LINEARIZATION OF POISSON ACTIONS AND SINGULAR VALUES OF MATRIX PRODUCTS

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Abstract. We prove that the linearization functor from the category of Hamiltonian K-actions with group-valued moment maps in the sense of Lu, to the category of ordinary Hamiltonian K-actions, preserves products up to symplectic isomorphism. As an application, we give a new proof of the Thompson conjecture on singular values of matrix products and extend this result to the case of real matrices. We give a formula for the Liouville volume of these spaces and obtain from it a hyperbolic version of the Duflo isomorphism.

1. Introduction

Poisson-Lie groups were introduced by Drinfeld [5] as semiclassical analogs of quantum groups. By definition, a Poisson-Lie group is a Lie group endowed with a Poisson structure such that group multiplication is a Poisson map. An important role in applications (for example [19, 12, 7, 4]) is played by the notion of a *moment map* for a Poisson action of a Poisson-Lie group, due to J.-H. Lu [17]. In contrast to ordinary moment maps taking values in the dual of the Lie algebra, moment maps in the sense of Lu take values in the dual Poisson-Lie group.

Compact Lie groups K carry a distinguished non-trivial Lie-Poisson structure known as the Lu-Weinstein [20] Poisson structure. For this case, the first author showed [1] that the categories of symplectic K-manifolds with moment maps in the dual group K^*, respectively dual of the Lie algebra $\mathfrak{k}^*$ are equivalent. That is, for every Poisson K-action on a symplectic manifold (M, Ω) with K^*-valued moment map Ψ, there is a different symplectic form ω for which the action is Hamiltonian in the usual sense, with a $\mathfrak{k}^*$-valued moment map Φ. Poisson reductions of (M, Ω, Ψ) are isomorphic to reductions of its *linearization* (M, ω, Φ) as (stratified) symplectic spaces.

The categories of symplectic K-manifolds with $\mathfrak{k}^*$- and K^*-valued moment maps have natural structures of *tensor categories*: There are operations of products, sums and conjugation satisfying the usual axioms. The first main result of this paper is that the linearization functor preserves these operations up to symplectomorphism. The proof is based on a simple Moser isotopy argument. As an application, we prove the Thompson conjecture on singular values of products of complex matrices, which was first established in a recent paper by Klyachko [15], and also the corresponding statement for real matrices

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(Theorem 4.2). Independently, a completely different proof of these results was obtained by Kapovich-Leeb-Millson [13].

The second main result is a formula comparing the Liouville volume forms defined by ω and Ω. This formula involves the modular function for K^* and a Duflo factor. As a corollary, we obtain a hyperbolic version of the Duflo isomorphism. That is, a certain linear map between spaces of compactly supported distributions on $\mathfrak{k}$ and K^* becomes a ring homomorphism (with respect to convolution) if restricted to K-invariants. As pointed out by the referee, this fact was proved in a more general setting by Rouvière [22, Theorem 7.4].

2. Moment maps for Poisson actions

In this Section we recall the theory of moment maps for Poisson actions of compact Poisson-Lie groups on symplectic manifolds developed by Lu [17].

2.1. **Poisson-Lie groups.** Recall that a Poisson-Lie group is a Lie group K together with a Poisson bivector π_K such that group multiplication is a Poisson map. This condition implies that the inversion map $K \to K,\ k \mapsto k^{-1}$ is anti-Poisson. The Poisson bivector π_K vanishes at the group unit of K, and its linearization $\delta:\ \mathfrak{k} \to \mathfrak{k} \otimes \mathfrak{k}$ is a 1-cocycle on $\mathfrak{k}$. The dual map δ^* defines a Lie algebra structure on $\mathfrak{k}^*$. The connected, simply-connected Lie group K^* with Lie algebra $\mathfrak{k}^*$ is called the Poisson dual of K. It is a Poisson-Lie group, with Poisson bracket induced by the Lie algebra structure on $\mathfrak{k}$.

Let the vector space $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^*$ be equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle$ for which $\mathfrak{k}$ and $\mathfrak{k}^*$ are isotropic and which extends the natural pairing between elements in $\mathfrak{k}$ and $\mathfrak{k}^*$. According to [20, Theorem 1.12] there is a unique Lie algebra structure on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^*$ for which $\mathfrak{k}$ and $\mathfrak{k}^*$ are subalgebras and the pairing $\langle \cdot, \cdot \rangle$ is $\mathfrak{g}$-invariant.

A Lie group G with Lie algebra $\mathfrak{g}$ is called a *double* for the Poisson-Lie group K if the subalgebras $\mathfrak{k}, \mathfrak{k}^* \to \mathfrak{g}$ exponentiate to closed subgroups $K, K^* \to G$, and the multiplication map $K^* \times K \to G,\ (l, k) \mapsto lk$ is a diffeomorphism. In this case, the left-action of G on itself induces an action on $K^* = G/K$. Its restriction $K \times K^* \to K^*,\ (k, l) \mapsto l^k$ is called the dressing action of K on K^*. Similarly, the right-action of G restricts to the dressing action $K^* \times K \to K,\ (l, k) \mapsto k^l$ on $K = K^* \backslash G$. The two actions are related by

$$kl = l^k k^l. \tag{1}$$

The classification of Poisson-Lie structures on compact, connected Lie groups K was carried out by Levendorskii and Soibelman [16]. Besides the trivial structure, there is a distinguished example called the Lu-Weinstein structure. Let $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$, viewed as a real Lie algebra, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ an Iwasawa decomposition.

For any invariant inner product B on $\mathfrak{k}$, with complexification $B^{\mathbb{C}}$, the bilinear form

$$\langle \cdot, \cdot \rangle = 2\,\mathrm{Im}\ B^{\mathbb{C}}$$

defines a non-degenerate pairing between $\mathfrak{k}$ and $\mathfrak{a} \oplus \mathfrak{n}$, identifying $\mathfrak{k}^* \cong \mathfrak{a} \oplus \mathfrak{n}$. The induced Lie algebra structure on $\mathfrak{k}^*$ defines the Lu-Weinstein Poisson structure on K, with Poisson dual $K^* = AN$ and double $G = K^{\mathbb{C}} = KAN$.

2.2. **Poisson actions.** Let (K, π_K) be a connected Poisson-Lie group, with Poisson-dual K^*, and suppose K admits a double $G = K^*K$. Denote by $\theta^L, \theta^R \in \Omega^1(K^*) \otimes \mathfrak{k}^*$ the left- and right-invariant Maurer-Cartan forms. [17, Corollary 3.6] states that for every Poisson map $\Psi : M \to K^*$ from a Poisson manifold (M, π) to K^*, the formula

$$\xi_M = \pi^\sharp \Psi^* \langle \theta^R, \xi \rangle, \quad \xi \in \mathfrak{k} \tag{2}$$

defines a Lie algebra action of $\mathfrak{k}$ on M, i.e. $[\xi_M, \eta_M] = [\xi, \eta]_M$. If this action integrates to a K-action, with generating vector fields $m \mapsto \frac{d}{dt}|_{t=0} \exp(-t\xi).m$ equal to $\xi_M(m)$, then the the triple (M, π, Ψ) is called a *Hamiltonian K-space with K^*-valued moment map Ψ.*

It follows from the moment map condition (2) that the action map $K \times M \to M$ is Poisson [17, Corollary 3.6] and that the moment map is K-equivariant [17, Theorem 4.8]. For $\pi_K = 0$ this reduces to the usual definition of a Hamiltonian K-space with $\mathfrak{k}^*$-valued moment map. In the special case where π_K is the inverse of a symplectic structure $\Omega \in \Omega^2(M)$, the moment map condition is equivalent to

$$\iota(\xi_M)\Omega = \Psi^* \langle \theta^R, \xi \rangle. \tag{3}$$

There are sum, product, and conjugation operations for Hamiltonian K-manifolds with K^*-valued moment maps, as follows. Sum is given by disjoint union. The product of two Hamiltonian K-manifolds with K^*-valued moment maps (M_1, π_1, Ψ_1) and (M_2, π_2, Ψ_2) is given by

$$(M_1 \times M_2, \pi_1 + \pi_2, \Psi_1 \Psi_2).$$

Indeed, by Flaschka-Ratiu [8, Lemma 22.3] the infinitesimal action generated by the Poisson map $\Psi_1 \Psi_2$ exponentiates to the following K-action on $M_1 \times M_2$,

$$k.(m_1, m_2) = (k.m_1, k^{\Psi_1(m_1)}.m_2).$$

The twist product is associative. It defines a tensor category structure on Hamiltonian K-manifolds, with morphisms given by equivariant Poisson isomorphisms preserving the moment map.

Lemma 2.1. *For any Hamiltonian K-manifolds with K^*-valued moment map (M, π, Ψ) the formula*

$$(k, m) \mapsto k^{\Psi(m)^{-1}}.m \tag{4}$$

defines a Poisson action on $(M, -\pi)$ with moment map Ψ^{-1}. We call $(M, -\pi, \Psi^{-1})$ the conjugate of (M, π, Ψ).

Proof. First, we check that (4) defines an action. Let K_L, K_R be two copies of K acting on G by $(k, g) \mapsto kg$ and $(k, g) \mapsto gk^{-1}$, respectively. Consider G as a K_L-equivariant principal K_R-bundle over $K^* = G/K_R$, and let Ψ^*G denote the pull-back to M. The action of K_L on Ψ^*G is free, and has

$$\iota: \; M \to \Psi^*G, \quad m \mapsto (m, \Psi(m))$$

as a cross-section. Using ι identify $\Psi^*G/K_L = M$. We claim that the induced action of K_R on M is the twisted K-action. Given $m \in M$ we compute

$$(m, \Psi(m)k^{-1}) = (m, (k^{\Psi^{-1}(m)})^{-1}((\Psi^{-1}(m))^k)^{-1})$$

The action of $k^{\Psi^{-1}(m)}$ takes this back to $\iota(M)$, which proves the claim.

Since the inversion map on K^* is anti-Poisson, Ψ^{-1} is a Poisson map for the reversed Poisson structure $-\pi$ on M. We check it is a moment map for the twisted action. Let $\mathrm{pr}_{\mathfrak{k}}: \mathfrak{g} \to \mathfrak{k}$ denote projection along $\mathfrak{k}^*$. Using the moment map condition for Ψ,

$$\begin{aligned} -\pi^\sharp(\Psi^{-1})^*\langle\theta^R, \xi\rangle(m) &= \pi^\sharp\Psi^*\langle\theta^L, \xi\rangle(m) \\ &= \pi^\sharp\Psi^*\langle\theta^R, \mathrm{pr}_{\mathfrak{k}}(\mathrm{Ad}_{\Psi(m)}\,\xi)\rangle(m) \\ &= (\mathrm{pr}_{\mathfrak{k}}(\mathrm{Ad}_{\Psi(m)}\,\xi))_M(m) \\ &= (\xi^{\Psi(m)^{-1}})_M(m), \end{aligned}$$

which are the generating vector fields for the twisted action. □

Symplectic reduction extends to the setting of Hamiltonian Poisson actions with K^*-valued moment maps. Suppose M is symplectic structure and that the action is proper. For any $l \in K^*$, define

$$M_l = \Psi^{-1}(l)/K_l \cong \Psi^{-1}(Kl)/K$$

where Kl is the orbit of l under the dressing action of K on K^*. Then M_l is a symplectic manifold, if the action of K on $\Psi^{-1}(Kl)$ is free [17, Theorem 4.12].

2.3. **Anti-Poisson involutions.** Recall the definition of compatible involutions from O'Shea-Sjamaar [21]. Let K be a connected Lie group, together with an involutive automorphism σ_K. Let $\sigma_{\mathfrak{k}}$ denote the corresponding Lie algebra involution, and define an involution on $\mathfrak{k}^*$ by $\sigma_{\mathfrak{k}^*} = -(\sigma_{\mathfrak{k}})^*$.

An involution $\sigma_M: M \to M$ of a symplectic manifold (M, ω) is called anti-symplectic if $\sigma_M^*\omega = -\omega$. If M carries a Hamiltonian K-action, with moment map $\Phi: M \to \mathfrak{k}^*$, then σ_M is called compatible with σ_K if

$$\Phi \circ \sigma_M = \sigma_{\mathfrak{k}^*} \circ \Phi.$$

As explained in [21], since K is connected this implies

$$\sigma_M(k.m) = \sigma_K(k).\sigma_M(m).$$

If $M \subset \mathfrak{k}^*$ is a coadjoint orbit with the Kirillov-Kostant-Souriau symplectic structure, such that M is invariant under $\sigma_{\mathfrak{k}^*}$, the involution $\sigma_M = \sigma_{\mathfrak{k}^*}|_M$ is compatible with σ_K.

Suppose K is compact. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ such that $\mathfrak{k}^\sigma \cap \mathfrak{t}$ has maximal dimension. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a positive Weyl chamber. The following theorem of O'Shea-Sjamaar describes the image of the fixed point manifold M^σ under the moment map. The special case where K is a torus and $\sigma_K(k) = k^{-1}$ is due to Duistermaat [6].

Theorem 2.2 (O'Shea-Sjamaar)**.** *Let (M, ω, Φ) be a symplectic Hamiltonian K-manifold with proper $\mathfrak{k}^*$-valued moment map, and let σ_M be a σ_K-compatible anti-symplectic involution on M. Suppose the fixed point set M^σ is non-empty. Then*

$$\Phi(M^\sigma) \cap \mathfrak{t}^*_+ = \Phi(M)^\sigma \cap \mathfrak{t}^*_+.$$

A theorem of Kirwan says that if M is compact and connected, $\Delta(M) = \Phi(M) \cap \mathfrak{t}^*_+$ is a convex polytope. By Theorem 2.2, $\Phi(M^\sigma) \cap \mathfrak{t}^*_+$ is also a polytope, obtained from the Kirwan polytope by intersecting with the subspace $(\mathfrak{k}^*)^\sigma$.

We generalize these definitions to Poisson actions and K^*-valued moment maps as follows. Let K be a connected Poisson-Lie group, together with an anti-Poisson involutive automorphism σ_K. Then $\sigma_{\mathfrak{k}^*}$ is a Lie algebra automorphism, and therefore exponentiates to a Lie group automorphism σ_{K^*} on the Poisson dual K^*. For any Hamiltonian Poisson K-manifold (M, π, Ψ) we say that an anti-Poisson involution σ_M of M is compatible with σ_K if

$$\Psi \circ \sigma_M = \sigma_{K^*} \circ \Psi. \tag{5}$$

Since K is connected, this implies $\sigma_M(k.m) = \sigma_K(k).\sigma_M(m)$. Indeed, for anti-Poisson involutions σ_M and σ_K, the composition $\sigma_{K^*} \circ \Psi \circ \sigma_M$ is Poisson, and is the moment map for the action, $(k, m) \mapsto \sigma_M(\sigma_K(k).\sigma_M(m))$. Condition (5) implies that these are the original moment map and action. Examples of Hamiltonian K-spaces with compatible involution are σ_{K^*}-invariant dressing orbits M for the action of K on K^*, with $\sigma_M = \sigma_{K^*}|_M$. If σ_M is a compatible involution of (M, π, Ψ) then it is also a compatible involution of the conjugate $(M, -\pi, \Psi^{-1})$. Similarly, if σ_{M_j} $(j = 1, 2)$ are compatible involutions of (M_j, π_j, Ψ_j), then $\sigma_{M_1} \times \sigma_{M_2}$ is a compatible involution of their product.

The fixed point set M^σ carries an action of the group K^σ. For $l \in (K^*)^\sigma$ we denote by M^σ_l the quotient

$$M^\sigma_l = \Psi^{-1}(l)^\sigma / K^\sigma_l. \tag{6}$$

2.4. **Examples of anti-Poisson involutions.** Anti-Poisson involutions of general Poisson Lie groups are studied by Hilgert-Neeb in [12, Section 3]. We will consider the special case of a compact, connected Lie group K, equipped with the Lu-Weinstein Poisson structure corresponding to an invariant inner product B on $\mathfrak{k}$. Let $\sigma_\mathfrak{g}$ be a complex anti-linear involutive automorphism of $\mathfrak{g} = \mathfrak{k}^\mathbb{C}$ preserving $\mathfrak{k}$ and $\mathfrak{k}^* = \mathfrak{a} \oplus \mathfrak{n}$, and σ_G the corresponding involution of $G = K^\mathbb{C}$. Since $\sigma_\mathfrak{g}$ is anti-linear, it preserves $\mathfrak{a} = i\mathfrak{k} \cap \mathfrak{k}^*$. In particular, $\sigma_\mathfrak{g}$ permutes the root spaces, hence preserves $\mathfrak{n}$.

Lemma 2.3. *Suppose the restriction $\sigma_\mathfrak{k}$ of $\sigma_\mathfrak{g}$ to $\mathfrak{k}$ is an isometry. Then the exponentiated automorphism σ_K of K is an anti-Poisson involution.*

We remark that if $\mathfrak{k}$ is simple then any involution preserves the Killing form, hence defines an isometry of $\mathfrak{k}$.

Proof. Since $\sigma_\mathfrak{k}$ preserves B, the anti-linear map $\sigma_\mathfrak{g}$ takes $B^\mathbb{C}$ to its complex conjugate and so changes the sign of $\langle\ ,\ \rangle = 2\,\mathrm{Im}(B^\mathbb{C})$. It follows that the involutions $\sigma_\mathfrak{k} = \sigma_\mathfrak{g}|_\mathfrak{k}$ and $\sigma_{\mathfrak{k}^*} = \sigma_\mathfrak{g}|_{\mathfrak{k}^*}$ are related by $\sigma_{\mathfrak{k}^*} = -(\sigma_\mathfrak{k})^*$. Therefore $\sigma_\mathfrak{k}$ changes the sign of the cocycle δ dual to the bracket on $\mathfrak{k}^*$. $\square$

Recall [10, p.322] that for $\mathfrak{k}$ semisimple with a given choice of Chevalley basis, any automorphism φ of the root system of $\mathfrak{k}$ defines an automorphism ζ of $\mathfrak{k}$. Let $\kappa:\ \mathfrak{g}\to\mathfrak{g}$ be the Cartan involution given by complex conjugation for $\mathfrak{g} = \mathfrak{k}^\mathbb{C}$. Then $\sigma_\mathfrak{g}^\varphi = \kappa\circ\zeta^\mathbb{C}:\ \mathfrak{g}\to\mathfrak{g}$ is an anti-linear involution preserving $\mathfrak{k},\mathfrak{k}^*$, if and only if φ takes the positive roots to the negative roots.

Lemma 2.4. *Any anti-linear involutive automorphism $\sigma_\mathfrak{g}$ of $\mathfrak{g} = \mathfrak{k}^\mathbb{C}$ preserving the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ is of the form $\sigma_\mathfrak{g} = \sigma_\mathfrak{g}^\varphi\circ\mathrm{Ad}_t$, where φ is a root system automorphism taking positive roots to negative roots, and $t\in T$ is such that $\sigma_G^\varphi(t)t$ is in the center of G.*

Proof. Since $\sigma_\mathfrak{g}$ preserves the sum $\mathfrak{n}$ of positive root spaces, the complex linear automorphism $\kappa\circ\sigma_\mathfrak{g}$ takes positive root spaces to negative roots spaces. It therefore induces a root system automorphism φ taking positive roots to negative roots. The composition $\sigma_\mathfrak{g}^\varphi\circ\sigma_\mathfrak{g}$ is a complex linear involutive automorphism acting trivially on $\mathfrak{t}$, and therefore equal to Ad_t for some $t\in T$ [10, Prop. 2.5, p. 334]. Since $\sigma_\mathfrak{g} = \sigma_\mathfrak{g}^\varphi\circ\mathrm{Ad}_t$ is an involution, $(\sigma_\mathfrak{g}^\varphi\circ\mathrm{Ad}_t)^2 = \mathrm{Ad}_{\sigma_G^\varphi(t)t}$ is the identity. $\square$

Any automorphism of the Dynkin diagram gives rise to an automorphism φ of the root system taking the positive roots to negative roots, by composing with the automorphism $\alpha\mapsto-\alpha$. Consider for example the case $G = \mathrm{Sl}(r,\mathbb{C})$ with $r\geq 3$. The trivial automorphism of the Dynkin diagram A_{r-1} induces complex conjugation on G, while the unique non-trivial automorphism induces

$$\sigma_G(g) = P(g^\dagger)^{-1}P, \tag{7}$$

where P is the anti-diagonal matrix $P_{ij} = \delta_{i,n+1-j}$. Note that the matrix P is $U(r)$-conjugate to $I_k\otimes -I_k$ if $r = 2k$, and to $I_{k+1}\otimes -I_k$ if $r = 2k+1$. Therefore, the restriction of σ_G to $K = SU(r)$ has fixed point group isomorphic to $S(U(k)\times U(k))$ if $r = 2k$, resp. $S(U(k+1)\times U(k))$ if $r = 2k+1$.

3. Linearization

In this Section we recall the notion of linearization for Lu-Weinstein moment maps, and then prove that linearization commutes with product and conjugation up to symplectomorphism. From now on, K denotes a compact, connected, Lie group with Lu-Weinstein Poisson-Lie structure, $K^* = AN$ denotes its Poisson dual, and $G = K^\mathbb{C} = KAN$ the double.

3.1. **Linearization Theorem.** In [1] the first author constructed a 1-1 correspondence between Hamiltonian K-manifolds with $\mathfrak{k}^*$-valued moment maps and with K^*-valued moment maps. To set up this correspondence one first needs an equivariant map from $\mathfrak{k}^*$ to K^*. Let $\kappa : \mathfrak{g} \to \mathfrak{g}$ be the Cartan involution given by complex conjugation of $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$, and let $\dagger : \mathfrak{g} \to \mathfrak{g}$ be the anti-involution

$$\xi^\dagger = -\kappa(\xi).$$

We also denote by $\dagger$ the induced anti-involution of G, considered as a real group. For $K = \mathrm{U}(r)$ and $G = \mathrm{Gl}(r, \mathbb{C})$, $g^\dagger = \overline{g}^t$. Let $B^\sharp : \mathfrak{k}^* \to \mathfrak{k}$ be the isomorphism given by B. For any $\mu \in \mathfrak{k}^*$, the element $g = \exp(iB^\sharp(\mu)) \in G$ admits a unique decomposition $g = ll^\dagger$, for some $l \in K^*$. It follows from the Iwasawa decomposition that the map

$$E : \mathfrak{k}^* \to K^*, \quad \mu \to l$$

is a diffeomorphism. It is K-equivariant with respect to the coadjoint action on $\mathfrak{k}^*$ and the left dressing action on K^*.

Next, we define a certain 1-form on $\mathfrak{k}^*$. Recall that $\theta^L \in \Omega^1(K^*) \otimes \mathfrak{k}^*$ is the left-invariant Maurer-Cartan form, and let $\theta^{\dagger L}$ be its image under the map $\dagger : \mathfrak{k}^* \subset \mathfrak{g} \to \mathfrak{g}$. Then $B^{\mathbb{C}}(\theta^L, \theta^{\dagger L}) \in \Omega^2(K^*)$ is imaginary-valued, and we can define a real-valued 1-form on $\mathfrak{k}^*$ by

$$\beta = \frac{1}{2i} \mathcal{H}\big(E^* B^{\mathbb{C}}(\theta^L, \theta^{\dagger L})\big) \in \Omega^1(\mathfrak{k}^*) \tag{8}$$

where $\mathcal{H} : \Omega^\star(\mathfrak{k}^*) \to \Omega^{\star-1}(\mathfrak{k}^*)$ is the standard homotopy operator for the de Rham differential.

Proposition 3.1. *The 1-form β has the following property:*

$$\iota(\xi_{\mathfrak{k}^*}) d\beta = E^* \langle \theta^R, \xi \rangle - d\langle \cdot, \xi \rangle, \quad \xi \in \mathfrak{k} \tag{9}$$

A proof of this Proposition will be given in the appendix. Suppose now that (M, Ω, Ψ) is a Hamiltonian K-space with K^*-valued moment map. Let

$$\Phi = E^{-1} \circ \Psi, \quad \omega = \Omega - \mathrm{d}\Phi^* \beta. \tag{10}$$

As an immediate consequence of Proposition 3.1, the moment map condition (3) for Ψ is equivalent to the moment map condition $\mathrm{d}\langle \Phi, \xi \rangle = \iota(\xi_M)\omega$ for the closed 2-form ω.

Theorem 3.2 (Linearization Theorem [1])**.** *Suppose M is a K-manifold. Let $\Omega, \omega \in \Omega^2(M)$ be two-forms and $\Psi : M \to K^*$, $\Phi : M \to \mathfrak{k}^*$ maps related by* (10)*. Then (M, Ω, Ψ) is a Hamiltonian K-space with K^*-valued moment map if and only if (M, ω, Φ) is a Hamiltonian K-space with $\mathfrak{k}^*$-valued moment map.*

We call (M, ω, Φ) the linearization of (M, Ω, Ψ). For example, linearization of a dressing orbit $\mathcal{D} \subset K^*$ gives the corresponding co-adjoint orbit $\mathcal{O} = E^{-1}(\mathcal{D}) \subset \mathfrak{k}^*$. Note also that since the pull-backs of Ω and ω to any level surface $\Phi^{-1}(\mu) = \Psi^{-1}(l)$ agree, for $\mu = E(l)$, there is a canonical isomorphism of symplectic quotients

$$M_\mu \cong M_l$$

of (M, ω, Φ) at μ and of (M, Ω, Ψ) at l.

Remark 3.3. A different linearization of a Hamiltonian Poisson K-space (M, Ω, Ψ) may be constructed from a Poisson diffeomorphism $E' : \mathfrak{k}^* \to K^*$ given by P. Boalch in [4]. The triple $(M, \Omega, E' \circ \Psi)$ is then an ordinary Hamiltonian K-space, but for a different K-action.

3.2. **Moser isotopy lemma.** We will need the following variation of Moser's argument.

Lemma 3.4. *Let (M, ω^s, Φ^s) be a family of compact Hamiltonian K-manifolds, $s \in [0, 1]$. For $\xi \in \mathfrak{k}$ let ξ_M^s denote the Hamiltonian vector field for (M, ω^s, Φ^s). Suppose ω^s and Φ^s depend smoothly on s and that there exists a smooth family of 1-forms α^s such that*

$$\dot{\omega}^s = d\alpha^s, \tag{11}$$

where the dot stands for $\frac{d}{ds}$. Assume that for all elements $\xi \in \mathfrak{k}^K$,

$$\langle \dot{\Phi}^s, \xi \rangle + \iota(\xi_M^s)\alpha^s = 0. \tag{12}$$

Then there is a smooth isotopy $\phi^s : M \to M$ which intertwines the K-actions for the parameters $0, s$ and which satisfies

$$(\phi^s)^*\omega^s = \omega^0, \quad (\phi^s)^*\Phi^s = \Phi^0.$$

Given a family of anti-symplectic involutions σ_M^s of (M, ω^s, Φ^s), such that each α^s is σ_M^s-anti-invariant, one can arrange that $\phi^s \circ \sigma_M^0 = \sigma_M^s \circ \phi^s$.

Proof. For each $s \in [0, 1]$ let $j^s : M \to \tilde{M} := [0, 1] \times M$ be the inclusion $j^s(m) = (s, m)$. Equip $\tilde{M}$ with the K-action such that the maps j^s are equivariant, with respect to the K-action on M defined by ω^s, Φ^s. Define $\Phi \in C^\infty(\tilde{M}) \otimes \mathfrak{k}^*$ by $(j^s)^*\Phi = \Phi^s$, and let

$$\tilde{\omega} = \omega + \mathrm{d}s \wedge \alpha \in \Omega^2(\tilde{M})$$

where ω, α pull-back to ω^s, α^s under j^s and vanish on $\frac{\partial}{\partial s}$. Then (11) is equivalent to

$$\mathrm{d}\tilde{\omega} = 0 \tag{13}$$

and (12) is equivalent to the moment map condition

$$\mathrm{d}\langle \Phi, \xi \rangle = \iota(\xi_{\tilde{M}})\tilde{\omega}, \ \xi \in \mathfrak{k}^K. \tag{14}$$

These two equations also hold for the average of $\tilde{\omega}$ under the K-action. Since

$$L(\xi_{\tilde{M}})\omega = \iota(\xi_{\tilde{M}})\mathrm{d}\omega + \mathrm{d}\iota(\xi_{\tilde{M}})\omega = -\mathrm{d}s \wedge \iota(\xi_{\tilde{M}})\dot{\omega},$$

the averaging process changes only α, but not ω. We may therefore assume that $\tilde{\omega}$ is K-invariant.

Let $\tilde{X}$ be the unique vector field on $\tilde{M}$ such that $\iota(\tilde{X})\tilde{\omega} = 0$ and $\iota(\tilde{X})\mathrm{d}s = 1$. It is K-invariant, preserves $\tilde{\omega}$, and its flow $\tilde{\phi}^s$ takes the slice at 0 to that at s. Let ϕ^s be the isotopy of M defined by $\tilde{\phi}^s \circ j^0 = j^s \circ \phi^s$. Then $(\tilde{\phi}^s)^*\tilde{\omega} = \tilde{\omega}$ implies $(\phi^s)^*\omega^s = \omega^0$. Similarly, for $\xi \in \mathfrak{k}^K$ we have

$$L(\tilde{X})\langle \Phi, \xi \rangle = \iota(\tilde{X})\mathrm{d}\langle \Phi, \xi \rangle = \iota(\tilde{X})\iota(\xi_{\tilde{M}})\tilde{\omega} = 0.$$

This shows $(\tilde{\phi}^s)^*\langle\Phi,\xi\rangle = \langle\Phi,\xi\rangle$, or equivalently $(\phi^s)^*\langle\Phi^s,\xi\rangle = \langle\Phi^0,\xi\rangle$. K-equivariance of the flow $\tilde{\phi}^s$ implies that ϕ^s intertwines the K-actions on M for the parameters $0, s$. Since the moment maps are determined up to a constant in $(\mathfrak{k}^*)^K$, this proves $(\phi^s)^*\Phi^s = \Phi^0$.

In the presence of a family of anti-symplectic involutions with $(\sigma_M^s)^*\alpha^s = -\alpha^s$, the 2-form $\tilde{\omega}$ changes sign under the corresponding involution $\sigma_{\tilde{M}}$ of $\tilde{M}$. The vector field $\tilde{X}$, and therefore its flow, are $\sigma_{\tilde{M}}$-invariant. Equivalently, $\phi^s \circ \sigma_M^0 = \sigma_M^s \circ \phi^s$. □

3.3. **Linearization commutes with products and conjugation.**

Theorem 3.5. *Let (M_j,Ω_j,Ψ_j) be two compact Hamiltonian K-spaces with K^*-valued moment maps and (M_j,ω_j,Φ_j) their linearizations. Consider the products*

$$\begin{aligned}(M,\Omega,\Psi) &= (M_1\times M_2,\Omega_1+\Omega_2,\Psi_1\Psi_2)\\ (M,\omega,\Phi) &= (M_1\times M_2,\omega_1+\omega_2,\Phi_1+\Phi_2).\end{aligned}$$

The Hamiltonian K-space (M,ω,Φ) is equivariantly symplectomorphic to the linearization of (M,Ω,Ψ). That is, there exists a diffeomorphism ϕ of M which takes the diagonal K-action to the twisted diagonal action, and satisfies

$$\phi^*\Omega = \omega + d\Phi^*\beta, \quad \phi^*\Psi = E\circ\Phi.$$

In particular, this implies that $M_1\times M_2$ is isomorphic as a Hamiltonian Poisson manifold to $M_2\times M_1$, which is not at all obvious from the definition. It would be interesting to know whether the category of Hamiltonian Poisson manifolds admits the structure of a *braided* tensor category.

Proof. Recall that the definition of a K^*-valued moment map depends on the inner product B on $\mathfrak{k}$. For any $s>0$ consider the rescaled inner product $B^s = s^{-1}B$, and let $\zeta^s : \mathfrak{k}^*\to\mathfrak{k}^*,\ \mu\mapsto s\mu$. Replacing B with B^s replaces the map E by $E^s = (\zeta^s)^*E$ and the form β by $\beta^s = s^{-1}(\zeta^s)^*\beta$. We obtain a family $(M_j,\Omega_j^s,\Psi_j^s)$ of Hamiltonian K-spaces with K^*-valued moment map (relative to B^s), with

$$\Omega_j^s = \omega_j + \mathrm{d}\Phi_j^*\beta^s, \quad \Psi_j^s = E^s\circ\Phi_j.$$

Taking the linearizations of their products

$$(M,\Omega^s,\Psi^s) = (M_1\times M_2,\Omega_1^s+\Omega_2^s,\Psi_1^s\Psi_2^s)$$

we obtain a family of Hamiltonian K-spaces (M,ω^s,Φ^s) where

$$\begin{aligned}E^s\circ\Phi^s &= (E^s\circ\Phi_1)(E^s\circ\Phi_2)\\ \omega^s &= \omega + \mathrm{d}(\Phi-\Phi^s)^*\beta^s\end{aligned}$$

Consider the limit $s\searrow 0$. The family of moment maps Φ^s extends smoothly to $s=0$ by $\Phi^0=\Phi$. Since the family of 1-forms β^s extends smoothly to $s=0$ by $\beta^0=0$, ω^s extends smoothly to $s=0$ by $\omega^0=\omega$. We thus have a family of compact Hamiltonian K-spaces,

(M, ω^s, Φ^s), $\quad s \in [0,1]$ connecting (M, ω, Φ) with the linearization of (M, Ω, Ψ). The proof is completed by an application of Lemma 3.4, with

$$\alpha^s = \frac{\mathrm{d}}{\mathrm{d}s}(\Phi - \Phi^s)^* \beta^s.$$

To check the condition (12) for $\xi \in \mathfrak{k}^K$, we note that the first term vanishes in our case since $\langle \Phi^s, \xi \rangle$ is independent of s. Since $\iota(\xi_M^s)(\Phi^s)^* \mathrm{d}\beta = 0$, also ξ_M^s is independent of s, and therefore

$$\iota(\xi_M^s)\alpha^s = \frac{\mathrm{d}}{\mathrm{d}s}\iota(\xi_M^s)(\Phi - \Phi^s)^* \beta^s = 0.$$

□

The following corollary of Theorem 3.5 is important for the proof of the Thompson conjecture in the next Section.

Corollary 3.6. *Under conditions of Theorem 3.5 the reduced spaces $(M_1 \times M_2)_l$ at $l \in K^*$ and of $(M_1 \times M_2)_\mu$ at $\mu = E^{-1}(l)$ are symplectomorphic.*

Theorem 3.7 (Linearization commutes with conjugation)**.** *Let (M, Ω, Ψ) be a compact Hamiltonian K-manifold with K^*-valued moment map, and (M, ω, Φ) its linearization. Consider the conjugates*

$$\begin{aligned} (M, \tilde{\Omega}, \tilde{\Psi}) &= (M, -\Omega, \Psi^{-1}), \\ (M, \tilde{\omega}, \tilde{\Phi}) &= (M, -\omega, -\Phi). \end{aligned}$$

There exists an equivariant symplectomorphism between $(M, \tilde{\omega}, \tilde{\Phi})$ and the linearization of $(M, \tilde{\Omega}, \tilde{\Psi})$. That is, there exists a diffeomorphism ϕ of M, which intertwines the twisted action with the original K-action on M, and satisfies

$$\begin{aligned} \phi^* \tilde{\Psi} &= E \circ \tilde{\Phi}, \\ \phi^* \tilde{\Omega} &= \tilde{\omega} + d\tilde{\Phi}^* \beta \end{aligned}$$

Proof. We proceed as in the proof of Theorem 3.2. Replacing B with B^s we obtain a family (M, Ω^s, Ψ^s) of Hamiltonian K-manifolds with K^*-valued moment maps (relative to B^s) with

$$\Psi^s = E^s \circ \Phi, \quad \Omega^s = \omega + \mathrm{d}\Phi^* \beta^s.$$

Conjugating and linearizing we obtain a family $(M, \tilde{\omega}^s, \tilde{\Phi}^s)$ of Hamiltonian K-manifolds with

$$E^s \circ \tilde{\Phi}^s = (E^s \circ \Phi)^{-1}$$

and

$$\tilde{\omega}^s = -\omega - \mathrm{d}(\Phi + \tilde{\Phi}^s)^* \beta^s$$

These families extend smoothly to $s = 0$ by $\tilde{\omega}^0 = -\omega$ and $\tilde{\Phi}^0 = -\Phi$, and connect the linearization of $(M, \tilde{\Omega}, \tilde{\Psi})$ with the space $(M, \tilde{\omega}, \tilde{\Phi})$. Therefore, the claim again follows from Lemma 3.4. □

3.4. **Linearization and anti-symplectic involutions.** Suppose σ_K is an involution of K of the type described in Section 2.4. That is, the corresponding Lie algebra involution $\sigma_{\mathfrak{k}}$ is an isometry with respect to B, and extends to a $\mathbb{C}$-anti-linear involution $\sigma_{\mathfrak{g}}$ preserving $\mathfrak{k}^*$. Letting σ_{K^*} be the induced involution of K^*, we have

$$E \circ \sigma_{\mathfrak{k}^*} = \sigma_{K^*} \circ E \tag{15}$$

by the calculation,

$$\exp(iB^\sharp(\sigma_{\mathfrak{k}^*}(\mu))) = \exp(-i\sigma_{\mathfrak{k}}B^\sharp(\mu)) = \sigma_G(\exp(iB^\sharp(\mu))) = \sigma_{K^*}(l)\sigma_{K^*}(l)^\dagger.$$

The 1-form β defined in (8) changes its sign,

$$(\sigma_{\mathfrak{k}^*})^*\beta = -\beta. \tag{16}$$

Suppose now that (M, Ω, Ψ) is a Hamiltonian K-space with K^*-valued moment map, and (M, ω, Φ) its linearization. Equations (15) and (16) show that an involution σ_M of M is anti-symplectic for Ω and σ_K-compatible with Ψ if and only if it is anti-symplectic for ω and σ_K-compatible for Φ. For $\mu \in (\mathfrak{k}^*)^\sigma$, $l = E^{-1}(\mu) \in (K^*)^\sigma$ one has a homeomorphism of quotients,

$$M_\mu^\sigma \cong M_l^\sigma.$$

Using the last part of Lemma 3.4 one obtains the following extensions of Theorem 3.5, 3.2. In Theorem 3.5, given σ_K-compatible anti-symplectic involutions $\sigma_{M_1}, \sigma_{M_2}$, the diffeomorphism ϕ can be chosen to be $\sigma_M = (\sigma_{M_1}, \sigma_{M_2})$-equivariant, and assuming $\mu \in (\mathfrak{k}^*)^\sigma$, one has a homeomorphism

$$(M_1 \times M_2)_l^\sigma \cong (M_1 \times M_2)_\mu^\sigma. \tag{17}$$

Similarly, in Theorem 3.7 the diffeomorphism ϕ can be chosen to be equivariant with respect to a given anti-symplectic involution σ_M.

4. The Thompson conjecture for complex and real matrices

In this Section we apply our results to give a new proof of the Thompson conjecture on singular values of complex matrices and to extend this result to real matrices.

4.1. **Moduli spaces for additive and multiplicative problems.** Let $\mathcal{O}_1, \ldots, \mathcal{O}_n \subset \mathfrak{k}^*$ be given coadjoint orbits, and $\mathcal{D}_j = E(\mathcal{O}_j) \subset K^*$ the corresponding dressing orbits. Also let $\mathcal{C}_i = \mathcal{D}_i K = K g_i K \subset G$ denote the double coset containing $\mathcal{D}_i$. Consider the following three moduli spaces,

$$\begin{aligned}
\mathcal{M}_\mathcal{O} &= \{(\xi_1, \ldots, \xi_n) \in \mathcal{O}_1 \times \ldots \times \mathcal{O}_n |\ \xi_1 + \cdots + \xi_n = 0\}/K, \\
\mathcal{M}_\mathcal{D} &= \{(g_1, \ldots, g_n) \in \mathcal{D}_1 \times \ldots \times \mathcal{D}_n |\ g_1 \cdots g_n = e\}/K, \\
\mathcal{M}_\mathcal{C} &= \{(g_1, \ldots, g_n) \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_n | g_1 \cdots g_n = e\}/K^n,
\end{aligned}$$

where in the last line K^n acts as follows,

$$(k_1, \ldots, k_n).(g_1, \ldots, g_n) = (k_1 g_1 k_2^{-1}, k_2 g_2 k_3^{-1}, \ldots, k_n g_n k_1^{-1}).$$

Lemma 4.1. *The natural map $\mathcal{M}_\mathcal{D} \to \mathcal{M}_\mathcal{C}$ is a homeomorphism.*

Proof. Given a solution $(g_1,\ldots,g_n)\in\mathcal{C}_1\times\ldots\times\mathcal{C}_n$ with product $\prod g_j=e$, define $k_j\in K$ recursively as follows: put $k_1=e$, let $k_2\in K$ be the unique element with $g_1k_2^{-1}\in K^*$, then let $k_3\in K$ the unique element with $k_2g_2k_3^{-1}\in K^*$, and so on. Let $(l_1,\ldots,l_n)\in G^n$ be the image of $(g_1,\ldots,g_n)$ by the action of $(k_1,\ldots,k_n)$. By construction $l_j\in K^*$ for $j<n$, and since the product is e we must have $l_n\in K^*$. This shows that the map $\mathcal{M}_\mathcal{D}\to\mathcal{M}_\mathcal{C}$ is surjective. Starting the recursion with $k_1=k$ rather than $k_1=e$ replaces $(l_1,\ldots,l_n)$ by its image under the diagonal dressing action of k. This shows that the map is a bijection. □

Corollary 3.6 shows that there exists a symplectomorphism between $\mathcal{M}_\mathcal{O}$ and $\mathcal{M}_\mathcal{D}$. It follows that the three moduli spaces are all homeomorphic:

$$\mathcal{M}_\mathcal{O}\cong\mathcal{M}_\mathcal{D}\cong\mathcal{M}_\mathcal{C}. \tag{18}$$

Given a $\mathbb{C}$-antilinear involution $\sigma_\mathfrak{g}$ of $\mathfrak{g}=\mathfrak{k}^\mathbb{C}$ preserving $\mathfrak{k},\mathfrak{k}^*$ and the inner product B on $\mathfrak{k}$, we can similarly consider moduli spaces

$$\begin{aligned}
\mathcal{M}_\mathcal{O}^\sigma &= \{(\xi_1,\ldots,\xi_n)\in\mathcal{O}_1^\sigma\times\ldots\times\mathcal{O}_n^\sigma|\ \xi_1+\cdots+\xi_n=0\}/K^\sigma,\\
\mathcal{M}_\mathcal{D}^\sigma &= \{(g_1,\ldots,g_n)\in\mathcal{D}_1^\sigma\times\ldots\times\mathcal{D}_n^\sigma|\ g_1\cdots g_n=e\}/K^\sigma,\\
\mathcal{M}_\mathcal{C}^\sigma &= \{(g_1,\ldots,g_n)\in\mathcal{C}_1^\sigma\times\ldots\times\mathcal{C}_n^\sigma|g_1\cdots g_n=e\}/(K^\sigma)^n.
\end{aligned}$$

Again we find $\mathcal{M}_\mathcal{D}^\sigma\cong\mathcal{M}_\mathcal{C}^\sigma$, and together with (17) we obtain homeomorphisms

$$\mathcal{M}_\mathcal{O}^\sigma\cong\mathcal{M}_\mathcal{D}^\sigma\cong\mathcal{M}_\mathcal{C}^\sigma. \tag{19}$$

4.2. **Thompson conjecture.** We now specialize to the case of $K=\mathrm{U}(r)$, $G=K^\mathbb{C}=\mathrm{Gl}(r,\mathbb{C})$. The Lie algebra $\mathfrak{k}$ consists of anti-Hermitian matrices. Identify $\mathfrak{k}^*$ with Hermitian matrices by the pairing,

$$\langle\mu,\xi\rangle=\frac{1}{i}\,\mathrm{tr}(\mu\xi).$$

The orbits $\mathcal{O}_j\subset\mathfrak{k}^*$ consist of Hermitian matrices with prescribed eigenvalues $\lambda_j^1,\ldots,\lambda_j^r$. On the other hand, the double coset spaces $\mathcal{C}_j\subset G$ consist of matrices with positive determinant and prescribed *singular* values $\Lambda_j^1,\cdots,\Lambda_j^r$. (Recall that the singular values of a matrix A are the eigenvalues of $AA^\dagger$.) Therefore, the equality of moduli spaces $\mathcal{M}_\mathcal{O}\cong\mathcal{M}_\mathcal{C}$ has the following consequence.

Theorem 4.2. *Let $\lambda_j^k\in\mathbb{R}$, be given real numbers, $1\le j\le n,\ 1\le k\le r$. The following four conditions are equivalent:*

(a) *there exist complex matrices A_j with singular values* $\exp(\lambda_j^k)$ *and product $A_1\cdots A_n=I$;*
(b) *there exist self-adjoint matrices B_j with eigenvalues λ_j^k and sum $B_1+\cdots+B_n=0$;*
(c) *there exist real matrices A_j with singular values* $\exp(\lambda_j^k)$ *and product $A_1\cdots A_n=I$;*
(d) *there exist real symmetric matrices B_j with eigenvalues λ_j^k and sum $B_1+\cdots+B_n=0$.*

Proof. The equivalence of (a) and (b), first proved by Klyachko in [15], follows from (18). The equivalence of (c) and (d) follows from (19). The equivalence of (b) and (d) follows from Theorem 2.2, since σ acts trivially on the Cartan in this case. It was proved independently by Fulton [9]. □

We note that in a different work Klyachko [14] gave an inequality description of the set of coadjoint orbits for which the additive problem admits a solution. This result was generalized to arbitrary compact Lie groups by Berenstein-Sjamaar [3]. Theorem 4.2 implies the same inequality description for the multiplicative problem for real matrices.

The more general involutions σ_K discussed in Section 2.4 yield "twisted" versions of the Thompson conjecture. For example, from the involution (7) we obtain

Theorem 4.3. *Let P be the anti-diagonal $n \times n$-matrix $P_{ij} = \delta_{i,n+1-j}$. Let $\lambda_j^k \in \mathbb{R}$, $j = 1, \ldots, n$, $k = 1, \ldots, r$ be given real numbers. Then the following two conditions are equivalent:*

(a) *there exist complex matrices A_j satisfying $PA_j^\dagger P = A_j^{-1}$, with singular values $\exp(\lambda_j^k)$ and product $A_1 \cdots A_n = I$;*
(b) *there exist self-adjoint matrices B_j anti-commuting with P, with eigenvalues λ_j^k and sum $B_1 + \cdots + B_n = 0$.*

Inequality descriptions for additive problems involving involutions σ_K are provided by O'Shea-Sjamaar [21].

5. Volume forms

Let (M, Ω, Ψ) be a Hamiltonian K-space with K^*-valued moment map. Since the symplectic form is not preserved by the K-action, the symplectic volume form $(\exp \Omega)_{[top]}$ is not K-invariant in general. We will show in this Section that one obtains a K-invariant volume form if one multiplies by the pull-back of a certain multiplicative character of K^*. Similar volume forms were studied by Lu in the context of Bruhat-Poisson structures on flag manifolds [18]. In the case of dressing orbits, the volumes agree with the ones considered by Klyachko.

Let $\delta : \mathfrak{k} \to \wedge^2 \mathfrak{k}$ be the co-bracket defining the Lu-Weinstein structure on K^*. It is a 1-cocycle for the adjoint representation of $\mathfrak{k}$:

$$[\xi, \delta(\eta)] - [\eta, \delta(\xi)] - \delta([\xi, \eta]) = 0, \tag{20}$$

using the Schouten bracket on $\wedge \mathfrak{k}$. The cocycle property (20) of δ implies that the operators

$$\mathbf{L}_\xi = L(\xi_M) - \tfrac{1}{2}\iota(\delta(\xi)_M), \quad \xi \in \mathfrak{k}$$

define a representation of $\mathfrak{k}$ on the space $\Omega(M)$ of differential forms. We will construct a differential form Γ on M which is invariant under this $\mathfrak{k}$-representation and such that the top degree part $\Gamma_{[top]}$ is a volume form. Since the operators $\iota(\delta(\xi)_M)$ lower the degree, $\Gamma_{[top]}$ is then invariant under the usual K-action.

The definition involves the modular function $\tau : K^* \to \mathbb{R}_{>0}$ for the group $K^* = AN$, i.e. $\tau(g)$ is the determinant of the adjoint representation of K^* on $\mathfrak{k}^*$. One finds

$$\tau(\exp \mu) = e^{-4\pi\langle \mu, \rho^\sharp \rangle}, \quad \mu \in \mathfrak{k}^*.$$

Here $\rho \in \mathfrak{t}^*$ is the half-sum of positive roots, and $\rho^\sharp = B^\sharp(\rho) \in \mathfrak{t}$.

Theorem 5.1. *Let (M, Ω, Ψ) be a K^*-valued Hamiltonian K-space. The differential form $\Gamma = \frac{\exp(\Omega)}{\Psi^* \tau^{1/2}}$ is invariant under the action of the operators $\mathbf{L}_\xi$. Hence its top form degree part*

$$\Gamma_{[top]} = \frac{\exp(\Omega)_{[top]}}{\Psi^* \tau^{1/2}}$$

is a K-invariant volume form on M.

Proof. The exterior differential of τ is given by

$$\mathrm{d}\tau = -\tau \, 4\pi \, \langle \theta^R, \rho^\sharp \rangle.$$

This shows $L(\xi_{K^*})\tau = -\tau \; 4\pi \, \iota(\xi_M)\langle \theta^R, \rho^\sharp\rangle$, and together with $\mathcal{L}(\xi_M)\Omega = \mathrm{d}\langle \Psi^*\theta^R, \xi\rangle$ yields,

$$\mathcal{L}(\xi_M)\Gamma = \Gamma \; \Psi^*\big(\mathrm{d}\langle \theta^R, \xi\rangle + 2\pi \, \iota(\xi_{K^*})\langle \theta^R, \rho^\sharp\rangle\big). \tag{21}$$

To compute $\iota(\delta(\xi)_M)\Gamma$, observe first that by the moment map condition, the contraction of $\exp(\Omega)$ with any bivector field of the form $(\xi_1 \wedge \xi_2)_M$ for $\xi_j \in \mathfrak{k}$ is given by

$$\iota((\xi_1 \wedge \xi_2)_M) \exp(\Omega) = \Big(\langle \Psi^*\theta^R, \xi_1\rangle \langle \Psi^*\theta^R, \xi_2\rangle + \iota((\xi_1 \wedge \xi_2)_M)\Omega \Big) \exp(\Omega).$$

The bivector field $\delta(\xi)_M$ is a linear combination of such terms. Using the defining property $\langle \mu_1 \wedge \mu_2, \delta(\xi)\rangle = \langle [\mu_1, \mu_2], \xi\rangle$ of the cocycle, the first summand simplifies and we obtain

$$\tfrac{1}{2}\iota(\delta(\xi)_M)\Gamma = \Big(\tfrac{1}{2}\Psi^*\langle [\theta^R, \theta^R], \xi\rangle + \tfrac{1}{2}\iota(\delta(\xi)_M)\Omega \Big)\Gamma \tag{22}$$

By the structure equation $\mathrm{d}\theta^R = \frac{1}{2}[\theta^R, \theta^R]$, the first terms in (21) and (22) agree. By the following Proposition 5.2 the second terms agree as well. $\square$

Proposition 5.2. *Let (M, Ω, Ψ) be a K^*-valued Hamiltonian K-space. For all $\xi \in \mathfrak{k}$, the contractions of Ω with the bivector field $\delta(\xi)_M$ are given by the formula,*

$$\iota(\delta(\xi)_M)\Omega = 4\pi \, \iota(\xi_M)\Psi^*\langle \theta^R, \rho^\sharp\rangle.$$

The proof of this Proposition is deferred to Appendix B. Now let $J_h^{1/2} : \mathfrak{k} \to \mathbb{R}_{>0}$ be the unique K-invariant function

$$J_h^{1/2}(\xi) = \prod_{\alpha \succ 0} \frac{\sinh \pi\langle \alpha, \xi\rangle}{\pi\langle \alpha, \xi\rangle} \tag{23}$$

for $\xi \in \mathfrak{t}$, where the product is over positive (real) roots of K. Recall that the Duflo factor $J^{1/2} : \mathfrak{k} \to \mathbb{R}$ (square root of the Jacobian of the exponential map) is given by a similar equation but with sin rather than sinh. We therefore call $J_h^{1/2}$ the *hyperbolic Duflo factor*. Using the isomorphism $B^\sharp : \mathfrak{k}^* \to \mathfrak{k}$ we will view $J_h^{1/2}$ as a function on $\mathfrak{k}^*$.

Theorem 5.3. *Let (M, Ω, Ψ) be a K^*-valued Hamiltonian K-space, and (M, ω, Φ) its linearization. The top form degree parts of $\Gamma = \Psi^* \tau^{-1} \exp(\Omega)$ and $\exp(\omega)$ are related by the hyperbolic Duflo factor:*

$$\Gamma_{[top]} = \Phi^* J_h^{1/2} \exp(\omega)_{[top]}$$

Proof. Since both sides are K-invariant, it suffices to verify the identity at points of $m \in \Phi^{-1}(\mathfrak{t}^*) = \Psi^{-1}(A)$. Let $\mu = \Phi(m) \in \mathfrak{t}^*$ and $g = \Psi(m) \in A$. Then $g = E(\mu) = \exp(i\zeta/2)$ where $\zeta = B^\sharp(\mu) \in \mathfrak{t}$, and we have

$$\tau(g)^{1/2} = e^{-2\pi\langle \mu, \rho^\sharp\rangle} = e^{-2\pi\langle \rho, \zeta\rangle}$$

Let $U \subset K^*$ be a slice at μ for the coadjoint-action on $\mathfrak{k}^*$. There is a splitting,

$$T_\mu \mathfrak{k}^* = T_\mu U \oplus T_\mu(G \cdot \mu) = T_\mu U \oplus \mathfrak{k}_\mu^\perp \tag{24}$$

where $\mathfrak{k}_\mu^\perp$ (the orthogonal complement of the isotropy algebra) is embedded via the generating vector fields. Let $Y = \Phi^{-1}(U)$. By the Guillemin-Sternberg symplectic cross-section theorem, Y is a symplectic submanifold, and the embedding $\mathfrak{k}_\mu^\perp \to T_m M$ given by the generating vector fields defines an ω-orthogonal splitting

$$T_m M = T_m Y \oplus \mathfrak{k}_\mu^\perp, \tag{25}$$

where the 2-form on $\mathfrak{k}_\mu^\perp$ is given by the Kirillov-Kostant-Souriau formula,

$$\omega(\xi_1, \xi_2) = \langle \mu, [\xi_1, \xi_2]\rangle.$$

Let $e_\alpha \in \mathfrak{n}$ be root vectors for the positive roots α, normalized by $B(e_\alpha, e_{-\alpha}) = 1$. Then $\mathrm{Re}(e_\alpha), \mathrm{Im}(e_\alpha)$ form a basis of $\mathfrak{t}^\perp$, and $\mathfrak{k}_\mu^\perp$ is the subspace corresponding to roots with $\langle \alpha, \zeta\rangle \neq 0$. By a short calculation,

$$\omega_\mu(\mathrm{Re}(e_\alpha), \mathrm{Im}(e_\alpha)) = \pi\langle \alpha, \zeta\rangle.$$

The splitting (25) is also Ω-orthogonal. The pull-backs Ω_Y and ω_Y to Y differ by the pull-back by $\Phi|_Y$ of the 2-form $\varpi = \mathrm{d}\beta$. Since $\ker(\mathrm{d}_m\Phi) \cap T_m Y$ is a co-isotropic subspace of $T_m Y$ and ω_Y, Ω_Y agree on that subspace, it follows that the top exterior powers of ω_Y and Ω_Y are equal. Therefore,

$$\exp(\Omega_m)_{[top]} = \exp(\omega_m)_{[top]} \prod_{\alpha \succ 0,\ \langle \alpha, \zeta\rangle \neq 0} \frac{\Omega_g(\mathrm{Re}(e_\alpha), \mathrm{Im}(e_\alpha))}{\omega_\mu(\mathrm{Re}(e_\alpha), \mathrm{Im}(e_\alpha))}.$$

Since

$$\Omega_g(e_\alpha, e_{-\alpha}) = e^{-\pi\langle \alpha, \zeta\rangle} \sinh(\pi\langle \alpha, \zeta\rangle)$$

this gives,

$$\exp(\Omega_m)_{[top]} = \exp(\omega_m)_{[top]}\ e^{-2\pi\langle \rho, \zeta\rangle} \prod_{\alpha \succ 0,\ \langle \alpha, \zeta\rangle \neq 0} \frac{\sinh(\pi\langle \alpha, \zeta\rangle)}{\pi\langle \alpha, \zeta\rangle}.$$

as required. □

Remark 5.4. The proof has not actually used non-degeneracy of the 2-forms ω resp. Ω. Since $\Gamma_{[top]}$ is a volume form if and only if Ω is non-degenerate, we have re-proved the second half of Theorem 3.2: The 2-form ω of the linearization is non-degenerate if and only if the 2-form Ω is non-degenerate.

6. DH-measures and the hyperbolic Duflo isomorphism

In this Section we identify $\mathfrak{k} \cong \mathfrak{k}^*$ using $B^\sharp$. In particular $E : \mathfrak{k} \to K^* = AN$ is the map such that

$$\exp(i\xi) = E(\xi)E(\xi)^\dagger$$

for $\xi \in \mathfrak{k}$. We will think of E as some kind of exponential map, and define a *hyperbolic Duflo map*

$$\mathbf{D}_h = E_* \circ J_h^{1/2} : \ \mathcal{E}'(\mathfrak{k}) \to \mathcal{E}'(K^*)$$

in analogy to the usual Duflo map $\mathbf{D} = \exp_* \circ J^{1/2} : \ \mathcal{E}'(\mathfrak{k}) \to \mathcal{E}'(K)$. (Here $\mathcal{E}'(\cdot)$ denotes the space of compactly supported distributions.) Recall that **DR** is a ring homomorphism if restricted to invariant distributions. Using Theorem 3.5 we will show that the same holds true for the hyperbolic Duflo map $\mathbf{D}_h$.

For any compact $\mathfrak{k}$-valued Hamiltonian K-space (M, ω, Φ), the Duistermaat-Heckman measure is the compactly supported distribution on $\mathfrak{k}$ given as a push-forward of the Liouville measure,

$$u := \Phi_* \ |(e^\omega)_{[top]}| \in \mathcal{E}'(\mathfrak{k})^K$$

where $\mathcal{E}'(\cdot)$ denotes the space of compactly supported distributions. Similarly, for the corresponding K^*-valued Hamiltonian K-space (M, Ω, Ψ) we define a DH-measure

$$\mathfrak{m} := \tau^{-1} \ \Psi_* \ |(e^\Omega)_{[top]}| \in \mathcal{E}'(K^*)^K.$$

It is an immediate consequence of 5.3 that the two measures are related by

$$\mathfrak{m} = \mathbf{D}_h(u). \tag{26}$$

Now suppose (M_j, Ω_j, Ψ_j) are two K^*-valued Hamiltonian K-spaces, and (M_j, ω_j, Φ_j) their linearizations. Let $\mathfrak{m}_j$, u_j denote the respective DH-measures, so that $\mathfrak{m}_j = \mathbf{D}_h(u_j)$. The DH-measures for the product $(M_1 \times M_2, \Omega_1 + \Omega_2, \Psi_1\Psi_2)$ is the convolution on the group K^*,

$$\mathfrak{m} = \mathfrak{m}_1 *_{K^*} \mathfrak{m}_2,$$

while the DH-measure for $(M_1 \times M_2, \omega_1 + \omega_2, \Phi_1 + \Phi_2)$ is a convolution on the vector space $\mathfrak{k}$,

$$u = u_1 *_{\mathfrak{k}} u_2.$$

Since products commute with linearizations up to symplectomorphism (Theorem 3.5), we conclude that $\mathfrak{m} = \mathbf{D}_h(u)$. Thus

$$\mathbf{D}_h(u_1) *_{K^*} \mathbf{D}_h(u_2) = \mathbf{D}_h(u_1 *_{\mathfrak{k}} u_2) \tag{27}$$

for any distributions u_1, u_2 given as DH-measures of Hamiltonian K-spaces. In particular, it holds for DH-measures of coadjoint orbits; this is one of the results of Klyachko [15]. Since linear combinations of delta distributions are dense in the space $\mathcal{E}'(\mathfrak{k})$ of compactly supported distributions, linear combinations of DH-measures of coadjoint orbits are dense in the space $\mathcal{E}'(\mathfrak{k})^K$ of invariant compactly supported distributions, by averaging. Therefore, (27) holds for *arbitrary* elements $u_j \in \mathcal{E}'(\mathfrak{k})^K$. This gives,

Theorem 6.1 (Hyperbolic Duflo theorem)**.** *The map*

$$\mathbf{D_h} = E_* \circ J_h^{1/2} : \mathcal{E}'(\mathfrak{k}) \to \mathcal{E}'(K^*)$$

is a ring isomorphism if restricted to K-invariant distributions. That is, (27) *holds for all $u_1, u_2 \in \mathcal{E}'(\mathfrak{k})^K$.*

As pointed out by the referee, this result was proved in a more general setting by Rouvière [22] (see in particular Theorem 7.4). Namely, for any symmetric space $S = G/K$ consider the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ into ± 1 eigenspaces of the involution. Assume there exists an invariant measure on G/K, and let $J_\mathfrak{s}^{1/2}$ denote the square root of the Jacobian of the exponential map $\exp_\mathfrak{s} : \mathfrak{s} \to S$. Rouvière proved that in a number of cases (including the case considered here, and also the case G solvable) the map $\mathbf{D}_\mathfrak{s} = (\exp_\mathfrak{s})_* \circ J_\mathfrak{s}^{1/2}$ takes convolution of K-invariant distributions on $\mathfrak{s}$ (of suitable support) to convolution on S.

If K is the maximal compact subgroup of a semisimple group G with finite center, the algebras of invariant compactly supported distributions on S and on $\mathfrak{s}$ are known to be isomorphic by theorems of Paley-Wiener type [11, Chapter 3]. If $\mathfrak{g}$ is quadratic, Torossian [23] proves an isomorphism between the algebras of K-invariant distributions on $\mathfrak{s}$ supported at 0 and on S supported at eK.

Appendix A. Proof of Proposition 3.1

In this section we prove the property (9) of the 1-form β used in the linearization construction:

$$\iota(\xi_{\mathfrak{k}^*})\mathrm{d}\beta = 2E^* \,\mathrm{Im}\, B^\mathbb{C}(\theta^R, \xi) - \mathrm{d}\langle \cdot, \xi\rangle, \quad \xi \in \mathfrak{k}. \tag{28}$$

Let

$$\Upsilon : \mathfrak{k}^* \to G, \ \mu \mapsto \exp(iB^\sharp(\mu))$$

so that $\Upsilon = EE^\dagger$. It is straightforward to check

$$6\mathrm{d}E^* B^\mathbb{C}(\theta^L, {\theta^\dagger}^L) = \Upsilon^* B^\mathbb{C}(\theta^L, [\theta^L, \theta^L]).$$

(Here and for the rest of this Section θ^L, θ^R denote the Maurer-Cartan forms for the group G. This does not conflict with our earlier notation, since the Maurer-Cartan forms for K^* are given simply by pull-back under the inclusion $K^* \hookrightarrow G$.) Hence,

$$\mathrm{d}\beta = -\frac{i}{12}\mathcal{H}\big(\Upsilon^* B^\mathbb{C}(\theta^L, [\theta^L, \theta^L])\big) + \frac{i}{2} E^* B^\mathbb{C}(\theta^L, {\theta^\dagger}^L).$$

Let us denote the first summand by ϖ_1 and the second summand by ϖ_2. The contractions of ϖ_2 with generating vector fields $\xi_{\mathfrak{k}^*}$ for $\xi \in \mathfrak{k}$ are calculated in [2, Lemma 10]

$$\iota(\xi_{\mathfrak{k}^*})\varpi_2 = \frac{i}{2}\Upsilon^* B^{\mathbb{C}}(\theta^L + \theta^R, \xi) + 2E^* \ \mathrm{Im}\ B^{\mathbb{C}}(\theta^R, \xi), \quad \xi \in \mathfrak{k}.$$

To find the contractions of $\iota(\xi_{\mathfrak{k}^*})\varpi_1$, we use the identity

$$\iota(\xi_G) B^{\mathbb{C}}(\theta^L, [\theta^L, \theta^L]) = -6\mathrm{d}B^{\mathbb{C}}(\theta^L + \theta^R, \xi).$$

Since $\mathcal{H}$ anti-commutes with $\iota(\xi_{\mathfrak{k}^*})$, this shows

$$\begin{aligned} \iota(\xi_{\mathfrak{k}^*})\varpi_1 &= -\frac{i}{2}\mathcal{H}\Upsilon^*\big(\mathrm{d}B^{\mathbb{C}}(\theta^L + \theta^R, \xi)\big) \\ &= -\frac{i}{2}\Upsilon^* B^{\mathbb{C}}(\theta^L + \theta^R, \xi) + \frac{i}{2}\mathrm{d}\mathcal{H}\big(\Upsilon^* B^{\mathbb{C}}(\theta^L + \theta^R, \xi)\big). \end{aligned}$$

¿From the definition of Υ and of the homotopy operator, one finds that

$$\mathcal{H}\big(\Upsilon^* B^{\mathbb{C}}(\theta^L, \xi)\big) = \mathcal{H}\big(\Upsilon^* B^{\mathbb{C}}(\theta^R, \xi)\big) = i\langle \cdot, \xi \rangle.$$

Hence

$$\iota(\xi_{\mathfrak{k}^*})\varpi_1 = \frac{i}{2}\Upsilon^* B^{\mathbb{C}}(\theta^L + \theta^R, \xi) - \mathrm{d}\langle \cdot, \xi \rangle$$

Summing with the expression for $\iota(\xi_{\mathfrak{k}^*})\varpi_2$, we obtain (28).

Appendix B. Proof of Proposition 5.2

It is convenient to introduce an orthonormal basis e_a of $\mathfrak{k}$. Let $\epsilon^a \in \mathfrak{k}^* \cong \mathfrak{a} \oplus \mathfrak{n}$ be the dual basis. We denote the structure constants of $\mathfrak{k}$ by f^c_{ab} and those of $\mathfrak{k}^*$ by F^{ab}_c. Thus

$$[e_a, e_b] = f^c_{ab} e_c, \quad [\epsilon^a, \epsilon^b] = F^{ab}_c \epsilon^c, \quad [e_a, \epsilon^b] = -f^b_{ac}\epsilon^c + F^{bc}_a e_c, \tag{29}$$

using summation convention. Let $v_a = (e_a)_{K^*}$ denote the dressing vector fields, and $(\epsilon^a)^R$ the right-invariant vector fields on K^*. Let $S_{ab} \in C^\infty(K^*)$ be defined by

$$v_a = S_{ab}(\epsilon^b)^R.$$

In terms of the right-invariant Maurer-Cartan forms, $S_{ab} = \iota(v_a)\theta^R_b$. Note that the restriction to any dressing orbit $\mathcal{D} \subset K^*$ is given in terms of the symplectic form Ω on $\mathcal{D}$ by $S_{ab}|_{\mathcal{D}} = -\frac{1}{2}\Omega(v_a, v_b)$. In particular, S_{ab} is anti-symmetric.

Recall that $\rho^\sharp = B^\sharp(\rho) \in \mathfrak{t}$ where ρ is the half-sum of positive roots, and write $\rho^\sharp = \rho^b e_b$.

Lemma B.1. $F^{ab}_a = 4\pi\rho^b$.

Proof. For all $\mu = \mu_a \epsilon^a \in \mathfrak{k}^*$, the number $F^{ab}_a \mu_b$ is the trace of the operator $-\mathrm{ad}(\mu)$ on $\mathfrak{k}^*$. For $\mu \in \mathfrak{n}$, the operator $-\mathrm{ad}(\mu)$ is nilpotent and therefore has zero trace. Suppose $\mu \in \mathfrak{a}$, and let $\zeta = B^\sharp(\mu) \in \mathfrak{t}$. Since the pairing between $\mathfrak{k}^* \supset \mathfrak{a} = i\mathfrak{t}$ and $\mathfrak{k}$ is given by $2\,\mathrm{Im}\, B^{\mathbb{C}}$, we have $\mu = \frac{i}{2}\zeta$.

On any root space $\mathrm{span}_{\mathbb{C}}(e_\alpha) \subset \mathfrak{n} \subset \mathfrak{k}^*$, $-\,\mathrm{ad}(\mu)$ acts as a scalar $-2\pi i\langle \alpha, \frac{i}{2}\zeta\rangle = \pi\langle\alpha,\zeta\rangle$, hence has trace $2\pi\langle\alpha,\zeta\rangle$. It follows that the trace of $-\,\mathrm{ad}(\mu)$ on $\mathfrak{k}^*$ is,

$$F_a^{ab}\mu_b = 2\pi\sum_\alpha\langle\alpha,\zeta\rangle = 4\pi\langle\rho,\zeta\rangle = 4\pi\langle\mu,\rho^\sharp\rangle = 4\pi\rho^b\mu_b.$$

□

Lemma B.2. $F_b^{ac}S_{ac} + 4\pi\rho^a S_{ab} = 0.$

Proof. We claim that the statement of the Lemma is equivalent to the equation,

$$(\epsilon^a)^R S_{ab} = 0. \tag{30}$$

Indeed, using the definition of S_{ab} we have

$$(\epsilon^a)^R S_{ab} = \iota([(\epsilon^a)^R, v_a])\theta_b^R + \iota(v_a)\ L((\epsilon^a)^R)\theta_b^R.$$

Since $L((\epsilon^a)^R)\theta_b^R = F_b^{ac}\theta_c^R$ the second term is $F_b^{ac}S_{ac}$. To compute the first term, note that the dressing vector fields v_a, together with *minus* the right-invariant vector fields $-(\epsilon^b)^R$, are the generators for the G-action on $K^* = G/K$. Therefore, using (29), and Lemma B.1, $[(\epsilon^a)^R, v_a] = F_a^{ac}v_c = 4\pi\rho^a v_a$ which identifies the first term with $4\pi\rho^a S_{ab}$.

It remains to show (30). This condition is equivalent to the vanishing of the second order differential operator $\Delta_{K^*} = v_b(\epsilon^b)^R + (\epsilon^a)^R v_a$ on K^*, because

$$\Delta_{K^*} = (\epsilon^a)^R S_{ab}(\epsilon^b)^R - S_{ab}(\epsilon^a)^R(\epsilon^b)^R = ((\epsilon^a)^R S_{ab})(\epsilon^b)^R.$$

Let $p:\ G \to K^* = G/K$ be the projection. Then $p^*\circ\Delta_{K^*} = \Delta_G\circ p^*$ where

$$\Delta_G = (e_a)^R(\epsilon^a)^R + (\epsilon^a)^R(e_a)^R$$

is the Casimir operator on G corresponding to the invariant bilinear form $\langle\cdot,\cdot\rangle$. Since Δ_G is $\mathrm{Ad}(G)$-invariant, we can replace the superscript "R" by a superscript "L". Hence

$$\Delta_G = (e_a)^L(\epsilon^a)^L + (\epsilon^a)^L(e_a)^L = 2(\epsilon^a)^L(e_a)^L + F_a^{ar}(e_r)^L$$

where we have used $f_{ab}^a = 0$. The vector fields $(e_a)^L$ generate the right-K action and therefore vanish on right-K-invariant functions. It follows that $p^*\circ\Delta_{K^*} = \Delta_G\circ p^* = 0$, so that $\Delta_{K^*} = 0$. □

Now let $S = \frac{1}{2}S_{ab}\epsilon^a\wedge\epsilon^b$. The cocycle $\delta(\xi)$ is given in terms of the basis by $\delta(\xi) = \frac{1}{2}F_c^{ab}\xi^c\, e_a\wedge e_b$.

Lemma B.3. *For all $\xi\in\mathfrak{k}$, $\langle S,\delta(\xi)\rangle = 2\pi\langle S,\,\xi\wedge\rho^\sharp\rangle$.*

Proof. Using Lemma B.2 we compute,

$$2\pi\langle S,\,\xi\wedge\rho^\sharp\rangle = 2\pi S_{cb}\,\rho^b\xi^c = \tfrac{1}{2}S_{ab}F_c^{ab}\xi^c = \langle S,\delta(\xi)\rangle.$$

□

Proposition 5.2 is now a direct consequence of Lemma B.3, together with the moment map condition.

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