LIE THEORY AND THE CHERN-WEIL HOMOMORPHISM

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Abstract. Let $P \to B$ be a principal G-bundle. For any connection θ on P, the Chern-Weil construction of characteristic classes defines an algebra homomorphism from the Weil algebra $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ into the algebra of differential forms $\mathcal{A} = \Omega(P)$. Invariant polynomials $(S\mathfrak{g}^*)_{\mathrm{inv}} \subset W\mathfrak{g}$ map to cocycles, and the induced map in cohomology $(S\mathfrak{g}^*)_{\mathrm{inv}} \to H(\mathcal{A}_{\mathrm{basic}})$ is independent of the choice of θ. The algebra $\Omega(P)$ is an example of a *commutative* $\mathfrak{g}$-differential algebra with connection, as introduced by H. Cartan in 1950. As observed by Cartan, the Chern-Weil construction generalizes to all such algebras.

In this paper, we introduce a canonical Chern-Weil map $W\mathfrak{g} \to \mathcal{A}$ for possibly *non-commutative* $\mathfrak{g}$-differential algebras with connection. Our main observation is that the generalized Chern-Weil map is an algebra homomorphism "up to $\mathfrak{g}$-homotopy". Hence, the induced map $(S\mathfrak{g}^*)_{\mathrm{inv}} \to H_{\mathrm{basic}}(\mathcal{A})$ is an algebra homomorphism. As in the standard Chern-Weil theory, this map is independent of the choice of connection.

Applications of our results include: a conceptually easy proof of the Duflo theorem for quadratic Lie algebras, a short proof of a conjecture of Vogan on Dirac cohomology, generalized Harish-Chandra projections for quadratic Lie algebras, an extension of Rouvière's theorem for symmetric pairs, and a new construction of universal characteristic forms in the Bott-Shulman complex.

Contents

1. Introduction

In an influential paper from 1950, H. Cartan [9] presented an algebraic framework for the Chern-Weil [10, 44] construction of characteristic classes in terms of differential forms. In

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Cartan's approach, the de Rham complex $\Omega(P)$ of differential forms on a principal G-bundle P is generalized to a differential algebra $\mathcal{A}$, together with algebraic counterparts of the Lie derivative and contraction operations for the action of the Lie algebra $\mathfrak{g}$ of G. We will refer to any such $\mathcal{A}$ as a $\mathfrak{g}$-*differential algebra.* Cartan introduced the notion of an *algebraic connection* on $\mathcal{A}$; $\mathfrak{g}$-differential algebras admitting connections are called *locally free* and are viewed as algebraic counterparts of principal bundles. A counterpart of the base of the principal bundle is the basic subcomplex $\mathcal{A}_{\mathrm{basic}}$. The *Weil algebra* $W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^*$ replaces the classifying bundle $EG \to BG$. The generators of $\wedge\mathfrak{g}^*$ are viewed as "universal connections", the generators of $S\mathfrak{g}^*$ as "universal curvatures". Cartan shows that if $\mathcal{A}$ is any (graded) *commutative* $\mathfrak{g}$-differential algebra with connection θ, there is a characteristic homomorphism

$$c^\theta : W\mathfrak{g} \to \mathcal{A} \tag{1}$$

sending the generators of $\wedge\mathfrak{g}^*$ to the connection variables of $\mathcal{A}$ and the generators of $S\mathfrak{g}^*$ to the curvature variables of $\mathcal{A}$. Passing to the cohomology of the basic subcomplex, this gives a homomorphism

$$(S\mathfrak{g}^*)_{\mathrm{inv}} = H((W\mathfrak{g})_{\mathrm{basic}}) \to H(\mathcal{A}_{\mathrm{basic}}) \tag{2}$$

from the algebra of invariant polynomials on the Lie algebra $\mathfrak{g}$ into the cohomology algebra of the basic subcomplex of $\mathcal{A}$. As in the usual Chern-Weil theory, this homomorphism is independent of the choice of θ.

The main theme of this paper is a generalization of Cartan's algebraic Chern-Weil construction to possibly non-commutative $\mathfrak{g}$-differential algebras. The idea in the general case is to define c^θ by a suitable ordering prescription. Recall that any linear map $E \to \mathcal{A}$ from a vector space to an associative algebra $\mathcal{A}$ extends to a linear map $S(E) \to \mathcal{A}$ from the symmetric algebra, by symmetrization. This also holds for $\mathbb{Z}_2$-graded vector spaces and algebras, using super symmetrization (i.e. taking signs into account). The Weil algebra may be viewed as the (super) symmetric algebra $S(E)$ over the space E spanned by the generators $\mu \in \mathfrak{g}^*$ of $\wedge\mathfrak{g}^*$ and their differentials $\overline{\mu} = \mathrm{d}\mu$, and a connection θ on a $\mathfrak{g}$-differential algebra $\mathcal{A}$ defines a linear map $E \to \mathcal{A}$. With $c^\theta : W\mathfrak{g} \to \mathcal{A}$ defined by symmetrization, we prove:

Theorem A. *For any $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection θ, the map $c^\theta : W\mathfrak{g} \to \mathcal{A}$ is a homomorphism of $\mathfrak{g}$-differential spaces. The induced homomorphism in basic cohomology $(S\mathfrak{g}^*)_{\mathrm{inv}} \to H(\mathcal{A}_{\mathrm{basic}})$ is independent of θ, and is an algebra homomorphism.*

Note that the Weil algebra could also be viewed as a super symmetric algebra over the subspace $E' \subset W\mathfrak{g}$ spanned by the universal connections and curvatures. However, the resulting symmetrization map $W\mathfrak{g} = S(E') \to \mathcal{A}$ would not be a chain map, in general.

Our first application of Theorem A gives a new perspective on the proof of [1] of the Duflo isomorphism for quadratic Lie algebras. Recall that a Lie algebra $\mathfrak{g}$ is called *quadratic* if it comes equipped with an invariant scalar product B. Let $\mathcal{W}\mathfrak{g}$ be the non-commutative super algebra generated by odd elements ξ and even elements $\overline{\xi}$ for $\xi \in \mathfrak{g}$, subject to relations

$$\bar{\xi}\bar{\xi}' - \bar{\xi}'\bar{\xi} = \overline{[\xi,\xi']_{\mathfrak{g}}}, \quad \bar{\xi}\xi' - \xi'\bar{\xi} = [\xi,\xi']_{\mathfrak{g}}, \quad \xi\xi' + \xi'\xi = B(\xi,\xi').$$

Using B to identify $\mathfrak{g}^*$ and $\mathfrak{g}$, we obtain a symmetrization map $W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$. The following is a fairly easy consequence of Theorem A:

Theorem B. *There is a commutative diagram*

$$\begin{tikzcd} W\mathfrak{g} \arrow[r] & \mathcal{W}\mathfrak{g} \\ S\mathfrak{g} \arrow[u] \arrow[r] & U\mathfrak{g} \arrow[u] \end{tikzcd}$$

in which the vertical maps are injective algebra homomorphisms and the horizontal maps are vector space isomorphisms. The lower map restricts an algebra isomorphism on invariants.

Recall that the Poincaré-Birkhoff-Witt symmetrization $S\mathfrak{g} \to U\mathfrak{g}$ does *not* restrict to an algebra homomorphism on invariants, in general. On the other hand, it was shown by Duflo that the PBW map does have this property if it is pre-composed with a certain infinite order differential operator known as the "Duflo factor".

Theorem C. *The lower horizontal map in the commutative diagram of Theorem B coincides with the Duflo map* [13].

That is, while the Duflo map is not a symmetrization map for $U\mathfrak{g}$, it may be viewed as the *restriction* of a symmetrization map of a super algebra containing $U\mathfrak{g}$! We stress that our theory only covers the case of quadratic Lie algebras – it remains a mystery how the general situation might fit into this picture.

Suppose $\mathfrak{k} \subset \mathfrak{g}$ is a quadratic subalgebra of $\mathfrak{g}$, i.e. that the restriction of $B|_\mathfrak{k}$ is non-degenerate. Let $\mathfrak{p}$ denote the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{p}$, and $\mathrm{Cl}(\mathfrak{p})$ its Clifford algebra. In [25], Kostant introduced a canonical element $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ of the algebra $(U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$ which he called the *cubic Dirac operator.* He showed that $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ squares to an element of the center of this algebra, so that the graded commutator $[\mathcal{D}_{\mathfrak{g},\mathfrak{k}}, \cdot]$ is a differential. The cohomology of this differential features in a conjecture of Vogan. Generalizing results of Huang-Pandzic [20] and Kostant [26] we prove:

Theorem D. *There is a natural algebra homomorphism $(U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \to (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$, taking values in cocycles and inducing an isomorphism in cohomology. The map $(U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ taking $z \in (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}}$ to the cohomology class of $z \otimes 1$ coincides with the restriction map $(S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ under the Duflo isomorphisms for $\mathfrak{g}$ and $\mathfrak{k}$.*

Our next result is a Harish-Chandra map for a quadratic Lie algebra $\mathfrak{g}$ with a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+$, where $\mathfrak{k}$ is a quadratic Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_\pm$ are $\mathfrak{k}$-invariant isotropic subalgebras (that is, the restriction of B to $\mathfrak{n}_\pm$ vanishes). By the Poincaré-Birkhoff-Witt theorem, the splitting of $\mathfrak{g}$ gives rise to a decomposition of the enveloping algebra $U\mathfrak{g}$, hence to a projection $\kappa_U : U\mathfrak{g} \to U\mathfrak{k}$. As for the usual Harish-Chandra projection, it is convenient to compose κ_U with a certain automorphism τ of $U\mathfrak{k}$ (the "ρ-shift").

Theorem E. *Under the Duflo isomorphisms for $\mathfrak{g}, \mathfrak{k}$, the composition $\tau \circ \kappa_U : (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ coincides with the projection $(S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$. In particular, it is an algebra homomorphism.*

We obtain Theorem E by studying the Harish-Chandra projection $\kappa_{\mathcal{W}} : \mathcal{W}\mathfrak{g} \to \mathcal{W}\mathfrak{k}$, and comparing to the natural projection $\kappa_W : W\mathfrak{g} \to W\mathfrak{k}$. It turns out that $\kappa_{\mathcal{W}}$ directly restricts to $\tau \circ \kappa_U$: That is, the shift τ emerges from the theory in a very natural way and need not be put in 'by hand'.

Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair, that is, $\mathfrak{k}$ is the fixed point set of an involutive automorphism ϵ on $\mathfrak{g}$. Let $\mathfrak{p}$ be its complement given as the eigenspace of ϵ for the eigenvalue -1. By results of Lichnerowicz [30] and Duflo [14], the algebra $(U\mathfrak{g}/U\mathfrak{g}\mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}}$ (where $\mathfrak{k}^f \hookrightarrow U\mathfrak{g}$ a suitable "twisted" inclusion of $\mathfrak{k}$) is *commutative.* Rouvière in his paper [37] introduced a map $(S\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}} \to (U\mathfrak{g}/U\mathfrak{g}\mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}}$ generalizing the Duflo isomorphism, and described conditions under which this map is an algebra isomorphism. We prove a similar result for the following new class of examples:

Theorem F. *Suppose $\mathfrak{g}$ carries an invariant scalar product B that changes sign under ϵ. Then the Duflo-Rouvière map $(S\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}} \to (U\mathfrak{g}/U\mathfrak{g}\mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}}$ is an algebra isomorphism.*

Anti-invariance of B under the involution ϵ implies that B vanishes on both $\mathfrak{k}$ and $\mathfrak{p}$, and gives a non-degenerate pairing between the two subspaces. In line with our general strategy, we prove this result by identifying the Duflo-Rouvière map as a Chern-Weil map, using the isomorphism $(S\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}} = (S\mathfrak{k}^*)_{\mathfrak{k}-\mathrm{inv}}$ given by the pairing.

Our final result is a new construction of universal characteristic forms in the Bott-Shulman complex. For any Lie group G, Bott and Shulman considered a double complex $\Omega^q(G^p)$ as a model for differential forms on the classifying space BG, and showed how to associate to any invariant polynomial on $\mathfrak{g}$ a cocycle for the total differential on this double complex. In our alternative approach, we observe that $\bigoplus_{p,q} \Omega^q(G^p)$ carries a non-commutative product, and obtain:

Theorem G. *The generalized Chern-Weil construction defines a linear map*

$$(S\mathfrak{g}^*)_{\mathrm{inv}} \to \bigoplus_{p,q} \Omega^q(G^p)$$

taking values in cocycles for the total differential. The image of a polynomial of degree r under this map has non-vanishing components only in bidegree $p + q = 2r$ with $p \leq r$. The map induces an algebra homomorphism in cohomology, and in fact an algebra isomorphism if G is compact.

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2. Non-commutative differential algebras

In this Section we review some material on symmetrization maps for super vector spaces and $\mathfrak{g}$-differential spaces. Our conventions for super spaces will follow [12]; in particular we take the categorical point of view that super vector spaces form a tensor category where the super sign convention is built into the isomorphism $E \otimes E' \to E' \otimes E$. The concept of $\mathfrak{g}$-differential spaces is due to Cartan [9], a detailed treatment can be found in the book [18].

2.1. **Conventions and notation.** Throughout, we will work over a field $\mathbb{F}$ of characteristic 0. A super vector space is a vector space over $\mathbb{F}$ with a $\mathbb{Z}_2$-grading $E = E^{\bar{0}} \oplus E^{\bar{1}}$. Super vector spaces form an $\mathbb{F}$-linear tensor category; algebra objects in this tensor category are called super algebras, Lie algebra objects are called super Lie algebras.

2.2. **Symmetrization maps.** Let $E = E^{\bar{0}} \oplus E^{\bar{1}}$ be a super vector space. The (super) symmetric algebra $S(E) = \bigoplus_{k=0}^{\infty} S^k(E)$ is the quotient of the tensor algebra $\mathcal{T}(E) = \bigoplus_{k=0}^{\infty} E^{\otimes k}$ by the two-sided ideal generated by all elements of the form $v \otimes w - (-1)^{|v||w|} w \otimes v$, for homogeneous elements $v, w \in E$ of $\mathbb{Z}_2$-degree $|v|, |w|$. Both $\mathcal{T}(E)$ and $S(E)$ are super algebras, in such a way that the inclusion of E is a homomorphism of super vector spaces. The tensor algebra $\mathcal{T}(E)$ is characterized by the universal property that any homomorphism of super vector spaces $E \to \mathcal{A}$ into a super algebra $\mathcal{A}$ extends uniquely to a homomorphism of super algebras $\mathcal{T}(E) \to \mathcal{A}$; the symmetric algebra has a similar universal property for commutative super algebras.

Given a super algebra $\mathcal{A}$, any homomorphism of super vector spaces $\phi: E \to \mathcal{A}$ extends to $S(E)$ by symmetrization

$$\mathrm{sym}(\phi): S(E) \to \mathcal{A}, \quad v_1 \cdots v_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{N_\sigma(v_1, \ldots, v_k)} \phi(v_{\sigma^{-1}(1)}) \cdots \phi(v_{\sigma^{-1}(k)}).$$

Here $\mathfrak{S}_k$ is the symmetric group, and $N_\sigma(v_1, \ldots, v_k)$ is the number of pairs $i < j$ such that v_i, v_j are odd elements and $\sigma^{-1}(i) > \sigma^{-1}(j)$.

Equivalently, the symmetrization map $S(E) \to \mathcal{A}$ may be characterized as the inclusion $S(E) \to \mathcal{T}(E)$ as "symmetric tensors", followed by the algebra homomorphism $\mathcal{T}(E) \to \mathcal{A}$ given by the universal property of $\mathcal{T}(E)$.

2.3. **Poincaré-Birkhoff-Witt symmetrization.** If $(E, [\cdot,\cdot]_E)$ is a super Lie algebra, one defines the enveloping algebra $U(E)$ as the quotient of $\mathcal{T}(E)$ by the relations $v_1 \otimes v_2 - (-1)^{|v_1||v_2|} v_2 \otimes v_1 - [v_1, v_2]_E$. By the Poincaré-Birkhoff-Witt theorem for super Lie algebras (Corwin-Neeman-Sternberg [11], see also [12]) the symmetrization map $S(E) \to U(E)$ is a linear isomorphism.

Similarly, if E is a super vector space with a skew-symmetric bi-linear form $\omega \in \mathrm{Hom}(E \otimes E, \mathbb{F})$ (i.e. $\omega(v, w) = -(-1)^{|v||w|} \omega(w, v)$), one defines the *Weyl algebra* $\mathrm{Weyl}(E)$ as the quotient of the tensor algebra by the ideal generated by elements $v \otimes w - (-1)^{|v||w|} w \otimes v - \omega(v, w)$. The corresponding symmetrization map $S(E) \to \mathrm{Weyl}(E)$ is an isomorphism of super vector spaces, known as the *quantization map* for the Weyl algebra. [1] In the purely odd case $E^{\bar{0}} = 0$, ω is a symmetric bilinear form B on $V = E^{\bar{1}}$ (viewed as an ungraded vector space), the Weyl algebra is the Clifford algebra of (V, B), and the symmetrization map reduces to the Chevalley quantization map $q: \wedge V \to \mathrm{Cl}(V)$.

[1] The fact that q is an isomorphism may be deduced from the PBW isomorphism for the Heisenberg Lie algebra $E \oplus \mathbb{F}\mathfrak{c}$, i.e. the central extension with bracket $[v, v'] = \omega(v, v')\mathfrak{c}$. Indeed, the symmetrization map $S(E \oplus \mathbb{F}\mathfrak{c}) \to U(E \oplus \mathbb{F}\mathfrak{c})$ restricts to an isomorphism between the ideals generated by $\mathfrak{c} - 1$, and so the claim follows by taking quotients by these ideals.

2.4. **Derivations.** Given a super algebra $\mathcal{A}$ we denote by $\mathrm{Der}(\mathcal{A}) \subset \mathrm{End}(\mathcal{A})$ the super Lie algebra of derivations of $\mathcal{A}$. Similarly, if E is a super Lie algebra we denote by $\mathrm{Der}(E) \subset \mathrm{End}(E)$ the super Lie algebra of derivations of E. For any super vector space E there is a unique homomorphism of super Lie algebras

$$\mathrm{End}(E) \to \mathrm{Der}(\mathcal{T}(E)),\ A \mapsto D_A^{\mathcal{T}(E)}$$

such that $D_A^{\mathcal{T}(E)}(v) = Av$ for $v \in E \subset \mathcal{T}(E)$. Similarly one defines $D_A^{S(E)} \in \mathrm{Der}(S(E))$ and, if A is a derivation for a super Lie bracket on E, $D_A^{U(E)} \in \mathrm{Der}(U(E))$. We will need the following elementary fact.

Lemma 2.1. *Let E be a super vector space, $\mathcal{A}$ a super algebra, and $\phi \in \mathrm{Hom}(E, \mathcal{A})$ a homomorphism of super vector spaces. Suppose we are given a linear map $A \in \mathrm{End}(E)$ and a derivation $D \in \mathrm{Der}(\mathcal{A})$, and that ϕ intertwines A and D. Then the extended map* $\mathrm{sym}(\phi) : S(E) \to \mathcal{A}$ *obtained by symmetrization intertwines $D_A^{S(E)} \in \mathrm{Der}(S(E))$ and D.*

Proof. Recall that $\mathrm{sym}(\phi)$ factors through the symmetrization map for the tensor algebra $\mathcal{T}(E)$. Since the map $\mathcal{T}(E) \to \mathcal{A}$ intertwines $D_A^{\mathcal{T}(E)}$ with D, it suffices to prove the Lemma for $\mathcal{A} = \mathcal{T}(E)$, $D = D^{\mathcal{T}(E)}$. The action of $D_A^{\mathcal{T}(E)}$ on $E^{\otimes k}$ commutes with the action of $\mathfrak{S}_k$, as one easily checks for transpositions. In particular $D_A^{\mathcal{T}(E)}$ preserves the $\mathfrak{S}_k$-invariant subspace. It therefore restricts to $D_A^{S(E)}$ on $S(E) \subset \mathcal{T}(E)$. □

2.5. **Differential algebras.** A *differential space* (ds) is a super vector space E, together with a differential, i.e. an odd endomorphism $\mathrm{d} \in \mathrm{End}(E)^{\bar{1}}$ satisfying $\mathrm{d} \circ \mathrm{d} = 0$. Morphisms in the category of differential spaces will be called chain maps or ds *homomorphisms*. The tensor product $E \otimes E'$ of two differential spaces is a differential space, with $\mathrm{d}(v \otimes v') = \mathrm{d}v \otimes v' + (-1)^{|v|} v \otimes \mathrm{d}v'$. Algebra objects in this tensor category are called *differential algebras* (da), Lie algebra objects are called *differential Lie algebras* (dl). Lemma 2.1 shows:

Lemma 2.2. *Let $\mathcal{A}$ be a differential algebra, and $\phi : E \to \mathcal{A}$ a* ds *homomorphism. Then the symmetrized map* $\mathrm{sym}(\phi) : S(E) \to \mathcal{A}$ *is a* ds *homomorphism.*

For any differential algebra $(\mathcal{A}, \mathrm{d})$, the unit $i : \mathbb{F} \hookrightarrow \mathcal{A}$ (i.e. the inclusion of $\mathbb{F}$ as multiples of the unit element) is a da homomorphism. By an *augmentation map* for $(\mathcal{A}, \mathrm{d})$, we mean a da homomorphism $\pi : \mathcal{A} \to \mathbb{F}$ such that $\pi \circ i$ is the identity map. For the tensor algebra, the natural projections onto $\mathbb{F} = E^{\otimes 0}$ is an augmentation map; this descends to augmentation maps for $S(E)$ and (in the case of a super Lie algebra) $U(E)$.

2.6. **Koszul algebra.** For any vector space V over $\mathbb{F}$, let E_V be the ds with $E_V^{\bar{0}} = V$ and $E_V^{\bar{1}} = V$, with differential d equal to 0 on even elements and given by the natural isomorphism $E_V^{\bar{1}} \to E_V^{\bar{0}}$ on odd elements. For $v \in V$ denote the corresponding even and odd elements in E_V by $\overline{v} \in E_V^{\bar{0}}$ and $v \in E_V^{\bar{1}}$, respectively. Thus

$$\mathrm{d}v = \overline{v}, \quad \mathrm{d}\overline{v} = 0.$$

The symmetric algebra $S(E_V)$ is known as the *Koszul algebra* over V. It is characterized by the universal property that if $\mathcal{A}$ is any commutative da, any vector space homomorphism $V \to \mathcal{A}^{\bar{1}}$ extends to a unique homomorphism of da's $S(E_V) \to \mathcal{A}$. We will also encounter a non-commutative version of the Koszul algebra, $\mathcal{T}(E_V)$. It has a similar universal property, but in the category of not necessarily commutative da's $\mathcal{A}$. The non-commutative Koszul algebra appears in a paper of Gelfand-Smirnov [17].

2.7. **$\mathfrak{g}$-differential spaces.** For any Lie algebra $\mathfrak{g}$, we denote by $\widetilde{\mathfrak{g}}$ the super Lie algebra

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes \mathfrak{g}$$

(semi-direct product) where the even part $\widetilde{\mathfrak{g}}^{\bar{0}} = \mathfrak{g}$ acts on the odd part $\widetilde{\mathfrak{g}}^{\bar{1}} = \mathfrak{g}$ by the adjoint representation. It is a differential Lie algebra under the identification $\widetilde{\mathfrak{g}} = E_{\mathfrak{g}}$: that is, d is a derivation for the Lie bracket. A $\mathfrak{g}$-*differential space* ($\mathfrak{g} - \mathrm{ds}$) is a differential space (E, d) together with a dl homomorphism, $\widetilde{\mathfrak{g}} \to \mathrm{End}(E)$. That is, it consists of a representation of the super Lie algebra $\widetilde{\mathfrak{g}}$ on E, where the operators $\iota_\xi, L_\xi \in \mathrm{End}(E)$ corresponding to $\xi, \overline{\xi} \in \widetilde{\mathfrak{g}}$ satisfy the relations

$$[\mathrm{d}, \iota_\xi] = L_\xi, \quad [\mathrm{d}, L_\xi] = 0. \tag{3}$$

The operators ι_ξ are called *contractions*, the operators L_ξ are called *Lie derivatives.* The tensor product of any two $\mathfrak{g} - \mathrm{ds}$, taking the tensor product of the $\widetilde{\mathfrak{g}}$-representations, is again a $\mathfrak{g} - \mathrm{ds}$. Hence $\mathfrak{g} - \mathrm{ds}$'s form an $\mathbb{F}$-linear tensor category; the algebra objects in this tensor category are called $\mathfrak{g}$-*differential algebras* ($\mathfrak{g} - \mathrm{da}$), the Lie algebra objects are called $\mathfrak{g}$-*differential Lie algebras* ($\mathfrak{g} - \mathrm{dl}$). This simply means that the representation should act by derivations of the product, respectively Lie bracket.

For any $\mathfrak{g} - \mathrm{ds}$ E, one defines the horizontal subspace $E_{\mathrm{hor}} = \bigcap \ker \iota_\xi$, the invariant subspace $E_{\mathrm{inv}} = \bigcap \ker L_\xi$, and the basic subspace $E_{\mathrm{basic}} = E_{\mathrm{hor}} \cap E_{\mathrm{inv}}$. That is, E_{basic} is the space of fixed vectors for $\widetilde{\mathfrak{g}}$. This subspace is stable under d, hence is a differential space. Any $\mathfrak{g} - \mathrm{ds}$ homomorphism $\phi : E \to E'$ restricts to a chain map between basic subspaces.

An example of a $\mathfrak{g} - \mathrm{dl}$ is $E = \widetilde{\mathfrak{g}}$ with the adjoint action. Another example is obtained by adjoining d as an odd element, defining a semi-direct product

$$\mathbb{F}\mathrm{d} \ltimes \widetilde{\mathfrak{g}} \tag{4}$$

where the action of d on $\widetilde{\mathfrak{g}}$ is as the differential, $\mathrm{d}\xi = \overline{\xi}$, $\mathrm{d}\overline{\xi} = 0$. (Note that a $\mathfrak{g} - \mathrm{ds}$ can be defined equivalently as a module for the super Lie algebra (4); this is the point of view taken in the book [18].) The symmetric and tensor algebras over a $\mathfrak{g} - \mathrm{ds}$ E are $\mathfrak{g} - \mathrm{da}$'s, and Lemma 2.1 shows:

Lemma 2.3. *If $\mathcal{A}$ is a $\mathfrak{g} - \mathrm{da}$, and $\phi : E \to \mathcal{A}$ is a $\mathfrak{g} - \mathrm{ds}$ homomorphism, then the symmetrization $\mathrm{sym}(\phi) : S(E) \to \mathcal{A}$ is a $\mathfrak{g} - \mathrm{ds}$ homomorphism.*

2.8. **Homotopy operators.** A $\mathfrak{g}$-homotopy between two $\mathfrak{g} - \mathrm{ds}$ homomorphisms $\phi_0, \phi_1 : E \to E'$ is an odd linear map $h : E \to E'$ such that

$$[\mathrm{d}, h] = \phi_0 - \phi_1, \ [\iota_\xi, h] = 0, \ [L_\xi, h] = 0.$$

A $\mathfrak{g}$-homotopy inverse to a $\mathfrak{g} - \mathrm{ds}$ homomorphism $\phi : E \to E'$ is a $\mathfrak{g} - \mathrm{ds}$ homomorphism $\psi : E' \to E$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are $\mathfrak{g}$-homotopic to the identity maps of E', E.

Lemma 2.4. *Let E be a $\mathfrak{g} - \mathrm{ds}$, and suppose there exists $s \in \mathrm{End}(E)^{\bar{1}}$ with*

$$[d, s] = \mathrm{id}_E, \ [\iota_\xi, s] = 0, \ [L_\xi, s] = 0. \tag{5}$$

Then the inclusion $i : \mathbb{F} \to \mathcal{T}(E)$ and the augmentation map $\pi : \mathcal{T}(E) \to \mathbb{F}$ are $\mathfrak{g}$-homotopy inverses. A similar statement holds for the symmetric algebra $S(E)$, as well as for the enveloping algebra $U(E)$ in case E is a $\mathfrak{g} - \mathrm{dl}$.

Proof. The derivation extension of $[\mathrm{d}, s]$ to $\mathcal{T}(E)$ is the Euler operator, equal to k on $E^{\otimes k}$. Hence $[\mathrm{d}, s] + i \circ \pi \in \mathrm{End}(E)^{\bar{0}}$ is an invertible $\mathfrak{g} - \mathrm{ds}$ homomorphism, and the calculation

$$I - i \circ \pi = [\mathrm{d}, s] \circ ([\mathrm{d}, s] + i \circ \pi)^{-1} = [\mathrm{d}, s \circ ([\mathrm{d}, s] + i \circ \pi)^{-1}]$$

shows that $h = s \circ ([\mathrm{d}, s] + i \circ \pi)^{-1} \in \mathrm{End}(E)^{\bar{1}}$ is a $\mathfrak{g}$-homotopy operator between I and $i \circ \pi$. The proof for the symmetric algebra is similar. The case of the enveloping algebra of a $\mathfrak{g} - \mathrm{dl}$ follows, since the symmetrization map $S(E) \to U(E)$ is a $\mathfrak{g} - \mathrm{ds}$ homomorphism intertwining the augmentation maps. $\square$

Example 2.5. Suppose V is a $\mathfrak{g}$-module. Then E_V becomes a $\mathfrak{g} - \mathrm{ds}$ with $\iota_\xi v = 0$, $\iota_\xi \overline{v} = L_\xi v$. The operator $s \in \mathrm{End}(E_V)^{\bar{1}}$ given by $s(\overline{v}) = v$ and $s(v) = 0$ has the desired properties (5). Hence, both the Koszul algebra $S(E_V)$ and its non-commutative counterpart $\mathcal{T}(E_V)$ are $\mathfrak{g}$-homotopic to $\mathbb{F}$. If V carries a Lie algebra structure, and $\mathfrak{g}$ acts by derivations of the bracket $[\cdot,\cdot]_V$, then E_V becomes a $\mathfrak{g} - \mathrm{dl}$ and its enveloping algebra $U(E_V)$ is homotopy equivalent to $\mathbb{F}$.

3. The Chern-Weil homomorphism

3.1. **Connection and curvature.** A *connection* on a $\mathfrak{g} - \mathrm{da}$ $\mathcal{A}$ is a linear map $\theta : \mathfrak{g}^* \to \mathcal{A}^{\bar{1}}$ with the properties,

$$\iota_\xi(\theta(\mu)) = \mu(\xi), \quad L_\xi(\theta(\mu)) = -\theta(\mathrm{ad}^*_\xi \mu).$$

A $\mathfrak{g} - \mathrm{da}$ admitting a connection is called *locally free*. The following equivalent definition of a connection will be useful in what follows. Let $\mathbb{F}\mathfrak{c}$ be the 1-dimensional space spanned by an even generator $\mathfrak{c}$, viewed as a differential space on which d acts trivially, and consider the direct sum $E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c}$ with $\widetilde{\mathfrak{g}}$-action,

$$L_\xi \overline{\mu} = -\overline{\mathrm{ad}^*_\xi \mu}, \ L_\xi \mu = -\mathrm{ad}^*_\xi \mu, \ \iota_\xi \overline{\mu} = -\mathrm{ad}^*_\xi \mu, \ \iota_\xi \mu = \mu(\xi)\mathfrak{c}. \tag{6}$$

Then, a connection on a $\mathfrak{g} - \mathrm{da}$ $\mathcal{A}$ is equivalent to a $\mathfrak{g} - \mathrm{ds}$ homomorphism

$$E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c} \to \mathcal{A} \tag{7}$$

taking $\mathfrak{c}$ to the unit of $\mathcal{A}$. (It may be verified that $E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c}$ is the *odd* dual space of the super Lie algebra $\mathbb{F}\mathrm{d} \ltimes \widetilde{\mathfrak{g}}$, i.e. the dual space with the opposite $\mathbb{Z}_2$-grading.)

3.2. **The Weil algebra.** The *Weil algebra* $W\mathfrak{g}$ is a commutative $\mathfrak{g} - \mathrm{da}$ with connection $\mathfrak{g}^* \to W\mathfrak{g}$, with the following universal property: For any commutative $\mathfrak{g} - \mathrm{da}$ $\mathcal{A}$ with connection $\theta : \mathfrak{g}^* \to \mathcal{A}$, there exists a unique $\mathfrak{g} - \mathrm{da}$ homomorphism $c^\theta : W\mathfrak{g} \to \mathcal{A}$ such that the following diagram commutes:

$$\begin{tikzcd} W\mathfrak{g} \arrow[r, "c^\theta"'] & \mathcal{A} \\ \mathfrak{g}^* \arrow[u] \arrow[ur, "\theta"'] & \end{tikzcd}$$

We will refer to c^θ as the *characteristic homomorphism* for the connection θ. The Weil algebra is explicitly given as a quotient

$$W\mathfrak{g} = S(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c}) / < \mathfrak{c} - 1 >, \tag{8}$$

where $< \mathfrak{c} - 1 >$ denotes the two-sided ideal generated by $\mathfrak{c} - 1$. From the description (7) of connections, it is obvious that $W\mathfrak{g}$ carries a "tautological" connection. If $\mathcal{A}$ is commutative, the homomorphism (7) extends, by the universal property of the symmetric algebra, to a $\mathfrak{g} -$da homomorphism $S(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c}) \to \mathcal{A}$. This homomorphism takes $< \mathfrak{c} - 1 >$ to 0, and therefore descends to a $\mathfrak{g} -$ da homomorphism $c^\theta : W\mathfrak{g} \to \mathcal{A}$.

As a differential algebra, the Weil algebra is just the Koszul algebra $W\mathfrak{g} = S(E_{\mathfrak{g}^*})$. The $\mathfrak{g} -$da-structure is given on generators $\mu, \overline{\mu}$ by formulas similar to (6), with $\mathfrak{c}$ replaced by 1, and the connection is the map sending $\mu \in \mathfrak{g}^*$ to the corresponding odd generator of $W\mathfrak{g}$. Introduce the *curvature variables*

$$\widehat{\mu} := \overline{\mu} - \lambda(\mu)$$

where

$$\lambda : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*, \; \iota_\xi \lambda(\mu) = -\operatorname{ad}^*_\xi \mu, \quad \xi \in \mathfrak{g}, \; \mu \in \mathfrak{g}^*$$

is the map dual to the Lie bracket, and we identify $\wedge \mathfrak{g}^*$ as the subalgebra of $W\mathfrak{g}$ defined by the odd generators μ. Since $\iota_\xi \widehat{\mu} = 0$, the curvature map extends to an algebra homomorphism $S\mathfrak{g}^* \to (W\mathfrak{g})_{\mathrm{hor}}$, which is easily seen to be an isomorphism. Thus

$$W\mathfrak{g} = S\mathfrak{g}^* \otimes \wedge \mathfrak{g}^*, \tag{9}$$

where $S\mathfrak{g}^* \cong (W\mathfrak{g})_{\mathrm{hor}}$ is generated by the "curvature variables " $\widehat{\mu}$, and the exterior algebra $\wedge \mathfrak{g}^*$ by the "connection variables" μ. The Weil differential vanishes on $(W\mathfrak{g})_{\mathrm{basic}} = (S\mathfrak{g}^*)_{\mathrm{inv}}$; hence the cohomology of the basic subcomplex coincides with $(S\mathfrak{g}^*)_{\mathrm{inv}}$.

3.3. The non-commutative Weil algebra $\tilde{W}\mathfrak{g}$.

Let $\tilde{W}\mathfrak{g}$ be the non-commutative $\mathfrak{g} -$ da, defined similar to (8) but with the tensor algebra in place of the symmetric algebra,

$$\tilde{W}\mathfrak{g} = \mathcal{T}(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c}) / < \mathfrak{c} - 1 > .$$

Then $\tilde{W}\mathfrak{g}$ has a universal property similar to $W\mathfrak{g}$ among *non-commutative* $\mathfrak{g} -$ da's $\mathcal{A}$, with connection θ. That is, θ extends uniquely to a $\mathfrak{g} -$ da homomorphism $\tilde{c}^\theta : \tilde{W}\mathfrak{g} \to \mathcal{A}$. As a differential algebra, $\tilde{W}\mathfrak{g}$ is the non-commutative Koszul algebra, $\tilde{W}\mathfrak{g} = \mathcal{T}(E_{\mathfrak{g}^*})$. In particular, it is acyclic, with a canonical homotopy operator. The following two results are well-known in the commutative case:

Proposition 3.1. *If θ_0, θ_1 are two connections on a $\mathfrak{g} -$ da $\mathcal{A}$, the corresponding $\mathfrak{g} -$ da homomorphisms $\tilde{c}^{\theta_0}, \; \tilde{c}^{\theta_1} : \tilde{W}\mathfrak{g} \to \mathcal{A}$ are $\mathfrak{g}$-homotopic.*

Proof. Let $\mathcal{S} = \mathbb{F}[s, \bar{s}]$ be the differential algebra with generators s of degree $\bar{0}$ and $\bar{s}$ of degree $\bar{1}$, with differential $\mathrm{d}s = \bar{s}$, $\mathrm{d}\bar{s} = 0$. Define $J : \mathcal{S} \to \mathbb{F}$ by $J(s^k) = 0, \; J(s^k \mathrm{d}s) = \frac{1}{k+1}$. The tensor product $\mathcal{S} \otimes \mathcal{A}$ carries a connection, $\theta(\mu) = (1 - s) \otimes \theta_0(\mu) + s \otimes \theta_1(\mu)$, extending to a $\mathfrak{g} -$ da-homomorphism $\tilde{c}^\theta : \tilde{W}\mathfrak{g} \to \mathcal{S} \otimes \mathcal{A}$. Let

$$h = (J \otimes 1) \circ \tilde{c}^\theta : \tilde{W}\mathfrak{g} \to \mathcal{A}.$$

Using 'Stokes theorem', one verifies $[\mathrm{d}, h] = \tilde{c}^{\theta_1} - \tilde{c}^{\theta_0}$. □

Proposition 3.2. *If $\mathcal{A}$ is a $\mathfrak{g}-\mathrm{da}$ with connection θ, the map*

$$\phi:\ \tilde{W}\mathfrak{g}\otimes\mathcal{A}\to\mathcal{A},\quad w\otimes x\mapsto\tilde{c}^{\theta}(w)x$$

is a $\mathfrak{g}$-homotopy equivalence, with $\mathfrak{g}$-homotopy inverse $\psi:\ \mathcal{A}\to\tilde{W}\mathfrak{g}\otimes\mathcal{A},\ x\mapsto 1\otimes x$

Proof. Clearly, $\phi\circ\psi=\mathrm{id}_{\mathcal{A}}$, while the opposite composition is

$$\psi\circ\phi(w\otimes x)=1\otimes c^{\theta}(w)x.$$

It hence suffices to show that the two maps $\tilde{W}\mathfrak{g}\to\tilde{W}\mathfrak{g}\otimes\mathcal{A}$, taking w to $w\otimes 1$ and $1\otimes c^{\theta}(w)$, are $\mathfrak{g}$-homotopic. But this follows from Proposition 3.1. □

Theorem 3.3. *The quotient map $\pi:\ \tilde{W}\mathfrak{g}\to W\mathfrak{g}$ is a $\mathfrak{g}$-homotopy equivalence, with homotopy inverse given by symmetrization, $\mathrm{sym}:\ W\mathfrak{g}\to\tilde{W}\mathfrak{g}$.*

Proof. Clearly, $\pi\circ\mathrm{sym}=\mathrm{id}_{W\mathfrak{g}}$. We need to show that conversely, the composition $\mathrm{sym}\circ\pi$ is homotopic to $\mathrm{id}_{\tilde{W}\mathfrak{g}}$. Let $f:\ \tilde{W}\mathfrak{g}\otimes W\mathfrak{g}\to\tilde{W}\mathfrak{g}$ be the map, $f(x\otimes x')=x\,\mathrm{sym}(x')$, and let

$$c_0,c_1:\ \tilde{W}\mathfrak{g}\longrightarrow\tilde{W}\mathfrak{g}\otimes W\mathfrak{g},$$

be the two maps, $c_0(x)=x\otimes 1,\ \ c_1(x)=1\otimes\pi(x)$. By Proposition 3.1, the two homomorphisms c_0,c_1 are $\mathfrak{g}$-homotopic. Hence, so are the compositions $f\circ c_0=\mathrm{id}$ and $f\circ c_1=\mathrm{sym}\circ\pi$. □

Corollary 3.4. *The symmetrization map $\mathrm{sym}:\ W\mathfrak{g}\to\tilde{W}\mathfrak{g}$ is an algebra homomorphism up to $\mathfrak{g}$-homotopy.*

Proof. By Theorem 3.3, there exists a $\mathfrak{g}$-homotopy operator $h:\ \tilde{W}\mathfrak{g}\to\tilde{W}\mathfrak{g}$ with $\mathrm{sym}\circ\pi-\mathrm{id}=[\mathrm{d},h]$. Let $m:\ W\mathfrak{g}\otimes W\mathfrak{g}\to W\mathfrak{g}$ be the multiplication map for the Weil algebra, and $\tilde{m}$ that for $\tilde{W}\mathfrak{g}$. Then

$$\mathrm{sym}\circ m-\tilde{m}\circ(\mathrm{sym}\otimes\mathrm{sym})=[\mathrm{d},h\circ\tilde{m}\circ(\mathrm{sym}\otimes\mathrm{sym})].$$

□

As an immediate consequence, we obtain:

Theorem 3.5. *Let $\mathcal{A}$ be a $\mathfrak{g}-\mathrm{da}$ with connection θ, and*

$$c^{\theta}=\tilde{c}^{\theta}\circ\mathrm{sym}:\ W\mathfrak{g}\to\mathcal{A}$$

the Chern-Weil map defined by symmetrization. Then c^{θ} is an algebra homomorphism up to $\mathfrak{g}$-homotopy. The induced algebra homomorphism, $(S\mathfrak{g}^*)_{\mathrm{inv}}\to H(\mathcal{A}_{\mathrm{basic}})$ is independent of θ.

4. A deformation of the Weil algebra

In this Section we construct, for any *quadratic* Lie algebra $\mathfrak{g}$, an interesting non-commutative deformation $\mathcal{W}\mathfrak{g}$ of the Weil algebra $W\mathfrak{g}$. In the non-degenerate case, Theorem 5.3 identifies $\mathcal{W}\mathfrak{g}$ with the Weil algebra from [1].

4.1. **Quadratic Lie algebras.** We begin by recalling some examples and facts about quadratic Lie algebras. From now on, we will refer to any non-degenerate symmetric bilinear form on a vector space as a *scalar product.* A Lie algebra $\mathfrak{g}$ with invariant scalar product B will be called a *quadratic Lie algebra.* First examples of quadratic Lie algebras are semi-simple Lie algebras, with B the Killing form. Here are some other examples:

Examples 4.1. (a) Let $\mathfrak{g}$ be any Lie algebra, with given symmetric bilinear form. Then the radical $\mathfrak{r}$ of the bilinear form is an ideal, and the quotient $\mathfrak{g}/\mathfrak{r}$ with induced bilinear form is quadratic.

(b) Let $\mathbb{F}^{2n}$ be equipped with the standard symplectic form, $\omega(e_{2i-1}, e_{2i}) = 1$. Recall that the Heisenberg Lie algebra H_n is the central extension

$$0 \longrightarrow \mathbb{F} \longrightarrow H_n \longrightarrow \mathbb{F}^{2n}$$

of the Abelian Lie algebra $\mathbb{F}^{2n}$ by $\mathbb{F}$, with bracket defined by the cocycle ω. Let c denote the basis vector for the center $\mathbb{F} \subset H_n$. Let another copy of $\mathbb{F}$, with basis vector r, act on $\mathbb{F}^{2n}$ by infinitesimal rotation in each e_{2i-1}–e_{2i}-plane: $r.e_{2i-1} = e_{2i},\ r.e_{2i} = -e_{2i-1}$. This action lifts to derivations of H_n, and we may form the semi-direct product

$$\mathfrak{g} = \mathbb{F} \ltimes H_n.$$

The Lie algebra $\mathfrak{g}$ is quadratic, with bilinear form given by $B(e_{2i-1}, e_{2i}) = B(c, r) = 1$, and all other scalar products between basis vectors equal to 0.

(c) Let $\mathfrak{s}$ be any Lie algebra, acting on its dual by the co-adjoint action. Viewing $\mathfrak{s}^*$ as an Abelian Lie algebra, form the semi-direct product $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{s}^*$. Then $\mathfrak{g}$ is a quadratic Lie algebra, with bilinear form B given by the natural pairing between $\mathfrak{s}$ and $\mathfrak{s}^*$. More generally, given an invariant element $C \in (\wedge^3 \mathfrak{s})_{\mathfrak{s}-\mathrm{inv}}$, one obtains a quadratic Lie algebra where the bracket between elements of $\mathfrak{s}^*$ is given by $[\mu, \mu']_{\mathfrak{g}} = C(\mu, \mu', \cdot) \in \mathfrak{s}$. See Section 8.2 below.

From a given quadratic Lie algebra $(\mathfrak{a}, B_{\mathfrak{a}})$, new examples are obtained by the *double extension construction* of Medina-Revoy [34]: Suppose a second Lie algebra $\mathfrak{s}$ acts on $\mathfrak{a}$ by derivations preserving the scalar product. Let ω be the following $\mathfrak{s}^*$-valued cocycle on $\mathfrak{a}$,

$$\langle \omega(a_1, a_2), \xi \rangle = B_{\mathfrak{a}}(a_1, \xi.a_2), \quad a_i \in \mathfrak{a},\ \xi \in \mathfrak{s}$$

and $\mathfrak{a} \oplus \mathfrak{s}^*$ the central extension of $\mathfrak{a}$ defined by this cocycle. The Lie algebra $\mathfrak{s}$ acts on $\mathfrak{a} \oplus \mathfrak{s}^*$ by derivations, hence we may form the semi-direct product $\mathfrak{g} = \mathfrak{s} \ltimes (\mathfrak{a} \oplus \mathfrak{s}^*)$. The given scalar product on $\mathfrak{a}$, together with the scalar product on $\mathfrak{s} \ltimes \mathfrak{s}^*$ given by the pairing, define a scalar product on $\mathfrak{g}$, which is easily checked to be invariant. Notice that Example 4.1(b) is a special case of this construction.

4.2. **The algebra $\mathcal{W}\mathfrak{g}$.** Consider the following example of the double extension construction (extended to super Lie algebras in the obvious way). Suppose $\mathfrak{g}$ is any Lie algebra with a (possibly degenerate) symmetric bilinear form B. Then the super Lie algebra $\widetilde{\mathfrak{g}}$ inherits an odd (!) symmetric bilinear form,

$$B_{\widetilde{\mathfrak{g}}}(\xi, \xi') = 0,\ B_{\widetilde{\mathfrak{g}}}(\overline{\xi}, \xi') = B(\xi, \xi'),\ B_{\widetilde{\mathfrak{g}}}(\overline{\xi}, \overline{\xi}') = 0, \quad \xi, \xi' \in \mathfrak{g}.$$

The action of $\mathfrak{s} = \mathbb{F}\mathrm{d}$ given by the differential on $\widetilde{\mathfrak{g}}$ preserves $B_{\widetilde{\mathfrak{g}}}$. The corresponding cocycle $\omega : \widetilde{\mathfrak{g}} \otimes \widetilde{\mathfrak{g}} \to \mathbb{F}$ is given by B on the odd part $\widetilde{\mathfrak{g}}^{\bar{1}} = \mathfrak{g}$ and vanishes on the even part. Thus, we obtain a central extension $\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}$ by an even generator $\mathfrak{c}$ dual to d; the new brackets between odd generators read,

$$[\zeta, \zeta']_{\widetilde{\mathfrak{g}}\oplus\mathbb{F}\mathfrak{c}} = B(\zeta, \zeta')\mathfrak{c}$$

while the brackets between even generators or between even and odd generators are unchanged. The second step of the double extension constructs the super Lie algebra

$$\mathbb{F}\mathrm{d} \ltimes (\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}), \tag{10}$$

together with an odd symmetric bilinear form. The latter is non-degenerate if and only if B is non-degenerate.

The super Lie algebra (10) is a $\mathfrak{g}-\mathrm{dl}$, where $\mathbb{F}\mathrm{d} \ltimes \widetilde{\mathfrak{g}}$ acts by inner derivations. It contains the central extension $\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}$ as a $\mathfrak{g}$-differential Lie subalgebra. Explicitly, the $\mathfrak{g}-\mathrm{ds}$ structure on $\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}$ is given by

$$L_\xi \overline{\zeta} = \overline{[\xi, \zeta]_{\mathfrak{g}}}, \quad L_\xi \zeta = [\xi, \zeta]_{\mathfrak{g}}, \quad \iota_\xi \overline{\zeta} = [\xi, \zeta]_{\mathfrak{g}}, \quad \iota_\xi \zeta = B(\xi, \zeta)\mathfrak{c},$$

while ι_ξ, L_ξ vanish on $\mathfrak{c}$.

Remark 4.2. It is instructive to compare the definition of the super Lie algebra (10) to the standard construction of affine Lie algebras. Let $\mathfrak{g}$ be a Lie algebra with invariant symmetric bilinear form B. Tensoring with Laurent polynomials, define an infinite dimensional Lie algebra $\mathfrak{g}[z, z^{-1}] = \mathfrak{g} \otimes \mathbb{F}[z, z^{-1}]$ with bilinear form $B'(x_1 \otimes f_1, x_2 \otimes f_2) = B(x_1, x_2)\,\mathrm{Res}(f_1 f_2)$ where the residue Res picks the coefficient of z^{-1}. The derivation $\partial(x \otimes f) = x \otimes \partial f/\partial z$ preserves the bilinear form since $\mathrm{Res}(\partial f/\partial z) = 0$. The double extension of $\mathfrak{g}[z, z^{-1}]$ with respect to the derivation ∂ is called an affine Lie algebra (at least if B is non-degenerate). In a similar fashion, letting u be an *odd* variable we may tensor with the algebra $\mathbb{F}[u] = \{a + bu |\, a, b \in \mathbb{F}\}$ to define $\mathfrak{g}[u] = \mathfrak{g} \otimes \mathbb{F}[u]$. It carries an odd symmetric bilinear form, defined similar to B' but with Res replaced by the Berezin integral $\mathrm{Ber}(a + bu) = b$. Again, the derivation $\mathrm{d}(x \otimes f) = x \otimes \partial f/\partial u$ preserves the inner product since $\mathrm{Ber}(\partial f/\partial u) = 0$. Then $\mathfrak{g}[u] \cong \widetilde{\mathfrak{g}}$, the derivation d is the Koszul differential, and the double extension yields (10).

The two constructions may be unified to define a super Lie algebra $\mathfrak{g}[z, z^{-1}, u]$ known as *super-affinization*, see [21].

We define the $\mathfrak{g}-\mathrm{da}$ $\mathcal{W}\mathfrak{g}$ as a quotient of the enveloping algebra,

$$\mathcal{W}\mathfrak{g} := U(\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}) / < \mathfrak{c} - 1 > . \tag{11}$$

Note that $\mathcal{W}\mathfrak{g}$ can be defined directly as a quotient of the tensor algebra $\mathcal{T}(\widetilde{\mathfrak{g}})$, in terms of generators $\xi, \overline{\xi}$ ($\xi \in \mathfrak{g}$) with relations $[\overline{\xi}, \overline{\xi}'] = \overline{[\xi, \xi']_{\mathfrak{g}}}$, $[\overline{\xi}, \xi'] = [\xi, \xi']_{\mathfrak{g}}$ and $[\xi, \xi'] = B(\xi, \xi')$. In particular there is a symmetrization map $S(\widetilde{\mathfrak{g}}) \to \mathcal{W}\mathfrak{g}$. Let $S(\widetilde{\mathfrak{g}})$ carry the structure of a $\mathfrak{g}-\mathrm{da}$, induced by its identification with $S(\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}) / < \mathfrak{c} - 1 >$.

Lemma 4.3. *The symmetrization map*

$$\mathcal{Q}_{\mathfrak{g}} : \; S(\tilde{\mathfrak{g}}) \to \mathcal{W}\mathfrak{g} \tag{12}$$

is a $\mathfrak{g}-\mathrm{ds}$ *isomorphism.*

Proof. By Lemma 2.3, the PBW isomorphism $S(\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}) \to U(\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c})$ is a $\mathfrak{g}-\mathrm{ds}$ isomorphism. After quotienting the ideal $< \mathfrak{c} - 1 >$ on both sides, the Lemma follows. $\square$

Consider now the special case $B = 0$. Then $\mathcal{W}\mathfrak{g} = U(\tilde{\mathfrak{g}})$ as a $\mathfrak{g}-$da. As a special case of Example 2.5, we have:

Proposition 4.4. *If $B = 0$, the inclusion map $\mathbb{F} \hookrightarrow \mathcal{W}\mathfrak{g}$ and the augmentation map $\mathcal{W}\mathfrak{g} \to \mathbb{F}$ are $\mathfrak{g}$-homotopy inverses. More generally, if $\mathfrak{k}$ acts on $\mathfrak{g}$ by Lie algebra automorphisms, these two maps are $\mathfrak{k}$-homotopy inverses.*

4.3. **The non-degenerate case.** Let us now assume that $\mathfrak{g}$ is a quadratic Lie algebra, i.e. that the bilinear B on $\mathfrak{g}$ is *non-degenerate.* The scalar product B defines an isomorphism $B^\sharp : \mathfrak{g}^* \to \mathfrak{g}$, and hence $E_{\mathfrak{g}^*} \cong E_{\mathfrak{g}} = \widetilde{\mathfrak{g}}$. This isomorphism identifies the $\mathfrak{g}-$ds structures on $E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c}$ and $\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}$, hence it identifies $S(\widetilde{\mathfrak{g}})$ with the Weil algebra $W\mathfrak{g} = S(E_{\mathfrak{g}^*})$. That is, for any invariant scalar product B, (12) becomes a $\mathfrak{g}-$ds isomorphism

$$\mathcal{Q}_{\mathfrak{g}} : \; W\mathfrak{g} \to \mathcal{W}\mathfrak{g}. \tag{13}$$

We will refer to $\mathcal{Q}_{\mathfrak{g}}$ as the *quantization map.*

Remark 4.5. On the subalgebras $\wedge\mathfrak{g} \subset W\mathfrak{g}$ resp. $\mathrm{Cl}(\mathfrak{g}) \subset \mathcal{W}\mathfrak{g}$ generated by odd elements ζ, the quantization map restricts to the usual quantization map (Chevalley symmetrization map)

$$q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$$

for Clifford algebras, while on the subalgebras generated by even elements $\overline{\zeta}$ it becomes the PBW symmetrization map $S\mathfrak{g} \to U\mathfrak{g}$.

Remark 4.6. The $\mathfrak{g}-$da $\mathcal{W}\mathfrak{g}$ carries a connection $\mathfrak{g}^* \cong \mathfrak{g} \to \mathcal{W}\mathfrak{g}$, induced from the inclusion of $\mathfrak{g} = \widetilde{\mathfrak{g}}^{\bar{1}}$. By construction, the quantization map (13) is the characteristic homomorphism for this connection.

As mentioned above, the horizontal subalgebra $(W\mathfrak{g})_{\mathrm{hor}}$ of the Weil algebra $W\mathfrak{g}$ is isomorphic to the symmetric algebra, and the differential d vanishes on $(W\mathfrak{g})_{\mathrm{basic}} = (S\mathfrak{g}^*)_{\mathrm{inv}}$. We will now show that similarly, the horizontal subalgebra $(\mathcal{W}\mathfrak{g})_{\mathrm{hor}}$ is $\mathfrak{g}$-equivariantly isomorphic to the enveloping algebra $U\mathfrak{g}$. Let $\gamma : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ be the map,

$$\gamma(\zeta) = \tfrac{1}{2} \sum_a [\zeta, e_a]_{\mathfrak{g}} e^a$$

where e_a is a basis of $\mathfrak{g}$ and e^a the dual basis with respect to the given scalar product. It is a standard fact that the map γ is a Lie algebra homomorphism, and that Clifford commutator with $\gamma(\zeta)$ is the generator for the adjoint action of ζ on the Clifford algebra:

$$[\gamma(\zeta), \cdot] = L_\zeta^{\mathrm{Cl}(\mathfrak{g})}$$

where the bracket denotes the super commutator in the Clifford algebra. Recall the definition of the map $\lambda : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ by $\iota_\xi\lambda(\mu) = -\operatorname{ad}_\xi^* \mu$. Identify $\mathfrak{g}^* \cong \mathfrak{g}$ by means of the scalar product, and let $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ the quantization map (i.e. symmetrization map) for the Clifford algebra. Then

$$\gamma(\zeta) = q(\lambda(\zeta)).$$

Introduce curvature variables $\widehat{\zeta} \in (\mathcal{W}\mathfrak{g})_{\mathrm{hor}}$ by

$$\widehat{\zeta} = \overline{\zeta} - \gamma(\zeta).$$

Theorem 4.7. *The super algebra $\mathcal{W}\mathfrak{g}$ is a tensor product*

$$\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g}) \tag{14}$$

where $U\mathfrak{g}$ is generated by the even variables $\widehat{\zeta}$ and the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ is generated by the odd variables ζ. Under this identification, the map $\mathcal{Q}_\mathfrak{g} : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ restricts to a vector space isomorphism

$$S\mathfrak{g} = (W\mathfrak{g})_{\mathrm{hor}} \to U\mathfrak{g} = (\mathcal{W}\mathfrak{g})_{\mathrm{hor}}. \tag{15}$$

In fact, (15) *is an algebra isomorphism on $\mathfrak{g}$-invariants.*

Proof. The elements $\widehat{\zeta} \in \mathcal{W}\mathfrak{g}$ are the images of the corresponding elements in $W\mathfrak{g}$ (denoted by the same symbol) under the quantization map $\mathcal{Q}_\mathfrak{g}$. The commutator of two such elements $\widehat{\zeta}, \widehat{\zeta}' \in \mathcal{W}\mathfrak{g}$ is given by

$$[\widehat{\zeta}, \widehat{\zeta}'] = \overline{[\zeta,\zeta']_\mathfrak{g}} - L_\zeta \gamma(\zeta') - L_{\zeta'}\gamma(\zeta) + \gamma([\zeta,\zeta']_\mathfrak{g}) = \widehat{[\zeta,\zeta']_\mathfrak{g}}.$$

Hence the variables $\widehat{\zeta}$ generate a copy of the enveloping algebra $U\mathfrak{g} \subset \mathcal{W}\mathfrak{g}$. On the other hand, the odd variables ζ generate a copy of the Clifford algebra. Since $[\zeta, \widehat{\zeta}'] = 0$ for $\zeta, \zeta' \in \mathfrak{g}$, the decomposition (14) follows. Since the map $\mathcal{Q}_\mathfrak{g} : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ is a $\mathfrak{g} - \mathrm{ds}$ isomorphism, and the Weil differential d vanishes on $(W\mathfrak{g})_{\mathrm{basic}}$, it also vanishes on $(\mathcal{W}\mathfrak{g})_{\mathrm{basic}}$. Hence $H((\mathcal{W}\mathfrak{g})_{\mathrm{basic}}) = (\mathcal{W}\mathfrak{g})_{\mathrm{basic}} = (U\mathfrak{g})_{\mathrm{inv}}$, and the last claim of the Theorem follows from Corollary 3.5. □

5. Duflo isomorphism

We will now show that the isomorphism (15) from $S\mathfrak{g}$ to $U\mathfrak{g}$ is exactly the Duflo isomorphism, for the case of quadratic Lie algebras. Thus Theorem 4.7 proves Duflo's theorem for this case. More generally, we will show that the map $\mathcal{Q}_\mathfrak{g} : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$ coincides with the quantization map introduced in [1].

Proposition 5.1. *Each of the derivations ι_ξ, d, L_ξ of $\mathcal{W}\mathfrak{g}$ is inner.*

Proof. By construction, $\iota_\xi = [\xi, \cdot]$ and $L_\xi = [\overline{\xi}, \cdot]$. To show that the differential d is inner, choose a basis e_a of $\mathfrak{g}$, and let e^a be the dual basis with respect to B. Then $\sum_a \overline{e_a} e^a \in \mathcal{W}\mathfrak{g}$ is an invariant element, independent of the choice of basis. We have

$$[\sum_a \overline{e_a} e^a, \overline{\xi}] = 0, \quad [\sum_a \overline{e_a} e^a, \xi] = \overline{\xi} - 2\gamma(\xi).$$

Let

$$\mathcal{D} := \sum_a \overline{e_a} e^a - \frac{2}{3} \sum_a \gamma(e_a) e^a.$$

Since $\mathcal{D}$ is invariant, $[\mathcal{D}, \overline{\xi}] = -L_\xi \mathcal{D} = 0 = \mathrm{d}\overline{\xi}$. The element $\phi = \frac{1}{3}\sum_a \gamma(e_a) e^a$ satisfies $[\phi, \xi] = \gamma(\xi)$. Therefore On the other hand, since $[\frac{1}{3}\sum_a \gamma(e_a)e^a, \xi] = \gamma(\xi)$, we have

$$[\mathcal{D}, \xi] = \overline{\xi} + \sum_a \mathrm{ad}_\xi(e_a) e^a - 2\gamma(\xi) = \overline{\xi} = \mathrm{d}\xi.$$

□

Remarks 5.2. (a) The cubic element $\mathcal{D}$ may be interpreted as a quantized chain of transgression. Indeed, it is easily checked that $\mathcal{Q}_\mathfrak{g}(D) = \mathcal{D}$ where

$$D := h(\sum_a \widehat{e_a}\widehat{e^a}) = \sum\nolimits_a \overline{e_a} e^a - \frac{2}{3}\sum_a \lambda(e_a)e^a \in W\mathfrak{g}$$

is the chain of transgression corresponding to the quadratic polynomial $\sum_a \widehat{e_a}\widehat{e^a} \in (S\mathfrak{g})_{\mathrm{inv}} \subset W\mathfrak{g}$. Here h is the standard homotopy operator for the Weil algebra.

(b) The fact that the derivation d is inner may also be formulated in terms of the quadratic super Lie algebra $\mathbb{F}\mathrm{d} \ltimes (\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c})$. Indeed, it may be verified that the cubic element

$$\mathfrak{c}^2\mathrm{d} - \mathfrak{c}\sum\nolimits_a \overline{e_a}e^a + \frac{2}{3}\sum\nolimits_a \gamma(e_a)e^a$$

in the enveloping algebra of $\mathbb{F}\mathrm{d} \ltimes (\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c})$ is a central element. Specializing to $\mathfrak{c} = 1$ we see that $\mathrm{d} - D$ is central in the quotient by $< \mathfrak{c} - 1 >$.

We now recall the definition of the Duflo map $S\mathfrak{g} \to U\mathfrak{g}$. Let $\overline{S\mathfrak{g}^*} = \coprod_{k=0}^\infty S^k\mathfrak{g}^*$ be the completion of the symmetric algebra, or equivalently the algebraic dual space to $S\mathfrak{g}$. Informally, we will view $\overline{S\mathfrak{g}^*}$ as Taylor series expansions of functions on $\mathfrak{g}$. There is an algebra homomorphism

$$\overline{S\mathfrak{g}^*} \to \mathrm{End}(S\mathfrak{g}),\ \ F \mapsto \widehat{F}$$

extending the natural action of $\mathfrak{g}^*$ by derivations, that is, $\hat{F}$ is an infinite order differential operator acting on polynomials. Let $j(z) = \frac{\sinh(z/2)}{z/2}$ and define $J \in \overline{S\mathfrak{g}^*}$ by

$$J(\xi) = \det(j(\mathrm{ad}_\xi)) = e^{\mathrm{tr}\left((\ln j)(\mathrm{ad}_\xi)\right)}.$$

The square root of J is a well-defined element of $\overline{S\mathfrak{g}^*}$. The Duflo map is the composition

$$\mathrm{sym}_{U\mathfrak{g}} \circ \widehat{J^{1/2}} : S\mathfrak{g} \to U\mathfrak{g}. \tag{16}$$

The quantization map in [1] is an extension of the Duflo map, for the case that $\mathfrak{g}$ is quadratic. Write $W\mathfrak{g} = S\mathfrak{g} \otimes \wedge\mathfrak{g}$, as in (9), and $\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$ as in (14). Let

$$(\ln j)'(z) = \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z}$$

be the logarithmic derivative of the function j, and let $\mathfrak{r} \in \overline{S\mathfrak{g}^*} \otimes \wedge^2\mathfrak{g}$ be given by $r(\xi) = (\ln j)'(\mathrm{ad}_\xi)$, where we identify skew-symmetric operators on $\mathfrak{g}$ with elements in $\wedge^2\mathfrak{g}$. Put

$$\mathcal{S}(\xi) = J^{1/2}(\xi)\exp(\mathfrak{r}(\xi))$$

and let $\widehat{\iota(\mathcal{S})} \in \mathrm{End}(W\mathfrak{g})$ denote the corresponding operator, where the $\overline{S\mathfrak{g}^*}$ factor acts as an infinite order differential operator on $S\mathfrak{g}$, and the $\wedge\mathfrak{g}$ factor acts by contraction on $\wedge\mathfrak{g}$. Let $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ be the Chevalley quantization map for the Clifford algebra. The tensor product of the PBW symmetrization map $\mathrm{sym}_{U\mathfrak{g}} : S\mathfrak{g} \to U\mathfrak{g}$ and the Chevalley quantization map $q : \wedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ define a linear isomorphism $\mathrm{sym}_{U\mathfrak{g}} \otimes q : W\mathfrak{g} \to \mathcal{W}\mathfrak{g}$. Put differently, this is the symmetrization map with respect to the generators $\xi, \widehat{\xi}$, rather than the generators $\xi, \overline{\xi}$ used in the definition of $\mathcal{Q}_\mathfrak{g}$.

Theorem 5.3. *Under the identification $W\mathfrak{g} = S\mathfrak{g}\otimes\wedge\mathfrak{g}$ and $\mathcal{W}\mathfrak{g} = U\mathfrak{g}\otimes\mathrm{Cl}(\mathfrak{g})$, the quantization map is given by the formula,*

$$\mathcal{Q}_\mathfrak{g} = (\mathrm{sym}_{U\mathfrak{g}}\otimes q)\circ\widehat{\iota(\mathcal{S})}:\ W\mathfrak{g}\to\mathcal{W}\mathfrak{g}.$$

In particular, its restriction to the symmetric algebra $S\mathfrak{g}$ is the Duflo map.

Proof. We use an alternative description of the symmetrization map $S(\widetilde{\mathfrak{g}})\to\mathcal{T}(\widetilde{\mathfrak{g}})$. Let $\nu^a\in E^{\bar 1}_{\mathfrak{g}^*}$ and $\mu^a\in E^{\bar 0}_{\mathfrak{g}^*}$ be "parameters". The symmetrization map is characterized by its property that for all p, the map $\mathrm{id}\otimes\mathcal{Q}_\mathfrak{g}$ takes the pth power of $\sum_a(\nu^a e_a+\mu^a\overline{e_a})$ in the algebra $S(E_{\mathfrak{g}^*})\otimes W\mathfrak{g}$ to the corresponding pth power in $S(E_{\mathfrak{g}^*})\otimes\mathcal{W}\mathfrak{g}$. These conditions may be combined into a single condition

$$\mathrm{id}\otimes\mathcal{Q}_\mathfrak{g}:\ \exp_{W\mathfrak{g}}(\sum\nolimits_a(\nu^a e_a+\mu^a\overline{e_a}))\mapsto\exp_{\mathcal{W}\mathfrak{g}}(\sum\nolimits_a(\nu^a e_a+\mu^a\overline{e_a}));$$

here the exponentials are well-defined in completions $\overline{S(E_{\mathfrak{g}^*})}\otimes W\mathfrak{g}$ and $\overline{S(E_{\mathfrak{g}^*})}\otimes\mathcal{W}\mathfrak{g}$, respectively.

We want to re-express the symmetrization map in terms of the new generators $e_a,\widehat{e_a}=\overline{e_a}-\lambda(e_a)$ of $W\mathfrak{g}$ and $e_a,\widehat{e_a}=\overline{e_a}-\gamma(e_a)$ of $\mathcal{W}\mathfrak{g}$. Using that e_a and $\widehat{e_a}$ commute in $\mathcal{W}\mathfrak{g}$, we may separate the $\mathrm{Cl}(\mathfrak{g})$ and $U\mathfrak{g}$-variables in the exponential and obtain:

$$\exp_{\mathcal{W}\mathfrak{g}}(\sum\nolimits_a(\nu^a e_a+\mu^a\overline{e_a}))=\exp_{\mathrm{Cl}(\mathfrak{g})}\big(\sum\nolimits_a(\nu^a e_a+\mu^a\gamma(e_a)\big)\exp_{U\mathfrak{g}}(\sum\nolimits_a\mu^a\widehat{e_a}).$$

The factor $\exp_{U\mathfrak{g}}(\sum_a\mu^a\widehat{e_a})$ is the image of $\exp_{S\mathfrak{g}}(\sum_a\mu^a\widehat{e_a})$ under the symmetrization map $\mathrm{sym}_{U\mathfrak{g}}:\ S\mathfrak{g}\to U\mathfrak{g}$. The other factor is the exponential of a quadratic expression in the Clifford algebra. Using [3, Theorem 2.1] such exponentials may be expressed in terms of the corresponding exponentials in the exterior algebra:

$$\exp_{\mathrm{Cl}(\mathfrak{g})}\sum\nolimits_a\big(\nu^a e_a+\mu^a\gamma(e_a)\big)=q\Big(\iota(\mathcal{S}(\mu))\exp_{\wedge\mathfrak{g}}\sum\nolimits_a\big(\nu^a e_a+\mu^a\lambda(e_a)\big)\Big),$$

where $\iota:\ \wedge\mathfrak{g}\to\mathrm{End}(\wedge\mathfrak{g})$ is contraction. This shows

$$\begin{aligned}
&\exp_{\mathcal{W}\mathfrak{g}}\sum\nolimits_a(\nu^a e_a+\mu^a\overline{e_a})\\
&=\ (\mathrm{sym}_{U\mathfrak{g}}\otimes q)\Big(\iota(\mathcal{S}(\mu))\exp_{W\mathfrak{g}}\sum\nolimits_a\big(\nu^a e_a+\mu^a(\widehat{e_a}+\lambda(e_a))\big)\Big)\\
&=\ \mathcal{Q}_\mathfrak{g}\Big(\exp_{W\mathfrak{g}}\sum\nolimits_a\big(\nu^a e_a+\mu^a(\widehat{e_a}+\lambda(e_a))\big)\Big)\\
&=\ \mathcal{Q}_\mathfrak{g}\Big(\exp_{W\mathfrak{g}}\sum\nolimits_a\big(\nu^a e_a+\mu^a\overline{e_a}\big)\Big).
\end{aligned}$$

□

A different proof of Theorem 5.3 will be given in Section 8. Our result shows that while the Duflo map itself is not a symmetrization map, it may be viewed as the restriction of a symmetrization map for a larger algebra. Using Theorem 5.3, we obtain a very simple proof of the following result from [1, 25]:

Proposition 5.4. *The square of the cubic element $\mathcal{D}$ is given by*

$$\mathcal{D}^2=\frac{1}{2}\,\mathrm{Cas}_\mathfrak{g}+\frac{1}{48}\,\mathrm{tr}_\mathfrak{g}(\mathrm{Cas}_\mathfrak{g}) \tag{17}$$

where $\mathrm{Cas}_{\mathfrak{g}} = \sum_a \widehat{e_a}\widehat{e^a} \in (U\mathfrak{g})_{\mathrm{inv}} \subset \mathcal{W}\mathfrak{g}$ *is the quadratic Casimir element, and* $\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}})$ *its trace in the adjoint representation.*

Outline of proof. Write $\mathcal{D} = \mathcal{Q}_{\mathfrak{g}}(D)$ as in Remark 5.2 (a). Then

$$\mathcal{D}^2 = \tfrac{1}{2}[\mathcal{D},\mathcal{D}] = \tfrac{1}{2}\mathrm{d}(\mathcal{D}) = \tfrac{1}{2}\mathcal{Q}_{\mathfrak{g}}(\mathrm{d}D) = \tfrac{1}{2}\mathcal{Q}_{\mathfrak{g}}(\sum_a \widehat{e_a}\widehat{e^a}).$$

Using the explicit formula (16) for the Duflo map, one finds that $\mathcal{Q}_{\mathfrak{g}}(\sum_a \widehat{e_a}\widehat{e^a})$ is equal to $\mathrm{Cas}_{\mathfrak{g}} + \frac{1}{24}\mathrm{tr}_{\mathfrak{g}}(\mathrm{Cas}_{\mathfrak{g}})$. □

Remarks 5.5. (a) The algebra $\mathcal{W}\mathfrak{g}$ carries a natural $\mathbb{Z}$-filtration, where the odd generators have degree 1 and the even generators have degree 2. Its associated graded algebra is the Weil algebra $W\mathfrak{g}$, with its standard grading. The filtration on $\mathcal{W}\mathfrak{g}$ is compatible with the $\mathbb{Z}_2$-grading in the sense of [27], and therefore induces the structure of a graded Poisson algebra on $W\mathfrak{g}$. On generators, the formulas for the graded Poisson bracket are given by $\{\xi,\zeta\} = B(\xi,\zeta)$, $\{\overline{\xi},\zeta\} = \mathrm{ad}_\xi\,\zeta$, $\{\overline{\xi},\overline{\zeta}\} = \overline{\mathrm{ad}_\xi\,\zeta}$. The differential may be written as $\mathrm{d} = \{D,\cdot\}$ with D as in Remark 5.2 (a).

(b) It is possible to re-introduce a grading on $\mathcal{W}\mathfrak{g}$, by adding an extra parameter. Let $\hbar$ be a variable of degree 2, and view $S(\mathbb{F}\cdot\hbar)$ as a $\mathfrak{g}-\mathrm{da}$ with contractions, Lie derivatives and differential all equal to zero. Define a $\mathfrak{g}-\mathrm{da}$ $\mathcal{W}\mathfrak{g}[\hbar]$ as the quotient of $\mathcal{T}(\widetilde{\mathfrak{g}})\otimes S(\mathbb{F}\cdot\hbar)$ by the relations,

$$[\zeta,\zeta'] = \hbar B(\zeta,\zeta'),\quad [\overline{\zeta},\zeta'] = \hbar[\zeta,\zeta']_{\mathfrak{g}},\quad [\overline{\zeta},\overline{\zeta'}] = \hbar\overline{[\zeta,\zeta']_{\mathfrak{g}}}$$

(on the left hand side, the bracket denotes super commutators). Note that the ideal generated by these relations is graded. Hence $\mathcal{W}\mathfrak{g}[\hbar]$ is graded and the symmetrization map $W\mathfrak{g}[\hbar] = W\mathfrak{g}\otimes S(\mathbb{F}\cdot\hbar) \to \mathcal{W}\mathfrak{g}[\hbar]$ preserves degrees. The Weil algebra $W\mathfrak{g}$ is obtained by dividing out the ideal $\langle\hbar\rangle$, while $\mathcal{W}\mathfrak{g}$ is obtained by dividing out $\langle\hbar-1\rangle$. This clearly exhibits $\mathcal{W}\mathfrak{g}$ as a deformation of the Weil algebra $W\mathfrak{g}$.

(c) Pavol Severa explained to us in the summer of 2001, that the quantization map from [1] is closely related to the exponential map for a central extension of the super Lie group $T[1]G$, as discussed in his paper [39]. Theorem 5.3 may be viewed as the algebraic version of Severa's observation.

6. Vogan conjecture

Suppose $\mathfrak{k}\subset\mathfrak{g}$ is a Lie subalgebra admitting a $\mathfrak{k}$-invariant complement $\mathfrak{p}$. Thus $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ with

$$[\mathfrak{k},\mathfrak{k}]_{\mathfrak{g}}\subset\mathfrak{k},\quad [\mathfrak{k},\mathfrak{p}]_{\mathfrak{g}}\subset\mathfrak{p}.$$

Any $\mathfrak{g}-\mathrm{ds}$ E becomes a $\mathfrak{k}-\mathrm{ds}$ by restricting the action to $\widetilde{\mathfrak{k}}\subset\widetilde{\mathfrak{g}}$. If $\mathcal{A}$ is a $\mathfrak{g}-\mathrm{da}$ with connection $\theta:\mathfrak{g}^*\to\mathcal{A}^{\bar{1}}$, then the restriction of θ to $\mathfrak{k}^*\subset\mathfrak{g}^*$ defines a connection for $\mathcal{A}$, viewed as a $\mathfrak{k}-\mathrm{da}$.

Lemma 6.1. *The projection* $W\mathfrak{g}\to W\mathfrak{k}$ *is a* $\mathfrak{k}$*-homotopy equivalence, with homotopy inverse* $W\mathfrak{k}\to W\mathfrak{g}$ *induced by the inclusion* $E_{\mathfrak{k}^*}\to E_{\mathfrak{g}^*}$.

Proof. We have $W\mathfrak{g} = W\mathfrak{k}\otimes S(E_{\mathfrak{p}^*})$, where the second factor carries the $\mathfrak{k}-\mathrm{da}$-structure induced by the $\mathfrak{k}$-action on $\mathfrak{p}^*$. According to that Example 2.5, the augmentation map $S(E_{\mathfrak{p}^*})\to\mathbb{F}$ is a $\mathfrak{k}$-homotopy equivalence, with $\mathfrak{k}$-homotopy inverse the inclusion. □

Suppose now that $\mathfrak{g}$ is a quadratic Lie algebra, and that the restriction of the scalar product B to the subalgebra $\mathfrak{k}$ is again non-degenerate. We will refer to $\mathfrak{k}$ as a *quadratic subalgebra.* In this case, we may take $\mathfrak{p}$ to be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.

Example 6.2. Let $\mathfrak{g}$ be a semi-simple Lie algebra, with B the Killing form, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition. Then $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, and B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. See [19, Chapter 3.7] or [24, Chapter VII.2].

Example 6.3. Suppose $\mathbb{F} = \mathbb{C}$. For any $\xi \in \mathfrak{g}$, the generalized eigenspace for the 0 eigenvalue of ad_ξ

$$\mathfrak{k} = \{\zeta \in \mathfrak{g} |\ \mathrm{ad}_\xi^N \zeta = 0 \text{ for } N >> 0\}$$

is a quadratic subalgebra, with $\mathfrak{p}$ the direct sum of generalized eigenspaces for nonzero eigenvalues. Indeed, given $\zeta_1 \in \mathfrak{k},\ \zeta_2 \in \mathfrak{p}$, let $N > 0$ with $\mathrm{ad}_\xi^N \zeta_1 = 0$. Since ad_ξ is invertible on $\mathfrak{p}$, we have

$$B(\zeta_1, \zeta_2) = (-1)^N B(\mathrm{ad}_\xi^N \zeta_1,\ \mathrm{ad}_\xi^{-N} \zeta_2) = 0.$$

This shows $\mathfrak{p} = \mathfrak{k}^\perp$.

The inclusion $\widetilde{\mathfrak{k}} \oplus \mathbb{F}\mathfrak{c} \to \widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}$ is a $\mathfrak{k}-$dl homomorphism, hence it extends to a $\mathfrak{k}-$da homomorphism

$$U(\widetilde{\mathfrak{k}} \oplus \mathbb{F}\mathfrak{c}) \to U(\widetilde{\mathfrak{g}} \oplus \mathbb{F}\mathfrak{c}).$$

Taking quotients by the ideals generated by $\mathfrak{c}-1$, we obtain a $\mathfrak{k}-$da homomorphism $\mathcal{W}\mathfrak{k} \to \mathcal{W}\mathfrak{g}$. We obtain a commutative diagram of $\mathfrak{k}-$ds homomorphisms,

(18)
$$\begin{tikzcd} W\mathfrak{g} \arrow[r, "\mathcal{Q}_\mathfrak{g}"'] & \mathcal{W}\mathfrak{g} \\ W\mathfrak{k} \arrow[u] \arrow[r, "\mathcal{Q}_\mathfrak{k}"'] & \mathcal{W}\mathfrak{k} \arrow[u] \end{tikzcd},$$

in which all maps are $\mathfrak{k}$-homotopy equivalences, and the induced maps in basic cohomology are all algebra isomorphisms. We will now interpret these maps in terms of the isomorphism $\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$. We have

$$(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{hor}} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p})$$

and therefore $(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}} = (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$. The Kostant cubic Dirac operator for the pair $\mathfrak{g}, \mathfrak{k}$ is defined to as the difference of the Dirac operators for $\mathfrak{g}$ and $\mathfrak{k}$:

$$\mathcal{D}_{\mathfrak{g},k} = \mathcal{D}_\mathfrak{g} - \mathcal{D}_\mathfrak{k}.$$

The first two parts of the following Proposition were proved by Kostant in [25]. Let

(19)
$$\chi :\ U\mathfrak{k} \hookrightarrow U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p})$$

be the map given by the inclusion $(W\mathfrak{k})_{\mathfrak{k}-\mathrm{hor}} \hookrightarrow (\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{hor}}$.

Proposition 6.4. (a) *The cubic Dirac operator $\mathcal{D}_{\mathfrak{g},k}$ lies in the algebra $(U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$.*

(b) *The square of $\mathcal{D}_{\mathfrak{g},k}$ is given by the formula,*

$$\mathcal{D}_{\mathfrak{g},\mathfrak{k}}^2 = \frac{1}{2} \mathrm{Cas}_\mathfrak{g} - \frac{1}{2}\chi(\mathrm{Cas}_\mathfrak{k}) + \frac{1}{48}(\mathrm{tr}_\mathfrak{g}(\mathrm{Cas}_\mathfrak{g}) - \mathrm{tr}_\mathfrak{k}(\mathrm{Cas}_\mathfrak{k})).$$

(c) *The restriction of the differential on $\mathcal{W}\mathfrak{g}$ to the subalgebra $(U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$ is a graded commutator $[\mathcal{D}_{\mathfrak{g},k}, \cdot]$.*

Proof. (a) It is clear that $\mathcal{D}_{\mathfrak{g},k}$ is $\mathfrak{k}$-invariant. Furthermore, for $\xi \in \mathfrak{k}$, we have

$$[\mathcal{D}_{\mathfrak{g},k}, \xi] = [\mathcal{D}_{\mathfrak{g}}, \xi] - [\mathcal{D}_{\mathfrak{k}}, \xi] = \overline{\xi} - \overline{\xi} = 0$$

so $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ is $\mathfrak{k}$-basic. (b) Since $\iota_\xi = [\xi, \cdot]$ and $L_\xi = [\overline{\xi}, \cdot]$, the basic subalgebra $(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}$ is exactly the commutant of the subalgebra $\mathcal{W}\mathfrak{k} \subset \mathcal{W}\mathfrak{g}$. Hence $\mathcal{D}_{\mathfrak{g},\mathfrak{k}} \in (\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}$ and $\mathcal{D}_{\mathfrak{k}} \in \mathcal{W}\mathfrak{k}$ commute, and the formula follows from (17) by squaring the identity $\mathcal{D}_{\mathfrak{g}} = \mathcal{D}_{\mathfrak{g},\mathfrak{k}} + \mathcal{D}_{\mathfrak{k}}$. (c) On elements of $(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}$, $[\mathcal{D}_{\mathfrak{k}}, \cdot]$ vanishes since $\mathcal{D}_{\mathfrak{k}} \in \mathcal{W}\mathfrak{k}$. Hence $[\mathcal{D}_{\mathfrak{g},\mathfrak{k}}, \cdot]$ coincides with $[\mathcal{D}_{\mathfrak{g}}, \cdot]$ on $(\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}$. □

The following theorem is a version of Vogan's conjecture (as formulated in Huang-Pandzic [20]) for quadratic Lie algebras. It was first proved by Huang-Pandzic [20, Theorems 3.4, 5.5] for symmetric pairs, and by Kostant [26, Theorem 0.2] for reductive pairs. In a recent paper, Kumar [28] interpreted the Vogan conjecture in terms of induction maps in non-commutative equivariant cohomology.

Theorem 6.5. *The map $\chi : (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \to (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$ takes values in cocycles for the differential $[\mathcal{D}_{\mathfrak{g},\mathfrak{k}}, \cdot]$, and descends to an algebra isomorphism from $(U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ to the cohomology of $(U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$. The map $(U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ taking $z \in (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}}$ to the cohomology class of $z \otimes 1 \in (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$ fits into a commutative diagram,*

$$\begin{tikzcd}
(S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \arrow[r] \arrow[d] & (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \arrow[d] \\
(S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \arrow[r] & (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}
\end{tikzcd}$$

where the horizontal maps are Duflo maps and the left vertical map is induced by the projection $\mathfrak{g} \to \mathfrak{k}$.

Proof. The commutative diagram (18) gives rise to a commutative diagram of $\mathfrak{k}$-basic subcomplexes,

$$\begin{tikzcd}
(S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \arrow[r] \arrow[d] & (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \arrow[d, "\chi"'] \\
(S\mathfrak{g} \otimes \wedge\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}} \arrow[r] & (U\mathfrak{g} \otimes \mathrm{Cl}\,\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}}
\end{tikzcd}$$

As mentioned after (18), all of the maps in this diagram induce algebra isomorphisms in cohomology. This proves the first part of the theorem. The second part follows by combining this diagram with a commutative diagram

$$\begin{tikzcd}
(S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \arrow[r] \arrow[d] & (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \arrow[d] \\
(S\mathfrak{g} \otimes \wedge\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}} \arrow[r] & (U\mathfrak{g} \otimes \mathrm{Cl}\,\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}}
\end{tikzcd}$$

given by the inclusion of $\mathfrak{g}$-basic subcomplexes of $W\mathfrak{g}, \mathcal{W}\mathfrak{g}$ into the $\mathfrak{k}$-basic subcomplexes. □

7. Harish-Chandra isomorphism

Let $\mathfrak{g}$ be a quadratic Lie algebra, with scalar product B, and $\mathfrak{k} \subset \mathfrak{g}$ a quadratic subalgebra, with orthogonal complement $\mathfrak{p} = \mathfrak{k}^\perp$. In the previous Section, we obtained an algebra homomorphism $(U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ which under the Duflo isomorphism corresponds to the natural projection $(S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \to (S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$. We will now describe an alternative construction of this map for enveloping algebras, generalizing the Harish-Chandra construction [24, Chapter V.5] As an extra ingredient, we need a $\mathfrak{k}$-invariant splitting $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{n}_+$ into Lie subalgebras of $\mathfrak{g}$ which are *isotropic*, i.e. such that B vanishes on $\mathfrak{n}_\pm$. Thus

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{k} \oplus \mathfrak{n}_+ \tag{20}$$

(direct sum of subspaces).

Examples 7.1. (a) In the standard setting of the Harish-Chandra theorem, $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{F} = \mathbb{C}$, with compact real form $\mathfrak{g}_\mathbb{R}$, $\mathfrak{k} = \mathfrak{t}^\mathbb{C}$ is the complexification of a maximal Abelian subalgebra of $\mathfrak{t} \subset \mathfrak{g}_\mathbb{R}$, and $\mathfrak{n}_\pm$ are nilpotent subalgebras given as sums of root spaces for the positive/negative roots. More generally, one could take $\mathfrak{k}$ to be the centralizer of some element $\xi \in \mathfrak{t}$, and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the direct sum of the positive (resp. negative) root spaces that are not contained in $\mathfrak{k}$.

(b) Suppose $\mathbb{F} = \mathbb{C}$. Consider the $2n+2$-dimensional nilpotent Lie algebra $\mathbb{C} \ltimes H_n$ from Example 4.1(b). We obtain a decomposition (20) by letting $\mathfrak{k}$ be the Abelian subalgebra spanned by r and c, and letting $\mathfrak{n}_\pm$ be the span of $e_{2i} \pm \sqrt{-1}e_{2i-1}$.

(c) Both of these examples are special case of the following set-up. Suppose that $\mathfrak{g}_\mathbb{R}$ is an arbitrary quadratic Lie algebra over $\mathbb{R}$, $\mathfrak{g}$ its complexification, and $\xi \in \mathfrak{g}_\mathbb{R}$. Since ad_ξ preserves the quadratic form B, all eigenvalues of $\frac{1}{i}\,\mathrm{ad}_\xi$ are real. Let $\mathfrak{g}_t \subset \mathfrak{g}$ denote the generalized eigenspace for the eigenvalue $t \in \mathbb{R}$. Then $[\mathfrak{g}_t, \mathfrak{g}_{t'}]_\mathfrak{g} \subset \mathfrak{g}_{t+t'}$, and $B(\mathfrak{g}_t, \mathfrak{g}_{t'}) = 0$ for $t + t' \neq 0$. A decomposition (20) is obtained by setting $\mathfrak{k} = \mathfrak{g}_0$, $\mathfrak{n}_- = \bigoplus_{t<0} \mathfrak{g}_t$, $\mathfrak{n}_+ = \bigoplus_{t>0} \mathfrak{g}_t$.

By the Poincaré-Birkhoff-Witt theorem, the decomposition (20) of $\mathfrak{g}$ yields a decomposition of the enveloping algebra $U\mathfrak{g}$,

$$U\mathfrak{g} = (\mathfrak{n}_-\, U\mathfrak{g} + U\mathfrak{g}\, \mathfrak{n}_+) \oplus U\mathfrak{k},$$

hence a (generalized) *Harish-Chandra projection*

$$\kappa_U : U\mathfrak{g} \to U\mathfrak{k}.$$

The projection κ_U is $\mathfrak{k}$-invariant, and restricts to an algebra homomorphism on the subalgebra $\mathfrak{n}_-\, U\mathfrak{g}\, \mathfrak{n}_+ \oplus U\mathfrak{k}$. Similar to κ_U, we define Harish-Chandra projections $\kappa_{\mathrm{Cl}} : \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{k})$ using the decomposition

$$\mathrm{Cl}(\mathfrak{g}) = (\mathfrak{n}_-\, \mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\, \mathfrak{n}_+) \oplus \mathrm{Cl}(\mathfrak{k})$$

and $\kappa_\mathcal{W} : \mathcal{W}\mathfrak{g} \to \mathcal{W}\mathfrak{k}$ using the decomposition

$$\mathcal{W}\mathfrak{g} = (\widetilde{\mathfrak{n}_-}\mathcal{W}\mathfrak{g} + \mathcal{W}\mathfrak{g}\, \widetilde{\mathfrak{n}_+}) \oplus \mathcal{W}\mathfrak{k}. \tag{21}$$

In Harish-Chandra's construction for enveloping algebras, it is necessary to compose the projection κ_U with a "shift". Consider the infinitesimal character on $\mathfrak{k}$,

$$\zeta \mapsto \mathrm{Tr}_{\mathfrak{n}_+} \mathrm{ad}_\zeta\,.$$

The map

$$\tau:\,\mathfrak{k}\to U\mathfrak{k},\ \zeta\mapsto\zeta+\tfrac12\operatorname{Tr}_{\mathfrak{n}_+}\operatorname{ad}_\zeta$$

is a Lie algebra homomorphism, hence it extends to an algebra automorphism, $\tau:\,U\mathfrak{k}\to U\mathfrak{k}$. In the standard case where $\mathfrak{k}$ is a Cartan subalgebra of a complex semi-simple Lie algebra, this is the "ρ-shift". Remarkably, the shift is already built into the projection $\kappa_{\mathcal{W}}$:

Proposition 7.2. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{W}\mathfrak{g} \arrow[r, "\cong"'] \arrow[d, "\kappa_W"'] & U\mathfrak{g}\otimes\mathrm{Cl}(\mathfrak{g}) \arrow[d, "(\tau\circ\kappa_U)\otimes\kappa_{\mathrm{Cl}}"] \\
\mathcal{W}\mathfrak{k} \arrow[r, "\cong"'] & U\mathfrak{k}\otimes\mathrm{Cl}(\mathfrak{k})
\end{tikzcd}$$

Proof. Observe that the two projections $\kappa_{\mathcal{W}}$ and $\kappa_U\otimes\kappa_{\mathrm{Cl}}$ both vanish on $E_{\mathfrak{n}_-}\mathcal{W}\mathfrak{g}+\mathcal{W}\mathfrak{g}E_{\mathfrak{n}_+}$. Hence it suffices to compare them on the subalgebra $\mathcal{W}\mathfrak{k}\subset\mathcal{W}\mathfrak{g}$. Elements in $\mathcal{W}\mathfrak{k}$ may be written in the form $\overline{\xi_{i_1}}\cdots\overline{\xi_{i_r}}\,x$ where $\xi_{i_1},\ldots,\xi_{i_r}\in\mathfrak{k}$ and $x\in\mathrm{Cl}(\mathfrak{k})\subset\mathcal{W}\mathfrak{k}$. For $\xi\in\mathfrak{k}$, we have $\overline{\xi}=\widehat{\xi}+\gamma(\xi)$, where $\gamma(\xi)\in\mathrm{Cl}(g)$ decomposes into parts $\gamma^{\mathfrak{k}}(\xi)\in\mathrm{Cl}(\mathfrak{k})$ and $\gamma^{\mathfrak{p}}(\xi)\in\mathrm{Cl}(\mathfrak{p})$. Let $b_i\in\mathfrak{n}_-$ and $c_j\in\mathfrak{n}_+$ be dual bases, i.e. $B(b_i,c_j)=\delta_{ij}$. We have,

$$\begin{aligned}
\gamma^{\mathfrak{p}}(\xi) &= \tfrac12\sum_j\big((\operatorname{ad}_\xi b_j)c_j+(\operatorname{ad}_\xi c_j)b_j\big)\\
&= \tfrac12\sum_j\big((\operatorname{ad}_\xi b_j)c_j-b_j(\operatorname{ad}_\xi c_j)\big)-\tfrac12\operatorname{tr}_{\mathfrak{n}_+}(\operatorname{ad}_\xi).
\end{aligned}$$

Here we have used $\sum_j[b_j,\operatorname{ad}_\xi c_j]=\operatorname{tr}_{\mathfrak{n}_+}(\operatorname{ad}_\xi)$, for all $\xi\in\mathfrak{k}$. Since $(\operatorname{ad}_\xi b_j)c_j-b_j(\operatorname{ad}_\xi c_j)\in\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+$, it follows that

$$\overline{\xi}=\widehat{\xi}-\tfrac12\operatorname{tr}_{\mathfrak{n}_+}(\operatorname{ad}_\xi)+\gamma^{\mathfrak{k}}(\xi)\mod\mathfrak{n}_-\mathrm{Cl}(\mathfrak{g})\mathfrak{n}_+.$$

Hence, the image of $\overline{\xi_{i_1}}\cdots\overline{\xi_{i_r}}\,x$ under the upper horizontal map takes the form,

$$\Big((\widehat{\xi}_{i_1}-\tfrac12\operatorname{tr}_{\mathfrak{n}_+}(\operatorname{ad}_{\xi_{i_1}})+\gamma^{\mathfrak{k}}(\xi_{i_1}))\cdots(\widehat{\xi}_{i_r}-\tfrac12\operatorname{tr}_{\mathfrak{n}_+}(\operatorname{ad}_{\xi_{i_r}})+\gamma^{\mathfrak{k}}(\xi_{i_r}))x\Big)\ +\ldots$$

where the terms $\ldots$ are in $E_{n_-}\mathcal{W}\mathfrak{g}+\mathcal{W}\mathfrak{g}E_{n_+}$. The term in the large parentheses lies in the image of the tensor products of inclusions $U\mathfrak{k}\hookrightarrow U\mathfrak{g}$, $\mathrm{Cl}(\mathfrak{k})\hookrightarrow\mathrm{Cl}(\mathfrak{g})$, and $\kappa_U\otimes\kappa_{\mathrm{Cl}}$ is the identity map on this image. Hence, the image under the projection $\kappa_U\otimes\kappa_{\mathrm{Cl}}$, followed by the shift $\tau\otimes1$, is given by

$$(\widehat{\xi}_{i_1}+\gamma^{\mathfrak{k}}(\xi_{i_1}))\cdots(\widehat{\xi}_{i_r}+\gamma^{\mathfrak{k}}(\xi_{i_r}))x$$

But this is exactly the image of $\overline{\xi_{i_1}}\cdots\overline{\xi_{i_r}}\,x$ under the lower horizontal map. $\square$

It is now easy to verify the following properties of the Harish-Chandra map for Weil algebras.

Theorem 7.3 (Harish-Chandra projection for Weil algebras)**.** *Suppose $\mathfrak{g}$ is a quadratic Lie algebra, and $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{k}\oplus\mathfrak{n}_+$ a decomposition into subalgebras (direct sum of vector spaces) where $\mathfrak{k}$ is quadratic and $\mathfrak{n}_\pm$ are $\mathfrak{k}$-invariant and isotropic.*

(a) *The maps $\kappa_W:\,W\mathfrak{g}\to W\mathfrak{k}$ and $\kappa_{\mathcal{W}}:\,\mathcal{W}\mathfrak{g}\to\mathcal{W}\mathfrak{k}$ are $\mathfrak{k}-\mathrm{ds}$ homomorphisms. In fact, they are $\mathfrak{k}$-homotopy equivalences, with homotopy inverse given by the inclusions, $W\mathfrak{k}\hookrightarrow W\mathfrak{g}$ and $\mathcal{W}\mathfrak{k}\hookrightarrow\mathcal{W}\mathfrak{g}$, respectively.*

(b) *The diagram*

$$\begin{tikzcd} W\mathfrak{g} \arrow[r, "\mathcal{Q}_\mathfrak{g}"'] \arrow[d, "\kappa_W"'] & \mathcal{W}\mathfrak{g} \arrow[d, "\kappa_\mathcal{W}"] \\ W\mathfrak{k} \arrow[r, "\mathcal{Q}_\mathfrak{k}"'] & \mathcal{W}\mathfrak{k} \end{tikzcd}$$

commutes up to $\mathfrak{k}$-chain homotopy.

(c) *The above diagram contains a sub-diagram,*

$$\begin{tikzcd} S\mathfrak{g} \arrow[r] \arrow[d, "\kappa_S"'] & U\mathfrak{g} \arrow[d, "\tau\circ\kappa_U"] \\ S\mathfrak{k} \arrow[r] & U\mathfrak{k} \end{tikzcd}$$

where $S\mathfrak{g}$ is identified with $(W\mathfrak{g})_{\mathfrak{g}-\mathrm{hor}}$, $U\mathfrak{g}$ with $(\mathcal{W}\mathfrak{g})_{\mathfrak{g}-\mathrm{hor}}$, and similarly for $S\mathfrak{k}$ and $U\mathfrak{k}$. In this sub-diagram the upper and lower horizontal map are Duflo maps, the left vertical map is κ_S, and the right vertical map is $\tau\circ\kappa_U$.

Proof. By Lemma 6.1, the map $\kappa_W : W\mathfrak{g}\to W\mathfrak{k}$ is a $\mathfrak{k}$-homotopy equivalence. On the other hand, using the Poincare-Birkhoff-Witt theorem, write $\mathcal{W}\mathfrak{g}$ as a direct sum of $\mathfrak{k}$-differential spaces,

$$\mathcal{W}\mathfrak{g}=\mathcal{W}\mathfrak{k}\oplus\Big((\mathcal{W}\mathfrak{n}_-)^+\otimes\mathcal{W}\mathfrak{k}\Big)\oplus\Big((\mathcal{W}\mathfrak{n}_-)^+\otimes\mathcal{W}\mathfrak{k}\otimes(\mathcal{W}\mathfrak{n}_+)^+\Big)\oplus\Big(\mathcal{W}\mathfrak{k}\otimes(\mathcal{W}\mathfrak{n}_+)^+\Big),$$

where the superscript $+$ denotes the augmentation ideal. (Note that $\mathcal{W}\mathfrak{n}_\pm\cong U(E_{\mathfrak{n}_\pm})$ since B vanishes on $\mathfrak{n}_\pm$.) By Proposition 4.4, $(\mathcal{W}\mathfrak{n}_\pm)^+$ are $\mathfrak{k}$-homotopy equivalent to 0. This proves (a). Part (b) follows from (a), since the diagram is obtained from the commutative diagram (18) by replacing the vertical arrows by $\mathfrak{k}$-homotopy inverses. We have already shown that the symmetrization maps for Weil algebras restrict to the Duflo maps, and it is clear that the left vertical map in (c) is just κ_S. The fact that the vertical map is $\tau\circ\kappa_U$ follows from Proposition 7.2. ∎

Theorem 7.3 implies the following generalization of the Harish-Chandra homomorphism for enveloping algebras.

Theorem 7.4. *The following diagram commutes:*

$$\begin{tikzcd} (S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \arrow[r] \arrow[d, "\kappa_S"'] & (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \arrow[d, "\tau\circ\kappa_U"] \\ (S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \arrow[r] & (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \end{tikzcd}$$

Here the horizontal maps are the Duflo isomorphisms for $\mathfrak{g}$ and $\mathfrak{k}$, respectively.

Proof. By part (b) of the above Theorem, the diagram obtained by passing to $\mathfrak{k}$-basic cohomology

$$\begin{tikzcd} H((W\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}) \arrow[r] \arrow[d] & H((\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}) \arrow[d] \\ (S\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \arrow[r] & (U\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}} \end{tikzcd}$$

commutes. (Moreover, all maps in this diagram are algebra isomorphisms.) On the other hand, the maps from $\mathfrak{g}$-basic cohomology to $\mathfrak{k}$-basic cohomology gives a commutative diagram of algebra homomorphisms,

$$\begin{array}{ccc} (S\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} & \longrightarrow & (U\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \\ \downarrow & & \downarrow \\ H((W\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}) & \longrightarrow & H((\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}}) \end{array}$$

Placing these two diagrams on top of each other, it follows that the diagram in Theorem 7.4 commutes. □

Proposition 7.5. *Under the assumptions of Theorem 7.4, the image of the cubic Dirac operator $\mathcal{D}_{\mathfrak{g}}$ under the Harish-Chandra projection is*

$$\kappa_{\mathcal{W}}(\mathcal{D}_{\mathfrak{g}}) = \mathcal{D}_{\mathfrak{k}},$$

the cubic Dirac operator $\mathcal{D}_{\mathfrak{k}}$ for the subalgebra.

Proof. Recall that $\mathcal{D}_{\mathfrak{g}} = \mathcal{D}_{\mathfrak{k}} + \mathcal{D}_{\mathfrak{g},\mathfrak{k}}$, where $\mathcal{D}_{\mathfrak{g},\mathfrak{k}} \in (U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{p}))_{\mathfrak{k}-\mathrm{inv}}$. The image of $\mathcal{D}_{\mathfrak{g},\mathfrak{k}}$ under the Harish-Chandra projection vanishes since it is $\mathfrak{k}$-basic and odd. Hence, $\kappa_{\mathcal{W}}(\mathcal{D}_{\mathfrak{g}}) = \kappa_{\mathcal{W}}(\mathcal{D}_{\mathfrak{k}}) = \mathcal{D}_{\mathfrak{k}}$. □

Remark 7.6. For semi-simple Lie algebras and $\mathfrak{k} = \mathfrak{h}$ a Cartan subalgebra, the Harish-Chandra projection κ_{Cl} for Clifford algebras was studied by Kostant. In particular Kostant showed that the image of a primitive generator of $\wedge\mathfrak{g} \cong \mathrm{Cl}(\mathfrak{g})$ is always *linear*, i.e. contained in $\mathfrak{h} \subset \mathrm{Cl}(\mathfrak{h})$. He made a beautiful conjecture relating these projections to the adjoint representation of the principal TDS; this conjecture was recently proved by Y. Bazlov. It would be interesting to understand these results within our framework.

8. Rouvière isomorphism

In his 1986 paper [37], F. Rouvière described generalizations of Duflo's isomorphism to a certain class of symmetric spaces G/K. In this Section, we will prove a Duflo-Rouvière isomorphism for quadratic Lie algebras $\mathfrak{g}$, with a scalar product that is anti-invariant under a given involution of $\mathfrak{g}$.

8.1. **Statement of the theorem.** Let $\epsilon : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the subalgebra fixed by ϵ, and $\mathfrak{p}$ is the -1 eigenspace of ϵ. We will refer to $(\mathfrak{g}, \mathfrak{k})$ as a symmetric pair. For any Lie algebra homomorphism $f : \mathfrak{k} \to \mathbb{F}$, define a twisted inclusion of $\mathfrak{k}$ in $U\mathfrak{k}$ by

$$\mathfrak{k}^f = \{\xi + f(\xi) | \, \xi \in \mathfrak{k}\}.$$

Using the embedding $U\mathfrak{k} \hookrightarrow U\mathfrak{g}$, we may view $\mathfrak{k}^f$ as a subspace of $U\mathfrak{g}$. The space

$$(U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}} \tag{22}$$

inherits an algebra structure from the enveloping algebra $U\mathfrak{g}$: Indeed, $U\mathfrak{g} \cdot \mathfrak{k}^f$ is a two-sided ideal in the subalgebra $\{z \in U\mathfrak{g} | \, L_\xi z \in U\mathfrak{g} \cdot \mathfrak{k}^f \text{ for all } \xi \in \mathfrak{k}\}$ of $U\mathfrak{g}$, and (22) is the quotient algebra. The following was proved by Duflo, generalizing a result of Lichnerowicz [31]:

Theorem 8.1 (Duflo [14]). *Let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair, and let $f:\mathfrak{k}\to\mathbb{F}$ be the character $f(\xi)=\frac{1}{2}\operatorname{tr}_{\mathfrak{k}}(\operatorname{ad}_\xi)$. Then the algebra (22) is commutative.*

The geometric interpretation of the algebra (22) is as follows. Suppose $\mathbb{F}=\mathbb{R}$, and let G be the connected, simply connected Lie group having $\mathfrak{g}$ as its Lie algebra. Assume that $\mathfrak{k}\subset\mathfrak{g}$ is the Lie algebra of a closed, connected subgroup of $K\subset G$. Taking $f=0$, a theorem of Lichnerowicz [30] shows that (22) is the algebra of G-invariant differential operators on the symmetric space G/K. The algebra (22) for $f(\xi)=\frac{1}{2}\operatorname{tr}_{\mathfrak{k}}(\operatorname{ad}_\xi)$ is interpreted as the algebra of G-invariant differential operators on G/K, acting on sections of the *half density bundle.*

Returning to the general case, we relate the algebra (22) to invariants in the symmetric algebra $S\mathfrak{p}$. Indeed, using a PBW basis one sees that the map

$$S\mathfrak{p}\oplus U\mathfrak{g}\cdot\mathfrak{k}^f\to U\mathfrak{g},\quad (x,z)\mapsto \operatorname{sym}_{U\mathfrak{g}}(x)+z$$

is a $\mathfrak{k}$-module isomorphism. We therefore obtain an isomorphism of $\mathfrak{k}$-modules

$$\operatorname{Sym}:\ S\mathfrak{p}\to U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f$$

taking $x\in S\mathfrak{p}$ to the image of $\operatorname{sym}_{U\mathfrak{g}}(x)$ under the quotient map. Let $J_{\mathfrak{p}}\in\overline{S\mathfrak{p}^*}$ be defined by the function

$$J_{\mathfrak{p}}(\zeta)=\det(j(2\operatorname{ad}_\zeta)|_{\mathfrak{p}})$$

with $j(z)=\frac{\sinh(z/2)}{z/2}$ as in Section 5. This is well-defined: For $\zeta\in\mathfrak{p}$, ad_ζ takes $\mathfrak{k}$ to $\mathfrak{p}$ and vice versa; since j is an even function, it follows that $j(2\operatorname{ad}_\zeta)$ preserves both $\mathfrak{k}$ and $\mathfrak{p}$. Let

$$\widehat{J_{\mathfrak{p}}^{1/2}}:\ S\mathfrak{p}\to S\mathfrak{p}$$

denote the infinite order differential operator defined by the square root of the function $J_{\mathfrak{p}}$. In Section 8.3 we will show:

Theorem 8.2. *Let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair, where $\mathfrak{k}$ is the fixed point set of an involutive automorphism ϵ. Suppose $\mathfrak{g}$ admits an invariant scalar product B with $\epsilon^*B=-B$. Then the composition*

$$\operatorname{Sym}\circ\widehat{J_{\mathfrak{p}}^{1/2}}:\ (S\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}}\to(U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}}$$

(where $\mathfrak{p}$ is the -1 eigenspace for ϵ) is an algebra isomorphism.

Remark 8.3.
(a) A result similar to Theorem 8.2 was first proved by Rouvière [37] for symmetric pairs $(\mathfrak{g},\mathfrak{k})$ satisfying one of the following two conditions: (i) $\mathfrak{g}$ is solvable, or (ii) $\mathfrak{g}$ satisfies the Kashiwara-Vergne conjecture [23] and $(\mathfrak{g},\mathfrak{k})$ is *very symmetric* in the sense that there is a linear isomorphism $A:\mathfrak{g}\to\mathfrak{g}$, $A(\mathfrak{k})=\mathfrak{p}$, $A(\mathfrak{p})=\mathfrak{k}$ such that $[A,\operatorname{ad}_\xi]=0$ for all $\xi\in\mathfrak{g}$. At the time of this writing the Kashiwara-Vergne conjecture is still open, but it has been established for solvable Lie algebras and for quadratic Lie algebras [43] (see also [2]). (A major consequence of the conjecture, regarding convolution of invariant distributions, was proved in the series of papers [7, 5, 6]. See [41] for more information on the Kashiwara-Vergne method.)
(b) After an earlier version of this paper had been posted on the arXiv, C. Torossian [42] gave an alternative proof of Theorem 8.2.
(c) It is known that for general symmetric pairs, the statement of Theorem 8.2 becomes false. A counter-example, examined in [15], is $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$, with $\mathfrak{k}=\mathfrak{so}(2)$ the subalgebra of diagonal matrices, and $\mathfrak{p}$ the subspace of matrices having 0 on the diagonal.

8.2. **Examples.** Suppose $\mathfrak{g}$ is a quadratic Lie algebra, with involution ϵ changing the sign of the scalar product B. Then B vanishes on both $\mathfrak{k}$ (the $+1$ eigenspace of ϵ) and $\mathfrak{p}$ (the -1 eigenspace of ϵ), and hence defines a non-singular pairing between $\mathfrak{k}$ and $\mathfrak{p}$. This identifies $\mathfrak{p}$ and $\mathfrak{k}^*$, and defines an element $C \in (\wedge^3\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ by

$$B([\mu,\mu']_{\mathfrak{g}},\mu'') = C(\mu,\mu',\mu''), \quad \mu,\,\mu',\,\mu'' \in \mathfrak{k}^*.$$

Conversely, given a Lie algebra $\mathfrak{k}$ and an invariant element $C \in \wedge^3\mathfrak{k}$, the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^*$ carries a unique Lie bracket such that $\mathfrak{k}$ is a Lie subalgebra, $[\xi,\mu]_{\mathfrak{g}} = -\operatorname{ad}^*_\xi \mu$ for $\xi \in \mathfrak{k}$, $\mu \in \mathfrak{k}^*$, and

$$[\mu,\mu']_{\mathfrak{g}} = C(\mu,\mu',\cdot) \in (\mathfrak{k}^*)^* = \mathfrak{k}, \quad \mu,\,\mu' \in \mathfrak{k}^*.$$

Furthermore, the symmetric bilinear form B given by the pairing between $\mathfrak{k}$ and $\mathfrak{k}^*$ is $\mathfrak{g}$-invariant, and changes sign under the involution ϵ given by -1 on $\mathfrak{k}^*$ and by 1 on $\mathfrak{k}$. Hence all examples for Theorem 8.2 may be described in terms of a Lie algebra $\mathfrak{k}$ with a given element $C \in (\wedge^3\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$.

Examples 8.4. (a) If $C = 0$, the Lie algebra $\mathfrak{g}$ is just the semi-direct product $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{k}^*$. In this case, one finds that $(U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}} = (S\mathfrak{k}^*)_{\mathfrak{k}-\mathrm{inv}}$ and the Duflo-Rouvière isomorphism is just the identity map.

(b) Suppose $(\mathfrak{k}, B_{\mathfrak{k}})$ is a quadratic Lie algebra over $\mathbb{F} = \mathbb{R}$. Then $C(\xi,\xi',\xi'') = \pm B_{\mathfrak{k}}([\xi,\xi']_{\mathfrak{k}},\xi'')$ defines an element $C \in (\wedge^3\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ (where we use $B_{\mathfrak{k}}$ to identify $\mathfrak{k}^*$ with $\mathfrak{k}$). For the minus sign, one obtains the example $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$. For the plus sign, one arrives at $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$, with $\mathfrak{k}$ embedded diagonally and $\mathfrak{p}$ embedded anti-diagonally. In this case, the Duflo-Rouvière isomorphism reduces to the usual Duflo isomorphism.

(c) Given $n \geq 3$ let $\mathfrak{k}$ be the nilpotent Lie algebra of strictly upper-triangular $n\times n$-matrices, and $\{E_{ij},\, i<j\}$ its natural basis, where E_{ij} is the matrix having 1 in the (i,j) position and zeroes elsewhere. Then $C = E_{1,n-1} \wedge E_{1,n} \wedge E_{2,n}$ lies in $(\wedge^3\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$. The resulting quadratic Lie algebra $\mathfrak{g}$ is solvable for $n = 3$, and nilpotent for $n \geq 4$.

(d) There are non-trivial examples of $C \in (\wedge^3\mathfrak{k})_{\mathfrak{k}-\mathrm{inv}}$ such that the resulting symmetric pair $(\mathfrak{g},\mathfrak{k})$ is not very symmetric in Rouvière's sense, and also $\mathfrak{g}$ not solvable.

Indeed if $(\mathfrak{g},\mathfrak{k})$ is a very symmetric pair with $[\mathfrak{k},\mathfrak{k}]_{\mathfrak{k}} = \mathfrak{k}$, then $[\mathfrak{p},\mathfrak{p}]_{\mathfrak{g}} = \mathfrak{k}$, or equivalently $\ker(C) \equiv \{\mu \in \mathfrak{k}^* |\, \iota_\mu C = 0\} = 0$. Take $\mathfrak{k} = \mathfrak{a} \ltimes \mathfrak{a}^*$ with $\mathfrak{a}$ semi-simple. Let $C \in \wedge^3\mathfrak{a}^* \subset \wedge^3\mathfrak{k}$ be defined by the Lie bracket and the Killing form on $\mathfrak{a}$. Since C has non-zero kernel, $(\mathfrak{g},\mathfrak{k})$ is not very symmetric. Furthermore, $\mathfrak{a} \subset \mathfrak{k} \subset \mathfrak{g}$ is the Levi factor of $\mathfrak{g}$ which shows that $\mathfrak{g}$ is not solvable.

8.3. **Proof of Theorem 8.2.** View $\mathcal{W}\mathfrak{g}$ as a $\mathfrak{k}$-differential algebra, with connection defined by the canonical $\mathfrak{g}$-connection and the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Recall that the contraction operators on $\widetilde{\mathfrak{g}} \subset \mathcal{W}\mathfrak{g}$ are given by $\iota_\xi\zeta = B(\xi,\zeta)$, $\iota_\xi\overline{\zeta} = L_\xi\zeta$. Since $\mathfrak{k}$ is isotropic, it follows that $\widetilde{\mathfrak{k}}$ is a $\mathfrak{k}$-differential subspace of $\mathcal{W}\mathfrak{g}$, and so is the left ideal $\mathcal{W}\mathfrak{g}\widetilde{\mathfrak{k}}$. The algebra structure on $\mathcal{W}\mathfrak{g}$ does not descend to the $\mathfrak{k}-\mathrm{ds}$ $\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}})$, in general. However, there is an induced algebra structure on the basic subcomplex $(\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}))_{\mathfrak{k}-\mathrm{basic}}$ since its pre-image in $\mathcal{W}\mathfrak{g}$ is a subalgebra containing $\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}$ as a 2-sided ideal.

Proposition 8.5. *The characteristic homomorphism* $W\mathfrak{k} \to \mathcal{W}\mathfrak{g} \cong U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$ *descends to a* $\mathfrak{k}-\mathrm{ds}$ *isomorphism*

$$W\mathfrak{k} \cong S(E_{\mathfrak{p}}) \to \mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}) \cong (U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f) \otimes \wedge\mathfrak{p}, \tag{23}$$

an isomorphism of $\mathfrak{k}$-modules

$$(W\mathfrak{k})_{\mathfrak{k}-\mathrm{hor}} \cong S\mathfrak{p} \to (\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}))_{\mathfrak{k}-\mathrm{hor}} \cong U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f, \tag{24}$$

and an isomorphism of algebras

$$(W\mathfrak{k})_{\mathfrak{k}-\mathrm{basic}} \cong (S\mathfrak{p})_{\mathfrak{k}-\mathrm{inv}} \to (\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}))_{\mathfrak{k}-\mathrm{basic}} \cong (U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}}. \tag{25}$$

Proof. By a PBW argument, the map

$$S(E_\mathfrak{p}) \oplus \mathcal{W}\mathfrak{g}\,\widetilde{\mathfrak{k}} \to \mathcal{W}\mathfrak{g},\ \ (x,z) \mapsto Q_\mathfrak{g}(x) + z$$

is a $\mathfrak{k}-\mathrm{ds}$ isomorphism. Thus, the quotient map $W\mathfrak{k} \cong S(E_\mathfrak{p}) \to \mathcal{W}\mathfrak{g}/\mathcal{W}\mathfrak{g}\widetilde{\mathfrak{k}}$ is again a $\mathfrak{k}-\mathrm{ds}$ isomorphism, and its restriction to horizontal subspaces is a $\mathfrak{k}$-module isomorphism. The map on basic subspaces is a composition

$$(W\mathfrak{k})_{\mathfrak{k}-\mathrm{basic}} \to (\mathcal{W}\mathfrak{g})_{\mathfrak{k}-\mathrm{basic}} \to (\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}))_{\mathfrak{k}-\mathrm{basic}},$$

where the second map is an algebra homomorphism, and the first map induces an algebra homomorphism in cohomology. Since the differential vanishes on $(W\mathfrak{k})_{\mathfrak{k}-\mathrm{basic}}$ and as a consequence vanishes on $(\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}))_{\mathfrak{k}-\mathrm{basic}}$, it follows that (25) is an algebra isomorphism.

It remains to identify $\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}})$ and its horizontal and basic subspaces in terms of the isomorphism $\mathcal{W}\mathfrak{g} = U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$. Observe that $\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}$ is the left ideal generated by elements $\zeta, \overline{\zeta}$ with $\zeta \in \mathfrak{k}$. We will show that

$$\overline{\zeta} = \widehat{\zeta} + \gamma^\mathfrak{g}(\zeta) = \widehat{\zeta} + \tfrac{1}{2}\,\mathrm{tr}_\mathfrak{k}(\zeta) \mod \mathcal{W}\mathfrak{g}\ \mathfrak{k}.$$

To see this, choose bases e_i of $\mathfrak{k}$ and e^j of $\mathfrak{p}$ such that $B(e_i, e^j) = \delta_i^j$. Then

$$\begin{aligned} \gamma^\mathfrak{g}(\zeta) &= \tfrac{1}{2}\sum_i (\mathrm{ad}_\zeta(e_i)e^i + \mathrm{ad}_\zeta(e^i)e_i) \\ &= \tfrac{1}{2}\sum_i (-e^i\,\mathrm{ad}_\zeta(e_i) + \mathrm{ad}_\zeta(e^i)e_i) + \tfrac{1}{2}\sum_i B(\mathrm{ad}_\zeta(e_i), e^i). \end{aligned}$$

The first sum lies in $\mathcal{W}\mathfrak{g}\ \mathfrak{k}$, while the second sum gives $\frac{1}{2}\,\mathrm{tr}_\mathfrak{k}(\mathrm{ad}_\zeta) \mod \mathcal{W}\mathfrak{g}\ \mathfrak{k}$. This proves $\mathcal{W}\mathfrak{g}/(\mathcal{W}\mathfrak{g}\ \widetilde{\mathfrak{k}}) = (U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f) \otimes \wedge\mathfrak{p}$, where the contractions ι_ξ are induced by the contractions on $\wedge\mathfrak{p} = \wedge\mathfrak{k}^*$. Hence the $\mathfrak{k}$-horizontal subspace is $(U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f)$, and the $\mathfrak{k}$-basic subcomplex is $(U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f)_{\mathfrak{k}-\mathrm{inv}}$. ☐

To complete the proof of Theorem 8.2, we have to identify the isomorphism (24) from $S\mathfrak{p}$ onto $U\mathfrak{g}/U\mathfrak{g}\mathfrak{k}^f$ with the map $\mathrm{Sym}\circ \widehat{J_\mathfrak{p}^{1/2}}$. Our calculation will require the following Lemma.

Lemma 8.6. *Let V be a vector space, and suppose $A : V \to V^*$ and $B : V^* \to V$ are linear maps with $A^* = -A, B^* = -B$. Let $\lambda(A) \in \wedge^2 V^*$ and $\lambda(B) \in \wedge^2 V$ be the skew-symmetric bilinear forms defined by A, B, i.e. in a basis e_a of V, with dual basis e^a of V^*,*

$$\lambda(A) = \tfrac{1}{2}\sum_a A(e_a) \wedge e^a,\ \ \lambda(B) = \tfrac{1}{2}\sum_a B(e^a) \wedge e_a.$$

Let $\iota : V^* \to \mathrm{End}(\wedge V)$ denote contraction, and denote by the same letter its extension to the exterior algebra $\wedge V^*$. Suppose $I + AB$ is invertible. Then

$$\iota(\exp(\lambda(A)))\ \exp(\lambda(B)) = \det{}^{1/2}(1 + AB)\ \ \exp(\lambda(B \circ (I + AB)^{-1}))$$

for a unique choice of square root of $\det(I+AB)$.

In particular, it follows that the map

$$\wedge^2 V^* \times \wedge^2 V \to \mathbb{F},\ (\lambda(A),\lambda(B)) \mapsto \det(I+AB)$$

admits a smooth square root, given by the degree zero part of $\iota(\exp(\lambda(A))\,\exp(\lambda(B))$.

Proof. This may be proved by methods similar to [3, Section 5], to which we refer for more details. Let $V\oplus V^*$ be equipped with the symmetric bilinear form given by the pairing between V and V^*, and $\mathrm{Spin}(V\oplus V^*)\to \mathrm{SO}(V\oplus V^*)$ be the corresponding Spin group. One has the following factorization in $\mathrm{SO}(V\oplus V^*)$,

$$\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}=\begin{pmatrix} I & D \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ E & I \end{pmatrix}\begin{pmatrix} R & 0 \\ 0 & (R^{-1})^* \end{pmatrix}$$

where

$$R=(I+BA)^{-1},\ E=AR^{-1},\ D=BR^*.$$

This factorization lifts to a factorization in $\mathrm{Spin}(V\oplus V^*)$. Consider now the spinor representation of $\mathrm{Spin}(V\oplus V^*)$ on $\wedge V^*$. In this representation, the lift of $\left(\begin{smallmatrix} R & 0 \\ 0 & (R^{-1})^* \end{smallmatrix}\right)$ acts as $\alpha\mapsto \det{}^{-1/2}(R)\,R.\alpha$, the lift of $\left(\begin{smallmatrix} I & 0 \\ E & I \end{smallmatrix}\right)$ acts by contraction with $\exp(\lambda(E))$, and the lift of the factor $\left(\begin{smallmatrix} I & D \\ 0 & I \end{smallmatrix}\right)$ acts by exterior product with $\exp(\lambda(D))$. The lemma follows by applying the factorization to the "vacuum vector" $1\in\wedge V^*$. ☐

Proposition 8.7. *The* $\mathfrak{k}$*-module isomorphism* $S\mathfrak{p}\to U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f$ *in* (24) *is equal to the map*

$$\mathrm{Sym}\circ\widehat{J_{\mathfrak{p}}^{1/2}}:\ S\mathfrak{p}\to U\mathfrak{g}/U\mathfrak{g}\ \mathfrak{k}^f.$$

Proof. We write (24) as a composition of maps

$$S\mathfrak{p}\xrightarrow{(i)} W\mathfrak{g}=S\mathfrak{g}^*\otimes\wedge\mathfrak{g}^*\xrightarrow{(ii)}\mathcal{W}\mathfrak{g}=U\mathfrak{g}\otimes\mathrm{Cl}(\mathfrak{g})\xrightarrow{(iii)}U\mathfrak{g}\xrightarrow{(iv)}U\mathfrak{g}/U\mathfrak{g}\,\mathfrak{k}^f$$

Here (i) is the restriction of the characteristic map $W\mathfrak{k}\to W\mathfrak{g}$ to $S\mathfrak{p}=S\mathfrak{k}^*$, (ii) is the quantization map, (iii) is the tensor product of the augmentation map for $\mathrm{Cl}(\mathfrak{g})$ with the identity map for $U\mathfrak{g}$, and (iv) is the quotient map. In terms of the generators $\widehat{\mu}=\mu-\lambda^{\mathfrak{k}}(\mu)$ of $S\mathfrak{k}^*$, and the corresponding generators of $S\mathfrak{g}^*$, (i) is given by

$$\mu\mapsto\mu,\quad \widehat{\mu}\mapsto\widehat{\mu}+\lambda^{\mathfrak{p}}(\mu)$$

where $\lambda^{\mathfrak{p}}(\mu)=\lambda^{\mathfrak{g}}(\mu)-\lambda^{\mathfrak{k}}(\mu)$ takes values in $\wedge^2\mathfrak{p}^*$. View $\lambda^{\mathfrak{p}}$ as a $\wedge^2\mathfrak{p}^*$-valued function on $\mathfrak{g}^*$, constant in $\mathfrak{p}^*$-directions. Then $\exp(\lambda^{\mathfrak{p}})$ defines an infinite order $\wedge\mathfrak{p}^*$-valued differential operator

$$\widehat{\exp(\lambda^{\mathfrak{p}})}:\ W\mathfrak{g}\to W\mathfrak{g},$$

and the characteristic homomorphism $W\mathfrak{k}\to W\mathfrak{g}$ is a composition $\widehat{\exp(\lambda^{\mathfrak{p}})}\circ i$ where $i: W\mathfrak{k}=S\mathfrak{k}^*\otimes\wedge\mathfrak{k}^*\to W\mathfrak{g}=S\mathfrak{g}^*\otimes\wedge\mathfrak{g}^*$ is the inclusion given on generators by $\mu\mapsto\mu,\widehat{\mu}\mapsto\widehat{\mu}$. Note that the image of $S\mathfrak{k}^*\subset W\mathfrak{k}$ under the composition lies in the subalgebra $S\mathfrak{k}^*\otimes\wedge\mathfrak{p}^*=S\mathfrak{p}\otimes\wedge\mathfrak{k}$ of $W\mathfrak{g}$. Hence, when we apply the map (ii)

$$\mathcal{Q}_{\mathfrak{g}}=(\mathrm{sym}\otimes q)\circ\iota(\widehat{\mathcal{S}}):\ W\mathfrak{g}\to\mathcal{W}\mathfrak{g},$$

we need only consider the "restriction" of $\mathcal{S}(\xi) = J^{1/2}(\xi)\exp(\mathfrak{r}(\xi))$, $\xi \in \mathfrak{g} \cong \mathfrak{g}^*$ to $\mathfrak{k}^* \cong \mathfrak{p}$. That is, we have to compute

$$J^{1/2}(\xi)\iota(\exp(\mathfrak{r}(\xi)))\exp(\lambda^{\mathfrak{p}}(\xi)) \in W\mathfrak{g} = S\mathfrak{g}\otimes\wedge\mathfrak{g} \tag{26}$$

for $\xi \in \mathfrak{p} = \mathfrak{k}^*$. In fact, we are only interested in the component of (26) in $S\mathfrak{g}\otimes\wedge^0\mathfrak{g}$, since all the other components will vanish under the projection (iii). Since $\mathfrak{k}$ and $\mathfrak{p}$ are dual $\mathfrak{k}$-modules,

$$J(\xi) = \det_{\mathfrak{p}}(j(\mathrm{ad}_\xi))\det_{\mathfrak{k}}(j(\mathrm{ad}_\xi)) = \det_{\mathfrak{p}}(j(\mathrm{ad}_\xi)^2).$$

Similarly, $r(\xi)$ splits into a sum $r'(\xi) + r''(\xi)$ where $r'(\xi) \in \wedge^2\mathfrak{p}$ and $r''(\xi) \in \wedge^2\mathfrak{k}$. We may replace $r(\xi)$ with $r'(\xi)$ in (26) since the $r''(\xi)$ part will not contribute to the contraction. Thus, we may calculate (26) using Lemma 8.6, with

$$V = \mathfrak{p},\ A = (\ln j)'(\mathrm{ad}_\xi)|_{\mathfrak{p}},\ B = \mathrm{ad}_\xi|_{\mathfrak{p}^*}.$$

Letting $(\cdot)_{[0]}$ denote the $\wedge^0\mathfrak{g}$ component, the Lemma gives,

$$J^{1/2}(\xi)\left(\iota(\exp(\mathfrak{r}(\xi)))\exp(\lambda^{\mathfrak{p}}(\xi))\right)_{[0]} = \det{}_{\mathfrak{p}}^{1/2}(h(\mathrm{ad}_\xi))$$

where

$$\begin{aligned} h(z) &= j(z)^2(1+z(\ln j)'(z)) \\ &= \Big(\frac{\sinh(z/2)}{z/2}\Big)^2\left(1+z(\tfrac12\coth(z/2)-z^{-1})\right) \\ &= \frac{\sinh(z/2)\cosh(z/2)}{z/2} \\ &= \frac{\sinh(z)}{z}. \end{aligned}$$

To summarize, the composition of the maps (i),(ii),(iii) is the map $S\mathfrak{p}\to U\mathfrak{g}$ given by application of an infinite-order differential operator $\widehat{J_{\mathfrak{p}}^{1/2}}$ on $S\mathfrak{p}$, followed by the PBW symmetrization map $\mathrm{sym}: S\mathfrak{p}\to U\mathfrak{g}$. Since the map $\mathrm{Sym}: S\mathfrak{p}\to U\mathfrak{g}/U\mathfrak{g}\,\mathfrak{k}^f$ is defined as PBW symmetrization followed by the quotient map (iv), the proof is complete. □

9. Universal characteristic forms

As a final application of our theory, we obtain a new construction of universal characteristic forms in the Bott-Shulman complex [8, 16, 35, 40]. We assume $\mathbb{F}=\mathbb{R}$, and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Recall that in the simplicial construction of the classifying bundle $EG\to BG$ [35, 38], one models the de Rham complex of differential forms on EG by a double complex $C^{p,q} = \Omega^q(G^{p+1})$ with differentials $\mathrm{d}: C^{p,q}\to C^{p,q+1}$ (the de Rham differential) and $\delta: C^{p,q}\to C^{p+1,q}$. Here δ is the alternating sum $\delta = \sum(-1)^i\delta_i$ where $\delta_i = \partial_i^*$ is the pull-back under the map,

$$\partial_i: G^{p+1}\to G^p,\quad (g_0,\ldots,g_p)\mapsto(g_0,\ldots,\widehat{g_i},\ldots,g_p)$$

omitting the ith entry. View each $E_pG = G^{p+1}$ as a principal G-bundle over $B_pG = G^p$, with action the diagonal G-action from the right, and quotient map $E_pG\to B_pG$, $(g_0,\ldots,g_p)\mapsto(g_0g_1^{-1}, g_1g_2^{-1},\ldots)$. Let

$$\iota_\xi: C^{p,q}\to C^{p,q-1},\ L_\xi: C^{p,q}\to C^{p,q}$$

be the corresponding contraction operators and Lie derivatives. Then the total complex

$$W = \bigoplus_{k=0}^{\infty} W^k, \quad W^k = \bigoplus_{p+q=k} C^{p,q}$$

with differential $D = \mathrm{d} + (-1)^q \delta$, contractions ι_ξ and Lie derivatives L_ξ becomes a $\mathfrak{g} - \mathrm{ds}$. (The $\mathbb{Z}_2$-grading is given by $W^{\bar{0}} = \bigoplus_{k=0}^{\infty} W^{2k}$ and $W^{\bar{1}} = \bigoplus_{k=0}^{\infty} W^{2k+1}$.) By the simplicial de Rham theorem [35, Theorem 4.3], the total cohomology of the basic subcomplex computes the cohomology of the classifying space BG, with coefficients in $\mathbb{R}$. Define a product structure on the double complex,

$$C^{p,q} \otimes C^{p',q'} \to C^{p+p',q} \otimes C^{p+p',q'} \to C^{p+p',q+q'}$$

where the first map is given by $(-1)^{qp'}$ times the tensor products of pull-backs to the first $p+1$, respectively last $p'+1$, G-factors in $G^{p+p'+1}$, and the second map is wedge product. (This formula is motivated by the usual formula for cup products of the singular cochain complex.) It is straightforward to verify that D, ι_ξ, L_ξ are derivations for the product structure, thus W becomes a $\mathfrak{g} - \mathrm{da}$. It is locally free, with a natural connection

$$\theta : \mathfrak{g}^* \to C^{0,1} = \Omega^1(G)$$

given by the left-invariant Maurer-Cartan form on G. Hence, by symmetrization we obtain a $\mathfrak{g} - \mathrm{ds}$ homomorphism $W\mathfrak{g} \to W$ which restricts to a map of basic subcomplexes. The resulting map

$$(S\mathfrak{g}^*)_{\mathrm{inv}} \to \bigoplus_{p,q} \Omega^q(G^{p+1})_{\mathrm{basic}} \cong \bigoplus_{p,q} \Omega^q(G^p) \tag{27}$$

takes an invariant polynomial of degree r to a D-cocycle of total degree $2r$. By our general theory, the induced map in cohomology is a ring homomorphism. As in the usual Bott-Shulman construction we have the following vanishing phenomenon:

Proposition 9.1. *The image of an invariant polynomial of degree r under the map* (27) *has non-vanishing components only in bidegrees (p, q) with $p + q = 2r$ and $p \le r$.*

Proof. The connection θ lives in bidegree $(0, 1)$, and its total differential $D\theta$ has non-vanishing components only in bi-degrees $(0, 2)$ and $(1, 1)$. Since the product structure is compatible with the bi-grading, it follows that the image of an element $\overline{\xi}_{i_1} \cdots \overline{\xi}_{i_l} \xi_{j_1} \cdots \xi_{j_m} \in W\mathfrak{g}$ under the symmetrization map only involves bi-degrees (p, q) with $p+q = 2l+m$ and $p \le l$. Any element in $S^r\mathfrak{g}^* \subset W\mathfrak{g}$ is a linear combination of such elements, with $2l + m = 2r$. □

In the appendix, we will describe a class of $\mathfrak{g} - \mathrm{da}$'s with the connection, with the property that the Chern-Weil homomorphism c^θ is a $\mathfrak{g}$-homotopy equivalence. The following result says that this applies to the $\mathfrak{g} - \mathrm{da}$ W, provided G is compact.

Let $W^L \subset W$ denote the direct sum over the subspaces $(C^{p,q})^L = \Omega^q(G^{p+1})^L$ of forms that are invariant under the left G^{p+1}-action. Clearly, W^L is a $\mathfrak{g}$-differential subalgebra of W. Since the connection is left-invariant, the Chern-Weil map takes values in the subalgebra W^L_{basic}.

Theorem 9.2. *The algebras W^L, W are acyclic. In fact, W^L is a $\mathfrak{g} - \mathrm{da}$ of Weil type in the sense of Definition A.1. If G is compact, the inclusion $W^L \hookrightarrow W$ a $\mathfrak{g}$-homotopy equivalence.*

Proof. We recall the standard proof that W is acyclic. Let $\Pi : W \to W$ be the projection operator, given on $C^{p,q}$ as pull-back under the map

$$\pi : G^{p+1} \to G^{p+1},\ (g_0, \dots, g_p) \mapsto (e, \dots, e).$$

The projection Π is naturally chain homotopic to the identity: To construct a homotopy operator, let

$$s_j : G^{p+1} \to G^{p+2},\quad (g_0, \dots, g_p) \mapsto (g_0, \dots, g_j, e, \dots, e),\quad i = 0, \dots, p$$

and set $s = \sum_{j=0}^{p} (-1)^j s_j^* : C^{p+1,q} \to C^{p,q}$. Note that on G^{p+1}, $\partial_{p+1} s_p = \mathrm{id}$, $\partial_0 s_0 = \pi$. A direct calculation (as in May [33]) shows that $[\delta, s] = \Pi - \mathrm{id}$ on $\bigoplus_{p=0}^{\infty} \Omega^q(G^{p+1})$, for any fixed q. Since $[\mathrm{d}, s] = 0$, it follows that $-(-1)^q s : \Omega^q(G^{p+1}) \to \Omega^q(G^{p+2})$ gives the desired homotopy between the identity and Π.

The image $\Pi(W) \subset W$ is isomorphic to the singular cochain complex of a point. Hence, composing with the standard homotopy operator for this complex we see that the inclusion $i : \mathbb{F} \hookrightarrow W$ is a homotopy equivalence. Let $h : W \to W$ denote the homotopy operator constructed in this way.

The map s does not commute with L_ξ since the maps s_j are not G-equivariant. However, on the left invariant subcomplex W^L this problem disappears, since $s_j^* \circ L_\xi = L_\xi \circ s_j^*$ on W^L. It follows that h restricts to a homotopy operator on W^L with $[h, L_\xi] = 0$. This shows that W^L is of Weil type.

Suppose now that G is compact. Then there is a projection $\Pi_1 : W \to W^L$, given on $\Omega^q(G^{p+1})$ as the averaging operator for the left G^{p+1}-action. It is well-known that the averaging operator is homotopic to the identity operator; the homotopy operator h_1 may be chosen to commute with L_ξ, by averaging under the right G-action. (In our case, one may directly construct h_1 using the Hodge decomposition for the bi-invariant Riemannian metric on G.) It follows that $\Pi|_{W^L} \circ \Pi_1$ is homotopic to the identity map, by a homotopy operator that commutes with L_ξ and lowers the total degree by 1. $\square$

Remark 9.3. For the classical groups, there is another model for differential forms on the classifying bundle, as an inverse limit of differential forms on Stiefel manifolds – e.g. for $G = \mathrm{U}(k)$, the inverse limit of $\Omega(\mathrm{St}(k, n))$ for $n \to \infty$. The resulting $\mathfrak{g} - \mathrm{da}$ carries a natural "universal" connection (see Narasimhan-Ramanan [36]). The characteristic homomorphism for this case was studied by Kumar in [29].

Appendix A. $\mathfrak{g}$-differential algebras of Weil type

In this Appendix we describe a class of locally free $\mathfrak{g} - \mathrm{da}$'s that are homotopy equivalent to $W\mathfrak{g}$. This notion of a $\mathfrak{g} - \mathrm{da}$'s of *Weil type* will not be needed for most of our applications, except in Section 9.

Definition A.1. Let W be a locally free $\mathfrak{g} - \mathrm{da}$, together with a ds homomorphism $\pi : W \to \mathbb{F}$ such that $\pi \circ i = \mathrm{id}$, where $i : \mathbb{F} \to W$ is the unit for W. Then W will be called *of Weil type* if there exists a homotopy operator h between $i \circ \pi$ and id, with $h \circ i = 0$, such that $[L_\xi, h] = 0$ and such that h and all ι_ξ have degree < 0 with respect to some filtration

$$W = \bigcup_{N \geq 0} W^{(N)},\ W^{(0)} \subset W^{(1)} \subset \cdots$$

of W.

The definition is motivated by ideas from Guillemin-Sternberg [18, Section 4.3]. In most examples, the $\mathbb{Z}_2$-grading on W is induced from a $\mathbb{Z}$-grading $W = \bigoplus_{n\geq 0} W^n$, $\mathbb{F} \subset W^0$, with d of degree $+1$, L_ξ of degree 0 and and ι_ξ of degree -1. If a homotopy operator h with $[L_\xi, h] = 0$ exists, its part of degree -1 is still a homotopy operator, and has the required properties. (Note that compatibility of the grading with the algebra structure is not needed.) In particular, the Weil algebra $W\mathfrak{g}$, and its non-commutative version $\tilde{W}\mathfrak{g}$ are of Weil type. The tensor product of two $\mathfrak{g}-$da's of Weil type is again of Weil type.

Theorem A.2. *Suppose W is a $\mathfrak{g}-\mathrm{da}$ of Weil type, and θ a connection on W. Then the $\mathfrak{g}-\mathrm{da}$-homomorphism $\tilde{c}^\theta : \tilde{W}\mathfrak{g} \to W$ is a $\mathfrak{g}$-homotopy equivalence.*

In particular, any two $\mathfrak{g}-$da's of Weil type are $\mathfrak{g}$-homotopy equivalent. A similar result was proved by Guillemin-Sternberg [18] for *commutative* $\mathfrak{g}-$da's.

By Theorem A.2, many of the usual properties of $W\mathfrak{g}$ extend to $\mathfrak{g}-$da's of Weil type. For instance, since the basic cohomology of $W\mathfrak{g}$ is $(S\mathfrak{g}^*)_{\mathrm{inv}}$, the same is true for any $\mathfrak{g}-$da of Weil type. The proof of Theorem A.2 will be given at the end of this Section, after establishing two rather technical Lemmas.

Recall that a linear operator C on a vector space E is *locally nilpotent* if $E = \bigcup_{N\geq 0} E^{(N)}$ where $E^{(N)}$ is the kernel of C^{N+1}. If C is locally nilpotent, the operator $I+C$ has a well-defined inverse, since the geometric series $I - C + C^2 - C^3 \pm \cdots$ is finite on each $E^{(N)}$.

Lemma A.3. *Suppose E is a $\mathfrak{g}$-differential space, and $h \in \mathrm{End}(E)^{\bar{1}}$ an odd linear operator with the following properties:*

$$[\iota_\xi, h] = 0,\ [L_\xi, h] = 0,\ [d, h] = I - \Pi + C$$

where Π is a projection operator, and C is locally nilpotent. Assume that h and C vanish on the range of Π. Then $\tilde{\Pi} = \Pi(I + C)^{-1}$ is a projection operator having the same range as Π. Furthermore, it is a $\mathfrak{g}-\mathrm{ds}$ homomorphism, and $\tilde{h} = h(I + C)^{-1}$ is a $\mathfrak{g}$-homotopy between I and $\tilde{\Pi}$:

$$[\iota_\xi, \tilde{h}] = 0,\ [L_\xi, \tilde{h}] = 0,\ [d, \tilde{h}] = I - \tilde{\Pi}.$$

Proof. Since $C\Pi = 0$, it is clear that $\tilde{\Pi}$ is a projection operator. We will check $[\mathrm{d}, \tilde{h}] = I - \tilde{\Pi}$ on any given $v \in E$. Choose N sufficiently large so that $v \in E^{(N)}$ and $\mathrm{d}v \in E^{(N)}$. On $E^{(N)}$, the operator $\tilde{h}$ is a finite series

$$\begin{aligned} \tilde{h} &= h(I - C + C^2 - \cdots + (-1)^N C^N) \\ &= h(I - (C - \Pi) + (C - \Pi)^2 + \cdots + (-1)^N (C - \Pi)^N), \end{aligned}$$

where we have used $h\Pi = 0$ and $C\Pi = 0$. The equation $[\mathrm{d}, h] = I - \Pi + C$ shows that $C - \Pi$ commutes with d. Thus

$$[\mathrm{d}, \tilde{h}] = [\mathrm{d}, h](I - (C - \Pi) + (C - \Pi)^2 + \cdots + (-1)^N (C - \Pi)^N)$$

on the subspace $\{v \in E^{(N)} |\, \mathrm{d}v \in E^{(N)}\}$. But $[\mathrm{d}, h] = I - \Pi + C$ also shows $[\mathrm{d}, h]\Pi = 0$. Hence we may replace $C - \Pi$ by C again, and get

$$\begin{aligned} [\mathrm{d}, \tilde{h}] &= [\mathrm{d}, h](I - C + C^2 + \cdots) \\ &= [\mathrm{d}, h](I + C)^{-1} \\ &= (I - \Pi + C)(I + C)^{-1} = I - \tilde{\Pi}. \end{aligned}$$

By a similar argument, since $[L_\xi, C - \Pi] = [L_\xi, [h, \mathrm{d}]] = 0$ and $[\iota_\xi, C - \Pi] = -[\iota_\xi, [h, \mathrm{d}]] = -[h, L_\xi] = 0$, one proves $[L_\xi, \tilde{h}] = 0$ and $[\iota_\xi, \tilde{h}] = 0$. □

Lemma A.4. *Suppose $\mathcal{A}$ is a $\mathfrak{g} - \mathrm{da}$, with augmentation map $\Pi : \mathcal{A} \to \mathbb{F} \subset \mathcal{A}$. Assume that $h : \mathcal{A} \to \mathcal{A}$ is a homotopy operator with $[L_\xi, h] = 0$ and $[d, h] = I - \Pi$, and that there exists an increasing filtration $\mathcal{A} = \bigcup_{N=0}^\infty \mathcal{A}^{(N)}$, such that ι_ξ and h have negative filtration degree. Then, for any locally free $\mathfrak{g}$-differential space $\mathcal{B}$, the inclusion map*

$$\mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$$

is a $\mathfrak{g}$-homotopy equivalence, with a homotopy inverse $\mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ that is equal to the identity on $\mathcal{B} \subset \mathbb{F} \otimes \mathcal{B}$.

Proof. The proof is inspired by an argument of Guillemin-Sternberg (see [18, Theorem 4.3.1]). If $[\iota_\xi^{\mathcal{A}}, h] = 0$, the projection operator $\Pi = \Pi \otimes I$ is the desired homotopy inverse to the inclusion map, with homotopy $h = h \otimes 1$. In the general case, we employ the Kalkman trick [22] to shift the contraction operators on $\mathcal{A} \otimes \mathcal{B}$ to the second factor. Let e_a be a basis of $\mathfrak{g}$, and e^a the dual basis of $\mathfrak{g}^*$. Choose a connection $\theta : \mathfrak{g}^* \to \mathcal{B}$. Then $\psi = \sum_a \theta(e^a)\iota_{e_a}^{\mathcal{A}}$ is an even, nilpotent operator on $\mathcal{A} \otimes \mathcal{B}$, with $[\psi, L_\xi^{\mathcal{A}} + L_\xi^{\mathcal{B}}] = 0$. Hence $\exp\psi$ is a well defined automorphism of $\mathcal{A} \otimes \mathcal{B}$, and commutes with the $\mathfrak{g}$-action. Let

$$\tilde{L}_\xi = \mathrm{Ad}(\exp\psi)(L_\xi^{\mathcal{A}} + L_\xi^{\mathcal{B}}), \quad \tilde{\iota}_\xi = \mathrm{Ad}(\exp\psi)(\iota_\xi^{\mathcal{A}} + \iota_\xi^{\mathcal{B}}), \quad \tilde{\mathrm{d}} = \mathrm{Ad}(\exp\psi)(\mathrm{d}^{\mathcal{A}} + \mathrm{d}^{\mathcal{B}})$$

denote the transformed Lie derivatives, contractions and differential on $\mathcal{A} \otimes \mathcal{B}$. A calculation using $\mathrm{Ad}(\exp\psi) = \exp(\mathrm{ad}\,\psi) = \sum_{n=0}^\infty \frac{1}{n!}(\mathrm{ad}\,\psi)^n$ shows

$$\begin{aligned} \tilde{\iota}_\xi &= \iota_\xi^{\mathcal{B}} \\ \tilde{L}_\xi &= L_\xi^{\mathcal{A}} + L_\xi^{\mathcal{B}} \\ \tilde{\mathrm{d}} &= \mathrm{d}^{\mathcal{A}} + \mathrm{d}^{\mathcal{B}} + \theta(e^a)L_{e_a}^{\mathcal{A}} + R \end{aligned}$$

where the remainder term R is a polynomial in contractions $\iota_{e_a}^{\mathcal{A}}$, with coefficients in $\mathcal{B}$ (with no constant term). The operator h commutes with the contractions $\tilde{\iota}_\xi$ and Lie derivatives $\tilde{L}_\xi$, and

$$[\tilde{\mathrm{d}}, h] = [\mathrm{d}^{\mathcal{A}}, h] + [R, h] = I - \Pi + [R, h].$$

Our assumptions imply that $C = [R, h]$ has negative filtration degree, and in particular is locally nilpotent. Thus Lemma A.3 applies and shows that $\tilde{\Pi} = \Pi(I + C)^{-1}$ is a projection operator with the same range $\mathbb{F} \otimes \mathcal{B}$, and is homotopic to the identity by a homotopy $\tilde{h} = h(I + C)^{-1}$ which commutes with all $\tilde{\iota}_\xi$, $\tilde{L}_\xi$. Since $\iota_\xi^{\mathcal{A}}$ vanishes on $\mathbb{F}$, the operator $\exp\psi$ acts trivially on $\mathbb{F} \otimes \mathcal{B}$ and therefore $\widehat{\Pi} = \tilde{\Pi} \circ \exp(\psi)$ is a projection onto $\mathbb{F} \otimes \mathcal{B}$. The operator $\widehat{h} = \mathrm{Ad}(\exp(-\psi))\tilde{h}$ gives the desired $\mathfrak{g}$-homotopy between $\widehat{\Pi}$ and the identity. □

Proof of Theorem A.2. The map $\tilde{c}^\theta$ may be written as a composition of the inclusion

$$\tilde{W}\mathfrak{g} \hookrightarrow \tilde{W}\mathfrak{g} \otimes W,\ \tilde{w} \mapsto \tilde{w} \otimes 1 \tag{28}$$

with the map $\phi : \tilde{W}\mathfrak{g} \otimes W \to W,\ \tilde{w} \otimes w \mapsto \tilde{c}^\theta(\tilde{w})w$. The map ϕ is a $\mathfrak{g}$-homotopy equivalence by Proposition 3.2, while (28) is a $\mathfrak{g}$-homotopy equivalence by the Lemma A.4. □

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