DUISTERMAAT-HECKMAN MEASURES AND MODULI SPACES OF FLAT BUNDLES OVER SURFACES

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ABSTRACT. We introduce Liouville measures and Duistermaat-Heckman measures for Hamiltonian group actions with group valued moment maps. The theory is illustrated by applications to moduli spaces of flat bundles on surfaces.

1. Introduction

One of the fundamental invariants of a Hamiltonian G-manifold M in symplectic geometry is the Duistermaat-Heckman (DH) measure on the dual of the Lie algebra $\mathfrak{g}^*$, defined as the push-forward of the Liouville measure under the moment map. The DH-measure encodes volumes of reduced spaces, and by the Duistermaat-Heckman theorem [12] its derivatives describe Chern numbers for the corresponding level set.

The purpose of this paper is to develop a theory of Liouville measures and DH measures for Hamiltonian G-manifolds (M, ω, Φ) with group valued moment maps $\Phi : M \to G$ as introduced in [1]. The basic examples of such spaces are conjugacy classes $\mathcal{C} \subset G$, with moment map the inclusion into G. We allow G to be disconnected, and this leads to many new examples. In particular, we show that all compact symmetric spaces are Hamiltonian G-manifolds with group-valued moment map. Another example is the space G^{2h}, with moment map the product of Lie group commutators, $(a_1, b_1, \ldots, a_h, b_h) \mapsto \prod_j [a_j, b_j]$. There is a notion of reduction for group valued moment maps, and the reduced spaces $M/\!\!/G = \Phi^{-1}(e)/G$ are symplectic. For instance, $G^{2h}/\!\!/G$ is the moduli space of flat G-bundles over a surface of genus h, with its natural structure of a (stratified) symplectic space [5, 6].

The 2-form ω for a space with group valued moment map is usually degenerate. The kernel of ω is controlled by the *minimal degeneracy* condition, involving the moment map Φ. If the group G is connected and simply connected, we obtain a volume form as the top degree part of

$$\Gamma = e^{\omega}\, \Phi^* \mathcal{T},$$

where $\mathcal{T}$ is a differential form on G constructed from the homomorphism of G to the Spin-group of $\mathfrak{g} := \mathrm{Lie}(\mathfrak{g})$. The group $\mathrm{Spin}(\mathfrak{g})$ is a double cover of the group of orthogonal transformations $\mathrm{SO}(\mathfrak{g})$ with respect to an $\mathrm{ad}(\mathfrak{g})$-invariant inner product on $\mathfrak{g}$. In [4], the formula for the volume form is heuristically obtained as a Feynman integral over the path fibration $P_0G \to G$. For general compact groups G, group valued Hamiltonian

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G-manifolds are not necessarily orientable: an example is the conjugacy class in SO(3) consisting of rotations by π, which is isomorphic to the real projective plane. In such cases, we can still define Γ as a form with values in the orientation bundle, and obtain a nowhere vanishing density on M. We call this density the Liouville measure, and its push-forward to G the DH measure of (M, ω, Φ). Just as in the $\mathfrak{g}^*$-valued theory, the DH-measures describe volumes of reduced spaces. One of our main results says that the Liouville measure for a product of two spaces with group valued moment map is equal to the direct product of the Liouville measures, and consequently the DH-measure is the convolution on G of the DH-measures of its factors.

For the example G^{2h}, we show that the Liouville measure coincides with Haar measure. The push-forward of Haar measure on G^{2h} under the map $(a_1, b_1, \ldots, a_h, b_h) \mapsto \prod_j [a_j, b_j]$ gives Witten's Fourier series [22, Equation (2.73)]. Witten, Bismut-Labourie [9] and Liu [18] use a Reidemeister torsion calculation to identify this expression with the symplectic volume of moduli spaces of flat G-bundles over surfaces. Our construction provides a purely symplectic approach to the Witten volume formulas, and gives an extension to disconnected compact Lie groups.

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Notation.

Throughout this paper G denotes a compact Lie group, and G^{o} its identity component. The left/right invariant Maurer-Cartan forms on G are denoted $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$. We fix an invariant inner product on the Lie algebra $\mathfrak{g}$ of G. The corresponding Riemannian measure on G will be denoted $\mathrm{d}\,\mathrm{vol}_G$, and the Riemannian volume $\mathrm{vol}\, G$.

A G-manifold is a manifold M together with a group homomorphism $\phi:\ G \to \mathrm{Diff}(M)$ such that the action map $G \times M \to M$, $(g, m) \mapsto g.m := \phi(g)(m)$ is smooth. We denote by G_m the stabilizer group of $m \in M$ and by $\mathfrak{g}_m$ its Lie algebra. Given $\xi \in \mathfrak{g}$, we denote by ξ_M the corresponding generating vector field: $\xi_M(m) = \frac{d}{dt}|_{t=0} \exp(-t\xi).m$. Let $\Omega_G(M)$ be the Cartan complex of equivariant differential forms on M. Thus $\Omega_G(M)$ is the space of equivariant maps $\alpha:\ \mathfrak{g} \to \Omega(M)$, with equivariant differential $(\mathrm{d}_G\alpha)(\xi) = (\mathrm{d} + \iota(\xi_M))\alpha(\xi)$.

In this paper, our sign conventions differ from those in [1]. To recover formulas of [1] one should change the signs of vector fields, $\xi_M \mapsto -\xi_M$, and of 2-forms, $\omega \mapsto -\omega$.

2. Group valued Hamiltonian G-manifolds

In this Section we recall the definition of a space with group valued moment map, and describe the main examples. We refer to [1] for proofs and further details.

2.1. **Definition of a group valued Hamiltonian G-manifold.** Let G be a compact Lie group, acting on itself by conjugation. The canonical closed 3-form

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \tag{1}$$

on G extends to an equivariantly closed equivariant 3-form,

$$\eta_G(\xi) = \eta + \frac{1}{2}(\theta^L + \theta^R) \cdot \xi. \tag{2}$$

Definition 2.1. *A group valued Hamiltonian G-manifold (called quasi-Hamiltonian in* [1]*) is a triple (M, ω, Φ) consisting of a G-manifold M, an invariant 2-form ω, and an equivariant map $\Phi \in C^\infty(M, G)^G$ such that*

$$d_G\omega = \Phi^*\eta_G, \tag{3}$$

and such that for all $m \in M$, $\ker \omega_m = \{\xi_M(m) |\, (\mathrm{Ad}_{\Phi(m)} + 1)\xi = 0\}$.

Condition (3) splits into $\mathrm{d}\omega = \Phi^*\eta$ and the *moment map condition*

$$\iota(\xi_M)\omega = \frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi. \tag{4}$$

Equation (4) requires $\ker \omega_m \supseteq \{\xi_M(m) |\, (\mathrm{Ad}_{\Phi(m)} + 1)\xi = 0\}$; for this reason the condition in Definition 2.1 is called the *minimal degeneracy condition.* If $G = T$ is a torus, one recovers the usual definition of a symplectic T-manifold with torus valued moment map.

2.2. **Conjugacy classes, symmetric spaces.** Basic examples for spaces with group valued moment maps are G-conjugacy classes $\mathcal{C}$. The moment map Φ is the embedding into G and the 2-form is given by

$$\omega(\xi_{\mathcal{C}}(g), \zeta_{\mathcal{C}}(g)) = -\frac{1}{2}\,(\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi \cdot \zeta \tag{5}$$

($g \in \mathcal{C}$). The homogeneous group valued Hamiltonian G-manifolds are exactly the G-equivariant covering spaces of conjugacy classes. More generally, given a Lie group automorphism $\psi \in \mathrm{Aut}(G)$ one defines *twisted conjugacy classes* to be the orbits of the action

$$\mathrm{Ad}^\psi_h(g) = \psi(h)gh^{-1}.$$

The twisted conjugacy classes are ψ-invariant since $\psi(g) = \mathrm{Ad}^\psi_g(g)$. Note $\mathrm{Ad}^\psi_G(e) = G/G^\psi$. Taking ψ to be an involution we see that every compact symmetric space for G is a twisted conjugacy class, up to finite cover.

Suppose ψ has finite order k, and embed $\mathbb{Z}_k \hookrightarrow \mathrm{Aut}(G)$ as the subgroup generated by ψ. For any $g \in G$, the map $G \to \mathbb{Z}_k \ltimes G,\ g \mapsto (\psi^{-1}, g)$ takes the twisted conjugacy class of G to conjugacy classes of the disconnected group $\mathbb{Z}_k \ltimes G$. In particular, if $\psi = \psi^{-1}$ is an involution, the identification of the symmetric space G/G^ψ as a conjugacy class of $\mathbb{Z}_2 \ltimes G$ is given by the map $\Phi : xG^\psi \mapsto (\psi, \psi(x)x^{-1})$. In this case, the 2-form ω given by (5) vanishes since $g = g^{-1}$ for all $g = (\psi, \psi(x)x^{-1})$. The group G itself is a symmetric space for $G \times G$, where $\psi(a, b) = (b, a)$. Thus G becomes a group valued

Hamiltonian $\mathbb{Z}_2 \ltimes (G \times G)$ space, with 2-form $\omega = 0$, moment map $a \mapsto (\psi, a, a^{-1})$ and action $(g_1, g_2).a = g_2 a g_1^{-1}$, $\psi.a = a^{-1}$.

2.3. **The four-sphere.** Let $G = \mathrm{SU}(2)$ and $M = S^4$ the unit sphere in $\mathbb{R}^5 \cong \mathbb{C}^2 \times \mathbb{R}$, with SU(2)-action induced from the defining action on $\mathbb{C}^2$. In Appendix A we show that M carries the structure of a group valued Hamiltonian SU(2)-manifold, with moment map $\Phi : S^4 \to \mathrm{SU}(2) \cong S^3$ the suspension of the Hopf fibration $S^3 \to S^2$.

2.4. **Exponentials.** Let (M, ω_0, Φ_0) be a Hamiltonian G-manifold in the usual sense. That is, ω_0 is a G-invariant symplectic form, and $\Phi_0 : M \to \mathfrak{g}^*$ is an equivariant map satisfying the moment map condition, $\iota(\xi_M)\omega_0 = \mathrm{d}\Phi_0 \cdot \xi$, using the inner product to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let $\varpi \in \Omega^2(\mathfrak{g})$ be the image of $\exp^* \eta$ under the de Rham homotopy operator $\Omega^\star(\mathfrak{g}) \to \Omega^{\star-1}(\mathfrak{g})$. The triple $(M, \omega_0 + \Phi_0^*\varpi, \exp(\Phi_0))$ satisfies the axioms for a group valued Hamiltonian G-manifold, except possibly for the minimal degeneracy condition. It turns out that $\omega = \omega_0 + \Phi_0^*\varpi$ is minimally degenerate if and only if the exponential map has maximal rank at all points of $\Phi_0(M)$. This construction takes (co-)adjoint G-orbits in $\mathfrak{g}^* \cong \mathfrak{g}$ to conjugacy classes for G.

2.5. **Products.** Suppose $(M, \omega, (\Phi_1, \Phi_2))$ is a group valued Hamiltonian $G \times G$-manifold. Then $\tilde{M} = M$ with diagonal action, moment map $\tilde{\Phi} = \Phi_1\Phi_2$, and 2-form

$$\tilde{\omega} = \omega - \frac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R$$

is a group valued Hamiltonian G-manifold. If $M = M_1 \times M_2$ is a direct product of two group valued Hamiltonian G-manifolds, we call $\tilde{M} = M_1 \circledast M_2$ the fusion product of M_1, M_2.

2.6. **The double.** View G as a group valued Hamiltonian $\mathbb{Z}_2 \ltimes (G \times G)$-manifold as above. The fusion product $G \circledast G$ has moment map

$$(a_1, a_2) \mapsto (\psi, a_1, a_1^{-1})(\psi, a_2, a_2^{-1}) = (e, a_1^{-1}a_2, a_1a_2^{-1}).$$

Since the moment map takes values in $G \times G \subset \mathbb{Z}_2 \ltimes (G \times G)$, we can restrict to the $G \times G$-action, and consider $G \circledast G$ as a group valued Hamiltonian $G \times G$-manifold. Denoting $a := a_1^{-1}, b := a_2$ the moment map reads

$$(a, b) \mapsto (ab, a^{-1}b^{-1})$$

and the $G \times G$-action is

$$(g_1, g_2).(a, b) = (g_1 a g_2^{-1}, g_2 b g_1^{-1}).$$

This is the *double* $D(G)$ introduced in [1]. Passing to the diagonal action, we obtain a group valued Hamiltonian G-manifold $\tilde{D}(G)$, with G acting by conjugation on each factor and moment map the Lie group commutator, $(a, b) \mapsto [a, b] = aba^{-1}b^{-1}$.

2.7. **Reduction.** Suppose $(M, \omega, (\Phi, \Psi))$ is a group valued Hamiltonian G-manifold. Then $e \in G$ is a regular value of Φ if and only if the action of G on $\Phi^{-1}(e)$ is locally free, and in this case the *symplectic quotient* $M/\!\!/G = \Phi^{-1}(e)/G$ is a symplectic orbifold, with symplectic form induced from ω. If one drops the regularity assumption, $M/\!\!/G$ acquires more serious singularities and is a stratified symplectic space in the sense of Sjamaar-Lerman. More generally, if $\mathcal{C} \subset G$ is a conjugacy class, let $\mathcal{C}^-$ be its image under the inversion map $g \mapsto g^{-1}$. Then e is a regular value for the action on $M \circledast \mathcal{C}^-$ if and only if $\mathcal{C}$ is contained in the set of regular values of Φ. We define $M_{\mathcal{C}} = (M \circledast \mathcal{C}^-)/\!\!/G \cong \Phi^{-1}(\mathcal{C})/G$.

3. Liouville measures

Since the 2-form ω of a group valued Hamiltonian G-manifold (M, ω, Φ) may be degenerate, its top exterior power does not define a volume form, in general. We will show in this Section that if the adjoint action $G \to \mathrm{O}(\mathfrak{g})$ lifts to the group $\mathrm{Pin}(\mathfrak{g})$ (e.g. if G is connected and simply connected), then M carries a volume form $(e^{\omega}\Phi^*\mathcal{T})_{[top]}$ where $\mathcal{T}$ is a certain differential form on the group G. In the absence of such a lift, one still obtains a nowhere vanishing measure ν on M, called the Liouville measure.

3.1. **Volume forms.** We recall the definition of the double covering $\mathrm{Pin}(\mathfrak{g}) \to \mathrm{O}(\mathfrak{g})$ as a subset of the Clifford algebra [7]. Let $\mathrm{Cl}(\mathfrak{g})$ be the Clifford algebra of $\mathfrak{g}$ with respect to $-1/2$ the inner product on $\mathfrak{g}$. If e_i is a given orthonormal basis of $\mathfrak{g}$, and x_i the corresponding generators of $\mathrm{Cl}(\mathfrak{g})$, the defining relations of $\mathrm{Cl}(\mathfrak{g})$ read, $x_i x_j + x_j x_i = \delta_{ij}$. The group $\mathrm{Cl}(\mathfrak{g})^\times$ of invertible elements acts on $\mathrm{Cl}(\mathfrak{g})$ by the twisted adjoint action

$$\widetilde{\mathrm{Ad}}(x)y = \alpha(x)yx^{-1}, \quad x \in \mathrm{Cl}(\mathfrak{g})^\times, \ y \in \mathrm{Cl}(\mathfrak{g})$$

where $\alpha : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ is the parity operator, i.e. $\alpha(x) = x$ for x even and $\alpha(x) = -x$ for x odd. Let $\mathrm{Pin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g})^\times$ be the subgroup generated by elements $\xi \in \mathfrak{g} \subset \mathrm{Cl}(\mathfrak{g})$ with $\xi \cdot \xi = 2$. The action $\widetilde{\mathrm{Ad}}$ restricts to an isometry of $\mathrm{Pin}(\mathfrak{g})$ on $\mathfrak{g} \subset \mathrm{Cl}(\mathfrak{g})$, and $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(\mathfrak{g}) \to \mathrm{O}(\mathfrak{g})$ is a double covering. The group $\mathrm{Pin}(\mathfrak{g})$ has two connected components, given as intersections $\mathrm{Spin}(\mathfrak{g}) = \mathrm{Pin}(\mathfrak{g}) \cap \mathrm{Cl}(\mathfrak{g})_{\mathrm{even}}$ and $\mathrm{Pin}(\mathfrak{g}) \cap \mathrm{Cl}(\mathfrak{g})_{\mathrm{odd}}$.

Let us assume that the adjoint action $\mathrm{Ad} : G \to \mathrm{O}(\mathfrak{g})$ lifts to a group homomorphism $\tau : G \to \mathrm{Pin}(\mathfrak{g})$:

$$\begin{tikzcd} & \mathrm{Pin}(\mathfrak{g}) \arrow[r] \arrow[d] & \mathrm{Cl}(\mathfrak{g}) \\ G \arrow[ur, dotted, "\tau"] \arrow[r, "\mathrm{Ad}"] & \mathrm{O}(\mathfrak{g}) & \end{tikzcd}$$

The restriction of τ to the identity component takes values in $\mathrm{Spin}(\mathfrak{g})$ and is given by the formula

$$\tau(\exp\,\mu) = \exp(-\frac{1}{2}\sum_{i,j,k} f_{ijk}\,\mu_i\,x_j\,x_k). \tag{6}$$

where $f_{ijk} = [e_i, e_j] \cdot e_k$ are the structure constants. Let $\wedge\mathfrak{g}$ be the exterior algebra, with generators y_i and relations $y_i y_j + y_j y_i = 0$, and $\sigma : \mathrm{Cl}(\mathfrak{g}) \to \wedge\mathfrak{g}$ the symbol map, taking $x_{i_1} \cdots x_{i_s}$ to $y_{i_1} \cdots y_{i_s}$ for $i_1 < \cdots < i_s$. Denote by $s_{1/2} : \wedge\mathfrak{g} \to \wedge\mathfrak{g}$ the linear map equal to multiplication by 2^{-l} on $\wedge^l\mathfrak{g}$. We define $\mathcal{T} \in \Omega(G)$ to be the differential form

$$\mathcal{T} := s_{1/2} \circ \sigma \circ \tau \in C^\infty(G, \wedge\mathfrak{g})^G \cong \Omega(G)^G, \tag{7}$$

using the isomorphism defined by right-invariant (or equivalently left-invariant) differential forms. That is, the value of $\mathcal{T}$ at g is obtained from $\sigma(\tau(g))$ by replacing the variables y_i with $\frac{1}{2}(\theta_i^R)_g$, where $\theta_i^R = \theta^R \cdot e_i$ are the components of the right-invariant Maurer-Cartan forms. Our main theorem reads:

Theorem 3.1. *For any group valued Hamiltonian G-manifold (M, ω, Φ), the top form degree part of the differential form*

$$\Gamma = \exp(\omega)\Phi^*\mathcal{T}. \tag{8}$$

is a volume form on M. On the subset where ω is non-degenerate, we have

$$(\Gamma)_{[\dim M]} = \pm \frac{\exp(\omega)_{[\dim M]}}{\left|\det(\frac{1+\mathrm{Ad}_\Phi}{2})\right|^{1/2}}. \tag{9}$$

The signs depend on the choice of lift τ.

The proof of this Theorem is given in Section 3.2.

Remark 3.2. The volume form depends not only on the 2-form but also on the moment map. If $c \in Z(G)$ is a central element, then $\Phi' = c\Phi$ is a moment map for the same group action and 2-form ω. Since τ is a group homomorphism, $\Gamma' = \tau(c)\Gamma$. We have $\tau(c) = \pm 1$ since $\mathrm{Ad}_c = 1$. Therefore, the new volume form $\Gamma'_{[\dim M]}$ differs from $\Gamma_{[\dim M]}$ by the sign of $\tau(c)$.

If G is connected, we can be more precise. Let T be a maximal torus of G, with Lie algebra $\mathfrak{t}$. Choose a system of positive (real) roots, and let $\rho \in \mathfrak{t}^*$ be their half-sum. It is known that the adjoint action $\mathrm{Ad} : G \to \mathrm{SO}(\mathfrak{g})$ lifts to a homomorphism $\tau : G \to \mathrm{Pin}(\mathfrak{g})$ if and only if ρ is in the lattice of weights for T. [1] The character χ_ρ of the ρ-representation V_ρ defines a smooth square root of $g \mapsto \det(\mathrm{Ad}_g + 1)$:

$$\frac{\chi_\rho(g)}{\dim V_\rho} = \det{}^{1/2}(\frac{\mathrm{Ad}_g + 1}{2}).$$

We have the following extension of Theorem 3.1:

Theorem 3.3. *Suppose G is a compact connected group, and assume that ρ is a weight for G. Let (M, ω, Φ) be a group valued Hamiltonian G-manifold (M, ω, Φ). Then M is*

[1] In fact, ρ is a weight if and only if G/T admits a G-equivariant Spin structure (cf. [13] for discussion and references).

even-dimensional, and carries a canonical volume form. On the subset where $\Phi^*\chi_\rho \neq 0$, the volume form is given by

$$\frac{\dim V_\rho}{\Phi^*\chi_\rho}\ \exp(\omega)_{[\dim M]}.$$

3.2. **Proof of Theorems 3.1 and 3.3.** We first give a more explicit description of the differential form $\mathcal{T}$. For any subspace $\mathfrak{s} \subset \mathfrak{g}$ with a given orientation let

$$\mathrm{d}\,\mathrm{vol}_{\mathfrak{s}} \in \wedge^{\dim \mathfrak{s}}\mathfrak{s}$$

denote the Riemannian volume form. Also, for any endomorphism A of $\mathfrak{g}$ preserving $\mathfrak{s}$ we denote $\det_{\mathfrak{s}}(A) := \det(A|_{\mathfrak{s}})$. Given $g \in G$ let $\mathfrak{g}^{\pm}$ denote the ± 1 eigenspaces of $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, and $\mathfrak{g}'$ the orthogonal complement of $\mathfrak{g}^- \oplus \mathfrak{g}^+$. Thus

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}^- \oplus \mathfrak{g}^+. \tag{10}$$

Accordingly, decompose $I = (1, \ldots, \dim \mathfrak{g})$ as $I = I' \cup I^- \cup I^+$ where I' denotes the first $\dim \mathfrak{g}'$ indices, I^- the following $\dim \mathfrak{g}^-$ indices, and I^+ the remaining $\dim \mathfrak{g}^+$ indices. Choose an orthonormal basis e_i, $i \in I$ of $\mathfrak{g}$ such that the basis vectors labeled by I', I^-, I^+ span $\mathfrak{g}', \mathfrak{g}^-, \mathfrak{g}^+$, respectively. For any linear map $A : \mathfrak{g} \to \mathfrak{g}$ we denote by $A_{ij} = e_i \cdot A(e_j)$ its components. We denote by $\theta_i^R = \theta^R \cdot e_i$ and $\theta_i^L = \theta^L \cdot e_i$ the components of the Maurer-Cartan forms. On $\mathfrak{g}'$ the operator $\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}$ is well-defined and invertible.

Lemma 3.4. *At any point $g \in G$, the form $\mathcal{T}$ is a product $\pm \mathcal{T}'\mathcal{T}^-$, where*

$$\mathcal{T}^- = 2^{-\dim \mathfrak{g}^-/2} \prod_{i \in I^-} \theta_i^R \tag{11}$$

and

$$\mathcal{T}' = \left|\det_{\mathfrak{g}'}\Big(\frac{\mathrm{Ad}_g + 1}{2}\Big)\right|^{1/2} \exp\Big(-\frac{1}{4}\sum_{i,j \in I'} \Big(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\Big)_{ij} \theta_i^R \theta_j^R\Big). \tag{12}$$

(The sign $\pm$ depends on the choice of lift and on the orientation on $\mathfrak{g}^-$ defined by the choice of basis.) If G is connected we have the formula, valid on the subset where $\chi_\rho \neq 0$,

$$\mathcal{T} = \frac{\chi_\rho}{\dim V_\rho}\ \exp\Big(-\frac{1}{4}\sum_{i,j} \Big(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\Big)_{ij} \theta_i^R \theta_j^R\Big). \tag{13}$$

Proof. Write $\mathrm{Ad}_g =: S = S'S^-$, where $S^- = -\mathrm{Id}_{\mathfrak{g}^-} \in \mathrm{O}(\mathfrak{g}^-)$ and $S' \in \mathrm{SO}(\mathfrak{g}')$. The product

$$\tilde{S}^- = 2^{\dim \mathfrak{g}^-/2} \prod_{i \in I^-} x_i$$

is a lift of S^-, with symbol $\sigma(\tilde{S}^-) = 2^{\dim \mathfrak{g}^-/2} \prod_{i \in I^-} y_i$. As a special case of [8, Proposition 3.13], the symbol of any lift $\tilde{S}' \in \mathrm{Spin}(\mathfrak{g}')$ of S' is given by a formula

$$\sigma(\tilde{S}') = \pm \left|\det\Big(\frac{S' + 1}{2}\Big)\right|^{1/2} \exp\Big(-\sum_{i,j \in I'} \Big(\frac{S' - 1}{S' + 1}\Big)_{ij} y_i y_j\Big),$$

where the sign depends on the choice of lift. Replacing the variables $y_i \in \mathfrak{g}$ with $\frac{1}{2}(\theta_i^R)$, the first part of the Lemma follows. Equation (13) is a special case since $\mathfrak{g}^- = \{0\}$ on the set where $\chi_\rho \neq 0$; the sign is verified by evaluating at $g = e$. □

Proof of Theorem 3.1. Given $m \in M$ let $g = \Phi(m)$. From (10) we obtain the following splitting of the tangent space

$$T_m M = E \oplus \mathfrak{g}_g^\perp = E \oplus \mathfrak{g}' \oplus \mathfrak{g}^- . \tag{14}$$

Here $\mathfrak{g}_g^\perp = \mathfrak{g}' \oplus \mathfrak{g}^-$ is the orthogonal complement of the stabilizer algebra $\mathfrak{g}_g$, embedded by the generating vector fields $\xi \mapsto \xi_M(m)$, and $E = \{v \in T_m M |\, \iota(v)\Phi^*\theta^R \in \mathfrak{g}_g\}$ is the pre-image under $\mathrm{d}_m\Phi$ of the tangent space of the stabilizer group G_g. The moment map condition shows that this splitting is ω-orthogonal and identifies $\ker \omega_m \cong \mathfrak{g}_-$. In particular, the 2-forms $\omega_E = \omega|_E$ and $\omega' = \omega|_{\mathfrak{g}'}$ are symplectic.

We will show that in terms of the splitting $T_m M = E \oplus \mathfrak{g}_g^\perp$, the value of Γ at $m \in M$ is given by the formula

$$(\Gamma_m)_{[\dim M]} = \pm |\det_{\mathfrak{g}_g^\perp}(\mathrm{Ad}_g - 1)|^{1/2}\ (\exp \omega_E)_{[\dim E]} \wedge \mathrm{d}\,\mathrm{vol}_{\mathfrak{g}_g^\perp} . \tag{15}$$

In particular, (15) shows that $\Gamma_{[\dim M]}$ is a volume form. Consider the splitting $T_m M = E \oplus \mathfrak{g}' \oplus \mathfrak{g}^-$ and the corresponding decomposition $\Gamma = \pm e^{\omega_E}\,\Gamma'\Gamma^-$ with $\Gamma' = (\exp \omega')\mathcal{T}'$ and $\Gamma^- = \mathcal{T}^-$. (We are dropping the base point $m \in M$ from the notation.) The form ω' is determined by the moment map condition, and is given by

$$\omega' = \frac{1}{4} \sum_{i,j \in I'} \big(\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}\big)_{ij} \theta_i^R \theta_j^R .$$

Indeed, since $\iota_i \theta_j^R = (\mathrm{Ad}_g - 1)_{ij}$, it is easily verified that $\iota_i \omega' = \frac{1}{2}(\theta^L + \theta^R)_i$ for all $i \in I'$, as required. Using (12) and the following equality of operators on $\mathfrak{g}'$,

$$\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1} - \frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1} = \frac{4}{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}},$$

we obtain

$$\Gamma' = \pm \big|\det_{\mathfrak{g}'}\ \big(\frac{\mathrm{Ad}_g + 1}{2}\big)\big|^{1/2} \exp\big(\sum_{i,j \in I'} \big(\frac{1}{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}\big)_{ij} \theta_i^R \theta_j^R \big).$$

Thus

$$\begin{aligned}
(\Gamma')_{[\dim \mathfrak{g}']} &= \pm \left|\det_{\mathfrak{g}'}\ \big(\frac{\mathrm{Ad}_g + 1}{2}\big)\right|^{1/2} \left|\det_{\mathfrak{g}'}\big(\frac{2}{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}\big)\right|^{1/2} \prod_{i \in I'} \theta_i^R \\
&= \pm \left|\det_{\mathfrak{g}'}\big(\mathrm{Ad}_g - 1\big)\right|^{-1/2} \prod_{i \in I'} \theta_i^R . \qquad (16)
\end{aligned}$$

This expression combines with $\Gamma^- = \mathcal{T}^-$ to a factor,

$$(\Gamma'\Gamma^-)_{[\dim \mathfrak{g}_g^\perp]} = \left|\det_{\mathfrak{g}_g^\perp}\big(\mathrm{Ad}_g - 1\big)\right|^{-1/2} \prod_{i \in I' \cup I^-} \theta_i^R = \left|\det_{\mathfrak{g}_g^\perp}\big(\mathrm{Ad}_g - 1\big)\right|^{1/2} \mathrm{d}\,\mathrm{vol}_{\mathfrak{g}_g^\perp},$$

where we have used (11) with $2^{\dim \mathfrak{g}^-} = |\det_{\mathfrak{g}^-}(\mathrm{Ad}_g - 1)|$, and $\iota_i \theta_j^R = (\mathrm{Ad}_g - 1)_{ij}$. This proves (15). For (9), assume that ω_m is non-degenerate, that is $\ker \omega_m \cong \mathfrak{g}^- = \{0\}$. We have

$$(\exp \omega')_{[\dim \mathfrak{g}']} = \pm \left|\det_{\mathfrak{g}'}\left(\frac{1}{2}\,\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}\right)\right|^{1/2} \prod_{i \in I'} \theta_i^R.$$

Comparing with (16), this shows

$$(\Gamma')_{[\dim \mathfrak{g}']} = \pm\, (\exp \omega')_{[\dim \mathfrak{g}']} \left|\det_{\mathfrak{g}'}\left(\frac{\mathrm{Ad}_g + 1}{2}\right)\right|^{-1/2}$$

which yields (9). □

Proof of Theorem 3.3. Since G is connected, the lift $\tau : G \to \mathrm{Spin}(\mathfrak{g})$ is unique. Moreover, since $\mathrm{Spin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g})_{\mathrm{even}}$, the form $\mathcal{T}$, hence also Γ, vanishes in odd degrees. The formula for $\Gamma_{[\dim M]}$ is obtained by using the expression (13) for $\mathcal{T}$ in the proof of Theorem 3.1, and keeping track of the signs. □

3.3. **Products.** From the differential form expression (8), it is not obvious how the volume forms behave under the fusion operation from Section 2.5. In particular, there is no simple relation of the product $\Phi_1^*\mathcal{T}\,\Phi_2^*\mathcal{T}$ with $(\Phi_1\Phi_2)^*\mathcal{T}$. It will be convenient to work with an equivalent expression for $\mathcal{T}$ involving the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Let $P^{\mathrm{Cl}}_{\mathrm{hor}} : \mathrm{Cl}(\mathfrak{g}) \to \mathbb{R}$ denote horizontal projection, i.e. the linear map defined by $P^{\mathrm{Cl}}_{\mathrm{hor}}(1) = 1$ and $P^{\mathrm{Cl}}_{\mathrm{hor}}(x_{j_1} \ldots x_{j_s}) = 0$, where $j_1 < j_2 < \cdots < j_s$. For any $\mathbb{Z}_2$-graded space $\mathcal{A}$ it extends to a linear map $P^{\mathrm{Cl}}_{\mathrm{hor}} :\ \mathrm{Cl}(\mathfrak{g}) \otimes \mathcal{A} \to \mathcal{A}$ (graded tensor product). Observe that for any $j_1 < j_2 < \ldots < j_s$, the operator $P^{\mathrm{Cl}}_{\mathrm{hor}} :\ \mathrm{Cl}(\mathfrak{g}) \otimes \wedge\mathfrak{g} \to \wedge\mathfrak{g}$ takes

$$e^{-2\sum_i x_i y_i} x_{j_1} \cdots x_{j_s} = \prod_i (1 - 2x_i y_i) x_{j_1} \cdots x_{j_s} = (x_{j_1} + y_{j_1}) \cdots (x_{j_s} + y_{j_s})$$

to $y_{j_1} \cdots y_{j_s} = \sigma(x_{j_1} \cdots x_{j_s})$. This shows $P^{\mathrm{Cl}}_{\mathrm{hor}}(e^{-2\sum_i x_i y_i} \tau) = \sigma(\tau)$. Replacing y_i with $\frac{1}{2}\theta_i^R$ we obtain,

$$\mathcal{T} = P^{\mathrm{Cl}}_{\mathrm{hor}}(e^{-\sum_i x_i \theta_i^R} \tau). \tag{17}$$

Theorem 3.5 (Products)**.** *Let $(M, \omega, (\Phi_1, \Phi_2))$ be a group valued Hamiltonian G-manifold and $(\tilde{M}, \tilde{\omega}, \tilde{\Phi})$ its fusion as defined in 2.5. The corresponding forms $\Gamma, \tilde{\Gamma}$ are related by*

$$\tilde{\Gamma} = \exp(-\frac{1}{2} \sum_i \iota_i^1 \iota_i^2)\, \Gamma. \tag{18}$$

where ι_i^1 are the contraction operators for the action of the first G-factor on M, and ι_i^2 those for the second G-factor. In particular $\tilde{\Gamma}_{[\dim M]} = \Gamma_{[\dim M]}$.

It follows that the volume form of a fusion product $M_1 \circledast M_2$ is the product of the volume forms of M_1, M_2.

Proof. We will need the following two identities from [2, Section 5.3], both of which are verified by straightforward calculation:

$$\Phi_1^*(e^{-\sum_i x_i\theta_i^R}\tau)\Phi_2^*(e^{-\sum_i x_i\theta_i^R}\tau) = e^{-\frac{1}{2}\Phi_1^*\theta^L\cdot\Phi_2^*\theta^R}(\Phi_1\Phi_2)^*(e^{-\sum_i x_i\theta_i^R}\tau) \tag{19}$$

$$(\iota_i^{\mathrm{Cl}} + \iota_i + \frac{1}{2}(\theta_i^L + \theta_i^R))\big(e^{-\sum_j x_j\theta_j^R}\tau\big) = 0. \tag{20}$$

Here $\iota_i : \Omega(G) \to \Omega(G)$ are the contraction operators for the conjugation action, and $\iota_i^{\mathrm{Cl}} = \mathrm{ad}(x_i)$ the contraction operators for the Clifford algebra. ¿From (20) and the moment map condition we obtain

$$(\iota_i^{\mathrm{Cl}} + \iota_i)\big(e^{\omega}\Phi^*(e^{-\sum_j x_j\theta_j^R}\tau)\big) = 0. \tag{21}$$

Since $\tilde{\omega} = \omega - \frac{1}{2}\Phi_1^*\theta^L\cdot\Phi_2^*\theta^R$, Equation (19) gives

$$\begin{aligned}\tilde{\Gamma} &= P_{\mathrm{hor}}^{\mathrm{Cl}}\big(e^{\omega}\Phi_1^*(e^{-\sum_i x_i\theta_i^R}\tau)\Phi_2^*(e^{-\sum_i x_i\theta_i^R}\tau)\big)\\ &= P_{\mathrm{hor}}^{\wedge\mathfrak{g}}\sigma\big(e^{\omega}\Phi_1^*(e^{-\sum_i x_i\theta_i^R}\tau)\Phi_2^*(e^{-\sum_i x_i\theta_i^R}\tau)\big)\end{aligned} \tag{22}$$

where $P_{\mathrm{hor}}^{\wedge\mathfrak{g}}$ is horizontal projection for the exterior algebra $\wedge\mathfrak{g}$. Let $\iota_i^{\wedge\mathfrak{g}} = \iota(e_i)$ be the contraction operators for the exterior algebra. The symbol map takes the Clifford product on $\mathrm{Cl}(\mathfrak{g})$ to the product on on $\wedge\mathfrak{g}$ given by application of the operator $\exp(-\frac{1}{2}\sum_i \iota_i^{\wedge\mathfrak{g},1}\iota_i^{\wedge\mathfrak{g},2})$ on $\wedge\mathfrak{g}\otimes\wedge\mathfrak{g}$ (where the superscripts refer to the $\wedge\mathfrak{g}$-factor), followed by wedge product (see [17, Theorem 16], or [2, Lemma 3.1]). Because of (21) we may replace $\exp(-\frac{1}{2}\sum_i \iota_i^{\wedge\mathfrak{g},1}\iota_i^{\wedge\mathfrak{g},2})$ with $\exp(-\frac{1}{2}\sum_i \iota_i^1\iota_i^2)$ everywhere, which then commutes with $P_{\mathrm{hor}}^{\wedge\mathfrak{g}}$. Equation (22) becomes

$$\tilde{\Gamma} = e^{-\frac{1}{2}\sum_i \iota_i^1\iota_i^2}P_{\mathrm{hor}}^{\wedge\mathfrak{g}}\big(e^{\omega}\sigma(\Phi_1^*(e^{-\sum_i x_i\theta_i^R}\tau))\sigma(\Phi_2^*(e^{-\sum_i x_i\theta_i^R}\tau))\big).$$

Since $P_{\mathrm{hor}}^{\wedge\mathfrak{g}}$ is a ring homomorphism, this is equal to $\exp(-\frac{1}{2}\sum_i \iota_i^1\iota_i^2)(e^{\omega}\Phi_1^*\mathcal{T}\Phi_2^*\mathcal{T})$, proving the Theorem. □

3.4. **General case.** For general compact Lie groups G, a lift $\tau : G \to \mathrm{Pin}(\mathfrak{g})$ of the adjoint action need not exist. We will now show that for any group valued Hamiltonian G-manifold (M, ω, Φ), the form Γ may be defined as a form with values in the orientation bundle o_M. Recall that o_M is the real line bundle associated to the oriented double cover of M. One defines the space of twisted differential forms $\Omega_t(M) = \Omega(M, o_M)$ as sections of $\wedge T^*M\otimes o_M$. The space $\Omega_t(M)$ is naturally isomorphic to the space of differential forms on the oriented double cover of M that are anti-invariant under deck transformations. The real line bundle $\wedge^{\dim M}T^*M\otimes o_M$ is isomorphic to the density bundle of M; hence the space $\Omega_t^{\dim M}(M)$ is just the space of densities (smooth measures) of M.

Define double coverings $\pi_G : \widehat{G} \to G$ and $\pi_M : \widehat{M} \to M$ by the pull-back diagram

$$\begin{tikzcd}
\widehat{M} \arrow[r, "\hat{\Phi}"] \arrow[d, "\pi_M"] & \widehat{G} \arrow[r, "\hat{\tau}"] \arrow[d, "\pi_G"] & \mathrm{Pin}(\mathfrak{g}) \arrow[d, "\pi"] \\
M \arrow[r, "\Phi"] & G \arrow[r, "\mathrm{Ad}"] & \mathrm{O}(\mathfrak{g})
\end{tikzcd}$$

Write $\ker \pi_G = \{e, c\}$. The conjugation action of G on $\widehat{G}$ and the define an action on the fiber product $\widehat{M}$. Let $\widehat{\omega}$ be the pull-back of ω. Then $(\widehat{M}, \widehat{\omega}, \widehat{\Phi})$ is a group valued Hamiltonian $\widehat{G}$ space, with $\ker \pi_G$ acting trivially. The construction in 3.2 defines a differential form $\widehat{\Gamma}$ on $\widehat{M}$, for which $\widehat{\Gamma}_{[\dim M]}$ is a volume form.

Let $S : \widehat{M} \to \widehat{M}$ be the non-trivial deck transformation. Then S is $\widehat{G}$-equivariant, preserves the 2-form $\widehat{\omega}$, and changes the moment map by $S^*\widehat{\Phi} = c\widehat{\Phi}$. Thus

$$S^*\widehat{\Gamma} = \widehat{\tau}(c)\widehat{\Gamma} = -\widehat{\Gamma}.$$

In particular, S changes the orientation of $\widehat{M}$. This identifies $\widehat{M}$ with the oriented double cover of M. The form $\widehat{\Gamma}$ descends to a form $\Gamma \in \Omega_t(M)$, with top degree part a strictly positive measure. We call $\nu := \Gamma_{[\dim M]}$ the *Liouville measure*, and its integral $\mathrm{Vol}(M) := \int_M \nu$ the *Liouville volume.*

The fusion formula 3.5 holds in this more general context; in particular, the Liouville measure $\tilde{\nu}$ for the "fused" space $\tilde{M}$ coincides with the original Liouville measure ν.

3.5. **Conjugacy classes.** Going back to the proof of Theorem 3.1, we obtain the following formula for the Liouville measure of a conjugacy class in G:

Proposition 3.6. *The Liouville volume of the conjugacy class $\mathcal{C} \subset G$ of $g \in G$ is given by the formula*

$$\mathrm{Vol}(\mathcal{C}) = |\det_{\mathfrak{g}_g^\perp}(\mathrm{Ad}_g - 1)|^{\frac{1}{2}} \frac{\mathrm{vol}\, G}{\mathrm{vol}\, G_g}. \tag{23}$$

where $\mathrm{vol}\, G$ *and* $\mathrm{vol}\, G_g$ *are the Riemannian volumes for the given inner product on* $\mathfrak{g}$*, and* $\mathfrak{g}_g^\perp$ *is the orthogonal complement of the stabilizer algebra of* g*.*

Proof. By passing to a cover if necessary, we may assume that G admits a lift $\tau : G \to \mathrm{Spin}(\mathfrak{g})$. The formula follows directly from Equation (15) in the proof of Theorem 3.1, since the subspace E is trivial in this case. ☐

The formula (23) is similar to the well-known formula for the symplectic volume of a coadjoint orbit $\mathcal{O} = G \cdot \mu \subset \mathfrak{g}^* \cong \mathfrak{g}$:

$$\mathrm{Vol}(\mathcal{O}) = |\det_{\mathfrak{g}_\mu^\perp}(\mathrm{ad}_\mu)|^{\frac{1}{2}} \frac{\mathrm{vol}\, G}{\mathrm{vol}\, G_\mu} \tag{24}$$

3.6. **Exponentials.** Let $J: \mathfrak{g} \to \mathbb{R}$ the determinant of the Jacobian of the exponential map $\exp: \mathfrak{g} \to G$,

$$J(\xi) = \det_{\mathfrak{g}_\xi^\perp}\big(\frac{e^{\mathrm{ad}_\xi} - 1}{\mathrm{ad}_\xi}\big),$$

and $J^{1/2}$ the unique smooth square root equal to 1 at the origin. Suppose $J(\xi) \neq 0$ so that exp has maximal rank at ξ. The conjugacy class $G.\exp(\xi)$ is the group valued Hamiltonian G-manifold corresponding (in the sense of 2.4) to the (co)-adjoint orbit $G.\xi$. Equations (23) and (24) show that

$$\frac{\mathrm{Vol}(G.\xi)}{\mathrm{Vol}(G.\exp\xi)} = |J(\xi)^{\frac{1}{2}}|.$$

This generalizes, as follows:

Proposition 3.7. *Let (M, ω_0, Φ_0) be a Hamiltonian G-manifold, with Liouville measure ν_0. Suppose $\Phi_0^* J \neq 0$ everywhere, and let (M, ω, Φ) the corresponding group valued Hamiltonian G-manifold. Then the Liouville measure of (M, ω, Φ) is given by*

$$\nu = \Phi_0^* |J^{1/2}| \nu_0.$$

Proof. We may assume that G is connected and that it admits a lift $\tau: G \to \mathrm{Spin}(\mathfrak{g})$. We compare the two measures at some point $m \in M$. Again we go back to Equation (15) from the proof of Theorem 3.1. Let $\xi = \Phi_0(m)$ and $\mathfrak{h} = \mathfrak{g}_\xi = \mathfrak{g}_{\exp\xi}$. The splitting $T_m M = E \oplus \mathfrak{h}^\perp$ is both ω_0-orthogonal and ω-orthogonal. By the moment map condition, the restriction of ω_0 to $\mathfrak{h}^\perp$ is given by the skew-adjoint operator ad_ξ. Hence, its top exterior power is $\pm|\det_{\mathfrak{h}^\perp}(\mathrm{ad}_\xi)|^{\frac{1}{2}}\, \mathrm{d}\,\mathrm{vol}_{\mathfrak{h}^\perp}$. Together with (15) this proves the Proposition. □

4. Duistermaat-Heckman measures

4.1. **Volumes of symplectic quotients.** For any compact, group valued Hamiltonian G-manifold (M, ω, Φ) we define the *Duistermaat-Heckman (DH) measure* as the push-forward of the Liouville measure $\nu = \Gamma_{[\dim M]}$ under the moment map,

$$\mathfrak{m} = \Phi_* \nu \in \mathcal{E}'(G)^G.$$

By general properties of push-forwards, the singular support of $\mathfrak{m}$ is contained in the set of singular values of Φ. Similar to the DH-measure for $\mathfrak{g}^*$-valued moment maps, the measure $\mathfrak{m}$ encodes volumes of reduced spaces.

Recall that is M is any G-manifold, an orbit $G.m \subset M$ is called a *principal orbit* if the stabilizer groups for all orbits in a neighborhood of $G.m$ are conjugate to G_m. The stabilizer group $H = G_m$ is called a *principal stabilizer.* The union M_{prin} of principal orbits is open and dense in M. If M/G is connected, then M_{prin}/G is connected and any two principal stabilizers are conjugate.

Theorem 4.1. *Let (M, ω, Φ) be a compact, group valued Hamiltonian G-manifold. Assume that the orbit space M/G is connected and that the principal stabilizer groups are finite. Suppose each component of $\Phi^{-1}(e)$ meets M_{prin}. Then $\mathfrak{m}$ is continuous on a* *neighborhood of e, and the volume of the (possibly singular) symplectic quotient $M/\!/G$ is given by*

$$\mathrm{Vol}(M/\!/G) = \frac{k}{\mathrm{vol}\, G} \frac{\mathfrak{m}}{d\,\mathrm{vol}_G}\Big|_e. \tag{25}$$

Here k is the cardinality of the principal stabilizer, and $\mathrm{Vol}(M/\!/G) := \mathrm{Vol}(M_{\mathrm{prin}}/\!/G)$.

Proof. We may assume that (M, ω, Φ) is the exponential of a Hamiltonian space (M_0, ω_0, Φ_0). The symplectic quotients $M/\!/G$ and $M_0/\!/G$ coincide. Since $\nu = \Phi_0^*|J^{1/2}|\,\nu_0$, the DH-measures are related by $\mathfrak{m} = |J^{1/2}| \exp_* \mathfrak{m}_0$. Since $J(0) = 1$, we have $\frac{\mathfrak{m}}{d\,\mathrm{vol}_G}\big|_e = \frac{\mathfrak{m}_0}{d\,\mathrm{vol}_{\mathfrak{g}}}\big|_0$. Hence, (25) follows from the well-known statement

$$\mathrm{Vol}(M/\!/G) = \frac{k}{\mathrm{vol}\, G} \frac{\mathfrak{m}_0}{\mathrm{d}\,\mathrm{vol}_{\mathfrak{g}^*}}\Big|_0.$$

for Hamiltonian G-manifolds (M, ω_0, Φ_0). (See [11, 16, 21] for details in the case that 0 is a regular value of Φ_0. The formula extends to the singular case by continuity, see the discussion in [19, Appendix A.1].) ☐

The right hand side of (25) can be re-expressed as a Fourier series. Let $\mathrm{Irr}(G)$ denote the set of equivalence classes of irreducible unitary G-representations. For $\lambda \in \mathrm{Irr}(G)$ let χ_λ denote the character of the corresponding irreducible representation V_λ. The measure $\mathfrak{m}$ is determined by its Fourier coefficients $\langle \mathfrak{m}, \chi_\lambda \rangle$,

$$\frac{\mathfrak{m}}{\mathrm{d}\,\mathrm{vol}_G} = \frac{1}{\mathrm{vol}\, G} \sum_{\lambda \in \mathrm{Irr}(G)} \langle \mathfrak{m}, \chi_\lambda \rangle \chi_\lambda^* \tag{26}$$

In general, this sum need not converge as a continuous function at $g = e$. We therefore proceed as in Liu's paper [18] and apply a smoothing operator $e^{t\Delta}$, $t > 0$ to $\mathfrak{m}$, where Δ is the Laplace-Beltrami operator on G. Then $\lim_{t \to 0^+} (e^{t\Delta}\mathfrak{m})|_e = \mathfrak{m}|_e$, so that

$$\mathrm{Vol}(M/\!/G) = \frac{k}{\mathrm{vol}\, G^2} \lim_{t \to 0^+} \sum_{\lambda \in \mathrm{Irr}(G)} e^{-tp(\lambda)} \langle \mathfrak{m}, \chi_\lambda \rangle \dim V_\lambda \tag{27}$$

where $p(\lambda)$ is the eigenvalue of $-\Delta$ on the character χ_λ. If G is connected, the irreducible representations are labeled by the set of dominant (real) weights $\lambda \in \mathfrak{t}^*$, and

$$p(\lambda) = \|\lambda + \rho\|^2 - \|\rho\|^2.$$

In the general case, V_λ splits into a direct sum of irreducible representations of the identity component G° with highest weights λ_i, and $p(\lambda) = \|\lambda_i + \rho\|^2 - \|\rho\|^2$ for any i.

If the Fourier coefficients $\langle \mathfrak{m}, \chi_\lambda \rangle$ decrease sufficiently fast as $p(\lambda) \to \infty$, the convergence factor may be omitted and one has a simpler formula

$$\mathrm{Vol}(M/\!/G) = \frac{k}{\mathrm{vol}\, G^2} \sum_{\lambda \in \mathrm{Irr}(G)} \langle \mathfrak{m}, \chi_\lambda \rangle \dim V_\lambda. \tag{28}$$

Suppose G is connected, and view λ as a weight. By the Weyl dimension formula, $\dim V_\lambda$ is a polynomial of degree $\frac{1}{2}(\dim G - \operatorname{rank} G)$ in λ. Hence a sufficient criterion for absolute convergence is $|\langle \mathfrak{m}, \chi_\lambda\rangle| \le C||\lambda||^{-\frac{1}{2}(\dim G+\operatorname{rank} G)-\epsilon}$ for some constants $C > 0, \epsilon > 0$.

4.2. **Fusion.** By Theorem 3.5, the Liouville measures ν and $\tilde{\nu}$ on a group valued Hamiltonian $G \times G$ space M and on its fusion $\tilde{M}$ coincide. Therefore, if M is compact, the DH-measures $\mathfrak{m} \in \mathcal{E}'(G \times G)$ and $\tilde{\mathfrak{m}} \in \mathcal{E}'(G)$ are related by push-forward under group multiplication $\mathrm{Mult}_G : G \times G \to G$, and the Fourier coefficients are

$$\langle \tilde{\mathfrak{m}}, \chi_\lambda\rangle = (\dim V_\lambda)^{-1}\langle \mathfrak{m}, \chi_\lambda \otimes \chi_\lambda\rangle. \tag{29}$$

In particular, the DH-measure for a fusion product $M_1 \circledast M_2 \cdots \circledast M_r$ is given by convolution, $\mathfrak{m}_1 * \mathfrak{m}_2 * \cdots * \mathfrak{m}_r$, with Fourier coefficients

$$\langle \mathfrak{m}_1 * \cdots * \mathfrak{m}_l, \chi_\lambda\rangle = (\dim V_\lambda)^{1-l} \prod_{j=1}^{l} \langle \mathfrak{m}_j, \chi_\lambda\rangle. \tag{30}$$

4.3. **Examples.** In this section we compute the DH-measures for a number of examples, including the space $G^{2h} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ related to the moduli space of flat bundles on a surface of genus h. We begin with twisted conjugacy classes.

Proposition 4.2. *Let $\psi \in \mathrm{Aut}(G)$ be an automorphism of finite order k, and use the same notation for the induced automorphism of $\mathfrak{g}$. Given $g \in G$, let H be the stabilizer of g under the twisted adjoint action. The Liouville measure of the twisted conjugacy class $\mathcal{C} = \mathrm{Ad}_G^\psi(g)$ is equal to the Riemannian measure of the homogeneous space G/H times a factor $|\det_\mathfrak{c}(\mathrm{Ad}_g - \psi)|^{\frac{1}{2}}$, where $\mathfrak{c} \subset \mathfrak{g}$ is the orthogonal complement of the kernel of $(\mathrm{Ad}_g - \psi)$. That is,*

$$\mathrm{Vol}(\mathcal{C}) = |\det\nolimits_\mathfrak{c}(\mathrm{Ad}_g - \psi)|^{\frac{1}{2}} \frac{\mathrm{vol}\, G}{\mathrm{vol}\, H}.$$

Proof. View $\mathcal{C}$ as a conjugacy class of (ψ^{-1}, g) for the semi-direct product $\mathbb{Z}_k \ltimes G$ defined by ψ. The adjoint action of (ψ^{-1}, g) on $\mathfrak{g}$ is given by $\mathrm{Ad}_{(\psi^{-1},g)}\,\xi = \psi^{-1}(\mathrm{Ad}_g(\xi))$. The automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ induces an isometry from the orthogonal complement of $\mathfrak{h} = \ker(\mathrm{Ad}_{(\psi^{-1},g)} - 1)$ onto $\mathfrak{c}$. Therefore,

$$\det\nolimits_{\mathfrak{h}^\perp}(\mathrm{Ad}_{(\psi^{-1},g)} - 1) = \det\nolimits_\mathfrak{c}(\mathrm{Ad}_g - \psi).$$

□

Proposition 4.3. *The Liouville measure of the symmetric space G/G^ψ is equal to its Riemannian measure times a factor, $2^{\dim(G/G^\psi)/2}$.*

Proof. In this case, $\mathfrak{c} = \ker(1-\psi)^\perp$ is the -1 eigenspace of ψ. Therefore $|\det_\mathfrak{c}(1-\psi)| = 2^{\dim(G/G^\psi)}$. □

Proposition 4.4. *The Liouville measure of G, viewed as a symmetric space for the involution $\psi(a, b) = (b, a)$ of $G \times G$, coincides with its Riemannian measure.*

Proof. By Proposition 4.3 the Liouville volume is equal to $2^{\dim G/2}\operatorname{vol}(G)^2$, divided by the Riemannian volume of the diagonal subgroup $(G\times G)^\psi$. Since the induced Riemannian metric of $(G\times G)^\psi \cong G$ is twice the Riemannian metric of G, its volume is $2^{\dim G/2}\operatorname{vol} G$. ☐

Proposition 4.5. *The Liouville measure for the double $D(G)$ and for its fusion $\tilde{D}(G)$ coincide with the Riemannian measure of $G\times G$. The Fourier coefficients of their DH-measures are*

$$\begin{aligned}\langle \mathfrak{m}_{D(G)}, \chi_\lambda\otimes\chi_\mu\rangle &= \operatorname{vol} G^2\ \delta_{\lambda,\mu}\\ \langle \mathfrak{m}_{\tilde{D}(G)}, \chi_\lambda\rangle &= \frac{\operatorname{vol} G^2}{\dim V_\lambda}\end{aligned}$$

Proof. The identification of Liouville measures and Riemannian measures follows from Proposition 4.4, since fusion does not change the Liouville measure (Theorem 3.5). Hence the DH-measure $\mathfrak{m}_{D(G)}$ is the push-forward of the Riemannian measure on G^2 under the moment map $(a,b)\mapsto (ab, a^{-1}b^{-1})$. Its Fourier coefficients are obtained from the calculation,

$$\langle \mathfrak{m}_{D(G)}, \chi_\lambda\otimes\chi_\mu\rangle = \int_{G\times G}\chi_\lambda(ab)\chi_\mu(a^{-1}b^{-1})\mathrm{d}\operatorname{vol}_{G\times G} = \operatorname{vol} G\ \delta_{\lambda,\mu}$$

where we have used conjugation invariance and the orthogonality relations for irreducible characters. The formula for the DH-measure of $\tilde{D}(G)$ follows from (29). ☐

Proposition 4.6. *Let $\mathcal{C}_1,\ldots,\mathcal{C}_r$ be a collection of conjugacy classes in G and $g_j\in\mathcal{C}_j$. The Liouville measure of the fusion product*

$$M = \tilde{D}(G)\circledast\cdots\circledast\tilde{D}(G)\circledast\mathcal{C}_1\circledast\cdots\circledast\mathcal{C}_r \tag{31}$$

(with $h\geq 0$ factors of $\tilde{D}(G)$) is equal to the Riemannian measure of $G^{2h}\times G/G_{g_1}\times\cdots\times G/G_{g_r}$, times a factor $\prod_{j=1}^r |\det_{\mathfrak{g}_{g_j}^\perp}(\operatorname{Ad}_{g_j}-1)|^{1/2}$. The Fourier coefficients of its DH-measure $\mathfrak{m}\in\mathcal{E}'(G)$ are given by the formula,

$$\langle \mathfrak{m},\chi_\lambda\rangle = \operatorname{vol} G^{2h}\ \frac{\prod_{j=1}^r \operatorname{Vol}(\mathcal{C}_j)\,\chi_\lambda(\mathcal{C}_j)}{(\dim V_\lambda)^{2h+r-1}} \tag{32}$$

Proof. This follows directly from Propositions 3.6 and 4.5, together with the formula (30) for the Fourier coefficients of convolutions of invariant measures. ☐

5. Moduli spaces of flat bundles over surfaces

5.1. **Connected semi-simple groups.** Let Σ be a compact, oriented surface of genus h with r boundary components, G be a connected semi-simple Lie group, and let $\underline{\mathcal{C}} = (\mathcal{C}_1,\ldots,\mathcal{C}_r)$ be a collection of conjugacy classes in G. Denote by $\mathcal{M}_G(\Sigma,\underline{\mathcal{C}})$ the space of isomorphism classes of flat G-bundles over Σ, with holonomy around the jth boundary circle in the conjugacy class $\mathcal{C}_j$. The Atiyah-Bott [5] construction shows that $\mathcal{M}_G(\Sigma,\underline{\mathcal{C}})$

is a compact symplectic space (sometimes singular). $\mathcal{M}_G(\Sigma,\underline{\mathcal{C}})$ may also be obtained [1] as a symplectic quotient of a group valued Hamiltonian G-manifold

$$\mathcal{M}_G(\Sigma,\underline{\mathcal{C}}) = M/\!\!/G \tag{33}$$

where (M,ω,Φ) is the group valued Hamiltonian G-manifold defined in (31). Recall that the G-action on M is given by conjugation on each factor in (31). In particular, the center $Z(G)\subset G$ acts trivially.

In the following discussion we will make the assumption $(*)$: Each component of $\Phi^{-1}(e)$ contains at least one point with stabilizer equal to $Z(G)$. Hence $Z(G)$ is the principal stabilizer, and the assumptions of Theorem 4.1 hold with $k=\#Z(G)$.

In the case of genus $h\geq 2$ the assumption $(*)$ is automatic:

Lemma 5.1. *If $h\geq 1$, the moment map $\Phi : M\to G$ is surjective and the principal stabilizer is equal to the center $Z(G)$. If $h\geq 2$, each component of each fiber $\Phi^{-1}(g)$ meets M_{prin}.*

Proof. The stabilizer of any point $(a,b)\in G^2$ equals the centralizer of the subgroup generated [2] by (a,b). Since G is compact, connected and semi-simple, it has a generating set consisting of two elements (see *e.g.* [14, Theorem 6.55]). It follows that the principal stabilizer for the conjugation action on G^2 is equal to the center $Z(G)$. Hence the principal stabilizer for the action on M equals $Z(G)$ for genus $h\geq 1$. Moreover, since the commutator map $G^2\to G$, $(a,b)\mapsto aba^{-1}b^{-1}$ is surjective (see e.g. [14, Corollary 6.56], or Lemma B.1 below), the moment map Φ is surjective.

Let $h\geq 2$. Suppose first that G is simply connected, Then the fibers of Φ are connected by Theorem 7.2 of [1]. Write M as a fusion product $M=M_1\circledast M_2$ where $M_1=\tilde{D}(G)$ and M_2 is defined by a formula similar to (31), but with only $h-1$ factors of $\tilde{D}(G)$. Pick $m_1\in(M_1)_{\mathrm{prin}}$. Since the moment map Φ_2 is surjective, $m_2\in M_2$ with $\Phi_2(m_2)=\Phi_1(m_1)^{-1}g$. Then $m=(m_1,m_2)\in M_{\mathrm{prin}}\cap\Phi^{-1}(g)$.

In the general case, let $\hat{G}$ be the simply connected universal cover of G, and $(\hat{M},\hat{\omega},\hat{\Phi})$ the group-valued Hamiltonian $\hat{G}$-space obtained from (31) by replacing G with $\hat{G}$ and $\mathcal{C}_j$ with $\hat{G}$-conjugacy classes $\hat{\mathcal{C}}_j$ covering $\mathcal{C}_j$. The natural projection $\pi:\hat{M}\to M$ takes $\hat{M}_{\mathrm{prin}}$ to M_{prin}, and $\Phi=\pi\circ\hat{\Phi}$. From the result for $\hat{M}$, it follows that each component of each fiber of Φ meets the principal stratum of M. □

For surfaces of genus ≤ 1, the validity of $(*)$ depends on the conjugacy classes $\mathcal{C}_j$. Indeed, $(*)$ never holds for $2h+r\leq 2$. However, one finds that $(*)$ holds for $h=1, r\geq 1$ if at least one of the conjugacy classes $\mathcal{C}_j$ is a principal orbit for the conjugation action. For a careful description of the stabilizer groups in general, see Bismut-Labourie [9, Section 5.5].

[2] A closed subgroup $H\subset G$ of a compact Lie group is *generated* by $S\subset G$ if it is the smallest closed subgroup containing S.

Using (32) we obtain, with the assumption $(*)$,

$$\mathrm{Vol}(\mathcal{M}_G(\Sigma,\underline{\mathcal{C}})) = \#Z(G)\ \mathrm{vol}\,G^{2h-2}\prod_{j=1}^{r}\mathrm{Vol}(\mathcal{C}_j)\ \lim_{t\to 0+}\sum_{\lambda\in\mathrm{Irr}(G)}\frac{e^{-tp(\lambda)}\prod_{j=1}^{r}\chi_\lambda(\mathcal{C}_j)}{(\dim V_\lambda)^{2h+r-2}}, \tag{34}$$

which is Witten's formula [23]. For $h\geq 2$, the sum is absolutely convergent for $t=0$, and the convergence factor $e^{-tp(\lambda)}$ may be omitted. This can be seen as follows: Using $|\chi_\lambda(\mathcal{C}_j)|\leq\dim V_\lambda$, we have

$$\frac{\prod_{j=1}^{r}|\chi_\lambda(\mathcal{C}_j)|}{(\dim V_\lambda)^{2h+r-2}}\leq(\dim V_\lambda)^{-2}.$$

Let $\mathcal{R}_+$ be the set of positive roots, and $\{\alpha_1,\ldots,\alpha_l\}$ the simple roots of G, where l is the rank. By the Weyl dimension formula,

$$\dim V_\lambda=\prod_{\alpha\in\mathcal{R}_+}\Big(1+\frac{\langle\lambda,\alpha\rangle}{\langle\rho,\alpha\rangle}\Big)\geq\prod_{i=1}^{l}\big(1+\lambda_i\big)$$

where $\lambda_i=\frac{\langle\lambda,\alpha_i\rangle}{\langle\rho,\alpha_i\rangle}$ are the Dynkin labels of λ. Thus

$$\sum_\lambda(\dim V_\lambda)^{-2}\leq\sum_{\lambda\in\Lambda^*_+}\prod_{i=1}^{l}\big(1+\lambda_i\big)^{-2}\leq\big(\sum_{n=1}^{\infty}n^{-2}\big)^l<\infty.$$

5.2. **Flat G-bundles of a given topological type.** As in the previous section we assume that G is connected and semi-simple. If Σ has a boundary ($r\geq 1$), every G-bundle over Σ is trivial, and so the moduli space of flat G-bundles $\mathcal{M}_G(\Sigma,\underline{\mathcal{C}})$ can also be viewed as a moduli space of flat connections on a trivial bundle. On the other hand an oriented surface without boundary ($r=0$) carries topologically non-trivial G-bundles: Let $D\subset\Sigma$ be a disk, and $\Sigma'=\Sigma\backslash D$. Any map $\gamma:\partial D\to G$ defines a G-bundle over Σ, by using γ as a gluing function for trivial G-bundles over D and $\Sigma\backslash D$. It is well-known that this construction gives a bijection between elements of $\pi_1(G)$ and isomorphism classes of G-bundles over Σ.

Let $\mathcal{M}_G^{(c)}(\Sigma)$ be the moduli space of flat connections on the G-bundle of toplogical type given by $c\in\pi_1(G)$. We compute the symplectic volume $\mathrm{Vol}(M_G^{(c)}(\Sigma)$ as follows. Let $\hat{G}$ be the universal cover of G and

$$\tilde{\Phi}^{(c)}:\ \tilde{D}(G)\circledast\ldots\circledast\tilde{D}(G)\to\hat{G}$$

the unique lift of Φ such that $\tilde{\Phi}^{(c)}(e,\ldots,e)=c^{-1}$. Then $(M,\omega,\hat{\Phi})$ is a group-valued Hamiltonian $\hat{G}$-space and $\mathcal{M}_G^{(c)}(\Sigma)=M/\!\!/\hat{G}$.

The DH-measure $\mathfrak{m}$ with respect to $\tilde{\Phi}^{(c)}$ has Fourier coefficients

$$\langle\mathfrak{m}^{(c)},\chi_\lambda\rangle=\mathrm{vol}\,G^{2h}\frac{\chi_\lambda(c^{-1})}{(\dim V_\lambda)^{2h-1}},\quad\lambda\in\mathrm{Irr}(\hat{G}).$$

Since the principal stabilizer for the $\hat{G}$-action has $\#Z(\hat{G}) = \#Z(G)\,\#\pi_1(G)$ elements, and $\mathrm{Vol}(\hat{G}) = \#\pi_1(G)\,\mathrm{vol}\,G$, Equation (27) yields

$$\mathrm{Vol}(\mathcal{M}_G^{(c)}(\Sigma)) = \mathrm{vol}\,G^{2h-2}\frac{\#Z(G)}{\#\pi_1(G)}\sum_{\lambda\in\mathrm{Irr}(\hat{G})}\frac{\chi_\lambda(c^{-1})}{(\dim V_\lambda)^{2h-1}}. \tag{35}$$

This agrees with Witten's result ([23], Section 4.1). Summing over all $c \in \pi_1(G)$ we recover the formula for $\mathrm{Vol}(\mathcal{M}_G(\Sigma))$ (as a sum over irreducible characters for G rather than $\hat{G}$).

5.3. **Two-component groups.** If G is disconnected, the orbit space M/G may have several connected components, with different principal stabilizer groups. In this section we consider moduli spaces of flat G-bundles over a closed surfaces of genus $h \geq 2$ where G is a compact Lie group which has exactly two connected components and the identity component G^{o} is semi-simple. This class of groups is small enough so that nice formulas exist, but large enough to include interesting examples, such as moduli spaces of Riemannian (non-oriented) vector bundles on surfaces.

Lemma 5.2. *If G has two components, the principal stabilizer for the G-action on $G\times G$ is equal to the centralizer $Z_G(G^{\mathrm{o}})$ on $G^{\mathrm{o}}\times G^{\mathrm{o}}$, and equal to the center $Z(G)$ on the other components of $G\times G$.*

Proof. It suffices to show G generated by elements $g_1 \in G\backslash G^{\mathrm{o}}$ and $g_2 \in G^{\mathrm{o}}$. Let T be a maximal torus of G^{o}, and $g_2 \in T$ a regular element generating T. Choose $g_1 \in G\backslash G^{\mathrm{o}}$ such that $\mathrm{Ad}_{g_1}(g_2)$ is not contained in a proper closed subgroup $H \subset G^{\mathrm{o}}$ containing T. This is possible since $\{\mathrm{Ad}_{g_1}(g_2)|\, g_1 \in G\backslash G^{\mathrm{o}}\}$ is a regular conjugacy class. Then $g_2, \mathrm{Ad}_{g_1}(g_2)$ generate G^{o} and therefore g_1, g_2 generate G. □

Let G be a compact Lie group with two connected components. If $Z(G) \subseteq G^{\mathrm{o}}$, then $Z_G(G^{\mathrm{o}}) = Z(G^{\mathrm{o}})$ while $Z(G) \subseteq Z(G^{\mathrm{o}})$ is the subgroup fixed under the action of the component group G/G^{o}. If $Z(G) \not\subseteq G^{\mathrm{o}}$, then $Z(G) = Z_G(G^{\mathrm{o}})$ contains $Z(G^{\mathrm{o}})$, and $Z(G)/Z(G^{\mathrm{o}}) = \mathbb{Z}_2$.

Examples 5.3.
(a) Let $G = \mathrm{O}(n)$, $n \geq 3$. Then $Z(G) = Z_G(G^{\mathrm{o}}) = \{I, -I\}$. The spaces $M_G(\Sigma, \underline{\mathcal{C}})$ are moduli spaces of flat Riemannian vector bundles.
(b) Let $G = \mathrm{O}(2)$. Then $Z(G) = \mathbb{Z}_2$ but $Z_G(G^{\mathrm{o}}) = \mathrm{SO}(2)$.
(c) Let G^{o} be a compact, connected Lie group, and $\psi \in \mathrm{Aut}(G^{\mathrm{o}})$ an involutive automorphism. Let $G = \mathbb{Z}_2 \ltimes G^{\mathrm{o}}$ be the corresponding semi-direct product. If ψ is inner, $Z(G) = Z_G(G^{\mathrm{o}}) = \mathbb{Z}_2 \times Z(G^{\mathrm{o}})$. If ψ is *not* inner, $Z_G(G^{\mathrm{o}}) = Z(G^{\mathrm{o}})$ while $Z(G) = Z(G^{\mathrm{o}})^{\psi}$.
(d) Let $G = \{A \in \mathrm{U}(n)|\, \det(A)^2 = 1\}$, $n \geq 2$. Note that G is not a semi-direct product. One finds $Z(G) = Z_G(G^{\mathrm{o}}) = \{aI|\, a^{2n} = 1\} = \mathbb{Z}_{2n}$. The spaces $M_G(\Sigma, \underline{\mathcal{C}})$ are moduli spaces of flat Hermitian bundles over Σ, such that the square of the canonical line bundle is trivial.

Let $\mathrm{Irr}(G, G^o) \subset \mathrm{Irr}(G)$ be the set of G-representations which are irreducible as G^o-representations. The image of the map $\mathrm{Irr}(G, G^o) \to \mathrm{Irr}(G^o)$ consists of irreducible representations that are fixed, up to isomorphism, by the action of the component group G/G^o.

Theorem 5.4. *Let G be a compact Lie group with two components, with G^o semi-simple, Σ be a closed surface of genus $h \geq 2$, $\mathcal{M}_G^o(\Sigma) \subset \mathcal{M}_G(\Sigma)$ be the equivalence classes of flat G-bundles that reduce to flat G^o-bundles, and $\mathcal{M}_G^1(\Sigma)$ the remaining components. Then,*

$$\mathrm{Vol}(\mathcal{M}_G^o(\Sigma)) = \frac{\# Z_G(G^o)}{\# Z(G^o)} \mathrm{Vol}(\mathcal{M}_{G^o}(\Sigma)),$$

and

$$\mathrm{Vol}(\mathcal{M}_G^1(\Sigma)) = (1 - 2^{-2h}) \# Z(G) \ \mathrm{vol}\, G^{2h-2} \sum_{\lambda \in \mathrm{Irr}(G, G^o)} (\dim V_\lambda)^{2-2h}.$$

Since we assume $h \geq 2$, the sum over λ is absolutely convergent.

Proof. Let $M = \tilde{D}(G) \circledast \cdots \circledast \tilde{D}(G)$ be the fusion product of h copies of $\tilde{D}(G)$, and $\Phi : M \to G$ the moment map. By Lemma B.3 from the Appendix B, all components of $\Phi^{-1}(e)$ meet the principal orbit type stratum of the corresponding component of M. Let $M^o = \tilde{D}(G^o) \circledast \cdots \circledast \tilde{D}(G^o)$ be a fusion product of h copies of $\tilde{D}(G^o)$. The principal stabilizer is equal to $Z(G)$ on $M^1 = M \backslash M^o$ and to $Z_G(G^o)$ on M^o. An open dense subset of the symplectic quotient $M^o /\!\!/ G^o$ fibers over $M^o /\!\!/ G$ with fiber $Z_G(G^o)/Z(G^o)$. Hence the volumes are related by a factor, $\frac{\# Z_G(G^o)}{\# Z(G^o)}$.

To compute $\mathrm{Vol}(M^1 /\!\!/ G)$, we calculate the Fourier coefficients $\langle \mathfrak{m}^1, \chi_\lambda \rangle$ for M^1. Since χ_λ vanishes on $G \backslash G^o$ for $\lambda \notin \mathrm{Irr}(G, G^o)$, we only need to consider $\lambda \in \mathrm{Irr}(G, G^o)$. We have $\langle \mathfrak{m}^1, \chi_\lambda \rangle = \langle \mathfrak{m}, \chi_\lambda \rangle - \langle \mathfrak{m}^o, \chi_\lambda \rangle$. The Fourier coefficients $\langle \mathfrak{m}, \chi_\lambda \rangle$ are given by (32) as usual. We claim that $\langle \mathfrak{m}^o, \chi_\lambda \rangle$ is given by a similar formula but with G^o in place of G:

$$\langle \mathfrak{m}^o, \chi_\lambda \rangle = 2^{-2h} \mathrm{vol}\, G^{2h} \ (\dim V_\lambda)^{1-2h}.$$

Indeed, $\langle \mathfrak{m}_{\tilde{D}(G^o)}, \chi_\lambda \rangle = \mathrm{vol}(G^o)^2 (\dim V_\lambda)^{-1}$ by the formula for G^o, since χ_λ restricts to an irreducible character of G^o. It follows that

$$\langle \mathfrak{m}^1, \chi_\lambda \rangle = (1 - 2^{-2h}) \mathrm{vol}\, G^{2h} \ (\dim V_\lambda)^{1-2h}.$$

These Fourier coefficients contribute with a factor $\# Z(G)$. $\square$

6. Mixed DH-distributions

In this Section we generalize the definition of DH-measures for Hamiltonian G-spaces (M, ω, Φ) with group valued moment map. We will associate to (M, ω, Φ) invariant distributions on G which are smooth at regular values of Φ and which encode certain intersection pairings on symplectic quotients. The results of this Section depend heavily on techniques developed in [2]. In [3, Theorem 5.2], it is shown how to calculate the Fourier coefficients of the mixed DH-distributions by localization techniques.

6.1. **Mixed DH-distributions for Hamiltonian G-manifolds.** For Hamiltonian G-manifolds (M,ω,Φ) with $\mathfrak{g}^*$-valued moment maps, mixed DH-distributions were first introduced by Jeffrey-Kirwan in [16], and studied in detail by Duistermaat [11], Vergne [21] and Paradan [20]. The definition of mixed DH-distributions uses the framework of equivariant cohomology. In contrast to the above references, we will use the Weil model of equivariant cohomology rather than the Cartan model.

As before we identify $\mathfrak{g}^* \cong \mathfrak{g}$ by a given invariant inner product, and let e_i be an orthonormal basis of $\mathfrak{g}$. The Weil algebra is the $W_G = S\mathfrak{g} \otimes \wedge\mathfrak{g}$, equipped with the G-action induced by the adjoint action on $\mathfrak{g}$. Let $L_i^W = L_i^{S\mathfrak{g}} \otimes 1 + 1 \otimes L_i^{\wedge\mathfrak{g}}$ be the generators for the G-action, $\iota_i^W = 1 \otimes \iota_i^{\wedge\mathfrak{g}}$ the contraction operators, and let

$$\mathrm{d}^W = \sum_i (L_i^{S\mathfrak{g}} \otimes 1) y_i + \sum_i (v_i - \frac{1}{2}\sum_{j,k} f_{ijk} y_j y_k)\iota_i^W. \tag{36}$$

Here v_i, y_i are the generators of $S\mathfrak{g}, \wedge\mathfrak{g}$ corresponding to the basis e_i, and $f_{ijk} = [e_i,e_j]\cdot e_k$ are the structure constants. Then W_G together with these derivations is a G-differential algebra. That is, the derivations ι_i^W, L_i^W and d^W satisfy (super-) bracket relations analogous to those for contractions, Lie derivatives, and exterior differential for a G-manifold M. Given a G-manifold M, the equivariant cohomology $H_G(M)$ is the cohomology of the basic subcomplex of $W_G \otimes \Omega(M)$, consisting of invariant elements that are annihilated by contractions $\iota_i^W \otimes 1 + 1 \otimes \iota_i$. We will also need the Weil algebra with distributional coefficients $\widehat{W}_G = \mathcal{E}'(\mathfrak{g}) \otimes \wedge\mathfrak{g}$, where $\mathcal{E}'(\mathfrak{g})$ is the convolution algebra of compactly supported distributions on $\mathfrak{g}$. The inclusion $S\mathfrak{g} \hookrightarrow \mathcal{E}(\mathfrak{g}), \quad v_i \mapsto \frac{d}{dt}|_{t=0}\delta_{te_i}$ makes W_G into a G-differential subalgebra of $\widehat{W}_G$.

Suppose now that (M,ω_0,Φ_0) is a Hamiltonian G-manifold. The *equivariant Liouville form* is a basic, closed element of $\widehat{W}_G \otimes \Omega(M)$ defined by

$$\mathcal{L}_0 = e^{\omega_0}\Phi_0^*\Lambda_0, \tag{37}$$

where

$$\Lambda_0 = e^{-\sum_i y_i d\mu_i} \exp(-\frac{1}{2}\sum_{i,j,k} f_{ijk} y_i y_j \mu_k)\ \delta_\mu \in \widehat{W}_G \otimes \Omega(\mathfrak{g}). \tag{38}$$

Notice that the horizontal projection $P_{\mathrm{hor}}^{\wedge\mathfrak{g}}\mathcal{L}_0$ is just $e^{\omega_0}\delta_{\Phi_0}$. Therefore, if M is compact, the integral of $\mathcal{L}_0$ over M coincides with the DH-measure:

$$\int_M \mathcal{L}_0 = \int_M P_{\mathrm{hor}}^{\wedge\mathfrak{g}}\mathcal{L}_0 = \int_M e^{\omega_0}\delta_{\Phi_0} = (\Phi_0)_*\nu_0 = \mathfrak{m}_0.$$

Generalizing this equation, one defines for any equivariant cocycle $\beta_0 \in (W_G \otimes \Omega(M))_{basic}$, the *mixed DH distribution* $\mathfrak{m}_0^{\beta_0} \in \mathcal{E}'(\mathfrak{g})^G$ by

$$\mathfrak{m}_0^{\beta_0} = \int_M \beta_0\mathcal{L}_0. \tag{39}$$

The map $\beta_0 \mapsto \mathfrak{m}_0^{\beta_0}$ descends to a map $H_G(M) \to \mathcal{E}'(\mathfrak{g})^G$. We quote from [16] the following properties of the mixed DH-distributions. $\mathfrak{m}_0^{\beta_0}$ is supported on the image of Φ_0 and has singular support in the set of singular values of Φ_0. If 0 is a regular value of Φ_0, let the *Kirwan map*

$$\kappa:\ H_G(M) \to H_G(\Phi_0^{-1}(0)) \cong H(M/\!/G)$$

be the map given by restriction to the zero level set. If M/G is connected and each connected component of $\Phi^{-1}(0)$ meets the principal orbit type stratum,

$$\int_{M/\!/G} \kappa(\beta_0)\exp(\omega_0)_{red} = \frac{k}{\operatorname{vol} G}\ \frac{\mathfrak{m}_0^{\beta_0}}{\mathrm{d}\operatorname{Vol}_{\mathfrak{g}}}\Big|_0. \tag{40}$$

Here $(\omega_0)_{red}$ the reduced symplectic form, and k is the number of elements in a principal stabilizer.

6.2. **Mixed DH-distributions for group valued moment maps.** To extend the construction of mixed DH-distributions to group-valued moment maps, we need a non-commutative version of the Weil algebra introduced in [2]. For simplicity, we will only discuss the case where G is connected and simply connected. The generalization to arbitrary compact groups (along the lines of Section 3.4) is straightforward.

Let $U(\mathfrak{g})$ be the universal enveloping algebra, with generators u_i. It embeds into the convolution algebra $\mathcal{E}'(G)$ of distributions on G by the map $u_i \mapsto \frac{d}{dt}|_{t=0}\delta_{\exp(te_i)}$. The non-commutative Weil algebra $\mathcal{W}_G$ and the non-commutative Weil algebra $\widehat{\mathcal{W}}_G$ with distributional coefficients are the $\mathbb{Z}_2$-graded algebras

$$\mathcal{W}_G := U(\mathfrak{g}) \otimes \operatorname{Cl}(\mathfrak{g}),\quad \widehat{\mathcal{W}}_G := \mathcal{E}'(G) \otimes \operatorname{Cl}(\mathfrak{g}).$$

The conjugation actions of G on $\mathfrak{g}$ and on G induce an action on $\mathcal{W}_G$, $\widehat{\mathcal{W}}_G$. Let $L_i^{\mathcal{W}}$ be the corresponding Lie derivatives, and $\iota_i^{\mathcal{W}} = \operatorname{ad}(x_i)$ (graded commutator with x_i). As shown in [2],

$$\mathrm{d}^{\mathcal{W}} = \operatorname{ad}\Big(\sum_i u_i x_i - \frac{1}{6}\sum_{i,j,k} f_{ijk} x_i x_j x_k\Big)$$

is a differential, and makes $\mathcal{W}_G$ and $\widehat{\mathcal{W}}_G$ into G-differential algebras. For any G-manifold M, we define $\widehat{\mathcal{H}}_G(M)$ (resp.$\mathcal{H}_G(M)$) as the cohomology of the basic subcomplex of $\widehat{\mathcal{W}}_G \otimes \Omega(M)$ (resp. $\mathcal{W}_G \otimes \Omega(M)$). Let

$$\Lambda = e^{-\sum_i x_i \theta_i^R}\,\tau(g)\delta_g \in \widehat{\mathcal{W}}_G \otimes \Omega(G).$$

Now let (M, ω, Φ) be a group valued Hamiltonian G-manifold. From the properties of the form Λ (see [2, Proposition 5.7]) and the axioms for a space with group-valued moment map, it is immediate that the *equivariant Liouville form*

$$\mathcal{L} = e^{\omega}\ \Phi^*\Lambda \in \widehat{\mathcal{W}}_G \otimes \Omega(M)$$

is basic and closed. Note that if we denote

$$\Pi^{\mathcal{W}} = \int_G \otimes P_{\mathrm{hor}}^{\mathrm{Cl}} : \widehat{\mathcal{W}}^G = \mathcal{E}'(G) \otimes \mathrm{Cl}(\mathfrak{g}) \to \mathbb{R}$$

then $(\Pi^{\mathcal{W}} \otimes 1)\Lambda = \mathcal{T}$, $(\Pi^{\mathcal{W}} \otimes 1)\mathcal{L} = \Gamma$, where $\mathcal{T} \in \Omega(G)$ and $\Gamma \in \Omega(M)$ are the differential forms introduced in Section 3.2. Again the integral of $\mathcal{L}$ over M reproduces the DH-measure $\mathfrak{m} = \Phi_*(\Gamma_{[\dim M]})$:

$$\int_M \mathcal{L} = \int_M P_{\mathrm{hor}}^{\mathrm{Cl}}(\mathcal{L}) = \int_M \Gamma \delta_\Phi = \Phi_* \nu = \mathfrak{m}.$$

Given any equivariant cocycle $\beta \in (\mathcal{W}_G \otimes \Omega(M))_{basic}$ we define

$$\mathfrak{m}^\beta = \int_M \beta \mathcal{L} \in \mathcal{E}'(G)^G. \tag{41}$$

The map $\beta \mapsto \mathfrak{m}^\beta$ vanishes on coboundaries, hence it descends to a map $\mathcal{H}_G(M) \to \mathcal{E}'(G)^G$.

6.3. **Exponentials.** To study the properties of the mixed DH-distributions $\mathfrak{m}^\beta$, we need to understand the relationship between the mixed DH-distributions of a Hamiltonian G-space (M, ω_0, Φ_0) and of its exponential. This involves the *quantization map* $\mathcal{Q}$ introduced in [2]. Let $f(s) = \frac{1}{s} - \frac{1}{2}\coth(\frac{s}{2})$, and let T be the skew-symmetric tensor field on $\mathfrak{g} \backslash J^{-1}(0)$ given by $T(\mu)_{ij} = f(\mathrm{ad}_\mu)_{ij}$. Then

$$\mathcal{Q} : \widehat{W}_G \to \widehat{\mathcal{W}}_G, \quad \beta_0 \mapsto \sigma^{-1} \circ \exp_* \left(J^{\frac{1}{2}} \exp(\frac{1}{2} \sum_{i,j} T_{ij} \iota_i^W \iota_j^W)(\beta_0) \right) \tag{42}$$

is a well-defined map, since the singularities of T are compensated by the zeroes of $J^{\frac{1}{2}}$. It was shown in [2] that $\mathcal{Q}$ is a homomorphism of G-differential spaces, and for any G-manifold M the induced map in cohomology $\widehat{H}_G(M) \to \widehat{\mathcal{H}}_G(M)$ is in fact a ring homomorphism. The quantization map $\mathcal{Q}$ has the property,

$$\mathcal{Q}(\Lambda_0) = e^{\varpi}(1 \otimes \exp^*)\Lambda. \tag{43}$$

where $\varpi \in \Omega^2(\mathfrak{g})$ is the 2-form from 2.4. Now let (M, ω_0, Φ_0) be a Hamiltonian G-space such that exp has maximal rank on $\Phi_0(M) \subset \mathfrak{g}$, and let (M, ω, Φ) be its exponential. Let $\mathcal{L}_0, \mathcal{L}$ be the equivariant Liouville forms. Since $\omega = \omega_0 + \Phi_0^* \varpi$, it follows immediately from (43) that

$$\mathcal{Q}(\mathcal{L}_0) = \mathcal{L}.$$

Hence if $\beta_0 \in (W_G \otimes \Omega(M))$ is an equivariant cocycle and $\beta = \mathcal{Q}(\beta_0)$, then $\mathcal{Q}(\beta_0 \mathcal{L}_0)$ is cohomologous to $\beta\mathcal{L}$. It follows that

$$\mathfrak{m}^\beta = J^{\frac{1}{2}} \exp_*(\mathfrak{m}_0^{\beta_0}). \tag{44}$$

6.4. **Intersection pairings on symplectic quotients.** Let (M, ω, Φ) be a compact group-valued Hamiltonian G-space. If e is a regular value of Φ let

$$\kappa:\ \mathcal{H}_G(M) \to H(M/\!/G)$$

be composition of the isomorphism $\mathcal{Q}^{-1}:\ \mathcal{H}_G(M) \cong H_G(M)$, pull-back to the level set $H_G(M) \to H_G(\Phi^{-1}(e))$, and isomorphism $H_G(\Phi^{-1}(e)) \cong H(\Phi^{-1}(e)/G)$.

Theorem 6.1. *For every equivariant cocycle $\beta \in (\mathcal{W}_G \otimes \Omega(M))_{basic}$ the support of $\mathfrak{m}^\beta$ is contained in the image of Φ, and the singular support in the set of singular values of Φ. If e is a regular value of Φ and if each component of $\Phi^{-1}(e)$ meets M_{prin}, then*

$$\int_{M/\!/G} \kappa(\beta) \exp(\omega_{\mathrm{red}}) = \frac{k}{\operatorname{vol} G}\ \frac{\mathfrak{m}^\beta}{d\operatorname{Vol}_G}\Big|_e, \tag{45}$$

using the notation of Theorem 4.1.

Proof. Write $P^{\mathrm{Cl}}_{\mathrm{hor}}(\beta) = \sum_J u^J \otimes \beta_J$ with $u^J \in U(\mathfrak{g})$ and $\beta_J \in \Omega(M)$. Let $\odot$ be the following product structure on $\Omega(M)$,

$$\gamma_1 \odot \gamma_2 = \operatorname{diag}_M^* \exp(-\frac{1}{2}\sum_i \iota_i^1 \iota_i^2)(\gamma_1 \otimes \gamma_2),$$

where $\operatorname{diag}_M:\ M \to M \times M$ is the diagonal embedding. Arguing as in the proof of Theorem 3.5, we have $P^{\mathrm{Cl}}_{\mathrm{hor}}(\beta\mathcal{L}) = P^{\mathrm{Cl}}_{\mathrm{hor}}(\beta) \odot P^{\mathrm{Cl}}_{\mathrm{hor}}(\mathcal{L}) = P^{\mathrm{Cl}}_{\mathrm{hor}}(\beta) \odot (\Gamma\delta_\Phi)$. Integrating over M, the terms involving contractions ι_i make no contribution, and we obtain

$$\int_M \beta\mathcal{L} = \int_M P^{\mathrm{Cl}}_{\mathrm{hor}}(\beta\mathcal{L}) = \int_M P^{\mathrm{Cl}}_{\mathrm{hor}}(\beta)\ \Gamma\delta_\Phi = \sum_J u^J \int_M (\beta_J\Gamma)\,\delta_\Phi = \sum_J u^J\ \Phi_*(\beta_J\ \Gamma)_{[\dim M]}.$$

Thus

$$\mathfrak{m}^\beta = \sum_J u^J\ \Phi_*(\beta_J\ \Gamma)_{[\dim M]}.$$

Since left convolution by u^J is a differential operator, the description of the support and singular support of $\mathfrak{m}^\beta$ follows. The interpretation of $\mathfrak{m}^\beta$ at the group unit e follows as in the proof of Theorem 4.1 by a reduction to the Hamiltonian case, using Equations (40),(44). ☐

Appendix A. The spinning 4-sphere

An interesting example of a Hamiltonian action with group valued moment map is the 4-sphere S^4. We let $G = \mathrm{SU}(2)$, with inner product on $\mathfrak{su}(2)$ given by $\xi_1 \cdot \xi_2 = -\frac{1}{2}\operatorname{tr}(\xi_1\xi_2)$. Cover S^4 by two coordinate charts $U^\pm$ given as open balls $\|z\|^2 < 2$ in $\mathbb{C}^2$ with transition map

$$\varphi:\ U^+\backslash\{0\} \to U^-\backslash\{0\},\quad (z_1, z_2) \mapsto (-\overline{z}_2, \overline{z}_1)\frac{\sqrt{2 - \|z\|^2}}{\|z\|}.$$

The map φ is equivariant for the standard $\mathrm{SU}(2)$-action on $U^\pm \subset \mathbb{C}^2$. The $\mathrm{SU}(2)$-action on S^4 obtained in this way may be identified with the action induced from the embedding of S^4 in $\mathbb{R}^5 \cong \mathbb{C}^2 \oplus \mathbb{R}$, with SU(2) acting on the first factor via the standard representation. Let $\Phi^\pm : U^\pm \to \mathrm{SU}(2)$ be given by

$$\Phi^+(z_1, z_2) = \cos\big(\frac{\pi\|z\|^2}{2}\big)I + \frac{i}{\|z\|^2}\sin\big(\frac{\pi\|z\|^2}{2}\big)\begin{pmatrix} |z_1|^2 - |z_2|^2 & 2z_1\overline{z}_2 \\ 2\overline{z}_1 z_2 & |z_2|^2 - |z_1|^2 \end{pmatrix}$$

and $\Phi^-(z_1, z_2) = -\Phi^+(z_1, z_2)$. It is straightforward to verify that $\varphi^*\Phi^- = \Phi^+$, so that $\Phi^\pm$ patch together to define an SU(2)-equivariant map $\Phi : S^4 \to \mathrm{SU}(2)$. Define

$$\omega^+ = \frac{\pi}{2\|z\|^2}\,\mathrm{Re}(z_1 \mathrm{d}\overline{z}_1 + \mathrm{d}z_2\overline{z}_2) \wedge \mathrm{d}\|z\|^2 - \frac{\sin(\pi\|z\|^2)}{\|z\|^4}\,\mathrm{Im}(\overline{z}_1 z_2 \mathrm{d}z_1 \mathrm{d}\overline{z}_2)$$

and let $\omega^- = \omega^+$. Again, one can check that $\omega^\pm$ are SU(2)-invariant and that $\varphi^*\omega^- = \omega^+$.

Theorem A.1. *The triple (S^4, ω, Φ) is a group valued Hamiltonian* SU(2)*-manifold. The volume form $\Gamma_{[4]}$ is given in terms of the standard volume form $d\mathrm{Vol}_{\mathbb{C}^2}$ on $U^+ \subset \mathbb{C}^2$ by*

$$\Gamma_{[4]}|_{U_+} = \frac{\sin(\frac{\pi}{2}\|z\|^2)}{\frac{\pi}{2}\|z\|^2}\, d\mathrm{Vol}_{\mathbb{C}^2}\,.$$

Outline of proof. We show that (U^+, ω^+, Φ^+) is the exponential (cf. Section 6.3) of a Hamiltonian SU(2)-manifold. Let $\omega_0 = \frac{i}{2}(\mathrm{d}z_1\mathrm{d}\overline{z}_1 + \mathrm{d}z_2\mathrm{d}\overline{z}_2)$ be the standard symplectic structure on $\mathbb{C}^2$. The defining SU(2)-action on $\mathbb{C}^2$ is Hamiltonian, with moment map

$$\Phi_0(z_1, z_2) = \frac{i\pi}{2}\begin{pmatrix} |z_1|^2 - |z_2|^2 & 2z_1\overline{z}_2 \\ 2\overline{z}_1 z_2 & |z_2|^2 - |z_1|^2 \end{pmatrix}.$$

To compute $\exp(\Phi_0)$ we use the formula

$$\exp(\xi) = \cos(\|\xi\|)I + \frac{\sin(\|\xi\|)}{\|\xi\|}\xi, \quad \xi \in \mathfrak{su}(2)$$

Putting $\xi = \Phi_0(z)$, and using that $\|\Phi_0(z)\| = \frac{\pi\|z\|^2}{2}$, we obtain $\exp(\Phi_0) = \Phi^+$. The symplectic form ω_0 can be written in the form

$$\omega_0 = \frac{\pi}{2\|z\|^2}\,\mathrm{Re}(z_1 \mathrm{d}\overline{z}_1 + \mathrm{d}z_2\overline{z}_2) \wedge \mathrm{d}\|z\|^2 - \frac{\pi}{\|z\|^2}\,\mathrm{Im}(\overline{z}_1 z_2 \mathrm{d}z_1 \mathrm{d}\overline{z}_2).$$

The term $-\frac{1}{\|z\|^4}\mathrm{Im}(\overline{z}_1 z_2 \mathrm{d}z_1 \mathrm{d}\overline{z}_2)$ appearing in the formulas for both ω^+ and ω_0 is the pull-back of the normalized volume form $\mathrm{d\,Vol}_{S^2}$ on S^2 under the quotient map $\psi : \mathbb{C}^2\backslash\{0\} \to \mathbb{C}P(1)$. Thus

$$\omega^+ - \omega_0 = (\sin(\pi\|z\|^2) - \pi\|z\|^2)\psi^*\mathrm{d\,Vol}_{S^2}\,.$$

We want to identify this expression with $\Phi_0^*\varpi$. Introduce polar coordinates by the map $[0, \infty) \times S^2 \to \mathfrak{su}(2), (t, \zeta) \mapsto t\zeta$ where S^2 is viewed as the unit sphere in $\mathfrak{su}(2)$. Then the form ϖ is a multiple of the normalized volume form on S^2:

$$\varpi = \big(\sin(2\|\xi\|) - 2\|\xi\|\big)\,\mathrm{d\,Vol}_{S^2}\,.$$

Using again $\|\Phi_0(z)\| = \frac{\pi\|z\|^2}{2}$, it follows that $\omega^+ = \omega_0 + \Phi_0^*\varpi$. This shows that (U^+, ω^+, Φ^+) is the exponential of the subset $\|z\|^2 < 2$ of the Hamiltonian SU(2)-manifold $(\mathbb{C}^2, \omega_0, \Phi_0)$. Multiplying the moment map Φ_+ by the central element $-I \in \mathrm{SU}(2)$ one obtains (U^-, ω^-, Φ^-). The volume form $\Gamma_{[4]}$ is obtained from Proposition 3.7, since $\Phi_0^* J^{1/2} = \frac{\sin(\|\Phi_0\|)}{\|\Phi_0\|}$. □

The spinning 4-sphere is an example of a multiplicity free space: All reduced spaces for the SU(2)-action are points. The SU(2)-action extends to an action of U(2), by letting the central U(1) act on $U^+ \subset \mathbb{C}^2$ by $e^{i\phi} \cdot (z_1, z_2) = (e^{i\phi} z_1, e^{i\phi} z_2)$ and on U^- by the opposite action, $e^{i\phi} \cdot (z_1, z_2) = (e^{-i\phi} z_1, e^{-i\phi} z_2)$. A moment map for the U(2) action is given by

$$\tilde{\Phi}^+(z_1, z_2) = \tilde{\Phi}^-(z_1, z_2) = e^{-\frac{i\pi}{2}\|z\|^2} \Phi^+(z_1, z_2).$$

The reduced spaces by the U(1)-action are conjugacy classes of SU(2). In Hurtubise-Jeffrey [15], the S^4 example appears as the *imploded cross-section* for the double $\tilde{D}(\mathrm{SU}(2))$.

Appendix B. Principal Stabilizers

The following Lemma is an extension of a well-known fact about compact connected semi-simple Lie groups (Goto's commutator theorem, [14, Theorem 6.55]):

Lemma B.1. *Suppose G is a compact Lie group, with G^{o} semi-simple, and assume that G/G^{o} is finite cyclic. Then the restriction of the commutator map $f : G \times G \to G,\ (a, b) \mapsto [a, b]$ to any component of $G \times G$ is onto G^{o}.*

Proof. Note first that since the component group G/G^{o} is abelian, the commutator map takes values in G^{o}. Let $xG^{\mathrm{o}} \in G/G^{\mathrm{o}}$ be a generator. We want to show that f is onto G^{o} on the component of (x^k, x^l), for any $k, l \in \mathbb{Z}_{\geq 0}$. Using $[a, b] = [b, a]^{-1}$, we may assume $k \geq l$. If $l > 0$, let m be the largest non-negative integer such that $k - ml \geq 0$. Using $[a, b] = [ab^{-m}, b]$ we see that f is onto G^{o} on the component of (x^k, x^l) if and only it is onto G^{o} on the component of (x^{k-ml}, x^l). Let $k_1 = l, l_1 = k - ml$. Iterating this procedure, we obtain a finite sequence $k_j \geq l_j \geq 0$ such that f is onto G^{o} on the component of (x^{k_j}, x^{l_j}). The sequence terminates when $l_j = 0$. We have thus reduced the problem to the case $l = 0$. Since f is equivariant, it suffices to show that some maximal torus $T \subset G^{\mathrm{o}}$ is in the image for this component. Lemma B.2 below shows that one can find $a \in N_{G^{\mathrm{o}}}(T)x^k$ such that Ad_a preserves $\mathfrak{t}$ and $\mathrm{Ad}_a|_{\mathfrak{t}}$ has no eigenvalue equal to 1. Given $t \in T$, choose $\xi \in \mathfrak{t}$ with $\exp((\mathrm{Ad}_a - 1)\xi) = t$. Then $b = \exp \xi$ satisfies

$$[a, b] = \exp(\mathrm{Ad}_a \xi) \exp(-\xi) = \exp(\mathrm{Ad}_a \xi - \xi) = t.$$

□

Lemma B.2. *Let $\mathfrak{g}$ be a compact, semi-simple Lie algebra, and $\mathfrak{t} \subset \mathfrak{g}$ a maximal abelian subalgebra. For any Lie algebra automorphism $\psi \in \mathrm{Aut}(\mathfrak{g})$, there exists an inner automorphism $\phi \in \mathrm{Int}(\mathfrak{g})$ such that $\phi \circ \psi$ preserves $\mathfrak{t}$, and its restriction to $\mathfrak{t}$ has no eigenvalue equal to 1.*

Proof. Let $\mathfrak{t}_+ \subset \mathfrak{t}$ be a positive Weyl chamber. By composing ψ with an inner automorphism, we may assume that ψ takes $\mathfrak{t}_+$ to itself, and in fact any element of $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ has a unique representative of this form. (That is, ψ is induced from an automorphism of the Dynkin diagram of $\mathfrak{g}$.) We have to find a Weyl group element $w \in W$ such that $w \circ \psi|_{\mathfrak{t}}$ has no eigenvalue equal to 1. Then any $\phi \in \mathrm{Int}(\mathfrak{g})$ preserving $\mathfrak{t}$ and acting as w on $\mathfrak{t}$ will have the required property. If $\psi = 1$, one can take w to be any Coxeter element, cf. [15, Chapter 3.16]. Suppose $\psi \neq 1$. By decomposing $\mathfrak{g}$ with respect to the action of ψ, we may assume that $\mathfrak{g}$ contains no proper ψ-invariant ideal. We consider two cases:

1) Assume that $\mathfrak{g}$ is not simple. Let $\mathfrak{h}$ be a simple ideal of $\mathfrak{g}$. Then $\mathfrak{g} = \oplus_{j=0}^{k-1} \psi_0^j(\mathfrak{h})$, where k is the order of ψ_0. Let w' be a Coxeter element of $\mathfrak{h}$. Then $w = (w', 1, \ldots, 1)$ has the required property.

2) We are left with the case of $\mathfrak{g}$ simple. Recall that $\mathrm{Out}(\mathfrak{g}) = \mathbb{Z}_2$ for the Lie algebras A_n $(n \geq 2)$, D_n $(n \geq 5)$ and E_6, $\mathrm{Out}(\mathfrak{g}) = S_3$ for D_4, and $\mathrm{Out}(\mathfrak{g}) = \{1\}$ in all other cases.

For the Lie algebras A_n and E_6, and $\psi \neq 1$, one can take w to be the longest element of the Weyl group. Indeed $w \circ \psi|_{\mathfrak{t}} = -\mathrm{id}_{\mathfrak{t}}$ in both cases (see [10], planche I (p.251), planche V (p.261)). For the Lie algebra D_n $(n \geq 5)$, consider the standard presentation of the root system as the set of lattice vectors in $\mathbb{R}^n$ of length $\sqrt{2}$. In the standard basis ϵ_i of $\mathbb{R}^n$, a system of simple roots is given by $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, $\alpha_n = \epsilon_{n-1} + \epsilon_n$. The automorphism ψ exchanges α_{n-1} and α_n, hence $\psi|_{\mathfrak{t}}$ is the linear map changing the sign of the last coordinate in $\mathbb{R}^n$. Take $w \in W$ to be a cyclic permutation of the coordinates. Then $w \circ \psi$ has no fixed vector. It remains to consider the case of D_4. If ψ is induced by a diagram automorphism of order 3, the map $\psi|_{\mathfrak{t}}$ has eigenvalues the third roots of unity. The longest Weyl group element w acts as $-\mathrm{id}$ on $\mathfrak{t}$, hence $w \circ \psi|_{\mathfrak{t}}$ has no eigenvalue equal to 1. Finally, if ψ is induced by the automorphism exchanging α_3, α_4, we can take w to be cyclic permutation of the coordinates as above, the other diagram automorphisms of order 2 are obtained from this by conjugation with a third order automorphism. ☐

We following result generalizes Lemma 5.1, in the case without boundary, to two-component groups:

Lemma B.3. *Let G be a compact Lie group with two components and with G° semi-simple. For all $h \geq 2$, every connected component of every fiber of the map $\Phi : G^{2h} \to G$, $(a_1, b_1, \ldots, a_h, b_h) \mapsto \prod_{j=1}^h [a_j, b_j]$ meets the principal orbit type stratum $(G^{2h})_{\mathrm{prin}}$ for the conjugation action of G.*

Proof. For $j = 1, \dots, h$ let $p_j : (G^2)^h \to G^2$ denote projection to the jth factor. Given a component $X \subset G^{2h}$ it is possible to choose j such that the principal stabilizer for $p_j(X)$ equals that of X. (The index j is arbitrary if $X = (G^{\mathrm{o}})^{2h}$; otherwise pick j such that $p_j(X) \neq (G^{\mathrm{o}})^2$.) Thus $x \in X_{\mathrm{prin}}$ whenever $p_j(x) \in (p_j(X))_{\mathrm{prin}}$. It hence suffices to show that for any given $y \in G^2$, $g \in G$, the fiber $p_j^{-1}(y)$ meets each component of $\Phi^{-1}(g)$. Interpret G^{2h} as a fusion product of h copies of the double $\tilde{D}(G)$, with Φ as its moment map. By Lemma B.1, the restriction of Φ to any component $X \subset G^{2h}$ is surjective, and by another application of this Lemma each intersection $\Phi^{-1}(g) \cap X$ meets $p_j^{-1}(y)$. If G^{o} is simply connected, this completes the proof since each X contains a *unique* component of $\Phi^{-1}(g)$. (Recall [1, Theorem 7.2] that the fibers of the moment map for a compact, connected, group-valued Hamiltonian space are connected, provided the group is connected and simply connected.) If G^{o} fails to be simply connected, we construct a finite cover $\hat{G} \to G$, with identity component $\widehat{G}^{\mathrm{o}}$ the universal cover of G^{o}: Choose $g_0 \in G$ such that $\psi := \mathrm{Ad}_{g_0} : \mathfrak{g} \to \mathfrak{g}$ preserves a maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Any such ψ has finite order $k > 0$, hence it defines an action of $\mathbb{Z}_k$ on $\hat{G}^{\mathrm{o}}$ by automorphisms. We define $\hat{G} = \mathbb{Z}_k \ltimes \hat{G}^{\mathrm{o}}$, with covering map $\hat{G} \to G$, $(\psi^{-l}, \hat{h}) \mapsto g_0^{-l} h$ where $h \in G^{\mathrm{o}}$ is the image of $\hat{h} \in \hat{G}^{\mathrm{o}}$. Let $\pi : \hat{G}^{2h} \to G^{2h}$ be the projection. The product of commutators, $\hat{\Phi} : \hat{G}^{2h} \to \hat{G}^{\mathrm{o}}$ is the moment map for the $\hat{G}^{\mathrm{o}}$-action. The fiber $\Phi^{-1}(g)$ is covered by the union of $\hat{\Phi}^{-1}(\hat{g})$, as $\hat{g} \in \hat{G}$ ranges over pre-images of g. Again, Lemma B.1 shows that the restriction of $\hat{\Phi}$ to any component $\hat{X} \subset \hat{G}^{2h}$ is surjective, and $\hat{\Phi}^{-1}(\hat{g}) \cap \hat{X}$ meets $(p_j \circ \pi)^{-1}(y)$. This proves the Lemma since $\hat{\Phi}^{-1}(\hat{g}) \cap \hat{X}$ is connected. $\square$

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