{[\nu]}$ as a scalar, $B(\mu + \rho, \mu + \rho) - B(\nu + \rho_\mathfrak{u}, \nu + \rho_\mathfrak{u})$. This shows that

$$\ker(\mathcal{D}_{L(\mu)})_{[\nu]} = {\bigoplus}'_\nu (L(\mu) \otimes \mathsf{S}_\mathfrak{p})_{[\nu]},$$

where the sum ${\bigoplus}'_\nu$ is over all $\nu \in \Delta_\mathfrak{u}$ satisfying $B(\mu + \rho, \mu + \rho) = B(\nu + \rho_\mathfrak{u}, \nu + \rho_\mathfrak{u})$. We want to identify this sum as a sum over $W_\mathfrak{p}$.

Suppose ν is any weight with $(L(\mu) \otimes \mathsf{S}_\mathfrak{p})_\nu \neq 0$. We will show $B(\nu + \rho_\mathfrak{u},\ \nu + \rho_\mathfrak{u}) \leq B(\mu + \rho,\ \mu + \rho)$. By [5, Prop. 11.4(b)], an element $\nu \in P_\mathfrak{u}$ for which equality holds is automatically in $P_\mathfrak{u}^+$, and the multiplicity of $M(\nu)$ in $L(\mu) \otimes \mathsf{S}_\mathfrak{p}$ is then equal to the dimension of the highest weight space $(L(\mu) \otimes \mathsf{S}_\mathfrak{p})_\nu$. Write $\nu = \nu_1 + \nu_2$ where $L(\mu)_{\nu_1}$ and $(\mathsf{S}_\mathfrak{p})_{\nu_2}$ are non-zero. By our description of the set of weights of $\mathsf{S}_\mathfrak{p}$, the element $\nu_2 + \rho_\mathfrak{u}$ is among the weights of the $\mathfrak{g}$-representation $L(\rho)$, and in particular lies in the dual Tits cone $X^\vee$ of $\mathfrak{g}$. Since the Tits cone is convex, and $\nu_1 \in X^\vee$, it follows that $\nu_1 + (\nu_2 + \rho_\mathfrak{u}) = \nu + \rho_\mathfrak{u} \in X^\vee$.

Consequently, there exists $w \in W$ such that $w^{-1}(\nu + \rho_\mathfrak{u}) \in C^\vee \subset \mathfrak{h}^*$. Since $\nu_2 + \rho_\mathfrak{u}$ is a weight of $L(\rho)$, so is its image under w^{-1}. Hence

$$\kappa_2 = \rho - w^{-1}(\nu_2 + \rho_\mathfrak{u}) \in Q^+.$$

On the other hand, since $w^{-1}\nu_1$ is a weight of $L(\mu)$, we also have $\kappa_1 = \mu - w^{-1}\nu_1 \in Q^+$. Adding, we obtain

$$\mu + \rho = \kappa + w^{-1}(\nu + \rho_\mathfrak{u}).$$

with $\kappa = \kappa_1 + \kappa_2 \in Q^+$. Since the pairing of κ with $w^{-1}(\nu + \rho_\mathfrak{u}) \in C^\vee$ is non-negative, the inequality $B(\mu + \rho, \mu + \rho) \geq B(\nu + \rho_\mathfrak{u}, \nu + \rho_\mathfrak{u})$ follows. Equality holds if and only if $\kappa = 0$, i.e. $\kappa_1 = 0$ and $\kappa_2 = 0$, i.e. $\nu_2 = w\rho - \rho_\mathfrak{u}$ and $\nu_1 = w\mu$. The $\mathfrak{h}$-weight spaces $(\mathsf{S}_\mathfrak{p})_{w\rho - \rho_\mathfrak{u}}$ and $L(\mu)_{w\mu}$ are 1-dimensional, hence so is their tensor product, $(L(\mu) \otimes \mathsf{S}_\mathfrak{p})_\nu$. It follows that ν appears with multiplicity 1.

This shows that $M(\nu)$ appears in $\ker(D_{L(\mu)})$ if and only if it can be written in the form $\nu = w(\mu + \rho) - \rho_\mathfrak{u}$, for some $w \in W_\mathfrak{p}$, and in this case it appears with multiplicity 1. Note finally that w with this property is unique, since $\mu + \rho$ is regular. The parity of the ν-isotypical component follows since $(\mathsf{S}_\mathfrak{p})_{w\rho - \rho_\mathfrak{u}}$ has parity equal to that of $l(w)$. □

The weights

$$\nu = w(\mu + \rho) - \rho_\mathfrak{u}, \; w \in W_\mathfrak{p}$$

are referred to as the *multiplet* corresponding to μ. Note that for given μ, the value of the quadratic Casimir $\mathrm{Cas}_\mathfrak{u}$ on the representations $M(w(\mu + \rho) - \rho_\mathfrak{u})$ is given by the constant value $B(\mu + \rho, \mu + \rho) - B(\rho_\mathfrak{u}, \rho_\mathfrak{u})$, independent of w.

6.5. **Characters.** For any weight $\nu \in \mathfrak{h}^*$, we write $\mathsf{e}(\nu)$ for the corresponding formal exponential. We will regard the spinor module as a super representation, using the usual $\mathbb{Z}_2$-grading of the exterior algebra. The even and odd part are denoted $\mathsf{S}_\mathfrak{p}^{\bar{0}}$ and $\mathsf{S}_\mathfrak{p}^{\bar{1}}$, and its formal character $\mathsf{ch}(\mathsf{S}_\mathfrak{p}) = \sum_\nu (\dim(\mathsf{S}_\mathfrak{p}^{\bar{0}})_\nu - \dim(\mathsf{S}_\mathfrak{p}^{\bar{1}})_\nu)\, \mathsf{e}(\nu)$. Here $(\mathsf{S}_\mathfrak{p}^{\bar{0}})_\nu$ and $(\mathsf{S}_\mathfrak{p}^{\bar{1}})_\nu$ are the $\mathfrak{h}$ weight spaces, and $\mathsf{e}(\nu)$ is the formal character defined by ν (cf. [5, §10.2]).

Proposition 6.6. *The super character of the spin representation of* $\mathfrak{u}$ *on* $\mathfrak{p}$ *is given by the formula*

$$\mathsf{ch}(\mathsf{S}_\mathfrak{p}) = \mathsf{e}(\rho_\mathfrak{p}) \prod_{\alpha \in \Delta_\mathfrak{p}^+} (1 - \mathsf{e}(-\alpha))^{n_\mathfrak{p}(\alpha)}.$$

Proof. For each root space $\mathfrak{p}_{-\alpha}$, the character of the adjoint action of $\mathfrak{h}$ on $\wedge\mathfrak{p}_{-\alpha}$ equals $(1 - \mathsf{e}(-\alpha))^{n_\mathfrak{p}(\alpha)}$. The character of the adjoint action on $\wedge\mathfrak{p}_- = \bigotimes_{\alpha \in \Delta_\mathfrak{p}^+} \wedge\mathfrak{p}_{-\alpha}$ is the product of the characters on $\wedge\mathfrak{p}_{-\alpha}$. By Proposition 6.4 the action of $\mathfrak{h}$ as a subalgebra of $\mathfrak{u}$ differs from the adjoint action by a $\rho_\mathfrak{p}$-shift accounting for an extra factor $\mathsf{e}(\rho_\mathfrak{p})$. □

Consider $L(\mu) \otimes \mathsf{S}_\mathfrak{p}$ as a super representation of $\mathfrak{u}$. Its formal super character is

$$\mathsf{ch}(L(\mu) \otimes \mathsf{S}_\mathfrak{p}) = \mathsf{ch}(L(\mu))\mathsf{ch}(\mathsf{S}_\mathfrak{p}).$$

On the other hand, since $D_{L(\mu)}$ is an odd skew-adjoint operator on this space, this coincides with

$$\mathsf{ch}(\ker(D_{L(\mu)})) = \sum_{w \in \mathfrak{p}} (-1)^{l(w)} \mathsf{ch}(M(w(\mu + \rho) - \rho_\mathfrak{u})).$$

This gives the generalized Weyl-Kac character formula,

$$\mathsf{ch}(L(\mu)) = \frac{\sum_{w\in W_{\mathfrak{p}}}(-1)^{l(w)}\mathsf{ch}(M(w(\mu+\rho)-\rho_{\mathfrak{u}}))}{\mathsf{e}(\rho_{\mathfrak{p}})\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}(1-\mathsf{e}(-\alpha))^{n_{\mathfrak{p}}(\alpha)}},$$

valid for quadratic subalgebras $\mathfrak{u}\subset\mathfrak{g}$ of the form considered above. For $\mathfrak{u}=\mathfrak{h}$ one recovers the usual Weyl-Kac character formula [5, §10.4] for symmetrizable Kac-Moody algebras. Note that the Weyl-Kac character formula also holds for the non-symmetrizable case, see Kumar [13, Chapter 3.2]. We do not know how to treat this general case using cubic Dirac operators.

Example 6.7. As a concrete example, consider the Kac-Moody algebra of hyperbolic type, associated to the generalized Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

(cf. [5, Exercise 5.28]). The Weyl group W is generated by the reflections r_1, r_2 corresponding to α_1, α_2. The set P^+ of dominant weights is generated by $\varpi_1 = -\frac{1}{5}(2\alpha_1 + 3\alpha_2)$ and $\varpi_2 = -\frac{1}{5}(2\alpha_2 + 3\alpha_1)$. One has $\rho = \varpi_1 + \varpi_2 = -(\alpha_1 + \alpha_2)$.

Put $\Pi_{\mathfrak{u}} = \{\beta_1, \beta_2\}$ with

$$\beta_1 = \alpha_1, \qquad \beta_2 = r_2(\alpha_1) = \alpha_1 + 3\alpha_2.$$

Since $\beta_2 - \beta_1 = 3\alpha_2$ is not a root, $\Pi_{\mathfrak{u}}$ is the set of simple roots for a Kac-Moody Lie subalgebra $\mathfrak{u}\subset\mathfrak{g}$. One finds that $\rho_{\mathfrak{u}} = \varpi_1$, and the fundamental $\mathfrak{u}$-weights spanning $P_{\mathfrak{u}}^+$ are $\tau_1 = \varpi_1 - \frac{1}{3}\varpi_2$ and $\tau_2 = \frac{1}{3}\varpi_2$.

The Weyl group $W_{\mathfrak{u}}$ is generated by the reflections defined by β_1, β_2, i.e. by r_1 and $r_2 r_1 r_2$. A general element of $W_{\mathfrak{u}}$ is thus a word in r_1, r_2, with an even number of r_2's. One has

$$W_{\mathfrak{p}} = \{1, r_2\},$$

giving duplets of $\mathfrak{u}$-representations. Write weights $\mu\in P^+$ in the form $\mu = k_1\varpi_1 + k_2\varpi_2$. Then the corresponding duplet is given by the weights

$$\begin{aligned}\mu+\rho-\rho_{\mathfrak{u}} &= k_1\varpi_1 + (k_2+1)\varpi_2 = k_1\tau_1 + (k_1+3k_2+3)\tau_2\\ r_2(\mu+\rho)-\rho_{\mathfrak{u}} &= (k_1+3(k_2+1))\varpi_1 - (k_2+1)\varpi_2 = (k_1+3k_2+3)\tau_1 + k_2\tau_2.\end{aligned}$$

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