THE ATIYAH ALGEBROID OF THE PATH FIBRATION OVER A LIE GROUP

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Abstract. Let G be a connected Lie group, LG its loop group, and $\pi\colon PG \to G$ the principal LG-bundle defined by quasi-periodic paths in G. This paper is devoted to differential geometry of the Atiyah algebroid $A = T(PG)/LG$ of this bundle. Given a symmetric bilinear form on $\mathfrak{g}$ and the corresponding central extension of $L\mathfrak{g}$, we consider the lifting problem for A, and show how the cohomology class of the Cartan 3-form $\eta \in \Omega^3(G)$ arises as an obstruction. This involves the construction of a 2-form $\varpi \in \Omega^2(PG)^{LG} = \Gamma(\wedge^2 A^*)$ with $\mathrm{d}\varpi = \pi^*\eta$. In the second part of this paper we obtain similar LG-invariant primitives for the higher degree analogues of the form η, and for their G-equivariant extensions.

Contents

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1. Introduction

Let G be a connected Lie group with loop group LG. Denote by $\pi\colon PG \to G$ the principal LG-bundle, given by the set of 'quasi-periodic' paths in G. Thus $\gamma \in C^\infty(\mathbb{R}, G)$ belongs to the fiber $(PG)_g$ if it has the property,

$$\gamma(t+1) = g\gamma(t)$$

for all t. The principal action of $\lambda \in LG$ reads $(\lambda.\gamma)(t) = \gamma(t)\lambda(t)^{-1}$; it commutes with the action of $a \in G$ given as $(a.\gamma)(t) = a\gamma(t)$.

We are interested in the differential geometry of the infinite-dimensional space $PG \to G$. Since all of our considerations will be LG-equivariant, it is convenient to phrase this discussion in terms of the *Atiyah algebroid* $A = T(PG)/LG \to G$. As explained below, the fiber of A at $g \in G$ consists of paths $\xi \in C^\infty(\mathbb{R}, \mathfrak{g})$ such that $\xi(t+1) - \mathrm{Ad}_g\, \xi(t) =: v_\xi$ is constant. We may directly write down the Lie algebroid bracket on sections of A, thus avoiding a discussion of Lie brackets of vector fields on infinite dimensional spaces. The Lie algebra bundle $L \subset A$, given as the kernel of the anchor map, has fibers the *twisted loop algebras* defined by the condition $\xi(t+1) = \mathrm{Ad}_g\, \xi(t)$.

An invariant symmetric bilinear form on $\mathfrak{g}$ defines a central extension $\widehat{L} \to L$ by the trivial bundle $G \times \mathbb{R}$. One may then ask for a lift $\widehat{A} \to A$ of the Atiyah algebroid to this central extension. More generally, we will study a similar lifting problem for any transitive Lie algebroid A over a manifold M. We will show that the choice of a connection on A, together with a 'splitting', define an element $\varpi \in \Gamma(\wedge^2 A^*)$ whose Lie algebroid differential is basic. The latter defines a closed 3-form $\eta \in \Omega^3(M)$, whose cohomology class turns out to be the obstruction to the lifting problem. In the case of the Atiyah algebroid over G, with suitable choice of connection, η is the Cartan 3-form, while ϖ is explicitly given as

$$\varpi(\xi,\zeta)=\int_0^1 \dot{\xi}\cdot\zeta-\frac{1}{2}v_\xi\cdot v_\zeta-\operatorname{Ad}_g\xi(0)\cdot v_\zeta$$

for $\xi,\zeta\in\Gamma(A)$. Similarly, the obstruction for the G-equivariant lifting problem is the equivariant Cartan 3-form η_G, while it turns out that $\varpi_G=\varpi$. Note that ϖ may be viewed as a $G\times LG$-invariant 2-form on PG.

The second part of this paper is devoted to 'higher analogues' of the equations $\mathrm{d}\varpi=\pi^*\eta$, respectively $\mathrm{d}_G\varpi=\pi^*\eta_G$. For any invariant polynomial $p\in(S\mathfrak{g}^*)^G$ of homogeneous degree k, the Bott-Shulman construction [2, 10, 13] defines closed forms

$$\eta^p\in\Omega^{2d-1}(G),\quad \eta^p_G\in\Omega^{2d-1}_G(G).$$

These become exact if pulled back to elements of $\Gamma(\wedge A^*)$, and we will construct explicit primitives

$$\varpi^p\in\Gamma(\wedge^{2d-2}A^*),\quad \varpi^p_G\in\Gamma_G(\wedge^{2d-2}A^*).$$

which may be viewed as LG-invariant differential forms on PG. We stress that while the existence of primitives of $\pi^*\eta^p$, $\pi^*\eta^p_G$ is fairly obvious, the existence of *LG-invariant* primitives is less evident. Pulling ϖ^p back to the fiber over the identity $LG=(PG)_e$, one recovers the closed invariant forms on the loop group LG discussed in Pressley-Segal [12].

2. Review of transitive Lie algebroids

In this Sections we collect some basic facts about connections and curvature on transitive Lie algebroids. Most of this material is due to Mackenzie, and we refer to his book [11] or to the lecture notes by Crainic-Fernandes [7] for further details.

2.1. **Lie algebroids.** A Lie algebroid is a smooth vector bundle $A\to N$, with a Lie bracket on the space of sections $\Gamma(A)$ and an *anchor map* $\mathsf{a}\colon A\to TN$ satisfying the Leibniz rule, $[\xi_1,f\xi_2]_A=f[\xi_1,\xi_2]_A+\mathsf{a}(\xi_1)(f)\,\xi_2$. This implies that a induces a Lie algebra homomorphism on sections. An example of a Lie algebroid is the Atiyah algebroid TP/H of a principal H-bundle $P\to N$, where $\Gamma(TP/H)=\mathfrak{X}(P)^H$ with the usual bracket of vector fields. A *representation* of a Lie algebroid A on a vector bundle $\mathcal{V}\to N$ is given by a *flat A-connection* on $\mathcal{V}$, i.e. by a $C^\infty(N)$-linear Lie algebra homomorphism $\Gamma(A)\to\operatorname{End}(\Gamma(\mathcal{V}))$, $\xi\mapsto\nabla_\xi$ satisfying the Leibnitz rule, $\nabla_\xi(f\sigma)=f\nabla_\xi\sigma+\mathsf{a}(\xi)(f)\sigma$. Given additional structure on $\mathcal{V}$ one can ask for the representation to preserve that structure: For instance, if $\mathcal{V}=L$ is a bundle of Lie algebras, one would impose that ∇_ξ acts by derivations of the bracket $[\cdot,\cdot]_L$. Tensor products and direct sums of A-representations are defined in the obvious way. The *trivial* A-representation is the bundle $\mathcal{V}=N\times\mathbb{R}$ with $\nabla_\xi=\mathsf{a}(\xi)$ given by the anchor map.

Suppose $\mathcal{V}$ is an A-representation, and consider the graded $\Gamma(\wedge A^*)$-module $\Gamma(\wedge A^* \otimes \mathcal{V})$. Generalizing from $A = TM$, we will think of the sections of $\wedge A^* \otimes \mathcal{V}$ as *$\mathcal{V}$-valued forms* on A. For $\xi \in \Gamma(A)$, the *Lie derivatives* $\mathcal{L}_\xi$ are the operators of degree 0 on $\Gamma(\wedge A^* \otimes \mathcal{V})$, defined inductively by

$$\iota_\zeta \circ \mathcal{L}_\xi = \mathcal{L}_\xi \circ \iota_\zeta - \iota_{[\xi,\zeta]_A}, \quad \zeta \in \Gamma(A),$$

with $\mathcal{L}_\xi \sigma = \nabla_\xi \sigma$ for $\sigma \in \Gamma(\mathcal{V})$. Here ι_ξ are the operators of *contraction* by ξ. Similarly, d is the operator of degree 1 on $\Gamma(\wedge A^* \otimes \mathcal{V})$ defined by Cartan's identity $\iota_\xi \circ \mathrm{d} = \mathcal{L}_\xi - \mathrm{d} \circ \iota_\xi$. The operators $\iota_\xi, \mathcal{L}_\xi, \mathrm{d}$ satisfy the usual commutation relations of contractions, Lie derivative and differential. In particular, d squares to zero.

2.2. **Transitive Lie algebroids.** A Lie algebroid A over N is called *transitive* if its anchor map $\mathsf{a}\colon A \to TN$ is surjective. In that case, the kernel of the anchor map is a bundle $L \to N$ of Lie algebras, and we have the exact sequence of Lie algebroids,

$$0 \to L \to A \to TN \to 0. \tag{1}$$

The *structure Lie algebra bundle* L carries an A-representation $\nabla_\xi \zeta = [\xi, \zeta]$ $(\xi \in \Gamma(A),\ \zeta \in \Gamma(L))$ by derivations of the Lie bracket.

Example 2.1. The Atiyah algebroid $A = TP/H$ of a principal bundle is a transitive Lie algebroid, with L the associated bundle of Lie algebras $L = P \times_H \mathfrak{h}$. The bracket on $\Gamma(A)$ is given by its identification with H-invariant vector fields on P. The induced bracket on $\Gamma(L)$ is *minus* the pointwise bracket on $C^\infty(P, \mathfrak{h})^H \cong \Gamma(L)$.

The dual $\mathsf{a}^*\colon T^*N \to A^*$ of the anchor map extends to the exterior algebras. Given an A-representation $\mathcal{V}$, it hence gives an injective map $\mathsf{a}^*\colon \wedge T^*N \otimes \mathcal{V} \to \wedge A^* \otimes \mathcal{V}$, defining a map on sections,

$$\mathsf{a}^*\colon \Omega(N, \mathcal{V}) \cong \Gamma(\wedge T^*N \otimes \mathcal{V}) \to \Gamma(\wedge A^* \otimes \mathcal{V}).$$

The image of this map is the *horizontal subspace* $\Gamma(\wedge A^* \otimes \mathcal{V})_{\mathrm{hor}}$, consisting of sections ϕ satisfying $\iota_\xi \phi = 0$ for all $\xi \in \Gamma(L)$. We will often view $\Omega(N, \mathcal{V})$ as a subspace of $\Gamma(\wedge A^* \otimes \mathcal{V})$, without always spelling out the inclusion map a^*. The *basic subcomplex* $\Gamma(\wedge A^* \otimes \mathcal{V})_{\mathrm{basic}}$ is the subspace of horizontal sections satisfying $\mathcal{L}_\xi \phi = 0$ for all $\xi \in \Gamma(L)$; it is stable under the differential d.

Lemma 2.2. *Suppose that the A-connection on $\mathcal{V}$ descends to a flat $TN = A/L$-connection, i.e. that $\nabla_\xi = 0$ for $\xi \in \Gamma(L)$. Then*

$$\Gamma(\wedge A^*, \mathcal{V})_{\mathrm{basic}} \cong \Gamma(\wedge A^*, \mathcal{V})_{\mathrm{hor}} \cong \Omega(N, \mathcal{V}).$$

Proof. Let $\xi \in \Gamma(L)$ so that $\mathsf{a}(\xi) = 0$, and let $\phi \in \Gamma(\wedge^k A^*, \mathcal{V})_{\mathrm{hor}}$. We will show $\mathcal{L}_\xi \phi = 0$ by induction on k. If $k = 0$, we have $\mathcal{L}_\xi \phi = \nabla_\xi \phi = 0$ by assumption. If $k > 0$, the induction hypothesis shows that for all $\zeta \in \Gamma(A)$, $\iota_\zeta \mathcal{L}_\xi \phi = \mathcal{L}_\xi \iota_\zeta \phi - \iota_{[\xi,\zeta]_A} \phi = 0$, hence $\mathcal{L}_\xi \phi = 0$. Here we used that $\Gamma(\wedge A^*, \mathcal{V})_{\mathrm{hor}}$ is stable under ι_ζ and that $[\xi, \zeta]_A \in \Gamma(L)$. $\square$

Remark 2.3. Lemma 2.2 applies in particular to the trivial A-representation $\mathcal{V} = N \times \mathbb{R}$. Thus $\Gamma(\wedge A^*)_{\mathrm{basic}} \cong \Omega(N)$. For general A-representations the space $\Gamma(\wedge A^* \otimes \mathcal{V})_{\mathrm{basic}}$ can be strictly smaller than $\Omega(N, \mathcal{V})$. For instance, if $N = \mathrm{pt}$, so that $A = \mathfrak{k}$ is a Lie algebra and $\mathcal{V} = V$ is a $\mathfrak{k}$-representation, the space $\Gamma(\wedge A^* \otimes \mathcal{V})_{\mathrm{basic}} = V^{\mathfrak{k}}$ is the space of $\mathfrak{k}$-invariants, while $\Omega(N, \mathcal{V}) = V$.

A *connection* on a transitive Lie algebroid is a left splitting $\theta\colon A \to L$ of the exact sequence (1). The corresponding right splitting $\mathrm{Hor}^\theta\colon TN \to A$ is called the *horizontal lift*. Dually, the connection defines a horizontal projection

$$\mathrm{Hor}^\theta_*\colon \Gamma(\wedge A^* \otimes \mathcal{V}) \to \Gamma(\wedge A^* \otimes \mathcal{V})_{\mathrm{hor}}.$$

One defines the *covariant derivative* by $\mathrm{d}^\theta = \mathrm{Hor}^\theta_* \circ \mathrm{d}$, and the *curvature* of θ is given as [1]

$$F^\theta = \mathrm{d}^\theta \theta = \mathrm{d}\theta - \frac{1}{2}[\theta, \theta]_A \in \Gamma(\wedge^2 A^* \otimes L)_{\mathrm{hor}} \tag{2}$$

2.3. **Pull-backs.** We recall the notion of *pull-back Lie algebroids*, due to Higgins-Mackenzie [9], for the special case of transitive Lie algebroids. Suppose $A \to N$ is a transitive Lie algebroid, and $\Phi\colon M \to N$ is a smooth map. Let $\Phi^! A \to M$ be the bundle [2] over M, defined by the fiber product diagram

$$\begin{array}{ccc}
\Phi^! A & \longrightarrow & A \\
\big\downarrow & & \big\downarrow{\scriptstyle a} \\
TM & \xrightarrow[d\Phi]{} & TN
\end{array}$$

That is, $\Phi^! A = (d\Phi)^* A$ if A is viewed as a bundle over TN. Then $\Phi^! A$ carries a natural structure of a transitive Lie algebroid, with the left vertical map $\Phi^! A \to TM$ as the anchor map, and the upper horizontal map is a morphism of Lie algebroids.

We refer to $\Phi^! A$ as the pull-back of A by the map Φ. It is a pull-back in the category of Lie algebroids, not to be confused with the pull-back $\Phi^* A$ of A as a vector bundle. For instance, taking $A = TN$ one has $\Phi^! TN = TM \neq \Phi^* TN$. Note that if $A = TP/H$ is the Atiyah algebroid of a principal H-bundle $P \to N$, then $\Phi^! A = T(\Phi^* P)/H$ is the Atiyah algebroid of the pull-back principal bundle.

The kernel of the anchor map of $\Phi^! A$ is $\Phi^* L$, the usual pull-back as a bundle of Lie algebras. For any A-representation $\mathcal{V}$, the pull-back $\Phi^* \mathcal{V}$ inherits a $\Phi^! A$-representation, and there is a natural cochain map $\Phi^!\colon \Gamma(\wedge A^* \otimes \mathcal{V}) \to \Gamma(\wedge(\Phi^! A) \otimes \Phi^* \mathcal{V})$. Given a connection $\theta\colon A \to L$, the pull-back algebroid inherits a pull-back connection $\Phi^! \theta\colon \Phi^! A \to \Phi^* L$. The curvature of the pull-back connection is $F^{\Phi^! \theta} = \Phi^! F^\theta$.

[1] The minus sign in this formula is consistent with Example 2.1.

[2] We remark that our use of the notation $\Phi^!$ is different from that in the book [11].

2.4. **Equivariant transitive Lie algebroids.** Suppose G is a Lie group acting on $A \to N$ by Lie algebroid homomorphisms. By *infinitesimal generators* for the action we mean a G-equivariant map

$$\mathfrak{g} \to \Gamma(A),\ x \mapsto x_A \tag{3}$$

with the property $[x_A, \xi] = \frac{\partial}{\partial u}\Big|_{u=0} \exp(ux).\xi$. It is then automatic that (3) is a Lie algebra homomorphism. For any G-equivariant A-representation $\mathcal{V}$, the complex $\wedge A^* \otimes \mathcal{V}$ becomes a G-differential space (cf. [8, 3]), with contraction operators $\iota_x = \iota_{x_A}$. One may hence introduce the equivariant complex

$$\Gamma_G(\wedge A^* \otimes \mathcal{V}) := (S\mathfrak{g}^* \otimes \Gamma(\wedge A^* \otimes \mathcal{V}))^G$$

with differential $\mathrm{d}_G = 1 \otimes \mathrm{d} - \sum_j e^j \otimes \iota_{e_j}$ for a basis e_j of $\mathfrak{g}$, with dual basis e^j of $\mathfrak{g}^*$. For $A = TN$ this complex is denoted $\Omega_G(N, \mathcal{V})$. Replacing d with d_G in the discussion above, one may introduce equivariant curvatures F_G^θ for G-invariant connections on A:

$$F_G^\theta = \mathrm{d}_G\theta - \frac{1}{2}[\theta,\theta]_A = F^\theta - \Psi \in \Gamma_G(\wedge^2 A^* \otimes L)_{\mathrm{hor}}.$$

Here $\Psi(x) = \iota_x\theta \in \Gamma(\wedge^0 A^* \otimes L)$ for $x \in \mathfrak{g}$.

3. A lifting problem for transitive Lie algebroids

Let H be a Lie group, and $\pi\colon P \to N$ a smooth principal H-bundle. Given a central extension $\widehat{H} \to H$ of the structure group by $\mathrm{U}(1)$, it is not always possible to lift P to a principal $\widehat{H}$-bundle. As is well-known, the obstruction class is an element of $H^3(N, \mathbb{Z})$. A construction of Brylinski [4] gives an explicit de Rham representative of the image of this class in $H^3(N, \mathbb{R})$. In this Section, we will develop the analogue of Brylinski's theory for transitive Lie algebroids.

3.1. **The lifting problem.** Let $A \to N$ be a transitive Lie algebroid with anchor map $\mathsf{a}\colon A \to TN$, and with structure Lie algebra bundle $L = \ker(\mathsf{a})$. Suppose that

$$0 \to N \times \mathbb{R} \to \widehat{L} \xrightarrow{p} L \to 0 \tag{4}$$

is a central extension, where $\widehat{L}$ carries an A-representation (by derivations of the Lie bracket on sections), lifting that on L. The lifting problem is to find a central extension of Lie algebroids

$$0 \to N \times \mathbb{R} \to \widehat{A} \xrightarrow{p} A \to 0 \tag{5}$$

such that $\widehat{L}$ is realized the kernel of the anchor map $\widehat{A} \to TN$. We may also consider the lifting problem for a given connection $\theta\colon A \to L$, where we declare that $(\widehat{A}, \widehat{\theta})$ lifts (A, θ) if $\widehat{A}$ lifts A and $p \circ \widehat{\theta} = \theta \circ p$.

Example 3.1 (Principal bundles I). In the principal bundle case, $A = TP/H$ is the Atiyah algebroid, $L = P \times_H \mathfrak{h}$, and one obtains a lifting problem $\widehat{L} = P \times_H \widehat{\mathfrak{h}}$ for any given central extension $0 \to \mathbb{R} \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0$ of Lie algebras. Suppose these integrate to an exact sequence $1 \to \mathrm{U}(1) \to \widehat{H} \to H \to 1$ on the group level. Then for any principal $\widehat{H}$-bundle $\widehat{P}$ lifting P, its Atiyah algebroid $\widehat{A} = T\widehat{P}/\widehat{H}$ is a lift of A in the above sense.

3.2. **Splittings.** The set of splittings $j\colon L \to \widehat{L}$ of the exact sequence (4) is an affine space, with underlying vector space $\Gamma(L^*)$. Any splitting j determines a cocycle $\sigma \in \Gamma(\wedge^2 L^*)$, where

$$\sigma(\xi,\zeta) = j([\xi,\zeta]_L) - [j(\xi), j(\zeta)]_{\widehat{L}}, \quad \xi,\zeta \in \Gamma(L).$$

(The right hand side lies in the kernel of p, hence it takes values in the trivial bundle $N \times \mathbb{R} \subset \widehat{L}$.) The bracket on $\widehat{L}$ is given in terms of this cocycle as

$$[\hat{\xi},\hat{\zeta}]_{\hat{L}} = j([\xi,\zeta]_L) - \sigma(\xi,\zeta) \tag{6}$$

where $\xi = p(\hat{\xi}),\ \zeta = p(\hat{\zeta})$. Let $\theta \in \Gamma(A^* \otimes L)$ be a principal connection, and consider the covariant derivative of $j \in \Gamma(\wedge^0 A^* \otimes (L^* \otimes \widehat{L}))$.

Proposition 3.2. *Both dj and $d^\theta j$ map to 0 under p. Thus*

$$dj \in \Gamma(\wedge^1 A^* \otimes L^*), \quad d^\theta j \in \Gamma(\wedge^1 A^* \otimes L^*)_{\mathrm{hor}}.$$

One has $d^\theta j = dj + \sigma(\theta,\cdot)$. The differential of σ is related to the differential of j by

$$(d\sigma)(\xi_1,\xi_2) = \langle dj,\ [\xi_1,\xi_2]_L\rangle$$

for $\xi_1,\xi_2 \in \Gamma(L)$.

Proof. The first claim follows since $p(j) = \mathrm{id}_L \in L^* \otimes L$, hence $p(\mathrm{d}j) = \mathrm{d}p(j) = 0$. To prove the formula for $\mathrm{d}^\theta j$ we compute, for $\xi \in \Gamma(L)$ and $\zeta \in \Gamma(A)$,

$$\iota_\zeta \langle \mathrm{d}j, \xi\rangle = \langle \mathcal{L}_\zeta j, \xi\rangle = \mathcal{L}_\zeta(j(\xi)) - j(\mathcal{L}_\zeta \xi).$$

For $\zeta \in \Gamma(L)$ the right hand side is equal to $-\sigma(\zeta,\xi)$, and we obtain $\iota_\zeta(\mathrm{d}j + \sigma(\theta,\cdot)) = 0$. This shows that $\mathrm{d}j + \sigma(\theta,\cdot)$ is horizontal. On the other hand, it is obvious that $\mathrm{d}j$ and $\mathrm{d}j + \sigma(\theta,\cdot)$ agree on horizontal vectors. Now let $\xi_1,\xi_2 \in \Gamma(L)$ and $\zeta \in \Gamma(A)$. We compute, using $(\mathcal{L}_\zeta j)(\xi_i) = (\iota_\zeta \mathrm{d}j)(\xi_i) \in C^\infty(N) \subset \Gamma(\widehat{L})$,

$$\begin{aligned}
\iota_\zeta(\mathrm{d}\sigma)(\xi_1,\xi_2) &= (\mathcal{L}_\zeta\sigma)(\xi_1,\xi_2)\\
&= \mathcal{L}_\zeta(\sigma(\xi_1,\xi_2)) - \sigma(\mathcal{L}_\zeta\xi_1,\xi_2) - \sigma(\xi_1, L_\zeta\xi_2)\\
&= \mathcal{L}_\zeta\big(j([\xi_1,\xi_2]_L) - [j(\xi_1), j(\xi_2)]_{\widehat{L}}\big) - j([\mathcal{L}_\zeta\xi_1,\xi_2]_L) + [j(\mathcal{L}_\zeta\xi_1), j(\xi_2)]_{\widehat{L}}\\
&\quad - j([\xi_1,\mathcal{L}_\zeta\xi_2]_L) + [j(\xi_1), j(\mathcal{L}_\zeta\xi_2)]_{\widehat{L}}\\
&= (\mathcal{L}_\zeta j)([\xi_1,\xi_2]_L) - [(\mathcal{L}_\zeta j)(\xi_1), j(\xi_2)]_{\widehat{L}} - [j(\xi_1), (\mathcal{L}_\zeta j)(\xi_2)]_{\widehat{L}}\\
&= (\mathcal{L}_\zeta j)([\xi_1,\xi_2]_L) = \iota_\zeta(\mathrm{d}j)([\xi_1,\xi_2]_L).
\end{aligned}$$

Hence $(\mathrm{d}\sigma)(\xi_1,\xi_2) = (\mathrm{d}j)([\xi_1,\xi_2]_L)$. □

3.3. **The form ϖ.** Let $F^{j(\theta)} \in \Gamma(\wedge^2 A^* \otimes \widehat{L})$ be the curvature-like expression,

$$F^{j(\theta)} = \mathrm{d}(j(\theta)) - \frac{1}{2}[j(\theta), j(\theta)]_{\widehat{L}}.$$

Since $p(F^{j(\theta)}) = F^\theta$ the difference

$$\varpi := F^{j(\theta)} - j(F^\theta)$$

is scalar-valued, i.e. it is an element of $\Gamma(\wedge^2 A^*)$.

Proposition 3.3. *The 2-form $\varpi \in \Gamma(\wedge^2 A^*)$ is given by the formula,*

$$\varpi = \langle dj, \theta\rangle + \frac{1}{2}\sigma(\theta,\theta) = \langle d^\theta j, \theta\rangle - \frac{1}{2}\sigma(\theta,\theta). \tag{7}$$

Its differential is basic, so that $d\varpi = \mathsf{a}^*\eta$ for a closed 3-form $\eta \in \Omega^3(N)$. We have

$$\mathsf{a}^*\eta = -\langle d^\theta j, F^\theta\rangle. \tag{8}$$

The contractions of ϖ with $\xi \in \Gamma(L)$ are given by $\iota_\xi \varpi = -\langle dj, \xi\rangle$.

Proof. We compute:

$$\begin{aligned}
\varpi &= \mathrm{d}(j(\theta)) - j(\mathrm{d}\theta) - \frac{1}{2}\big([j(\theta), j(\theta)]_{\widehat{L}} - j([\theta,\theta]_L)\big)\\
&= \langle \mathrm{d}j, \theta\rangle + \frac{1}{2}\sigma(\theta,\theta)\\
&= \langle \mathrm{d}^\theta j, \theta\rangle - \frac{1}{2}\sigma(\theta,\theta),\\
\mathrm{d}\varpi &= -\langle \mathrm{d}j, \mathrm{d}\theta\rangle - \sigma(\theta, \mathrm{d}\theta) + \frac{1}{2}(\mathrm{d}\sigma)(\theta,\theta)\\
&= -\langle \mathrm{d}^\theta j, \mathrm{d}\theta\rangle + \frac{1}{2}(\mathrm{d}\sigma)(\theta,\theta)\\
&= -\langle \mathrm{d}^\theta j, F^\theta\rangle.
\end{aligned}$$

Here we have used $\sigma(\theta, [\theta,\theta]) = 0$ and Proposition 3.2. Since $\mathrm{d}\varpi \in \Gamma(\wedge^3 A^*)$ is horizontal, by Lemma 2.2 it is also basic. Hence, it is an image of a unique closed 3-form $\eta \in \Omega^3(N)$ under the map a^*.

Finally, for the contractions of ϖ with elements $\xi \in \Gamma(L)$ we find,

$$\iota_\xi \varpi = -\langle \mathrm{d}^\theta j, \xi\rangle + \sigma(\theta, \xi) = -\langle \mathrm{d}j, \xi\rangle.$$

∎

The next Proposition describes the dependence of η on the choice of splitting and connection.

Proposition 3.4. *Let $j' = j + \beta$ be a new splitting, where $\beta \in \Gamma(L^*)$, and $\theta' = \theta + \lambda$ a new connection, with $\lambda \in \Gamma(A^* \otimes L)_{\mathrm{hor}}$. Then $\eta' - \eta = d\gamma$ where $\gamma \in \Omega^2(N)$ is given by the following element of $\Gamma(\wedge^2 A^*)_{\mathrm{basic}}$,*

$$\mathsf{a}^*\gamma = \langle d^\theta j, \lambda\rangle + \frac{1}{2}\sigma(\lambda,\lambda) - \langle \beta, F^\theta + d^\theta\lambda\rangle + \frac{1}{2}\beta([\lambda,\lambda]_L).$$

In particular, the cohomology class $[\eta] \in H^3(N,\mathbb{R})$ is independent of the choices of j, θ.

Proof. From its defining formula, we see that the cocycle σ changes by a coboundary: $\sigma'(\xi_1,\xi_2) = \sigma(\xi_1,\xi_2) + \beta([\xi_1,\xi_2]_L)$. Hence,

$$\varpi' = \langle \mathrm{d}j', \theta'\rangle + \frac{1}{2}\sigma(\theta',\theta') + \frac{1}{2}\beta([\theta',\theta']_L).$$

First, consider terms which do not involve β:

$$\langle \mathrm{d}j, \theta+\lambda\rangle + \frac{1}{2}\sigma(\theta+\lambda,\theta+\lambda) = \varpi + \langle \mathrm{d}^\theta j, \lambda\rangle + \frac{1}{2}\sigma(\lambda,\lambda).$$

The remaining terms may be written as

$$\begin{aligned}\langle \mathrm{d}\beta, \theta+\lambda\rangle + \frac{1}{2}\beta([\theta+\lambda,\theta+\lambda]_L) &= \mathrm{d}\langle\beta,\theta+\lambda\rangle - \langle\beta, \mathrm{d}\theta + \mathrm{d}\lambda\rangle + \frac{1}{2}\beta([\theta+\lambda,\theta+\lambda]_L)\\ &= \mathrm{d}\langle\beta,\theta+\lambda\rangle - \langle\beta, F^\theta + \mathrm{d}^\theta\lambda\rangle + \frac{1}{2}\beta([\lambda,\lambda]_L).\end{aligned}$$

Hence, $\mathrm{d}(\varpi' - \varpi) = \mathrm{d}\sigma$, where

$$\sigma = \langle \mathrm{d}^\theta j, \lambda\rangle + \frac{1}{2}\sigma(\lambda,\lambda) - \langle\beta, F^\theta + \mathrm{d}^\theta\lambda\rangle + \frac{1}{2}\beta([\lambda,\lambda]_L).$$

Since $\sigma \in \Gamma(\wedge^2 A^*)_{\mathrm{hor}}$, by Lemma 2.2 it is basic, and we conclude that $\gamma \in \Omega^2(N)$ defined by equality $\mathsf{a}^*\gamma = \sigma$ satisfies $\eta' - \eta = \mathrm{d}\gamma$. □

3.4. **The cohomology class $[\eta]$ as an obstruction class.** We will now show that the cohomology class of η is precisely the obstruction class for our lifting problem.

Theorem 3.5. *Suppose that θ is a connection on A and $j\colon L \to \widehat{L}$ is a splitting. Then there is a 1-1- correspondence between:*

(a) *isomorphism classes of lifts $(\widehat{A}, \widehat{\theta})$ of the data (A,θ), and*
(b) *2-forms $\omega \in \Omega^2(N)$ such that $d\omega = -\eta$.*

It follows that $[\eta] = 0$ precisely if the lifting problem (4), (5) admits a solution.

Proof. We first show how to construct a solution of the lifting problem, provided η is exact. Let $A = L \oplus TN$ be the decomposition defined by the connection θ, and put

$$\widehat{A} := \widehat{L} \oplus TN$$

with the obvious projection $p\colon \widehat{A} \to A$, with the connection $\hat{\theta}$ the projection to the first summand, and with anchor map $\widehat{a}$ the projection to the second summand. Let $j_A = j \oplus \mathrm{id}_{TN}\colon A \to \widehat{A}$. We want to consider Lie brackets $[\cdot,\cdot]_{\hat{A}}$ on $\Gamma(\widehat{A})$, extending the bracket on $\Gamma(\widehat{L})$, and such that p induces a Lie algebra homomorphism $\Gamma(\widehat{A}) \to \Gamma(A)$. If $\hat{\zeta} \in \Gamma(\hat{L})$, we have

$$-[\hat{\zeta},\hat{\xi}]_{\hat{A}} = [\hat{\xi},\hat{\zeta}]_{\hat{A}} = \mathcal{L}_\xi\hat{\zeta}, \quad \xi = p(\hat{\xi})$$

using the A-representation on $\hat{L}$. Hence, the bracket is determined if one of the entries is a section of $\widehat{L}$. Consequently we only need to specify the bracket on horizontal sections. For $X, Y \in \mathfrak{X}(N)$ these brackets will have the form

$$\begin{aligned}[\mathrm{Hor}^{\hat{\theta}}(X), \mathrm{Hor}^{\hat{\theta}}(Y)]_{\hat{A}} &= j_A([\mathrm{Hor}^{\theta}(X), \mathrm{Hor}^{\theta}(Y)]_A) - \omega(X,Y)\\ &= \mathrm{Hor}^{\hat{\theta}}([X,Y]) + j(F^{\theta}(X,Y)) - \omega(X,Y)\end{aligned}$$

for some 2-form $\omega \in \Omega^2(N)$. Having defined the bracket in this way, consider the Jacobi identity

$$[\hat{\xi}_1, [\hat{\xi}_2, \hat{\xi}_3]_{\hat{A}}]_{\hat{A}} + \text{cycl.} = 0.$$

If $\hat{\xi}_3$ lies in $\Gamma(\widehat{L})$, this identity is equivalent to the representation property, $\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2}\hat{\xi}_3 - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}\hat{\xi}_3 - \mathcal{L}_{[\xi_1,\xi_2]}\hat{\xi}_3 = 0$, where $\xi_i = p(\hat{\xi}_i)$. Hence, the Jacobi identity is automatic if one of the entries lies in $\widehat{L}$. It remains to consider the case that $\xi_i = \mathrm{Hor}^{\hat{\theta}}(X_i)$ for $i = 1, 2, 3$. Separating terms according to the decomposition $\widehat{A} = (N \times \mathbb{R}) \oplus A$, we have,

$$\begin{aligned}&[\mathrm{Hor}^{\hat{\theta}}(X_1), [\mathrm{Hor}^{\hat{\theta}}(X_2), \mathrm{Hor}^{\hat{\theta}}(X_3)]_{\hat{A}}]_{\hat{A}}\\ &= -\mathcal{L}_{X_1}\omega(X_2, X_3) + \omega(X_1, [X_2, X_3]) + \langle d^{\theta} j(X_1),\ F^{\theta}(X_2, X_3)\rangle + \cdots\end{aligned}$$

where $\cdots$ indicates sections of $j_A(A) \cong 0 \oplus A$. So the scalar part of the Jacobi identity reads,

$$-\mathcal{L}_{X_1}\omega(X_2, X_3) + \omega(X_1, [X_2, X_3]) + \langle d^{\theta} j(X_1),\ F^{\theta}(X_2, X_3)\rangle + \text{cycl.} = 0$$

Equivalently, $\mathrm{d}\omega - \langle \mathrm{d}^{\theta} j, F^{\theta}\rangle = 0$. By Equation (8), we have $\mathsf{a}^*\eta = -\langle \mathrm{d}^{\theta} j, F^{\theta}\rangle$. We hence conclude that the bracket $[\cdot,\cdot]_{\hat{A}}$ defined by ω is a Lie bracket if and only if $\mathrm{d}\omega = -\eta$.

Conversely, if $p\colon \widehat{A} \to A$ is a solution of the lifting problem, choose a connection $\widehat{\theta}$ lifting θ. This gives a splitting $\widehat{A} = \widehat{L} \oplus TN$ lifting the splitting of A. Define ω as the scalar component of the bracket on $\Gamma(\hat{A})$, restricted to horizontal sections. The calculation above shows that $\mathrm{d}\omega = -\eta$. □

Example 3.6 (Principal bundles II). This is a continuation of Example 3.1, where we considered the lifting problem for a principal H-bundle $P \to N$. It was shown by Brylinski that $[\eta] \in H^3(N, \mathbb{R})$ is the image of the obstruction class under the coefficient homomorphism $H^3(N, \mathbb{Z}) \to H^3(N, \mathbb{R})$. Given a solution $\widehat{P} \to N$ of the lifting problem, the connection $\theta \in \Gamma(A^* \otimes L) \cong \Omega^1(P, \mathfrak{h})^H$ is an ordinary principal connection on P, and $\widehat{\theta}$ is a lift of θ to $\widehat{P}$.

3.5. **The equivariant lifting problem.** Suppose now that the sequence (4) is G-equivariant, with the trivial action on the bundle $N \times \mathbb{R}$, and that the A-representation on $\Gamma(\widehat{L})$ is G-equivariant. We may then consider the G-equivariant version of the lifting problem: Thus, we are looking for a G-equivariant lift $\widehat{A} \to A$, such that the action on $\widehat{A}$ has infinitesimal generators $x_{\widehat{A}}$ satisfying $p(x_{\widehat{A}}) = x_A$.

Suppose there is a G-equivariant splitting j of the sequence (4), and a G-invariant connection θ on A. Replacing d with the equivariant differential in the discussion above, we find that the 2-form ϖ coincides with its equivariant extension. Its equivariant differential is $\mathrm{d}_G\varpi = \mathsf{a}^*\eta_G$ where $\mathsf{a}^*\eta_G = -\langle \mathrm{d}^\theta j, F^\theta_G\rangle$. Thus

$$\mathsf{a}^*(\eta_G - \eta) = \langle \mathrm{d}^\theta j, \Psi\rangle$$

where $\Psi \in \Omega^0(N, L)$ was defined in 2.4. To address the equivariant lifting problem, we use the notation from the proof of Theorem 3.5. Let $\widehat{A} = \widehat{L} \oplus TN$ carry the diagonal G-action, the map $p\colon \widehat{A} \to A$ is G-equivariant. The bracket $[\cdot,\cdot]_{\hat{A}}$ defined by ω with $\mathrm{d}\omega = -\eta$, is G-invariant provided that ω is G-invariant.

Theorem 3.7. *Let (A, θ, x_A) be a G-equivariant transitive Lie algebroid, with invariant connection θ and with equivariant generators x_A. Let $\widehat{L} \to L$ be a G-equivariant central extension, together with a G-equivariant splitting j, defining $\eta_G \in \Omega^3_G(N)$ as above. Then there is a 1-1 correspondence between*

(a) *isomorphism classes of equivariant lifts $(\hat{A}, \hat{\theta}, x_{\hat{A}})$ of the data (A, θ, x_A).*
(b) *equivariant 2-forms $\omega_G \in \Omega^2_G(N)$ such that $d_G\omega_G = -\eta_G$.*

Proof. To describe generators for the action, it suffices to describe their scalar part. Thus write

$$x_{\widehat{A}} = j(x_A) + \Phi(x)$$

for some G-equivariant map $\Phi\colon N \to \mathfrak{g}^*$. Then $x_{\widehat{A}}$ are generators for the $\mathfrak{g}$-action if and only if the map $x \mapsto x_{\hat{A}}$ defines the $\mathfrak{g}$-representation on $\hat{A}$, i.e.

$$[x_{\hat{A}}, \hat{\xi}]_{\hat{A}} = \mathcal{L}_{x_A}\hat{\xi}$$

for $\hat{\xi} \in \Gamma(\hat{A})$. For $\hat{\xi} \in \Gamma(\widehat{L})$, this property is automatic. It is hence enough to consider the condition

$$[x_{\hat{A}}, \mathrm{Hor}^{\hat{\theta}}(X)]_{\hat{A}} = \mathcal{L}_{x_A}\mathrm{Hor}^{\hat{\theta}}(X) = \mathrm{Hor}^{\hat{\theta}}([x_N, X]).$$

Writing $x_{\hat{A}} = j_A(x_A) + \Phi(x) = \mathrm{Hor}^{\hat{\theta}}(x_N) + j(\Psi(x)) + \Phi(x)$, we find,

$$\begin{aligned}
&[x_{\hat{A}}, \mathrm{Hor}^{\hat{\theta}}(X)]_{\hat{A}} - \mathrm{Hor}^{\hat{\theta}}([x_N, X]) \\
&= j(F^\theta(x_N, X)) - \omega(x_N, X) - \mathcal{L}_X\Phi(x) + \mathcal{L}_{\mathrm{Hor}^\theta(X)}j(\Psi(x)) \\
&= -\omega(x_N, X) - \iota_X\mathrm{d}\Phi(x) + \langle \mathrm{d}^\theta j(X), \Psi(x)\rangle + \ldots
\end{aligned}$$

where $\ldots$ indicates sections of $j_A(A) = 0 \oplus A$. The $\ldots$ terms have to cancel (by considering their image under p), hence we obtain the condition

$$\omega(x_N, \cdot) + \mathrm{d}\Phi(x) = \langle \mathrm{d}^\theta j(X), \Psi(x)\rangle.$$

Since $\mathsf{a}^*\eta_G = \mathsf{a}^*\eta + \langle \mathrm{d}^\theta j(X), \Psi\rangle$, this is the component of form degree 1 of the equation $\mathrm{d}_G\omega_G = -\eta_G$, where $\omega_G = \omega - \Phi$ is an equivariant extension of ω. $\square$

If G is compact (so that invariant connections, splittings etc. can be obtained by averaging), it follows that $[\eta_G] \in H^3_G(N,\mathbb{R})$ is precisely the obstruction for the equivariant lifting problem.

Below, we will encounter situations where $\Phi = 0$, so that ω coincides with its equivariant extension. Equivalently, $j(x_A)$ are generators for the action. Here is a first example.

Example 3.8. Suppose $N = G/K$ for a compact subgroup K. Since $H_G(N,\mathbb{R}) = H_K(\mathrm{pt},\mathbb{R})$ vanishes in odd degrees, the class $[\eta_G]$ is necessarily trivial. Moreover, it is easy to see that $\eta_G = -\mathrm{d}_G\omega$ for a unique invariant 2-form ω. Hence, one obtains a solution of the lifting problem with $x_{\hat{A}} = j(x_A)$.

3.6. **Relation with Courant algebroids.** In the previous sections, we explained how the lifting problem for a transitive Lie algebroid defines an obstruction class in $H^3(N,\mathbb{R})$. By a well-known result of Ševera, the group $H^3(N,\mathbb{R})$ classifies exact Courant algebroids over N. We will explain now how to give a direct description of this Courant algebroid. As before, we start out by choosing a connection θ on A, as well as a splitting j. These data define a 2-form $\varpi \in \Gamma(\wedge^2 A^*)$, such that $\mathrm{d}\varpi = \mathsf{a}^*\eta$ is basic.

Let $A \oplus A^*$ carry the symmetric bilinear form extending the pairing between A and A^*, and the standard Courant bracket,

$$[\![(v_1,\alpha_1),\ (v_2,\alpha_2)]\!] = ([v_1,v_2]_A,\ \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2}\mathrm{d}\alpha_1).$$

Proposition 3.9. *The map $f\colon L \to A\oplus A^*$, $\xi \mapsto (\xi, \iota_\xi\varpi)$ defines an isotropic L-action on $A\oplus A^*$. That is, its image is isotropic, and the induced map on sections preserves brackets.*

Proof. Since $\mathrm{d}\varpi$ is basic, we have $\mathrm{d}\iota_\xi\varpi = L_\xi\varpi$. Hence

$$\begin{aligned}
[\![f(\xi_1), f(\xi_2)]\!] &= [\![(\xi_1,\iota_{\xi_1}\varpi),(\xi_2,\iota_{\xi_2}\varpi)]\!]\\
&= ([\xi_1,\xi_2]_A,\ L_{\xi_1}\iota_{\xi_2}\varpi - \iota_{\xi_2}\mathrm{d}\iota_{\xi_1}\varpi)\\
&= ([\xi_1,\xi_2]_A,\ \iota_{[\xi_1,\xi_2]_A}\varpi) = f([\xi_1,\xi_2]_A).
\end{aligned}$$

The property $\langle f(\xi), f(\xi)\rangle = 0$ is straightforward. $\square$

As in Bursztyn-Cavalcanti-Gualtieri [5] we may consider the reduction of A by the isotropic L-action.

Proposition 3.10. *The reduced Courant algebroid $f(L)^\perp/f(L)$ is canonically isomorphic to $TN \oplus T^*N$ with the η-twisted Courant bracket.*

Proof. Let $f\colon A \to A\oplus A^*$, $v \mapsto (v, \iota_v\varpi)$ be the obvious extension of the action map. Then $f(L)^\perp = f(A) + T^*N$, where T^*N is embedded as the annihilator of L in A^*, and hence $f(L)^\perp/f(L) = f(A)/f(L)\oplus T^*N = TN\oplus T^*N$. For $v_1, v_2 \in \Gamma(A)$ and if $\alpha_1,\alpha_2 \in \Gamma(T^*N) \cong \Gamma(A^*)_{\mathrm{basic}}$ we have

$$\begin{aligned}
[\![f(v_1)+\alpha_1, f(v_2)+\alpha_2]\!] &= ([v_1,v_2]_A, \mathcal{L}_{v_1}\iota_{v_2}\varpi - \iota_{v_2}\mathrm{d}\iota_{v_1}\varpi + L_{v_1}\alpha_2 - \iota_{v_2}\mathrm{d}\alpha_1)\\
&= ([v_1,v_2]_A,\ \iota_{[v_1,v_2]_A}\varpi + \iota_{v_2}\iota_{v_1}\mathsf{a}^*\eta + L_{v_1}\alpha_2 - \iota_{v_2}\mathrm{d}\alpha_1)\\
&= f([v_1,v_2]_A) + \iota_{v_2}\iota_{v_1}\mathsf{a}^*\eta + L_{v_1}\alpha_2 - \iota_{v_2}\mathrm{d}\alpha_1.
\end{aligned}$$

This shows that the Courant bracket on $\Gamma(f(L)^\perp/f(L))$ is the η-twisted Courant bracket on $TN \oplus T^*N$. $\square$

4. The Atiyah algebroid $A \to G$

4.1. **The bundle of twisted loop algebras.** Let G be a Lie group. For $g \in G$ define the twisted loop algebra

$$L_g = \{\xi \in C^\infty(\mathbb{R}, \mathfrak{g}) |\ \xi(t+1) = \mathrm{Ad}_g\, \xi(t)\},$$

with bracket $[\xi_1, \xi_2]_L(t) = -[\xi_1(t), \xi_2(t)]_{\mathfrak{g}}$ *minus* [3] the pointwise Lie bracket on $C^\infty(\mathbb{R}, \mathfrak{g})$. Let $L \to G$ be the Lie algebra bundle with fibers L_g. (The isomorphism type of the fiber L_g may depend on the connected component of G containing g.)

Remark 4.1. Let us discuss briefly the local triviality of L. Consider a connected component of G, with base point g_0. For any g in the same connected component, the choice of any path $\gamma = \gamma_g \in C^\infty([0,1], G)$ from $\gamma(0) = g_0$ to $\gamma(1) = g$, with γ constant near $t = 0, 1$, defines a Lie algebra isomorphism

$$L_{g_0} \to L_g,\ \xi \mapsto \tilde{\xi}$$

where $\tilde{\xi}(t) = \mathrm{Ad}_{\gamma(t)}\, \xi(t)$ for $t \in [0,1]$. One may take $\gamma_g(t)$ to depend smoothly on g, t (as g varies in a small open subset), thus obtaining local trivializations of L. The smooth sections of L are thus functions $\xi \in C^\infty(G \times \mathbb{R}, \mathfrak{g})$ satisfying $\xi(g, t+1) = \mathrm{Ad}_g\, \xi(g, t)$.

4.2. **The Lie algebroid $A \to G$.** Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan forms on G. We will work with the right trivialization of the tangent bundle, $TG \to G \times \mathfrak{g},\ X \mapsto \iota_X \theta^R$. Note that if $X, Y \in \mathfrak{X}(G)$ correspond to $v = \iota_X \theta^R,\ w = \iota_Y \theta^R$, then $[X, Y]$ corresponds to

$$\iota_{[X,Y]} \theta^R = -[v, w]_{\mathfrak{g}} + Xv - Yw$$

where the subscript $\mathfrak{g}$ indicates the pointwise bracket. For $g \in G$ let

$$A_g = \{\xi \in C^\infty(\mathbb{R}, \mathfrak{g}) |\ \exists v_\xi \in \mathfrak{g} \colon \xi(t+1) = \mathrm{Ad}_g\, \xi(t) + v_\xi\}.$$

We obtain an exact sequence

$$0 \to L_g \to A_g \xrightarrow{\mathsf{a}} T_g G \to 0$$

where the anchor map a is defined by $\iota_{\mathsf{a}(\xi)} \theta^R = v_\xi$. Let $A \to G$ be the bundle with fibers A_g.

Proposition 4.2. *The bundle A with anchor map a is a transitive Lie algebroid over G, with bracket on*

$$\Gamma(A) = \{\xi \in C^\infty(G \times \mathbb{R}, \mathfrak{g}) |\ \exists v_\xi \in C^\infty(G, \mathfrak{g}) \colon \xi(t+1) = \mathrm{Ad}_g\, \xi(t) + v_\xi\}.$$

given by

$$[\xi, \zeta]_A = -[\xi, \zeta]_{\mathfrak{g}} + X\zeta - Y\xi.$$

[3] The sign change will be convenient for what follows. It is related to the appearance of the minus sign in Example 3.1.

Here $X, Y \in \mathfrak{X}(G)$ *are determined by* $v_\xi = \iota_X \theta^R,\ v_\zeta = \iota_Y \theta^R$.

Proof. To show that a is surjective, fix $f \in C^\infty([0,1],\mathbb{R})$ with $f(0) = 0, f(1) = 1$, and constant near $t = 0, 1$. For $X \in T_g G$ let $\xi(t) = f(t)\iota_X\theta^R$ for $t \in [0,1]$. Then $\xi(1) - \xi(0) = \iota_X\theta^R$, so ξ extends uniquely to an element $\xi \in A_g$ with $\mathsf{a}(\xi) = X$. This argument also verifies that A is locally trivial, in fact $A \cong L \oplus TG$.

To check that $[\cdot,\cdot]_A$ preserves the space $\Gamma(A) \subset C^\infty(G\times\mathbb{R},\mathfrak{g})$, we calculate (at any given $g \in G$)

$$[\xi(t+1),\zeta(t+1)]_\mathfrak{g} = \mathrm{Ad}_g[\xi(t),\zeta(t)]_\mathfrak{g} + [\mathrm{Ad}_g\,\xi(t), v_\zeta]_\mathfrak{g} + [v_\xi, \mathrm{Ad}_g\,\zeta(t)]_\mathfrak{g} + [v_\xi, v_\zeta]_\mathfrak{g}.$$

On the other hand,

$$\begin{aligned}(X\zeta)(g,t+1) &= \frac{\partial}{\partial s}|_{s=0}\Big(\zeta\big(\exp(sv_\xi(g))g, t+1)\big)\Big) \\ &= \frac{\partial}{\partial s}|_{s=0}\Big(\mathrm{Ad}_{\exp(sv_\xi(g))g}\,\big(\zeta(\exp(sv_\xi(g))g,t)\big) + v_\zeta(\exp(sv_\xi(g))g)\Big) \\ &= [v_\xi, \mathrm{Ad}_g\,\zeta(t)]_\mathfrak{g} + \mathrm{Ad}_g(X\zeta)(t) + Xv_\zeta,\end{aligned}$$

with a similar expression for $(Y\xi)(g,t+1)$. This verifies

$$[\xi,\zeta]_A(t+1) = \mathrm{Ad}_g([\xi,\zeta]_A(t)) + v_{[\xi,\zeta]_\mathfrak{X}}.$$

This shows that $[\cdot,\cdot]_A$ takes values in A and also that $\mathsf{a}([\xi,\zeta]_A) = [\mathsf{a}(\xi),\mathsf{a}(\zeta)]$. It is straightforward to check that $[\cdot,\cdot]_A$ obeys the Jacobi identity. □

Proposition 4.3. *The Lie group G acts on $A \to G$ by Lie algebroid automorphisms covering the conjugation action on G. This action is given on sections $\xi \in \Gamma(A)$ by*

$$(k.\xi)(g,t) = \mathrm{Ad}_k\,\xi(\mathrm{Ad}_{k^{-1}}\,g, t),\ \ \xi \in \Gamma(A),\ k \in G,$$

and has infinitesimal generators

$$\mathfrak{g} \to \Gamma(A), \quad x \mapsto x_A = -x.$$

Proof. Let $X \in \mathfrak{X}(G)$ be the vector field such that $\iota_X\theta^R = v_\xi$, and let $k.X := (\mathrm{d}\,\mathrm{Ad}_k)(X)$ its push-forward under the conjugation action. Then

$$(\iota_{k.X}\theta^R)(g) = \mathrm{Ad}_k\,\big((\iota_X\theta^R)(\mathrm{Ad}_{k^{-1}}\,g)\big).$$

The following calculation shows that the action is well-defined, and that the anchor map is equivariant:

$$\begin{aligned}(k.\xi)(g,t+1) &= \mathrm{Ad}_k\,\big(\xi(\mathrm{Ad}_{k^{-1}}\,g, t+1)\big) \\ &= \mathrm{Ad}_k\,\big(\mathrm{Ad}_{k^{-1}gk}\,\xi(\mathrm{Ad}_{k^{-1}}\,g,t) + (\iota_X\theta^R)(\mathrm{Ad}_{k^{-1}}\,g)\big) \\ &= \mathrm{Ad}_g\,\big((k.\xi)(g,t)\big) + (\iota_{k.X}\theta^R)(g)\end{aligned}$$

It is straightforward to check that $k.[\xi,\zeta]_A = [k,\xi,k.\zeta]_A$. For $x \in \mathfrak{g}$ the generating vector field $x_G = x^L - x^R$ for the conjugation action on G satisfies

$\iota_{x_G}\theta^R = \mathrm{Ad}_g(x) - x$. Hence $x_A(g,t) := -x$ defines a section of A, with $\mathsf{a}(x_A) = x_G$. For all $\xi \in \Gamma(A)$,

$$[x_A, \xi]_A(g,t) = [x, \xi(g,t)]_{\mathfrak{g}} + x_G\xi = \frac{\partial}{\partial u}\Big|_{u=0}(\exp(ux).\xi)(g,t),$$

confirming that the map $x \mapsto x_A$ gives generators for the action. □

4.3. **The bundle $PG \to G$.** We will now interpret $A \to G$ as the Atiyah algebroid of a principal bundle over G. Suppose first that G is connected. Let $\pi\colon PG \to G$ be the bundle with fibers,

$$(PG)_g = \{\gamma \in C^\infty(\mathbb{R}, G)|\ \gamma(t+1) = g\gamma(t)\}.$$

By an argument similar to that for L_g, one sees that $PG \to G$ is a locally trivial bundle. In fact it is a principal bundle with fiber the loop group $LG = \pi^{-1}(e)$. We will argue that $A \to G$ may be regarded as the Atiyah algebroid of the principal LG-bundle $PG \to G$. Let $\gamma \in (PG)_g$. Given a family of paths $\gamma_s \in PG$ with $\gamma_0 = \gamma$, let $\zeta\colon \mathbb{R} \to \mathfrak{g}$ be defined as

$$\zeta(t) = \frac{\partial}{\partial s}|_{s=0}\big(\gamma_s(t)\gamma(t)^{-1}\big).$$

Put $g_s = \pi(\gamma_s)$, so that $g_s = \gamma_s(t+1)\gamma_s(t)^{-1}$ for all t. We may write $\gamma_s(t) = \exp(s\zeta_s(t))\gamma(t)$, so that $\zeta_0(t) = \zeta(t)$. Then

$$\begin{aligned} g_s &= \exp(s\zeta_s(t+1))\gamma(t+1)\gamma(t)^{-1}\exp(-s\zeta_s(t)) \\ &= \exp(s\zeta_s(t+1))g\exp(-s\zeta_s(t)). \end{aligned}$$

We find,

$$\frac{\partial}{\partial s}|_{s=0}\big(g_s g^{-1}\big) = \zeta(t+1) - \mathrm{Ad}_g\,\zeta(t).$$

This identifies A_g as the space of maps for which $\zeta(t+1) - \mathrm{Ad}_g\,\zeta(t)$ is constant. The formula for $[\cdot,\cdot]_A$ is the expected bracket on LG-invariant vector fields on PG. However, rather than attempting to construct Lie brackets of vector fields on infinite-dimensional manifolds, we will take this formula simply as a definition.

Remark 4.4. If G is disconnected, the condition $\gamma(t+1) = g\gamma(t)$ implies that g is in the identity component. One may however extend the definition, as follows: For any given component of G, pick a base point g_0, and take $(PG)_g$ (with g in the component of g_0 to consist of paths γ such that $\gamma(t+1) = g\gamma(t)g_0^{-1}$. Then $PG \to G$ is a principal L_{g_0}-bundle over the given component.

4.4. **Connections on $A \to G$.** Let us next discuss connections on the Atiyah algebroid over G. It will be convenient to describe θ in terms of the horizontal lift, $\mathrm{Hor}^\theta\colon TG \to A \subset C^\infty(\mathbb{R}, \mathfrak{g})$. Write $\mathrm{Hor}^\theta = -\alpha$, and think of α as a family of 1-forms $\alpha_t \in \Omega^1(G, \mathfrak{g})$.

Lemma 4.5. *A family of 1-forms α_t defines a horizontal lift $TG \to A$ if and only if*

$$\alpha_{t+1} = \mathrm{Ad}_g\, \alpha_t - \theta^R =: g \bullet \alpha_t. \tag{9}$$

Here $\bullet$ denotes the 'gauge action' of the identity map $g \in C^\infty(G,G)$ on $\Omega^1(G,\mathfrak{g})$. The resulting connection is G-equivariant if and only if $\alpha_t \in \Omega^1(G,\mathfrak{g})^G$.

Proof. The condition for $\mathrm{Hor}^\theta = -\alpha$ to define a horizontal lift is that for all $X \in \mathfrak{X}(G)$,

$$-\iota_X \alpha_{t+1} = -\mathrm{Ad}_g(\iota_X\alpha_t) + \iota_X\theta^R = -\iota_X(\mathrm{Ad}_g(\alpha_t) - \theta^R|_g)$$

for all such X. This gives the condition on α_t. It is clear that Hor^θ is G-equivariant exactly if α is G-equivariant. □

The connection $\theta = \theta^\alpha \colon A \to L$ defined by α is $\theta(\xi) = \xi + \alpha(\mathsf{a}(\xi))$. Let $F^{\alpha_t} = \mathrm{d}\alpha_t + \frac{1}{2}[\alpha_t, \alpha_t]_\mathfrak{g}$ be the curvature of α_t. By the property of the curvature under gauge transformations,

$$F^{\alpha_{t+1}} = F^{g\bullet\alpha_t} = \mathrm{Ad}_g\, F^{\alpha_t}.$$

Proposition 4.6. *The curvature $F^\theta \in \Omega^2(G,L)$ of the connection $\theta(\xi) = \xi + \alpha(\mathsf{a}(\xi))$ is given by*

$$F^\theta(X,Y)(t) = F^{\alpha_t}(X,Y), \quad X, Y \in \mathfrak{X}(G).$$

If α is G-invariant, then the corresponding map $\Psi\colon \mathfrak{g} \to \Gamma(L)$ (cf. 3.5) is

$$\Psi(x) = -x + \iota(x_N)\alpha.$$

Proof. This follows from the definition of the curvature in terms of horizontal lifts:

$$\begin{aligned} F^\theta(X,Y) &= \mathrm{Hor}_\theta([X,Y]) - [\mathrm{Hor}_\theta(X), \mathrm{Hor}^\theta(Y)]_A \\ &= -\alpha([X,Y]) - [\alpha(X), \alpha(Y)]_A \\ &= -\alpha([X,Y]) + [\alpha(X), \alpha(Y)]_\mathfrak{g} + X\alpha(Y) - Y\alpha(X) \\ &= (\mathrm{d}\alpha + \frac{1}{2}[\alpha,\alpha])(X,Y). \end{aligned}$$

If α is G-invariant, so that θ is G-equivariant, the map $\Psi(x) = -\iota_{x_A}\theta$ is given as

$$\Psi(x) = -\iota_{x_A}\theta = x_A - \mathrm{Hor}^\theta(x_N) = -x + \iota(x_N)\alpha.$$

□

To construct a family of 1-forms $\alpha_t \in \Omega^1(G,\mathfrak{g})$ with the transformation property (9), take any α_0 (for example $\alpha_0 = 0$), and put $\alpha_n = g^n \bullet \alpha_0$. Pick a smooth function $f\colon [0,1] \to \mathbb{R}$ such that $f(t) = 0$ near $t = 0$ and $f(t) = 1$ near $t = 1$, and let

$$\alpha_t = \alpha_n + f(t-n)(\alpha_{n+1} - \alpha_n) \tag{10}$$

for $n \le t \le n+1$. The resulting α_t is smooth, and has the desired transformation property. If $\alpha_0 \in \Omega^1(G, \mathfrak{g})$ is G-invariant, then

$$k^* \alpha_n = (\mathrm{Ad}_k(g))^n \bullet k^* \alpha_0 = (\mathrm{Ad}_k(g^n)) \bullet \mathrm{Ad}_k(\alpha_0) = \mathrm{Ad}_k(\alpha_n),$$

hence $\alpha_t \in \Omega^1(G, \mathfrak{g})$ is G-invariant for all t.

5. The lifting problem for $A \to G$

An invariant inner product on $\mathfrak{g}$ defines central extensions $\widehat{L}_g$ of the twisted loop algebras L_g. In this Section, we will work out the 2-form $\varpi \in \Gamma(\wedge^2 A^*)$ defined by the lifting problem, and discuss some of its properties.

5.1. **Central extensions.** Suppose the Lie algebra $\mathfrak{g}$ carries an invariant symmetric bilinear form $\cdot$ (possibly indefinite, or even degenerate). This then defines a central extension

$$0 \to G \times \mathbb{R} \to \widehat{L} \to L \to 0, \tag{11}$$

where $\widehat{L}_g = L_g \oplus \mathbb{R}$ with bracket,

$$[(\xi_1, s_1), (\xi_2, s_2)]_{\widehat{L}} = \Big(- [\xi_1, \xi_2]_{\mathfrak{g}},\ \int_0^1 \dot{\xi}_1 \cdot \xi_2 \Big).$$

Here $\dot{\xi} = \frac{\partial \xi}{\partial t}$, and the integral is relative to the measure $\mathrm{d}t$. The G-action on L lifts to an action on $\widehat{L}$, by $k.(\xi, s) = (k.\xi, s)$.

Proposition 5.1. *The representation of A on L (given by $\nabla_\xi \zeta = [\xi, \zeta]_A$) lifts to the Lie algebra bundle $\widehat{L}$, by the formula,*

$$\widehat{\nabla}_\xi(\zeta, s) = \Big(\nabla_\xi \zeta,\ \mathsf{a}(\xi) s + \int_0^1 \dot{\xi} \cdot \zeta \Big)$$

for $\xi \in \Gamma(A)$, $(\zeta, s) \in \Gamma(\widehat{L})$. This representation is equivariant relative to the G-actions on A, $\widehat{L}$.

Proof. We first verify that this formula defines an A-representation. Clearly $\xi \mapsto \widehat{\nabla}_\xi$ is $C^\infty(N)$-linear. For $\xi_1, \xi_2 \in \Gamma(A)$ and $\zeta \in \Gamma(L)$, we have

$$\int_0^1 [\xi_1, \dot{\xi}_2]_A \cdot \zeta = - \int_0^1 \dot{\xi}_2 \cdot [\xi_1, \zeta]_A + \mathsf{a}(\xi_1) \int_0^1 \dot{\xi}_2 \cdot \zeta,$$

by the definition of the bracket on A and the Ad-invariance of $\cdot$. Note that $\mathsf{a}(\dot{\xi}) = 0$ for all $\xi \in \Gamma(A)$. Subtracting a similar equation with $1 \leftrightarrow 2$ interchanged, one obtains

$$\int_0^1 \frac{\partial}{\partial t}([\xi_1, \xi_2]_A) \cdot \zeta = \mathsf{a}(\xi_1) \int_0^1 \dot{\xi}_2 \cdot \zeta - \mathsf{a}(\xi_2) \int_0^1 \dot{\xi}_1 \cdot \zeta + \int_0^1 (\dot{\xi}_1 \cdot \nabla_{\xi_2} \zeta - \dot{\xi}_2 \cdot \nabla_{\xi_1} \zeta)$$

which easily implies the property $\widehat{\nabla}_{\xi_1}\widehat{\nabla}_{\xi_2} - \widehat{\nabla}_{\xi_2}\widehat{\nabla}_{\xi_1} = \widehat{\nabla}_{[\xi_1,\xi_2]_A}$. We next check that this representation acts by derivations of the Lie bracket on $\Gamma(\widehat{L})$. We have

$$[\widehat{\nabla}_\xi(\zeta_1,s_1),(\zeta_2,s_2)]_{\widehat{L}} = \Big(-[\nabla_\xi\zeta_1,\zeta_2]_{\mathfrak{g}},\ \int_0^1 \big(-\frac{\partial}{\partial t}[\xi,\zeta_1]_{\mathfrak{g}} + \mathsf{a}(\xi)\dot{\zeta}_1\big)\cdot\zeta_2\Big),$$

$$[(\zeta_1,s_1),\widehat{\nabla}_\xi(\zeta_2,s_2)]_{\widehat{L}} = \Big(-[\zeta_1,\nabla_\xi\zeta_2]_{\mathfrak{g}},\ \int_0^1 \dot{\zeta}_1\cdot\big(-[\xi,\zeta_2]_{\mathfrak{g}} + \mathsf{a}(\xi)\zeta_2\big)\Big),$$

which adds up to

$$\widehat{\nabla}_\xi[(\zeta_1,s_1),(\zeta_2,s_2)]_{\widehat{L}} = \Big(-\nabla_\xi[\zeta_1,\zeta_2]_{\mathfrak{g}},\ \mathsf{a}(\xi)\int_0^1 \dot{\zeta}_1\cdot\zeta_2 - \int_0^1 \dot{\xi}\cdot[\zeta_1,\zeta_2]_{\mathfrak{g}}\Big)$$

as required. Equivariance of the action is clear. □

By definition, $\widehat{L}$ comes with the G-equivariant splitting $j\colon L\to\widehat{L},\ \xi\mapsto(\xi,0)$, with associated cocycle

$$\sigma(\xi_1,\xi_2) = -\int_0^1 \dot{\xi}_1\cdot\xi_2.$$

Let $\alpha_t\in\Omega^1(G,\mathfrak{g})$ be a family of 1-forms with the transformation property (9) and let $\theta^\alpha\colon A\to L$ the corresponding connection. Using the results from the last Section, we obtain a 2-form $\varpi^\alpha\in\Gamma(\wedge^2A^*)$ and a closed 3-form $\eta^\alpha\in\Omega^3(G)$, whose cohomology class is the obstruction to the existence of a lift $\widehat{A}$. If α is G-equivariant, we also obtain an equivariant extension η^α_G of the 3-form. We will now derive explicit formulas.

5.2. **The 2-form ϖ^α.** To begin, we need the covariant derivative $\mathrm{d}^{\theta^\alpha}j\in\Omega^1(G,L^*)$ of the splitting. Note that the derivative $\dot{\alpha}_t$ satisfies $\dot{\alpha}_{t+1}=\mathrm{Ad}_g\,\dot{\alpha}_t$, so it defines an element $\dot{\alpha}\in\Omega^1(G,L)$.

Lemma 5.2. *For $\zeta\in\Gamma(L)$ one has*

$$\langle \mathrm{d}^{\theta^\alpha}j,\zeta\rangle = -\int_0^1 \dot{\alpha}\cdot\zeta\ .$$

Proof. Recall that $\langle\mathrm{d}^{\theta^\alpha}j,\zeta\rangle = \langle\mathrm{d}j,\zeta\rangle + \sigma(\theta^\alpha,\zeta)$. For $\xi\in\Gamma(A)$ we compute

$$\iota_\xi\langle\mathrm{d}j,\zeta\rangle = \mathcal{L}_\xi j(\zeta) - j(\mathcal{L}_\xi\zeta) = \int_0^1 \dot{\xi}\cdot\zeta,$$

$$\sigma(\iota_\xi\theta^\alpha,\zeta) = \sigma(\xi+\iota_{\mathsf{a}(\xi)}\alpha,\zeta) = -\int_0^1 \iota_{\mathsf{a}(\xi)}\dot{\alpha}\cdot\zeta - \int_0^1\dot{\xi}\cdot\zeta. \qquad\square$$

Equation (7) together with this Lemma shows that

$$\varpi^\alpha = -\int_0^1 \dot{\alpha}\cdot\theta^\alpha - \frac{1}{2}\sigma(\theta^\alpha,\theta^\alpha).$$

It is convenient to introduce the forms $\kappa_t\in(\Gamma(A^*)\otimes\mathfrak{g})^G$,

$$\kappa_t(\xi) = -\xi_t,\quad \xi\in\Gamma(A).$$

Lemma 5.3. *The forms κ_t satisfy $F_G^{\kappa_t}(x)+x=0$, and*

$$\kappa_{t+1}=\mathrm{Ad}_g(\kappa_t)-\mathsf{a}^*\theta^R =: g\bullet\kappa_t.$$

Proof. We have

$$\mathrm{d}\kappa_t(\xi,\zeta)=-\kappa_t([\xi,\zeta]_A)+\mathsf{a}(\xi)\kappa_t(\zeta)-\mathsf{a}(\zeta)\kappa_t(\xi)=-[\xi,\zeta]_{\mathfrak{g}}=-\frac{1}{2}[\kappa_t,\kappa_t](\xi,\zeta).$$

This shows $F^{\kappa_t}=0$. Furthermore, for $x\in\mathfrak{g}$ we have $\iota(x_A)\kappa_t=-x$, by definition of x_A. Hence $F_G^{\kappa_t}(x)+x=0$. The transformation property $\kappa_{t+1}=\mathrm{Ad}_g\,\kappa_t-\mathsf{a}^*\theta^R$ follows from the definition of A. □

Let $Q^\alpha\in\Omega^2(G)$ be the 2-form (see Section A.1)

$$Q^\alpha=\frac{1}{2}\theta^L\cdot\alpha_0+\frac{1}{2}\int_0^1\alpha_t\cdot\dot{\alpha}_t, \tag{12}$$

and define $Q^\kappa\in\Gamma(\wedge^2A^*)$ by a similar expression, with α_t replaced by κ_t.

Proposition 5.4. *We have $\varpi^\alpha=\mathsf{a}^*Q^\alpha-Q^\kappa$.*

Proof. By definition, $\theta^\alpha=\mathsf{a}^*\alpha-\kappa$. To simplify notation, we omit the pull-back a^* in the following computation, i.e. we view $\Omega(G)$ as a subspace of $\Gamma(\wedge A^*)$:

$$\begin{aligned}
\varpi^\alpha&=-\int_0^1\dot{\alpha}\cdot(\alpha-\kappa)+\frac{1}{2}\int_0^1(\dot{\alpha}-\dot{\kappa})\cdot(\alpha-\kappa)\\
&=\frac{1}{2}\int_0^1\alpha\cdot\dot{\alpha}-\frac{1}{2}\int_0^1\kappa\cdot\dot{\kappa}-\frac{1}{2}\int_0^1\frac{\partial}{\partial t}(\kappa\cdot\alpha)\\
&=\frac{1}{2}\int_0^1\alpha\cdot\dot{\alpha}-\frac{1}{2}\int_0^1\kappa\cdot\dot{\kappa}-\frac{1}{2}(\mathrm{Ad}_g\,\kappa_0-\theta^R)\cdot(\mathrm{Ad}_g\,\alpha_0-\theta^R)+\frac{1}{2}\kappa_0\cdot\alpha_0\\
&=Q^\alpha-Q^\kappa.
\end{aligned}$$

□

Lemma 5.5. *For α as in* (10)*, one has*

$$Q^\alpha=\Big(\frac{\theta^L+\theta^R}{2}\Big)\cdot\alpha_0+\frac{1}{2}\alpha_0\cdot\mathrm{Ad}_g\,\alpha_0.$$

In particular, $Q^\alpha=0$ for $\alpha_0=0$.

Proof. By assumption, $\alpha_t=\alpha_0+f(t)(g\bullet\alpha_0-\alpha_0)$ for $0\le t\le 1$, where $f(0)=0$ and $f(1)=1$. Hence $\alpha_t\cdot\dot{\alpha}_t=\dot{f}\alpha_0\cdot(g\bullet\alpha_0)$, and therefore

$$\frac{1}{2}\int_0^1\alpha_t\cdot\dot{\alpha}_t=\frac{1}{2}\alpha_0\cdot(g\bullet\alpha_0)=\frac{1}{2}\alpha_0\cdot\mathrm{Ad}_g\,\alpha_0+\frac{1}{2}\theta^R\cdot\alpha_0.$$

Adding $\frac{1}{2}\theta^L\cdot\alpha_0$, the formula for Q^α follows. □

For the rest of this paper, we will write $\varpi:=-Q^\kappa\in\Gamma(\wedge^2A^*)$, that is

$$\varpi=-\frac{1}{2}\int_0^1\kappa\cdot\dot{\kappa}-\frac{1}{2}\mathsf{a}^*\theta^L\cdot\kappa_0. \tag{13}$$

Thus $\varpi^\alpha = \varpi$ for any choice of α with $Q^\alpha = 0$. More explicitly, for $\xi \in \Gamma(A)$ we have

$$\begin{aligned}
\varpi(\xi, \cdot) &= \frac{1}{2}\int_0^1 (\xi \cdot \dot{\kappa} - \dot{\xi} \cdot \kappa) - \frac{1}{2} v_\xi \cdot \mathrm{Ad}_g \kappa_0 - \frac{1}{2}\mathrm{Ad}_g \xi_0 \cdot \mathsf{a}^*\theta^R \\
&= -\int_0^1 \dot{\xi} \cdot \kappa + \frac{1}{2}(\xi_1 \cdot \kappa_1 - \xi_0 \cdot \kappa_0) - \frac{1}{2} v_\xi \cdot \mathrm{Ad}_g \kappa_0 - \frac{1}{2}\mathrm{Ad}_g \xi_0 \cdot \mathsf{a}^*\theta^R \\
&= -\int_0^1 \dot{\xi} \cdot \kappa - \mathrm{Ad}_g(\xi_0) \cdot \mathsf{a}^*\theta^R - \frac{1}{2} v_\xi \cdot \mathsf{a}^*\theta^R
\end{aligned}$$

Taking another contraction with $\zeta \in \Gamma(A)$,

$$\varpi(\xi, \zeta) = \int_0^1 \dot{\xi} \cdot \zeta - \mathrm{Ad}_g(\xi_0) \cdot v_\zeta - \frac{1}{2} v_\xi \cdot v_\zeta.$$

5.3. **The 3-form η^α.** Let $\eta \in \Omega^3(G)$ be the *Cartan 3-form* on G given as

$$\eta = \frac{1}{12}\theta^L \cdot [\theta^L, \theta^L] \in \Omega^3(G),$$

and let $\eta_G \in \Omega^3_G(G)$ be its equivariant extension

$$\eta_G(x) = \eta - \frac{1}{2}(\theta^L + \theta^R) \cdot x.$$

The 2-form $\varpi = -Q^\kappa \in \Gamma(\wedge^2 A^*)$ obeys

$$\mathrm{d}_G \varpi(x) = -\mathrm{d}_G Q^\kappa(x) = \mathsf{a}^*\eta_G(x) - \int_0^1 \dot{\kappa}_t \cdot (F_G^{\kappa_t}(x) + x) = \mathsf{a}^*\eta_G(x),$$

in particular $\mathrm{d}\varpi = \mathsf{a}^*\eta$. We obtain:

Theorem 5.6. *We have $d\varpi^\alpha = \mathsf{a}^*(\eta + dQ^\alpha)$, and if α is G-invariant, $d_G \varpi^\alpha = \mathsf{a}^*(\eta_G + d_G Q^\alpha)$. In particular, taking an invariant α with $Q^\alpha = 0$, the 2-form $\varpi \in \Gamma(\wedge^2 A^*)^G$ defined in* (13) *satisfies*

$$d_G \varpi = \mathsf{a}^*\eta_G.$$

6. Fusion

In this Section, we will study multiplicative properties of the Atiyah algebroid over G, and of the forms ϖ. We begin by introducing a (partial) multiplication on A, using concatenation of paths. Let $\xi' \in A_{g'},\ \xi'' \in A_{g''}$, with

$$\xi'_1 = \xi''_0.$$

The *concatenation* $\xi'' * \xi'$ is defined as follows:

$$(\xi'' * \xi')_t = \begin{cases} \xi'_{2t} & \text{if } 0 \le t \le \frac{1}{2} \\ \xi''_{2t-1} & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

extended to all t by the property,

$$(\xi'' * \xi')_{t+1} = \mathrm{Ad}_{g''g'}(\xi'' * \xi')_t + (\mathrm{Ad}_{g''} v_{\xi'} + v_{\xi''}).$$

This is consistent since, putting $t=0$,

$$\xi''_1 = \mathrm{Ad}_{g''}\,\xi''_0 + v_{\xi''} = \mathrm{Ad}_{g''g'}\,\xi'_0 + (\mathrm{Ad}_{g''}\,v_{\xi'} + v_{\xi''}).$$

Then $\xi'' * \xi' \in A_{g''g'}$ provided the concatenation is *smooth*. The concatenation is smooth if, for example, ξ'', ξ' are constant near $t=0$. Let

$$A^{[2]} \subset A \times A$$

be the sub-bundle of *composable paths*, with fiber at (g'', g') the set of pairs $(\xi'', \xi') \in A_{g''} \times A_{g'}$ such that $\xi'_1 = \xi''_0$ and such that $\xi'' * \xi'$ is smooth. One easily checks that $A^{[2]}$ is a Lie subalgebroid of $A\times A$, i.e. that the bracket on $\Gamma(A\times A)$ restricts to $\Gamma(A^{[2]})$. The kernel of its anchor map $\mathsf{a}^{[2]}\colon A^{[2]} \to TG^2$ is denoted $L^{[2]}$; it is a sub Lie algebra bundle of $L \times L$.

Concatenation gives a bundle map $\mathrm{mult}_A\colon A^{[2]} \to A$, covering the group multiplication $\mathrm{mult}_G\colon G\times G \to G$. That is, we have a commutative diagram,

$$\begin{array}{ccc} A^{[2]} & \xrightarrow[\mathrm{mult}_A]{} & A \\ \big\downarrow & & \big\downarrow \\ G\times G & \xrightarrow[\mathrm{mult}_G]{} & G \end{array}$$

We have three transitive Lie algebroids over G^2, with inclusion maps

(14) $$A^2 \leftarrow A^{[2]} \to \mathrm{mult}_G^!\,A,$$

Here the left map is given by the definition of $A^{[2]}$, while the right map is concatenation. The two maps correspond to reductions of the structure Lie algebroids to $L^{[2]}$, [4]

$$L^2 \leftarrow L^{[2]} \to \mathrm{mult}_G^*\,L.$$

We are interested in compatible principal connections on the three transitive Lie algebroids (14) over $G\times G$. Write the elements of G^2 as (g'', g'), and use the similar notation to indicate projections to the two factors. Let $\alpha', \alpha''\colon \mathbb{R} \to \Omega^1(G^2, \mathfrak{g})^G$ be smooth families of 1-forms with

$$\alpha'_{t+1} = g' \bullet \alpha'_t, \quad \alpha''_{t+1} = g'' \bullet \alpha''_t.$$

Assume both of these are constant near $t=0$ (hence near any integer $t=n$), and that $\alpha'_1 = \alpha''_0$. The concatentation (cf. Prop. A.3) $\alpha'' * \alpha'\colon \mathbb{R} \to \Omega^1(G^2, \mathfrak{g})^G$ defines a connection $\theta^{\alpha''*\alpha'}$ on $m_G^! A$, while the pair α'', α' defines a connection $\theta^{\alpha'',\alpha'}$ on $A\times A$. These two connections are compatible, in the sense that they restrict to the same connection on $A^{[2]}$. For the corresponding forms $\varpi^{\alpha'}$ etc. this implies

$$\varpi^{\alpha''*\alpha'}\Big|_{A^{[2]}} = \varpi^{\alpha'',\alpha'}\Big|_{A^{[2]}},$$

and hence the 3-forms satisfy $\mathrm{mult}_G^*\,\eta^{\alpha''*\alpha'} = \eta^{\alpha'',\alpha'}$.

[4] By analogy, one may think of $L^{[2]}$ as ‘figure eight’ loops. The two maps correspond to viewing the figure eight either as a single loop or as a pair of two loops.

Let $\varpi \in \Gamma(\wedge^2 A^*)$ be the 2-form defined in (13). Then

$$\varpi^{\alpha''*\alpha'} = \mathrm{mult}_A^! \varpi + Q^{\alpha''*\alpha'},$$
$$\varpi^{\alpha',\alpha''} = \mathrm{pr}_1^! \varpi + \mathrm{pr}_2^! \varpi + Q^{\alpha'} + Q^{\alpha''}.$$

Using the property (22) of Q^α under concatenation, we obtain:

Proposition 6.1. *The 2-form ϖ satisfies, over $A^{[2]} \subset A \times A$,*

$$\mathrm{mult}_A^! \varpi = \mathrm{pr}_1^! \varpi + \mathrm{pr}_2^! \varpi - \lambda$$

Here $\lambda \in \Omega^2(G \times G)$ is the 2-form, $\lambda = \frac{1}{2} \mathrm{pr}_1^* \theta^L \cdot \mathrm{pr}_2^* \theta^R$.

This 'lifts' the property of the Cartan 3-form, $\mathrm{mult}_G^* \eta = \mathrm{pr}_1^* \eta + \mathrm{pr}_2^* \eta - \mathrm{d}\lambda$.

7. Pull-backs

7.1. **The lifting problem for $\Phi^! A$.** Given a G-equivariant map $\Phi \colon M \to G$, consider the pull-back algebroid $A_M = \Phi^! A \to M$. Sections of A_M are pairs (X, ξ), where $X \in \mathfrak{X}(M)$ and $\xi \in C^\infty(M \times \mathbb{R}, \mathfrak{g})$ such that for all t,

$$\xi_{t+1} = \mathrm{Ad}_\Phi \xi_t + \iota_X \Phi^* \theta^R.$$

The bracket between two such sections reads,

$$[(X, \xi), (Y, \zeta)]_{A_M} = ([X, Y],\ -[\xi, \zeta]_{\mathfrak{g}} + X\zeta - Y\xi),$$

and the anchor map is $\mathsf{a}_M(X, \xi) = X$. The sections $x_{A_M} = \Phi^! x_A \in \Gamma(A_M)$ are generators for the G-action on A_M.

Suppose $\alpha_t \in \Omega^1(G, \mathfrak{g})^G$ is a family of 1-forms as in (10), with $Q^\alpha = 0$, thus $\varpi^\alpha = \varpi$ and $\eta_G^\alpha = \eta_G$. Let $\varpi_M = \Phi^! \varpi \in \Gamma(\wedge^2 A_M^*)$. Suppose

$$\Phi^* \eta_G = -\mathrm{d}_G \omega.$$

for an invariant 2-form ω. As shown in Section 3.5, this gives an equivariant solution of the lifting problem for A_M, relative to the central extension $\widehat{L}_M = \Phi^* \widehat{L} \to L_M = \Phi^* L$. Since we are assuming $\omega_G = \omega$, this solution will have the additional property that $j_{A_M}(x_{A_M})$ are generators for the action on $\hat{A}_M$. Since $\mathrm{d}_G \varpi_M = a_M^* \Phi^* \eta_G$, the sum

$$\mathsf{a}_M^* \omega + \varpi_M \in \Gamma(\wedge^2 A_M^*)$$

is equivariantly closed. Let us compute its kernel. For the following theorem, we assume that the inner product on $\mathfrak{g}$ is non-degenerate.

Theorem 7.1. *Suppose $\Phi \colon M \to G$ is a G-equivariant map, and $\omega \in \Omega^2(M)$ is an invariant 2-form such that $d_G \omega = -\Phi^* \eta_G$.*

At any point $m \in M$, the kernel of $\mathsf{a}_M^* \omega + \varpi_M \in \Gamma(\wedge^2 A_M^*)$ admits a direct sum decomposition,

$$\ker(\mathsf{a}_M^* \omega + \varpi_M) = \mathfrak{g} \oplus (\ker(\omega) \cap \ker(d\Phi)). \tag{15}$$

Here elements $v \in T_m M \cap \ker(d_m \Phi) \subset T_m M$ are embedded in $\ker(\mathsf{a}_M^* \omega + \varpi_M) \subset A_M \subset TM \oplus A$ as elements of the form $(v, 0)$, while $\mathfrak{g}$ is embedded diagonally as generators for the action, $x \mapsto (x_M, x_A)$.

Proof. By definition, the fiber of $\Phi^! A = A_M$ at $m \in M$ is the subspace of $T_mM \oplus A_{\Phi(m)}$, consisting of pairs (v, ξ) such that $(d_m\Phi)(v) = \mathsf{a}(\xi)$.

The property $\mathrm{d}_G(\mathsf{a}_M^*\omega + \varpi_M) = 0$ means in particular that elements of the form (x_M, x_A) are in the kernel of $\omega + \varpi_M$. On the other hand, elements of the form $(v, 0)$ with $v \in \ker(d_m\Phi)$ are contained in A_M, and they are in the kernel of $\omega + \varpi_M$ if and only if $v \in \ker(\omega)$. This proves the inclusion $\supseteq$ in (15).

For the opposite inclusion, consider a general element $(w, \xi) \in A_M \subset TM \oplus A$ in the kernel of $\mathsf{a}_M^*\omega + \varpi_M$ at $m \in M$. We have $\iota_{(w,\xi)}\varpi_M = \Phi^! \iota_\xi \varpi$, where $\iota_\xi\varpi$ is given by the calculation following (13). We thus obtain the condition

$$\mathsf{a}_M^*(\iota_w\omega) - \int_0^1 \dot{\xi} \cdot \kappa_M - \mathrm{Ad}_g(\xi_0) \cdot \mathsf{a}_M^*\theta^R - \frac{1}{2} v_\xi \cdot \mathsf{a}_M^*\theta^R = 0,$$

where $\kappa_M = \Phi^!\kappa$. Taking a contraction with $\zeta \in \ker(\mathsf{a}_M) \cong L_{\Phi(m)}$, we obtain

$$\int_0^1 \dot{\xi} \cdot \zeta = 0.$$

Since this is true for all $\zeta \in L_{\Phi(m)}$, the non-degeneracy of the inner product implies $\dot{\xi} = 0$. Thus ξ is a constant path. Letting $x = -\xi \in \mathfrak{g}$, it follows that $(v, 0)$ with $v = w - x_M$ lies in the kernel. As seen above, this means that $v \in \ker(\omega) \cap \ker(\mathrm{d}\Phi)$. □

The conditions, $\mathrm{d}_G\omega = -\Phi^*\eta_G$ and $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$ are exactly the defining conditions for a *q-Hamiltonian G-space* [1]. [5] That is, for a q-Hamiltonian G-space the kernel of $\mathsf{a}_M^*\omega + \varpi_M \in \Gamma(\wedge^2 A_M^*)$ is the action Lie algebroid for the G-action, embedded as the Lie subalgebroid of A_M spanned by the generators of the G-action x_{A_M}.

7.2. **The subalgebroid A' and its pull-back $\Phi^! A'$.** Let $A' \subset A$ be the G-invariant subalgebroid, consisting of $\xi \in A$ with $\xi_0 = 0$. Then A' is again a transitive Lie algebroid, and $A = \mathfrak{g} \ltimes A'$, where $\mathfrak{g}$ is embedded by the generators of the G-action. The Lie algebroid A' may be viewed as the Atiyah algebroid of the principal L_eG-bundle $P_eG \to G$, where the subscript indicates paths based at the group unit e. In turn, P_eG may be identified with the space $\Omega^1(S^1, \mathfrak{g})$ of connections on $S^1 = \mathbb{R}/\mathbb{Z}$, where the identification is given by the map $\gamma \mapsto \gamma^{-1}\mathrm{d}\gamma$. (Conversely, γ is recovered by parallel transport.) There is a natural projection $q\colon A \to A'$, $\xi \mapsto \xi - \xi(0)$, with $\mathsf{a}(q(\xi)) = \mathsf{a}(\xi) + \xi(0)_G$. Of course, q does not preserve brackets.

Suppose now that $\Phi\colon M \to G$ is a G-equivariant map, and let $A'_M = \Phi^! A'$. The projection q induces a projection map $q_M\colon A_M \to A'_M$, given on sections by

$$q_M(\xi, X) = (\xi - \xi(0),\ X + \xi(0)_M).$$

[5]In [1], the second condition was stated in the form $\ker(\omega) = \{\xi_M |\ \mathrm{Ad}_{\Phi(m)}\,\xi = -\xi\}$. The equivalence to $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$ was observed independently by Bursztyn-Crainic [6] and Xu [15].

Its kernel is the trivial bundle $\mathfrak{g}_M = M \times \mathfrak{g} \subset A_M$, embedded by the map $x \mapsto (-x, x_M)$ generating the G-action. Even though q_M does not preserve brackets, we have:

Corollary 7.2. *If $E \subset A_M$ is a G-invariant Lie subalgebroid, transverse to $\mathfrak{g}_M$, then $q_M(E) \subset A'_M$ is a Lie subalgebroid of A'_M.*

Proof. The transversality implies that $q_M(E)$ is a sub-bundle of A'_M, of the same rank as E. Letting $\mathfrak{g}_M = M \times \mathfrak{g} \subset A_M$ be the embedding given by the generators of the $\mathfrak{g}$-action, we have

$$q_M(E) = q_M(E + \mathfrak{g}_M) = (E + \mathfrak{g}_M) \cap A'_M.$$

But the sections of $E + \mathfrak{g}_M$ are closed under $[\cdot,\cdot]_A$, as are the sections of A'_M. ☐

Example 7.3. In this example, we assume that G is compact and that the inner product $\cdot$ on $\mathfrak{g}$ is positive definite. Let $\Phi\colon \mathcal{C} \subset G$ be the inclusion of a conjugacy class. Then $A_{\mathcal{C}} = \Phi^! A$ is a sum

$$A_{\mathcal{C}} = L + \mathfrak{g}_{\mathcal{C}}.$$

The intersection $L \cap \mathfrak{g}_{\mathcal{C}}$ is the sub-bundle of $\mathfrak{g}_{\mathcal{C}}$, spanned by $(-x, x_{\mathcal{C}}|_g)$ with $x \in \ker(\mathrm{Ad}_g - 1)$. Let the fibers L_g carry the inner product defined by the integration pairing. Then, after an appropriate Hilbert space completion (for instance, using the Sobolev space $W^{1,2}$), we obtain

$$L_g^{\mathbb{C}} = L_g^+ \oplus \ker(\mathrm{Ad}_g - 1)^{\mathbb{C}} \oplus L_g^-$$

where $L_g^{\pm}$ are the direct sum of the eigenspaces for the positive/negative part of the spectrum of $\frac{1}{\sqrt{-1}}\frac{\partial}{\partial t}$, and $\ker(\mathrm{Ad}_g - 1)^{\mathbb{C}} \cong L_g^0$ is embedded as the kernel. Consequently,

$$A_{\mathcal{C}}^{\mathbb{C}} = \Phi^! A = L^+ \oplus \mathfrak{g}_{\mathcal{C}} \oplus L^-.$$

Since $L^{\pm}$ are Lie algebra sub-bundles of L, their integrability is automatic, and hence

$$(A'_{\mathcal{C}})^{\mathbb{C}} = q(L^+) \oplus q(L^-)$$

is an integrable polarization of $A'_{\mathcal{C}}$. Letting $\mathcal{O}$ be the coadjoint LG-group orbit corresponding to $\mathcal{C} = \mathcal{O}/L_0 G$, the bundle $\mathcal{A}'_{\mathcal{C}}$ is interpreted as $T\mathcal{O}/L_0 G$, and its polarization is the standard Kähler structure.

8. Higher analogues of the form ϖ

We had remarked above that the Cartan form η may be viewed as a Chern-Simons form, and similarly η_G as an equivariant Chern-Simons form. For any invariant polynomial $p \in (S^m \mathfrak{g}^*)^G$, we may define 'higher analogues' η^p, η^p_G of the Cartan form using the theory of Bott forms. We will not assume the existence of an invariant inner product on $\mathfrak{g}$.

8.1. **Bott forms.** Let N be a manifold. Suppose $\beta \in \Omega^1(N,\mathfrak{g})$, and that $p \in (S^m\mathfrak{g}^*)^G$ is an invariant polynomial of degree m. Then $p(F^\beta)$ is closed, as an application of the Bianchi identity $\mathrm{d}F^\beta + [\beta, F^\beta] = 0$. [6] Given $\beta_0,\ldots,\beta_k \in \Omega^1(N,\mathfrak{g})$ we define *Bott forms* $\Upsilon^p(\beta_0,\ldots,\beta_k) \in \Omega^{2m-k}(N)$

$$\Upsilon^p(\beta_0,\ldots,\beta_k) = (-1)^{[\frac{k+1}{2}]}\int_{\Delta^k} p(F^\beta).$$

Here $\Delta^k = \{s \in \mathbb{R}^{k+1} |\ s_i \geq 0,\ \ \sum_{i=0}^k s_i = 1\}$ is the standard k-simplex, and $\beta = \sum_{i=0}^k s_i\beta_i$, viewed as a form $\beta \in \Omega^1(N\times\Delta^k,\mathfrak{g})$. For a detailed discussion of Bott forms, see [14, Chapter 4]. The Bott forms satisfy

$$\mathrm{d}\Upsilon^p(\beta_0,\ldots,\beta_k) = \sum_{i=0}^k(-1)^i\Upsilon^p(\beta_0,\ldots,\widehat{\beta}_i,\ldots\beta_k),$$
$$\Upsilon^p(\Phi\bullet\beta_0,\ldots,\Phi\bullet\beta_k) = \Upsilon^p(\beta_0,\ldots,\beta_k),\quad \Phi\in C^\infty(N,G).$$

The first identity follows from Stokes' theorem [14, Theorem 4.1.6], while the second identity comes from the gauge equivariance of the curvature, $F^{\Phi\bullet\beta} = \mathrm{Ad}_\Phi(F^\beta)$.

Consider the special case $N = G$. For any $p \in (S^m\mathfrak{g}^*)^G$ we define

$$\eta^p = \Upsilon^p(0,\theta^L) \in \Omega^{2m-1}(G).$$

Then $\mathrm{d}\eta^p = \Upsilon^p(\theta^L) - \Upsilon^p(0) = 0$, using that $F^\beta = 0$ for both $\beta = 0,\theta^L$. For G compact, the classes $[\eta^p]$ are known to generate the cohomology ring $H^*(G,\mathbb{R})$.

8.2. **Equivariant Bott forms.** With small modifications, the definition of Bott forms carries over to equivariant 1-forms $\beta \in \Omega^1(N,\mathfrak{g})^G$ for a given G-action on N, and for the adjoint action of G on $\mathfrak{g}$. For any such form, and an invariant polynomial p, the equivariant Bianchi identity $\mathrm{d}_G F_G^\beta + [\beta, F_G^\beta(x) + x] = 0$ implies that $p(F_G^\beta(x)+x)$ is equivariantly closed. Given $\beta_0,\ldots,\beta_k \in \Omega^1(N,\mathfrak{g})^G$ we define equivariant Bott forms $\Upsilon_G^p(\beta_0,\ldots,\beta_k) \in \Omega_G(N)$ by

$$\Upsilon_G^p(\beta_0,\ldots,\beta_k)(x) = (-1)^{[\frac{k+1}{2}]}\int_{\Delta^k} p(F_G^\beta(x)+x),$$

with $\beta = \sum_{i=0}^k s_i\beta_i$ as above. Then

$$\mathrm{d}_G\Upsilon_G^p(\beta_0,\ldots,\beta_k) = \sum_{i=0}^k(-1)^i\Upsilon_G^p(\beta_0,\ldots,\widehat{\beta}_i,\ldots\beta_k),$$
$$\Upsilon_G^p(\Phi\bullet\beta_0,\ldots,\Phi\bullet\beta_k) = \Upsilon_G^p(\beta_0,\ldots,\beta_k),\qquad \Phi\in C^\infty(N,G)^G.$$

Again this follows from Stokes' theorem, respectively from the property $F_G^{\Phi\bullet\beta_i}(x) + x = \mathrm{Ad}_\Phi(F_G^{\beta_i}(x)+x)$ of the equivariant curvature.

[6] For any polynomial $p \in S\mathfrak{g}^*$, we define its derivative $p' \in S\mathfrak{g}^*\otimes\mathfrak{g}^*$ by $\langle p'(v), w\rangle = \frac{\partial}{\partial t}\Big|_{t=0} p(v+tw)$. If p is G-invariant, then $[x,y]\cdot p'(y) = 0$ for all $x,y\in\mathfrak{g}$. Thus $\mathrm{d}p(F^\beta) = \mathrm{d}F^\beta\cdot p'(F^\beta) = -[\beta,F^\beta]\cdot p'(F^\beta) = 0$.

If $N = G$ with conjugation action, and $p \in (S^m\mathfrak{g}^*)^G$ we define [10]

$$\eta^p_G = \Upsilon^p_G(0, \theta^L) \in \Omega^{2m-1}_G(G).$$

Since $F^{\theta^L}_G(x) + x = \mathrm{Ad}_{g^{-1}}(x)$, we have

$$\mathrm{d}_G\eta^p_G(x) = \Upsilon^p_G(\theta^L) - \Upsilon^p_G(0) = p(\mathrm{Ad}_{g^{-1}}(x)) - p(x) = 0.$$

Thus η^p_G are closed equivariant extensions of η^p.

8.3. **Families of flat connections.** Suppose that $\beta_t \in \Omega^1(N, \mathfrak{g})^G$ is a family of invariant 1-forms, such that $F^{\beta_t}_G(x) + x = 0$ for all t. Then

$$\mathrm{d}_G\Upsilon^p_G(0, \beta_t)(x) = -p(x)$$

for all t, and so the difference $\Upsilon^p_G(0, \beta_t) - \Upsilon^p_G(0, \beta_0)$ is equivariantly closed. We will construct an equivariant primitive. Let $\beta \in \Omega^1(N \times \Delta^1 \times I, \mathfrak{g})^G$ be given as

$$\beta_{s,t} = s\beta_t, \quad t \in I = [0, 1], \quad s \in \Delta^1 \cong [0, 1].$$

We set

$$I^p_G(\{\beta_t\})(x) = \int_{\Delta^1\times I} p(F^\beta_G(x) + x).$$

Lemma 8.1. *If $m = \deg(p) \geq 2$,*

$$\Upsilon^p_G(0, \beta_1) - \Upsilon^p_G(0, \beta_0) = d_G I^p_G(\{\beta_t\}),$$

Proof. We compute $\mathrm{d}_G I^p_G(\{\beta_t\})(x)$ by Stokes' theorem. There will be four boundary contributions, corresponding to the four sides $s = 0,\ s = 1,\ t = 0,\ t = 1$ of the square $\Delta^1 \times I$. The boundary contribution for $s = 1$ is given as the integral of

$$p(\mathrm{d}t \wedge \dot\beta_t + F^{\beta_t}_G(x) + x).$$

But $F^{\beta_t}_G(x) + x = 0$ by assumption, and hence $p(\mathrm{d}t \wedge \dot\beta_t) = 0$ since $\deg(p) \geq 2$. The boundary contribution of $s = 0$ vanishes as well, since the pull-back of $p(F^\beta_G(x) + x)$ has no $\mathrm{d}t$-component there. The remaining two boundary contributions are $\Upsilon^p_G(0, \beta_1)$ and $-\Upsilon^p_G(0, \beta_0)$ as desired. □

The discussion for the non-equivariant case is essentially the same: Given a family $\beta_t \in \Omega^1(N, \mathfrak{g})$ with $F^{\beta_t} = 0$, the integral $I^p(\{\beta_t\}) = \int_{\Delta^1\times I} p(F^\beta)$ has the property $\Upsilon^p(0, \beta_1) - \Upsilon^p(0, \beta_0) = \mathrm{d}I^p(\{\beta_t\})$. Writing $F^\beta = \mathrm{d}s \wedge \beta_t + s\mathrm{d}t \wedge \dot\beta_t + \frac{s(s-1)}{2}[\beta_t, \beta_t]$, we may carry out the s-integration in the definition of Υ^p, and find that Υ^p is explicitly given as a rational multiple of

$$\int_0^1 p(\beta_t,\ \dot\beta_t,\ [\beta_t, \beta_t],\ \ldots,\ [\beta_t, \beta_t]). \tag{16}$$

Here we have associated to $p \in (S^m\mathfrak{g}^*)^G$ the multilinear form (again denoted p) such that $p(x, \ldots, x) = p(x)$.

8.4. **The form ϖ_G^p.** The theory described above works equally well for $\Omega(N)$ replaced with $\Gamma(A)$, for $A \to N$ a Lie algebroid. In the G-equivariant case, one has to require that the G-action on A admits infinitesimal generators x_A. As before, we will view $\Omega(N) \subset \Gamma(\wedge A^*)$ respectively $\Omega_G(N) \subset \Gamma_G(\wedge A^*)$ as the basic subcomplexes.

Our goal is to construct primitives of $\mathsf{a}^*\eta_G^p \in \Gamma_G(\wedge A^*)$, where $A \to G$ is the Atiyah algebroid over G. Let $\kappa_t \in \Gamma(A^*) \otimes \mathfrak{g}$ as in Section 5.2. With $I_G^p(\{\kappa_t\}) \in \Gamma_G(\wedge A^*)$ as above, put

$$\varpi_G^p = I_G^p(\{\kappa_t\}) - \Upsilon_G^p(0, \mathsf{a}^*\theta^L, \kappa_0).$$

Theorem 8.2. *The forms ϖ_G^p are equivariant primitives of $\mathsf{a}^*\eta_G^p$:*

$$d_G\varpi_G^p(x) = \mathsf{a}^*\eta_G^p(x).$$

Proof. Since $\kappa_1 = g \bullet \kappa_0$ by Lemma 5.3, we have

$$\Upsilon_G^p(0, \kappa_1) = \Upsilon_G^p(0, g \bullet \kappa_0) = \Upsilon_G^p(g^{-1} \bullet 0, \kappa_0) = \Upsilon_G^p(\mathsf{a}^*\theta^L, \kappa_0).$$

Lemma 5.3 also shows that $F_G^{\kappa_t}(x) + x = 0$. Hence Lemma 8.1 applies and gives

$$\begin{aligned} \mathrm{d}_G I_G^p(\{\kappa_t\}) &= \Upsilon_G^p(0, \kappa_1) - \Upsilon_G^p(0, \kappa_0) \\ &= \Upsilon_G^p(\mathsf{a}^*\theta^L, \kappa_0) + \Upsilon_G^p(\kappa_0, 0) \\ &= \Upsilon_G^p(\mathsf{a}^*\theta^L, 0) + \mathrm{d}_G\Upsilon_G^p(0, \mathsf{a}^*\theta^L, \kappa_0). \end{aligned}$$

□

Setting the equivariant parameter equal to 0, i.e. defining $\varpi^p = \varpi_G^p(0)$, this also gives in particular non-equivariant primitives, $\mathrm{d}\varpi^p = \mathsf{a}^*\eta^p$.

8.5. **The case $p(x) = \frac{1}{2}x \cdot x$.** If p is homogeneous of degree $\deg(p) = 2$, the formulas simplify. With $\beta_{s,t} = s\kappa_t$, the definition of $I_G^p(\{\kappa_t\}(x)$ gives

$$I_G^p(\{\kappa_t\})(x) = \int_{\Delta^1 \times I} p(F_G^\beta(x) + x) = \int_{\Delta^1 \times I} p(\mathrm{d}s \wedge \kappa_t + s\mathrm{d}t \wedge \dot{\kappa}_t).$$

Indeed, only the coefficient of $\mathrm{d}s \wedge \mathrm{d}t$ in $p(F_G^\beta(x) + x)$ will contributes to the integral. Hence

$$I_G^p(\{\kappa_t\})(x) = \int_0^1 p(\kappa_t, \dot{\kappa}_t),$$

where we associated to p a symmetric bilinear form, again denote p, with $p(x, x) = p(x)$. In particular, $I_G^p(\{\kappa_t\}) = I^p(\{\kappa_t\})$. A similar discussion applies to the 2-dimensional integral defining $\Upsilon_G^p(0, \mathsf{a}^*\theta^L, \kappa_0)$. One obtains

$$\Upsilon_G^p(0, \mathsf{a}^*\theta^L, \kappa_0)(x) = p(\mathsf{a}^*\theta^L, \kappa_0),$$

which again is independent of x. We conclude that if $p(x) = \frac{1}{2}x \cdot x$ for an invariant inner product $\cdot$ on $\mathfrak{g}$, then ϖ_G^p coincides with ϖ^p, and is given by the Formula (13).

8.6. **Pull-back to the group unit.** The inclusion map $\iota\colon \{e\} \to G$ is G-equivariant, and lifts to a morphism of Lie algebroids, $L\mathfrak{g} \to A$. (In fact, $L\mathfrak{g} = \iota^! A$.) Let

$$\sigma^p = \iota^! \varpi^p, \quad \sigma^p_G = \iota^! \varpi^p_G$$

be the resulting elements of $\Gamma(\wedge L\mathfrak{g}^*)$, resp. $\Gamma_G(\wedge L\mathfrak{g}^*)$. Since $\iota^* \eta^p_G = 0$, it is immediate that these forms are closed (resp. equivariantly closed) for the Lie algebra differential.

The pull-back of $\kappa^{L\mathfrak{g}} := \iota^! \kappa$ may be viewed as minus the right-invariant Maurer-Cartan forms for the group LG. Since the pull-back of $\Upsilon^p(0, \mathsf{a}^*\theta^L, \kappa_0)$ vanishes, Equation (16) shows that σ^p is a rational multiple of

$$p(\kappa^{L\mathfrak{g}}_t,\, \dot\kappa^{L\mathfrak{g}}_t,\, [\kappa^{L\mathfrak{g}}_t,\, \kappa^{L\mathfrak{g}}_t], \ldots, [\kappa^{L\mathfrak{g}}_t,\, \kappa^{L\mathfrak{g}}_t]).$$

These forms are discussed by Pressley-Segal in [12, Chapter 4.11], who prove that for compact G the cohomology ring $H^*(LG)$ is generated by the left-invariant forms, and is in fact isomorphic to the Lie algebra cohomology of $L\mathfrak{g}$. The forms σ^p arise as some of the generators of the cohomology. (The remaining generators are obtained by pull-back under the evaluation map $LG \to G,\ \gamma \mapsto \gamma_0$). Our theory thus provides closed G-equivariant extensions of the Pressley-Segal generators, and gives an explicit transgressions of these forms to η^p, η^p_G.

Appendix A. Chern-Simons forms on Lie algebroids

In this appendix, we extend some formulas for Chern-Simons forms to the case of Lie algebroids. We omit proofs, which are all given by straightforward calculations (extending the well-known case $A = TN$).

A.1. **Non-equivariant Chern-Simons forms.** Suppose $A \to N$ is a Lie algebroid. We will consider the elements of $\Gamma(\wedge A^*)$ as 'forms on A'. For any $\mathfrak{g}$-valued 1-form $\beta \in \Gamma(A^*) \otimes \mathfrak{g}$ with 'curvature' $F^\beta = \mathrm{d}\beta + \frac{1}{2}[\beta, \beta]_\mathfrak{g}$, the 4-form $\frac{1}{2} F^\beta \cdot F^\beta \in \Gamma(\wedge^4 A^*)$ is exact. A primitive is given by the *Chern-Simons form* $\mathrm{CS}(\beta) = \Upsilon^p(0, \beta)$ for $p(x) = \frac{1}{2} x \cdot x$, where we have used the notation from Section 8.1. Thus $\mathrm{d}\,\mathrm{CS}(\beta) = p(F^\beta) = \frac{1}{2} F^\beta \cdot F^\beta$. A short calculation gives

$$\mathrm{CS}(\beta) = \frac{1}{2}(\mathrm{d}\beta)\cdot\beta + \frac{1}{6}\beta \cdot [\beta, \beta]_\mathfrak{g} \in \Gamma(\wedge^3 A^*).$$

For $\Phi \in C^\infty(N, G)$ let $\Phi \bullet \beta = \mathrm{Ad}_\Phi(\beta) - \Phi^*\theta^R$ be the gauge transform of β. (Here the last term is viewed as an element of of $\Gamma(A^*)$, by the pull-back map $\Omega(N) \to \Gamma(\wedge A^*)$.

Proposition A.1. *For $\beta \in \Gamma(A^*) \otimes \mathfrak{g}$ and $\Phi \in C^\infty(N, G)$, we have*

$$\mathrm{CS}(\Phi \bullet \beta) = \mathrm{CS}(\beta) + \Phi^*\eta - \frac{1}{2} d\,(\beta \cdot \Phi^*\theta^L). \tag{17}$$

Given a smooth family β_t one has the transgression formula,

$$\frac{\partial}{\partial t}\,\mathrm{CS}(\beta_t) = \dot{\beta}_t \cdot F^{\beta_t} - \frac{1}{2}d\,(\beta_t \cdot \dot{\beta}_t). \tag{18}$$

Suppose $\beta_{t+1} = \Phi \bullet \beta_t$ for some given gauge transformation $\Phi \in C^\infty(N,G)$. Integrating (18) over $[0,1]$, and using the property of Chern-Simons forms under gauge transformations, we obtain

$$\int_0^1 \dot{\beta}_t \cdot F^{\beta_t} = \Phi^*\eta + \mathrm{d}Q^\beta \tag{19}$$

where $Q^\beta \in \Gamma(\wedge^2 A^*)$ is the 2-form,

$$Q^\beta = \frac{1}{2}\Phi^*\theta^L \cdot \beta_0 + \frac{1}{2}\int_0^1 \beta_t \cdot \dot{\beta}_t.$$

A.2. **G-equivariant Chern-Simons forms.** Suppose that the group G acts on $A \to N$, with infinitesimal generators $x \mapsto x_A$. Then we can consider the complex $\Gamma_G(\wedge A^*)$ of G-equivariant forms.

Suppose $\beta \in (\Gamma(A^*) \otimes \mathfrak{g})^G$, and let $F_G^\beta = \mathrm{d}_G\beta + \frac{1}{2}[\beta,\beta]$ be its 'equivariant curvature'. We have

$$\mathrm{d}_G F_G^\beta + [\beta, F_G^\beta(x) + x] = 0.$$

As a consequence, the equivariant 4-form $p(F_G^\beta(x) + x) - p(x)$ for $p(x) = \frac{1}{2}x \cdot x$ is equivariantly closed. [7] Let $\mathrm{CS}_G(\beta) = \Upsilon_G^p(0,\beta) \in \Gamma_G(\wedge^3 A^*)$, with differential $p(F_G^\beta(x) + x) - p(x)$. One finds

$$\mathrm{CS}_G(\beta)(x) = \frac{1}{2}\mathrm{d}_G\beta(x) \cdot \beta + \frac{1}{6}\beta \cdot [\beta,\beta]_{\mathfrak{g}} + \beta \cdot x.$$

Proposition A.2. *For $\beta \in (\Gamma(A^*) \otimes \mathfrak{g})^G$ and $\Phi \in C^\infty(N,G)^G$,*

$$\mathrm{CS}_G(\Phi \bullet \beta) = \mathrm{CS}_G(\beta) + \Phi^*\eta_G - \frac{1}{2}d_G(\beta \cdot \Phi^*\theta^L). \tag{20}$$

Given a smooth family $\beta_t \in (\Gamma(A^*) \otimes \mathfrak{g})^G$, one has

$$\frac{\partial}{\partial t}\,\mathrm{CS}_G(\beta_t)(x) = \dot{\beta}_t \cdot (F_G^{\beta_t}(x) + x) - \frac{1}{2}d\,(\beta_t \cdot \dot{\beta}_t).$$

Hence, if $\beta_t \in (\Gamma(A^*) \otimes \mathfrak{g})^G$ is a family of invariant forms with $\beta_1 = \Phi \bullet \beta_0$, and letting Q^β be defined as above, one finds

$$\int_0^1 \dot{\beta}_t \cdot (F_G^{\beta_t}(x) + x) = \Phi^*\eta_G + \mathrm{d}_G Q^\beta. \tag{21}$$

[7]In the case $A = TN$, the form β may be regarded as the restriction to $N \times \{e\}$ of a principal connection on $N \times G$, invariant relative to the diagonal action $k.(n,u) = (k.n, ku)$. The pull-back of the G-equivariant curvature $F_G^\theta(x)$ to $N \times \{e\}$ is $F_G^\beta(x) + x$.

A.3. **Properties of the functional Q.** Here are some properties of the functional $Q(\beta) = Q^\beta$.

Proposition A.3 (Properties of the functional Q)**.**

(a) **Reparametrization invariance.** *Let $\beta_t \in \Gamma(A^*) \otimes \mathfrak{g}$ be a smooth family of forms with $\beta_{t+1} = \Phi \bullet \beta_t$, and suppose $\phi\colon \mathbb{R} \to \mathbb{R}$ is an orientation preserving diffeomorphism such that $\phi(t+1) = \phi(t) + 1$. Then $Q(\beta \circ \phi) = Q(\beta)$.*

(b) **Multiplicative property.** *Let $\beta', \beta''\colon \mathbb{R} \to \Gamma(A^*) \otimes \mathfrak{g}$ be two maps such that $\beta'_{t+1} = \Phi' \bullet \beta'_t$, $\beta''_{t+1} = \Phi'' \bullet \beta''_t$. Suppose $\beta'_1 = \beta''_0$, and let the concatenation be defined for $0 \le t \le 1$ by*

$$(\beta'' * \beta')_t = \begin{cases} \beta'_{2t} & 0 \le t \le \frac{1}{2} \\ \beta''_{2t-1} & \frac{1}{2} \le t \le 1. \end{cases}$$

and extend to all t by the property, $(\beta'' * \beta')_{t+1} = (\Phi''\Phi') \bullet (\beta'' * \beta')_t$. (The resulting β is piecewise smooth, and it is smooth e.g. if β', β'' are constant near $t = 0$.) Then

$$Q(\beta'' * \beta') = Q(\beta') + Q(\beta'') + (\Phi', \Phi'')^* \lambda \tag{22}$$

where $\lambda \in \Omega^2(G \times G)$ is the 2-form, $\lambda = \frac{1}{2} \operatorname{pr}_1^* \theta^L \cdot \operatorname{pr}_2^* \theta^R$.

(c) **Inversion.** *Let $\beta\colon \mathbb{R} \to \Omega^1(N, \mathfrak{g})$ with $\beta_{t+1} = \Phi \bullet \beta_t$, and define $\beta^-_t = \beta_{-t}$. Then $\beta^-_{t+1} = \Phi^{-1} \bullet \beta^-_t$, and we have $Q(\beta^-) = -Q(\beta)$.*

Proof. (a) The claim is obvious if $\phi(0) = 0$, since both the integral and the term $\frac{1}{2}\Phi^*\theta^L \cdot \beta_0$ are unchanged in this case. It remains to check the case $\phi(t) = t + u$, for some fixed $u \in \mathbb{R}$. It is enough to consider the case $0 \le u \le 1$. We have

$$\begin{aligned} \int_0^1 \beta_{t+u} \cdot \dot\beta_{t+u} &= \int_u^{1+u} \beta_t \cdot \dot\beta_t \\ &= \int_u^1 \beta_t \cdot \dot\beta_t + \int_0^u (\operatorname{Ad}_\Phi \beta_t - \Phi^*\theta^R) \cdot \operatorname{Ad}_\Phi \dot\beta_t \\ &= \int_0^1 \beta_t \cdot \dot\beta_t - \int_0^u \Phi^*\theta^L \cdot \dot\beta_t \\ &= \int_0^1 \beta_t \cdot \dot\beta_t - \Phi^*\theta^L \cdot (\beta_u - \beta_0). \end{aligned}$$

□

(b) In calculating $Q(\beta) - Q(\beta') - Q(\beta'')$, the integral contributions cancel out, and we are left with

$$Q(\beta) - Q(\beta') - Q(\beta'') = \frac{1}{2}\big((\Phi''\Phi')^*\theta^L \cdot \beta_0 - (\Phi')^*\theta^L \cdot \beta_0 - (\Phi'')^*\theta^L \cdot \beta_{1/2}\big).$$

Since $\beta_{1/2} = \beta'_t = \Phi' \bullet \beta_0 = \operatorname{Ad}_{\Phi'} \beta_0 - (\Phi')^*\theta^R$ and $(\Phi''\Phi')^*\theta^L = (\Phi')^*\theta^L + \operatorname{Ad}_{(\Phi')^{-1}}(\Phi'')^*\theta^L$, we are left with $\frac{1}{2}(\Phi')^*\theta^L \cdot (\Phi'')^*\theta^R$.

(c) is a straightforward calculation.

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