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Clifford algebras and the Duflo isomorphism

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Abstract

This article summarizes joint work with A. Alekseev (Geneva) on the Duflo isomorphism for quadratic Lie algebras. We describe a certain quantization map for Weil algebras, generalizing both the Duflo map and the quantization map for Clifford algebras. In this context, Duflo's theorem generalizes to a statement in equivariant cohomology.

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1. Introduction

The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the relations, $\xi\xi' - \xi'\xi = [\xi, \xi']_{\mathfrak{g}}$. The inclusion of the symmetric algebra $S(\mathfrak{g})$ into $T(\mathfrak{g})$ as totally symmetric tensors, followed by the quotient map, gives an isomorphism of $\mathfrak{g}$-modules

$$\mathrm{sym}:\ S(\mathfrak{g}) \to U(\mathfrak{g}) \tag{1.1}$$

called the *symmetrization map*. The restriction of sym to $\mathfrak{g}$-invariants is a vector space isomorphism, but not an algebra isomorphism, from invariant polynomials to the center of the enveloping algebra. Let $J \in C^\infty(\mathfrak{g})$ be the function

$$J(\xi) = \det(j(\mathrm{ad}_\xi)), \quad j(z) = \frac{\sinh(z/2)}{z/2},$$

and $J^{1/2}$ its square root (defined in a neighborhood of $\xi = 0$). Denote by $\widehat{J^{1/2}}$ the infinite order differential operator on $S\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$, obtained by replacing the

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variable $\xi \in \mathfrak{g}$ with a directional derivative $\frac{\partial}{\partial \mu}$, where μ is the dual variable on $\mathfrak{g}^*$. Duflo's celebrated theorem says that the composition

$$\mathrm{sym} \circ \widehat{J^{1/2}}:\ S\mathfrak{g} \to U(\mathfrak{g})$$

restricts to an algebra isomorphism, $(S\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cent}(U(\mathfrak{g}))$. In more geometric language, Duflo's theorem gives an isomorphism between the algebra of invariant constant coefficient differential operators on $\mathfrak{g}$ and bi-invariant differential operators on the corresponding Lie group G.

The purpose of this note is to give a quick overview of joint work with A. Alekseev [1, 2], in which we obtained a new proof and a generalization of Duflo's theorem for the special case of a *quadratic* Lie algebra. That is, we assume that $\mathfrak{g}$ comes equipped with an invariant, non-degenerate, symmetric bilinear form B. Examples of quadratic Lie algebras include semi-simple Lie algebras, or the semi-direct product $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{s}^*$ of a Lie algebra $\mathfrak{s}$ with its dual. Using B we can define the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Duflo's factor $J^{1/2}(\xi)$ arises as the Berezin integral of $\exp(q(\lambda(\xi))) \in \mathrm{Cl}(\mathfrak{g})$, where $q:\ \wedge(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ is the quantization map, and $\lambda:\ \mathfrak{g} \to \wedge^2\mathfrak{g}$ is the map dual to the Lie bracket.

2. Clifford algebras

Let V be a finite-dimensional real vector space, equipped with a non-degenerate symmetric bilinear form B. Fix a basis $e_a \in V$ and let $e^a \in V$ be the dual basis. We denote by $\mathfrak{o}(V) \subset \mathrm{End}(V)$ the space of endomorphisms A of V that are skew-symmetric with respect to B. For any $A \in \mathfrak{o}(V)$ we denote its components by $A_{ab} = B(e_a, Ae_b)$. Consider the function $\mathcal{S}:\ \mathfrak{o}(V) \to \wedge(V)$ given by

$$\mathcal{S}(A) = \det{}^{1/2}\,(j(A))\ \exp_{\wedge(V)}\left(\frac{1}{2}f(A)_{ab}e^a \wedge e^b\right)$$

(using summation convention), where

$$f(z) = (\ln j)'(z) = \frac{1}{2}\coth(\frac{z}{2}) - \frac{1}{z}. \tag{2.1}$$

In turns out that, despite the singularities of the exponential, $\mathcal{S}$ is a global analytic function on all of $\mathfrak{o}(V)$. It has the following nice property. Let $\mathrm{Cl}(V)$ denote the Clifford algebra of V, defined as a quotient of the tensor algebra $T(V)$ by the relations $vv' + v'v = B(v, v')$. The inclusion of $\wedge(V)$ into $T(V)$ as totally anti-symmetric tensors, followed by the quotient map to $\mathrm{Cl}(V)$, gives a vector space isomorphism

$$q:\ \wedge(V) \to \mathrm{Cl}(V)$$

known as the *quantization map*. Then $\mathcal{S}(A)$ relates the exponentials of quadratic elements $1/2A_{ab}e^a \wedge e^b$ in the exterior algebra with the exponentials of the corresponding elements $1/2A_{ab}e^ae^b$ in the Clifford algebra:

$$\exp_{\mathrm{Cl}(V)}(1/2A_{ab}e^ae^b) = q\left(\iota(\mathcal{S}(A))\exp_{\wedge(V)}(1/2A_{ab}e^a \wedge e^b)\right). \tag{2.2}$$

Here $\iota : \wedge(V) \to \mathrm{End}(V)$ is the contraction operator. In fact, one may add linear terms to the exponent: Let E be some vector space of "parameters", and $\phi^a \in E$. Then the following identity holds in the $\mathbb{Z}_2$-graded tensor product $\mathrm{Cl}(V) \otimes \wedge(E)$:

$$\exp(1/2 A_{ab} e^a e^b + e_a \otimes \phi^a) = q\Big(\iota(\mathcal{S}(A)) \exp(1/2 A_{ab} e^a \wedge e^b + e_a \otimes \phi^a)\Big).$$

3. Quadratic Lie algebras

Let us now consider the case $V = \mathfrak{g}$ of a quadratic Lie algebra. Invariance of the bilinear form B means that the the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ takes values in $\mathfrak{o}(\mathfrak{g})$, or equivalently that the structure constants $f_{abc} = B(e_a, [e_b, e_c])$ are invariant under cyclic permutations of the indices a, b, c. We specialize (2.2) to $A = \mathrm{ad}_\xi$ for $\xi \in \mathfrak{g}$, so that $\lambda^{\mathfrak{g}} : \mathfrak{g} \to \wedge^2 \mathfrak{g}$,

$$\lambda^{\mathfrak{g}}(\xi) = 1/2 (\mathrm{ad}_\xi)_{ab} e^a \wedge e^b$$

is the map dual to the Lie bracket. Also, take $E = T^*_\xi \mathfrak{g}$ and $\phi^a = -d\,\xi^a$, where $\xi^a = B(\xi, e^a)$ are the coordinate functions. Then our formula become the following identity in $\mathrm{Cl}(\mathfrak{g}) \otimes \Omega(\mathfrak{g})$

$$\exp\big(q(\lambda^{\mathfrak{g}}) - e_a\, d\xi^a\big) = q\Big(\iota(\mathcal{S}^{\mathfrak{g}}) \exp(\lambda^{\mathfrak{g}} - e_a\, d\xi^a)\Big), \tag{3.1}$$

where $\mathcal{S}^{\mathfrak{g}} = \mathcal{S} \circ \mathrm{ad} : \mathfrak{g} \to \wedge \mathfrak{g}$. Consider now the following cubic element in the Clifford algebra,

$$\mathcal{C} = \frac{1}{6} f_{abc} e^a\, e^b\, e^c \in \mathrm{Cl}(\mathfrak{g}).$$

A beautiful observation of Kostant-Sternberg [8] says that $\mathcal{C}$ squares to a *constant*:

$$\mathcal{C}^2 = -\frac{1}{48} f_{abc} f^{abc}.$$

It follows that the graded commutator $\mathrm{d}^{\mathrm{Cl}\,\mathfrak{g}} := [\mathcal{C}, \cdot\,]$ defines a differential on $\mathrm{Cl}(\mathfrak{g})$. This *Clifford differential* is compatible with the filtration of $\mathrm{Cl}(\mathfrak{g})$, and the induced differential $\mathrm{d}^{\wedge\mathfrak{g}}$ on the associated graded algebra $\mathrm{gr}(\mathrm{Cl}(\mathfrak{g})) = \wedge\mathfrak{g}$ is nothing but the Lie algebra differential. Let d^{Rh} denote the exterior differential on the deRham complex $\Omega(\mathfrak{g})$.

It is easily verified that $\lambda^{\mathfrak{g}} - e_a\, d\xi^a \in \wedge\mathfrak{g} \otimes \Omega(\mathfrak{g})$ is closed for the differential $\mathrm{d}^{\wedge\mathfrak{g}} + \mathrm{d}^{\mathrm{Rh}}$, while $q(\lambda^{\mathfrak{g}}) - e_a\, d\xi^a \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(\mathfrak{g})$ is closed under $\mathrm{d}^{\mathrm{Cl}(\mathfrak{g})} + \mathrm{d}^{\mathrm{Rh}}$. Together with (3.1), this leads to a number of consistency conditions for the function $\mathcal{S}^{\mathfrak{g}}$. One of these conditions gives a solution of the *classical dynamical Yang-Baxter equation* (CDYBE): Let $\mathfrak{r} : \mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$ be the meromorphic function $\mathfrak{r}^{\mathfrak{g}}(\xi) = f(\mathrm{ad}_\xi)$ appearing in the exponential factor of $\mathcal{S}^{\mathfrak{g}}$. Then

$$\mathrm{cycl}_{abc}\left(\frac{\partial \mathfrak{r}_{ab}}{\partial \xi^c} - \mathfrak{r}_{ak} f_b^{kl} \mathfrak{r}_{lc}\right) = -\frac{1}{4} f_{abc} \tag{3.2}$$

where cycl_{abc} denotes the sum over cyclic permutations of a, b, c. This solution of the CDYBE was obtained by Etingof-Varchenko [5] and in [1] by different methods.

In Etingof-Schiffmann [6], it is shown that $\mathfrak{r}_{ab}$ is in fact the *unique* solution of this particular CDYBE, up to gauge transformation. More general CDYBE's are associated to a pair $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras, here $\mathfrak{h} = \mathfrak{g}$. The proof sketched above can be modified to produce some of these more general solutions.

4. The non-commutative Weil algebra

Using B to identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, the Weil algebra of $\mathfrak{g}$ is the $\mathbb{Z}$-graded $\mathfrak{g}$-module given as a tensor product

$$W\mathfrak{g} = S\mathfrak{g} \otimes \wedge\mathfrak{g},$$

where generators of $S\mathfrak{g}$ are assigned degree 2. Let L_ξ^W for $\xi \in \mathfrak{g}$ denote the generators for the $\mathfrak{g}$-action on $W\mathfrak{g}$, and $\iota_\xi^W = 1 \otimes \iota_\xi$ the contraction operators. The *Weil differential* d^W is a derivation of degree 1, uniquely characterized by its properties $\mathrm{d}^W \circ \mathrm{d}^W = 0$ and $\mathrm{d}^W(1 \otimes \xi) = \xi \otimes 1$ for $\xi \in \mathfrak{g}$. The Weil algebra $W\mathfrak{g}$ with these three types of derivations is an example of a $\mathfrak{g}$-differential algebra: That is, $L_\xi^W, \iota_\xi^W, \mathrm{d}^W$ satisfy relations similar to contraction operators, Lie derivatives, and de Rham differential for a manifold with group action.

In [1], we introduced the following non-commutative version of the Weil algebra,

$$\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}).$$

It carries a $\mathbb{Z}$-filtration, where generators of $U(\mathfrak{g})$ are assigned filtration degree 2, with associated graded algebra $\mathrm{gr}(\mathcal{W}\mathfrak{g}) = W\mathfrak{g}$. Moreover, it carries a $\mathbb{Z}_2$-grading, compatible with the $\mathbb{Z}$-filtration in the sense of [8]. Define contraction operators as $\mathbb{Z}_2$-graded commutators $\iota_\xi^{\mathcal{W}} = [1 \otimes \xi, \cdot]$, let $L_\xi^{\mathcal{W}}$ be the generators for the natural $\mathfrak{g}$-module structure, and set $\mathrm{d}^{\mathcal{W}} = [\mathcal{D}, \cdot]$ where

$$\mathcal{D} = e_a \otimes e^a - 1 \otimes \mathcal{C} \in \mathcal{W}\mathfrak{g}$$

is the *cubic Dirac operator* [7]. Its square

$$\mathcal{D}^2 = \frac{1}{2} e_a e^a \otimes 1 - \frac{1}{48} f_{abc} f^{abc}$$

is in the center of $\mathcal{W}\mathfrak{g}$, hence $\mathrm{d}^{\mathcal{W}}$ is a differential. As it turns out, $\mathcal{W}\mathfrak{g}$ is again a $\mathfrak{g}$-differential algebra. The derivations $\mathrm{d}^{\mathcal{W}}, \iota_\xi^{\mathcal{W}}, L_\xi^{\mathcal{W}}$ respect the $\mathbb{Z}$-filtration, and the induced derivations on the associated graded algebra are just the standard derivations for the Weil algebra $W\mathfrak{g}$.

The vector space isomorphism $\mathrm{sym} \otimes q : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ intertwines the contraction operators and Lie derivatives, but not the differentials. There does exist, however, a better "quantization map" $\mathcal{Q} : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ that is also a chain map. Using our function $\mathcal{S}^\mathfrak{g} \in C^\infty(\mathfrak{g}) \otimes \wedge\mathfrak{g}$, let $\iota(\widehat{\mathcal{S}^\mathfrak{g}})$ denote the operator on $W\mathfrak{g}$, where the $\wedge\mathfrak{g}$-factor acts by contraction on $\wedge\mathfrak{g}$ and the $C^\infty(\mathfrak{g})$-factor as an infinite order differential operator.

Theorem [1]). *The quantization map*

$$\mathcal{Q} := (\mathrm{sym} \otimes q) \circ \iota(\widehat{\mathcal{S}^{\mathfrak{g}}}) : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$$

intertwines the contraction operators, Lie derivatives, and differentials on $W\mathfrak{g}$ and on $\mathcal{W}\mathfrak{g}$.

The fact that $\mathcal{Q}$ intertwines the two differentials d^W, $\mathrm{d}^{\mathcal{W}}$ relies on a number of special properties of the function $\mathcal{S}^{\mathfrak{g}}$, including the CDYBE.

Put differently, the quantization map $\mathcal{Q}$ defines a new, graded non-commutative ring structure on the Weil algebra $W\mathfrak{g}$, in such a way that the derivations $\iota^W_\xi, L^W_\xi, \mathrm{d}^W$ are still derivations for the new ring structure, and in fact become *inner* derivations. Notice that $\mathcal{Q}$ restricts to the quantization map for Clifford algebras $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ on the second factor and to the Duflo map on the first factor, but is not just the product of these two maps.

5. Equivariant cohomology

H. Cartan in [3] introduced the Weil algebra $W\mathfrak{g}$ as an algebraic model for the algebra of differential forms on the classifying bundle EG, at least in the case G compact.

In particular, it can be used to compute the equivariant cohomology $H_G(M)$ (with real coefficients) for any G-manifold M. Let $\iota^{Rh}_\xi, L^{Rh}_\xi, \mathrm{d}^{Rh}$ denote the contraction operators, Lie derivatives, and differential on the de Rham complex $\Omega(M)$ of differential forms. Let

$$H_{\mathfrak{g}}(M) = H((W\mathfrak{g} \otimes \Omega(M))_{\mathrm{basic}}, \mathrm{d}^W + \mathrm{d}^{Rh})$$

where $(W\mathfrak{g} \otimes \Omega(M))_{\mathrm{basic}}$ is the subspace annihilated by all Lie derivatives $L^W_\xi + L^{Rh}_\xi$ and all contraction operators $\iota^W_\xi + \iota^{Rh}_\xi$. Cartan's result says that $H_{\mathfrak{g}}(M) = H_G(M, \mathbb{R})$ provided G is compact.

More generally, we can define $H_{\mathfrak{g}}(\mathcal{A})$ for any $\mathfrak{g}$-differential algebra $\mathcal{A}$. Let $\mathcal{H}_{\mathfrak{g}}(\mathcal{A})$ be defined by replacing $W\mathfrak{g}$ with $\mathcal{W}\mathfrak{g}$. The quantization map $\mathcal{Q} : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ induces a map $\mathcal{Q} : H_{\mathfrak{g}}(\mathcal{A}) \to \mathcal{H}_{\mathfrak{g}}(\mathcal{A})$.

Theorem [1]). *For any $\mathfrak{g}$-differential algebra $\mathcal{A}$, the vector space isomorphism $\mathcal{Q} : H_{\mathfrak{g}}(\mathcal{A}) \to \mathcal{H}_{\mathfrak{g}}(\mathcal{A})$ is in fact an algebra isomorphism.*

Our proof is by construction of an explicit chain homotopy between the two maps $W\mathfrak{g} \otimes W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ given by "quantization followed by multiplication" and "multiplication followed by quantization", respectively. Taking $\mathcal{A}$ to be the trivial $\mathfrak{g}$-differential algebra (i.e. $\mathcal{A} = \Omega(\mathrm{point})$), the statement specializes to Duflo's theorem for quadratic $\mathfrak{g}$.

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