THE MONOPOLIST'S FREE BOUNDARY PROBLEM IN THE PLANE

ROBERT J. MCCANN, CALE RANKIN, AND KELVIN SHUANGJIAN ZHANG

Abstract. We study the Monopolist's problem with a focus on the free boundary separating bunched from unbunched consumers, especially in the plane, and give a full description of its solution for the family of square domains $\{(a, a+1)^2\}_{a \geq 0}$. The Monopolist's problem is fundamental in economics, yet widely considered analytically intractable when both consumers and products have more than one degree of heterogeneity. Mathematically, the problem is to minimize a smooth, uniformly convex Lagrangian over the space of nonnegative convex functions. What results is a free boundary problem between the regions of strict and nonstrict convexity. Our work is divided into three parts: a study of the structure of the free boundary problem on convex domains in $\mathbf{R}^n$ showing that the product allocation map remains Lipschitz up to most of the fixed boundary and that each bunch extends to this boundary; a proof in $\mathbf{R}^2$ that the interior free boundary can only fail to be smooth in one of four specific ways (cusp, high frequency oscillations, stray bunch, nontranversal bunch); and, finally, the first complete solution to Rochet and Choné's example on the family of squares $\Omega = (a, a+1)^2$, where we discover bifurcations first to targeted and then to blunt bunching as the distance $a \geq 0$ to the origin is increased. We use techniques from the study of the Monge–Ampére equation, the obstacle problem, and localization for measures in convex-order.

Contents

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1. Introduction

The Monopolist's problem is a principal-agent model for making decisions facing asymmetric information; it has fundamental importance in microeconomic theory. A simple form from [51] capturing multiple dimensions of heterogeneity that we rederive below is to

$$\begin{cases} \text{minimize} \quad L[u] := \int_\Omega \left(\frac{1}{2}|Du - x|^2 + u\right) dx, \\ \text{over} \quad \mathcal{U} := \{u : \overline{\Omega} \to \mathbf{R} \ ; \ u \text{ is nonnegative and convex}\}. \end{cases} \tag{1}$$

We always take $\Omega \subset \mathbf{R}^n$ to be open and convex with compact closure $\overline{\Omega}$. Our goal in this paper is to elucidate the properties of solutions to this minimization problem, with an eventual focus on the two-dimensional setting. Because the minimization takes place over the set of convex functions the problem has a free boundary structure. The free boundary separates the region where the function is convex, but not strictly convex, from the region where the function is strictly convex. In this paper we refine our understanding of the free boundary structure in all dimensions by showing regions where u is not strictly convex always extend to the fixed boundary $\partial\Omega$. We show the regularity known for u often extends from the interior to the fixed boundary. In two-dimensions, we show in a neighbourhood of a certain class of free boundary points (that we call tame), the minimizer solves the classical obstacle problem. From this we obtain the tame free boundary is locally piecewise Lipschitz except at accumulation points of its local maxima; it has Hausdorff dimension strictly less than two and is the graph of a continuous function. We also establish a boot strapping procedure: if the free boundary is suitably Lipschitz, it is C^∞. As an application of our techniques we completely describe the solution on the square domains $\Omega = (a, a+1)^2 \subset \mathbf{R}^2$ with $a \geq 0$. Despite significant numerical [7, 18, 19, 20, 26, 43, 45, 49], and analytic attempts [51, 36, 42] the description of the solution on the square has previously remained incomplete, and has come to be regarded as analytically intractable [7, 20, 35]. At least on the plane, we rebut this view by confirming for $a \geq \frac{7}{2} - \sqrt{2}$ the solution recently hypothesized by McCann and Zhang [42]. We show how their solution can also be modified to accommodate smaller values of a and other convex, planar domains. We show that the nature of the bunching undergoes unanticipated changes — from absent to targeted to blunt — as $a \geq 0$ is increased. We rigorously prove the support $Du(\overline{\Omega})$ of the unknown distribution of products consumed has a lower boundary which is concave nondecreasing — as the above-mentioned numerics and stingray description suggest — and that all products selected by more than one type of consumer lie on this boundary or its reflection through the diagonal.

The problem (1) arises from the question of how a monopolist should price goods for optimal profit in the face of information asymmetry. Here is a simple derivation of (1). We assume a closed set of products $\Omega^* \subset \mathbf{R}^n$ where each coordinate represents some attribute of the product, and an open set of consumers $\Omega \subset \mathbf{R}^n$ where each coordinate represents some attribute of the consumer. Consumers are distributed according to a Borel probability measure $\mu \in \mathcal{P}(\Omega)$. The monopolist's goal is to determine a price $v(y)$ at which to sell product y in a way which maximizes their profit. If consumer $x \in \Omega$ attains benefit $b(x, y)$ from product $y \in \Omega^*$ then the consumer will choose the product y which maximizes their utility

$$u(x) := \sup_{y \in \Omega^*} b(x, y) - v(y). \tag{2}$$

Provided it exists, we denote the y which realizes this supremum by $Yu(x)$. Assuming the monopolist pays cost $c(y)$ for product y, then the monopolist's goal is to maximize their profit, the integral of price they sell for minus cost they pay,

$$\int_\Omega [v(Yu(x)) - c(Yu(x))]\, d\mu(x).$$

The problem has been considered in this generality in e.g. [3] [13] [31] [39], and for even more general (non quasilinear) utility functions in [48], [40]. For this paper, to highlight the mathematical properties of most interest, we adopt several standard simplifying assumptions proposed by Rochet and Choné [51]: that products lie in the nonnegative orthant $\Omega^* = [0, \infty)^n$ and the monopolist's direct cost to produce them is quadratic $c(y) = |y|^2/2$, furthermore, that consumers are uniformly distributed on their domain $\Omega \subset \mathbf{R}^n$ and their product preferences are bilinear $b(x, y) = x \cdot y$. In this case (2) implies u is the Legendre transform of v (and thus a convex function), $Yu(x) = \partial u(x)$, and when $\Omega \subset [0, \infty)^n$ the Monopolist's goal becomes to maximize

$$\int_\Omega \left(x \cdot Du(x) - u(x) - \frac{|Du(x)|^2}{2} \right) dx,$$

over nonnegative convex functions which, up to an irrelevant constant, is the problem (1). Since convex functions are differentiable almost everywhere the integrand is well-defined. The nonnegativity constraint on u represents the additional requirement that $v(0) = 0$, meaning consumers need not consume if the monopolist raises prices too high, or equivalently, are always free to pay nothing by choosing the zero product as an outside option.

This problem was first considered by Mussa and Rosen in the one-dimensional setting [47], (after related models of taxation [46], matching [4], and signaling [54] were introduced and analyzed by Mirrlees, Becker and Spence). The multidimensional problem was considered by Wilson [56] and Armstrong [2], while our formulation above is essentially that of Rochet and Choné [51]. Although this model is of significant importance to economists it presents serious mathematical difficulties. Indeed, were there only the nonnegativity constraint in (1) we would have a variant of the obstacle problem [9, 30]; with the *nonnegativity and convexity constraint* we have a free boundary problem for three different regions. In the region where the function is positive but not strictly convex, the fundamental tool of two-sided perturbation by an arbitrary test function is no longer applicable. As a result, until recent work of the first and third authors [42] it has not even been possible to write down the Euler–Lagrange equation in the region of nonstrict convexity. Despite this, other aspects of the problem have been studied, notably by Rochet and Choné [51], who derived a necessary and sufficient condition for optimality in terms of convex-ordering between the positive and negative parts of the variational derivative of the objective functional, Basov [3] who advanced a control theoretic approach to such problems, Carlier [13, 14] who considered existence and first-order conditions for the minimizer, and Carlier and Lachand-Robert [15, 21, 17] who studied regularity and gave a description of the polar cone.

In this paper we prove results of mathematical and economic interest. We invoke tools from diverse areas of mathematics: the theory of sweeping and convex orders of measures, Monge–Ampére equations, regularity theory for the obstacle problem, and the theory of optimal transport (which has deep links to the Monopolist's problem). We also indicate a striking connection to the classical obstacle problem: Locally the minimizer u solves an obstacle problem where the obstacle is the minimal convex extension of u from its region of nonstrict convexity. We now outline our results.

If u solves (1) and Ω is a convex open subset of $\mathbf{R}^n$ it is known from the work of Rochet–Choné and Carlier–Lachand-Robert that $u \in C^1(\overline{\Omega})$ and from the work of Caffarelli and Lions [10] (see [39]) that $u \in C^{1,1}_{\text{loc}}(\Omega)$. Any convex function $u \in C^1(\overline{\Omega})$ partitions Ω according to its sets of contact with supporting hyperplanes; these sets are convex. Namely for each $x_0 \in \overline{\Omega}$ set

$$p_{x_0}(x) = u(x_0) + Du(x_0) \cdot (x - x_0), \tag{3}$$

$$\text{and} \quad \tilde{x_0} = \{x \in \overline{\Omega}\ ;\ u(x) = p_{x_0}(x)\}.$$

$$= (Du)^{-1}(Du(x_0)) \tag{4}$$

Here $\tilde{x}$ is the equivalence class of x under the equivalence relation $x_0 \sim x_1$ if and only if $Du(x_0) = Du(x_1)$. We call an equivalence class trivial if $\tilde{x} = \{x\}$, in which case we say u is strictly convex at x. We call equivalence classes *leaves*, since they foliate the interior of Ω_i. They are also called *isochoice sets* [24] or *bunches* if nontrivial [51]. We also call one-dimensional leaves *rays*. We set

$$\Omega_i = \{x \in \overline{\Omega}\ ;\ \tilde{x} \text{ is } (n-i)\text{-dimensional}\}. \tag{5}$$

Thus, for example, Ω_n consists of all points at which u is strictly convex and Ω_0 consists of all points x lying in the closure of some open set on which

u is affine. These disjoint sets partition $\overline{\Omega}$ and our first result describes the qualitative behavior in each set.

Theorem 1.1 (Partition into foliations by leaves that extend to the boundary)**.** *Let u solve* (1) *where $\Omega \subset \mathbf{R}^n$ is bounded, open and convex. Then*

(1) *If $\Omega_0 \neq \emptyset$ then $\Omega_0 = \{x \in \overline{\Omega}\ ;\ u(x) = 0\}$ hence is closed and convex;*
(2) *$\Omega_1, \dots, \Omega_{n-1}$ are a union of equivalence classes on which u is affine and each such equivalence class intersects the boundary $\partial\Omega$;*
(3) *$\Omega_n \cap \Omega$ is an open set on which $u \in C^\infty(\Omega_n)$ solves $\Delta u = n + 1$.*

Remark 1.2 (An analytic interface)**.** *Any portion of $\partial\Omega_0$ lying outside $\overline{\Omega_1 \cup \cdots \cup \Omega_{n-1} \cup \partial\Omega}$ is smooth — in fact locally analytic — by the theory of the obstacle problem* [11, 9]*. Indeed, on any open ball in Ω exhausted by Ω_0 and Ω_n, u is a convex solution of $\frac{1}{n+1}\Delta u = 1_{\{u>0\}}(x)$. Thus Ω_0 is convex, and subsequently has Lipschitz boundary which improves to locally analytic by the regularity for the obstacle problem (see also* [30, Theorem 7.3]*). Theorem 1.5 below gives examples of square domains in the plane for which this portion is nonempty.*

It is immediate from the definition that Ω_i for $1 \leq i \leq n-1$ are a union of nontrivial equivalence classes; the key conclusion is these extend to the boundary (i.e. if $x \in \Omega_i$ for $1 \leq i \leq n-1$ then $\tilde{x} \cap \partial\Omega \neq \emptyset$). The PDE in point (3) has been considered in more generality by Rochet and Choné [51]. Note the economic interpretation of (1): no bunches of positive measure are sold apart from the null product; as we shall see during the proof of (2), any product sold to more than one consumer lies on the boundary of the set of products sold. Thus the entire interior of the set of products sold consists of individually customized products. Our proof of Theorem 1.1 and some subsequent results requires a new proposition asserting that a.e. on the boundary of a convex domain Ω, the minimizer of (1) satisfies the boundary condition $(Du(x) - x) \cdot \mathbf{n} \geq 0$ where $\mathbf{n}$ is the outer unit normal to Ω. Established in Proposition 2.3, it can be interpreted to mean that the normal component of any boundary distortion in product selected can never be inward. Moreover, in convex polyhedral domains and certain other situations, we are able to extend the interior regularity $u \in C^{1,1}_{\text{loc}}(\Omega)$ of Caffarelli and Lions [10] [39] to the smooth parts of the fixed boundary, Theorem 4.1.

The remainder of our results are restricted to the planar case $\Omega \subset \mathbf{R}^2$. Theorem 1.1 provides a complete description of the solution in Ω_0 and Ω_2: it remains to better understand the behavior of the solution in Ω_1 as well as the properties of the domains Ω_1 and Ω_2, (noting Ω_0 is, by Theorem 1.1, a closed convex set).

By Theorem 1.1, the free boundary $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \cap \Omega$ between Ω_1 and Ω_2 consists only of points in rays which extend to $\partial\Omega$. In §5 we prove that the Neumann condition

$$(Du(x_0) - x_0) \cdot \mathbf{n} = 0, \tag{6}$$

where $\mathbf{n}$ is the outer unit normal to the fixed boundary $\partial\Omega$, can be used to characterize the presence of these rays. Namely if (6) is not satisfied at $x_0 \in \partial\Omega$ then $\tilde{x_0} \neq \{x_0\}$ (and nearby rays foliate the interior of Ω_1 locally). Conversely, if $x \in \partial\Omega$ lies in a boundary neighbourhood $B_\varepsilon(x) \cap \partial\Omega$ on which the Neumann condition (6) is satisfied then x is a point of strict convexity for u. The remaining case — rays $\tilde{x}_0 \neq \{x_0\}$ which satisfy (6) — is subtle: we call such rays *stray* and conjecture the union $\mathcal{S}$ of stray rays has zero area in general. Remark 8.4 shows polygonal domains $\Omega \subset [0,\infty)^2$ admit at most countably many stray rays; the squares $\Omega = (a, a+1)^2$ admit none.

Let x_1 be a point in the free boundary Γ which, necessarily, lies on the ray $\tilde{x_1}$. Let $x_0 = \tilde{x_1} \cap \partial\Omega$ be the (fixed) boundary endpoint of $\tilde{x_1}$ provided by Theorem 1.1. Note that if $\partial\Omega$ is C^1 in a neighbourhood of x_0 and $(Du(x_0) - x_0) \cdot \mathbf{n} > 0$, the same inequality holds for all $x \in B_\varepsilon(x_0) \cap \partial\Omega$ and thus such x are also the boundary endpoints of nontrivial rays. In this case we call x_1 a *tame* free boundary point (and $\tilde{x}_1$ a *tame* ray); we denote the set of tame free boundary points by $\mathcal{T}$.

Theorem 1.3 (Regularity results for the free boundary)**.** *Let u solve* (1) *where $\Omega \subset \mathbf{R}^2$ is a bounded, open, and convex with smooth boundary except possibly at finitely many points. For every $x_1 \in \mathcal{T}$ and $x_0 \in \tilde{x}_1 \cap \partial\Omega$ there is $\varepsilon > 0$ such that*
(1) $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \setminus \partial\Omega$ has Hausdorff dimension less than 2 in $B_\varepsilon(x_1)$;
(2) the function $D(x) :=$ diam$(\tilde{x})$ is continuous in $B_\varepsilon(x_0) \cap \partial\Omega$;
(3) if $A \subset \partial\Omega$ denotes the closure of all points where D attains its local maxima, then $\Gamma \cap \{x' \in \tilde{x} : x \in B_\varepsilon(x_0) \cap \partial\Omega \setminus A\}$ is Lipschitz;
(4) if D is Lipschitz on $\partial\Omega$ near x_0, then a bootstrapping procedure yields $\Gamma \cap B_\varepsilon(x_1)$ is a C^∞ curve and $u \in C^\infty(B_\varepsilon(x_1) \cap$ int $\Omega_1)$.

Establishing the Lipschitz regularity of D, which permits the above-mentioned bootstrapping to a C^∞ free boundary, remains an interesting open problem.

Remark 1.4 (Lipschitzianity, convexity, and smoothness)**.** *Note the Lipschitz requirement on $D|_{B_\varepsilon(x_0)\cap\partial\Omega}$ from (4) is not necessarily satisfied even when the corresponding portion of Γ lies in a Lipschitz submanifold given by (3). If $\{x' \in \tilde{x} : x' \in B_\varepsilon(x_0) \cap \partial\Omega\}$ happens to be convex this distinction disappears for every smaller value of ε; simulations* [45] *suggest this occurs in the square examples from Theorem 1.5. Thus if the region Ω_1^+ depicted in Figure 1 is convex, as Mirebeau's simulations lead us to conjecture, Theorem 1.3 guarantees the curved portion of its boundary is smooth (away from $\partial\Omega$). See alternately Remark 7.5 below.*

Theorem 1.3 is proved using new coordinates for the problem, new Euler–Lagrange equations, and a new observation: That in a neighbourhood of a tame free boundary point the difference between the minimizer u and the minimal convex extension of $u|_{\Omega_1}$ solves the classical obstacle problem. A priori, the obstacle is $C^{1,1}$, i.e. has a merely L^∞ Laplacian, and thus, without first improving the regularity of $u|_{\partial\Omega_1}$, the above results are the best one can obtain from the theory of the obstacle problem.

For general convex domains it is difficult to study the structure of the stray set $\mathcal{S} = \Gamma \setminus \mathcal{T}$ which may include points in the relative interior of rays. We also have not ruled out, in general, that the relative boundary of

$$\partial\Omega_{\neq} := \{x \in \partial\Omega \; ; \; (Du - x) \cdot \mathbf{n} \neq 0\},$$

in $\partial\Omega$ might have positive $\mathcal{H}^1$-measure and the corresponding free boundary be nonsmooth. However, in specific cases, a more complete description is possible. For example, on the squares $\Omega = (a, a+1)^2$, we show $\partial\Omega_{\neq}$ is a single connected component of $\partial\Omega$. In fact, we are able to provide the first explicit and complete description of the solution on $\Omega = (a, a+1)^2$, including an unexpected trichotomy for $a = 0$, a sufficiently small, and a sufficiently large. To describe the solution we label the edges of Ω by their compass direction and set

$$\Omega_N = [a, a+1] \times \{a+1\} \qquad \Omega_E = \{a+1\} \times [a, a+1]$$
$$\Omega_W = \{a\} \times [a, a+1] \qquad \Omega_S = [a, a+1] \times \{a\}.$$

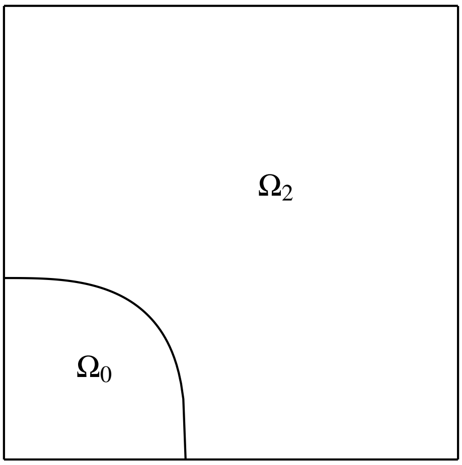
The minimizer is described by the following bifurcation theorem (illustrated schematically in Figure 1).

Theorem 1.5 (Blunt bunching is a symptom of a seller's market)**.** *Let u solve* (1) *with* $\Omega = (a, a+1)^2$ *where* $a \geq 0$. *Then*

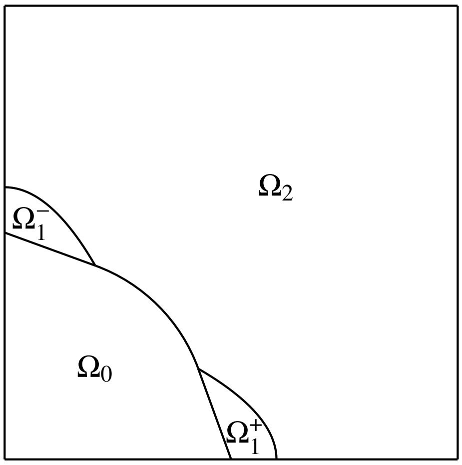
(1) Ω_0 *is a convex set which includes a neighbourhood of* (a, a) *in* $\overline{\Omega}$.
(2) *The portion of* Ω_1 *consisting of rays having both endpoints on the boundary* $\partial\Omega$ *is connected and denoted by* Ω_1^0. *It is nonempty when* $a \geq \frac{7}{2} - \sqrt{2} \approx 2.1$. *All rays in* Ω_1^0 *are orthogonal to the diagonal and have one endpoint on* Ω_W *and the other on* Ω_S. *On the other hand there is* $\varepsilon_0 > 0$ *such that* Ω_1^0 *is empty when* $a < \varepsilon_0$.
(3) *For all* $a > 0$ *there are exactly two disjoint connected components of* $\Omega_1 \setminus \Omega_1^0$. *In these regions, each ray has only one endpoint* $x_0 \in \partial\Omega$ *on the boundary; it lies in* $\Omega_S \cup \Omega_W$*, violates the Neumann condition* (6)*, and the solution u is described by the Euler–Lagrange equations of McCann and Zhang* [42]*; c.f.* (8)–(15). *For* $a = 0$, Ω_1 *is empty.*
(4) *The set* Ω_2 *of strict convexity of u contains* $\Omega_E \cup \Omega_N$ *and the Neumann condition* (6) *holds at each* $x_0 \in \Omega_2 \cap \partial\Omega$ *apart from the 3 vertices.*

The following corollary may be of purely mathematical interest from the point of view of calculus of variations and partial differential equations. The smoothness asserted follows from Remark 1.2.

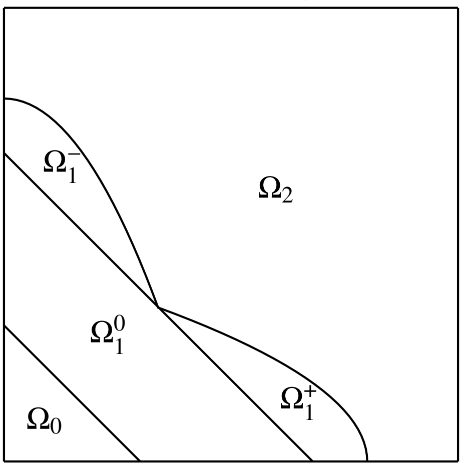
Corollary 1.6 (Convexity of solution to, and contact set for, an obstacle problem)**.** *For* $\Omega = (-1, 1)^2$, *the minimizer of* $L(u)$ *over non-negative functions* $0 \leq u \in W^{1,2}(\overline{\Omega})$ *is convex. Its zero set is smooth, convex, has positive area, and is compactly contained in the centered square* Ω.



(a) $a = 0$



(b) $a > 0$ sufficiently small



(c) $a \geq \frac{7}{2} - \sqrt{2}$

FIGURE 1. Bifurcation to targeted then blunt bunching (Thm 1.5) as distance $a \geq 0$ of $\Omega = (a, a+1)^2$ to zero is increased.

Remark 1.7 (Concave nondecreasing profile of stingray's tail)**.** *Numerical simulations of the square example show the region $Du(\Omega)$ of products consumed to be shaped like a stingray, e.g. Figure 1 of* [20]*. Theorem 1.5 combines with Lemma 8.8 below to provide a rigorous proof that the lower edge of stingray is concave non-decreasing — as the simulations suggest — while Theorem 1.1 shows that every product selected by more than one type of consumer lies on this boundary or its mirror image across the diagonal.*

Remark 1.8 (Absence and ordering of blunt vs targeted bunching)**.** *The potential absence of blunt bunching from the square example — established on a nonempty interval $a \in (0, \varepsilon_0)$ by the preceding theorem — has been overlooked in all previous investigations that we are aware of. It can be understood as a symptom of a buyers' market, in which a lack of enthusiasm on the part of qualified buyers incentivizes the monopolist to sell to fewer buyers but cater more to the tastes of those who do buy. The persistence of targeted bunching $\Omega_1^\pm \neq \emptyset$ for all $a > 0$ reflects the need to transition continuously from vanishing Neumann condition* (6) *— satisfied on the customization region $\Omega_2 \cap (\Omega_S \cup \Omega_W)$ where u is strictly convex — to the uniformly positive Neumann condition $(Du - x) \cdot \mathbf{n} = a$ on the exclusion region Ω_0 where u vanishes, in light of the known regularity $u \in C^1(\overline{\Omega})$* [16][51]*. Such bunching is neither needed nor present when $a = 0$: in this case $x_0 \cdot \mathbf{n} = a$ on $\Omega_S \cup \Omega_W$ shows the Neumann conditions in Ω_0 and Ω_2 coincide. When the blunt bunching region Ω_1^0 is present, our proof of Theorem 1.5 shows it separates Ω_0 from $\Omega_1^\pm$, which in turn separate all but one point of Ω_1^0 from Ω_2. In particular, blunt bunching implies Ω_0 is a triangle, which is exceedingly rare in its absence.*

Assuming Ω_2 is Lipschitz (or at least has finite perimeter), we arrive at a characterization of the solution to (1) on $\Omega = (a, a+1)^2$ for every value of $a \geq 0$. Namely $u \in \mathcal{U}$ minimizes (1) if and only if (A) bunching is absent ($\Omega_1 = \emptyset$, as for $a = 0$), in which case u solves $\frac{1}{3}\Delta u = 1_{\{u>0\}}$, i.e the classical obstacle problem [50, 30, 28] and $(Du - x) \cdot \mathbf{n} = 0$ on $\partial\Omega$ (Figure 1a), or (B) bunching is present but blunt bunching is absent, ($\Omega_1^0 = \emptyset \neq \Omega_1$, as for $a \ll 1$) in which case we derive below necessary conditions, whose sufficiency can be confirmed as in [42] (Figure 1b), or (C) blunt bunching is present, ($\Omega_1^0 \neq \emptyset$, as for $a \geq \frac{7}{2} - \sqrt{2}$) (Figure 1c), in which case the sufficient conditions for a minimum established by two of us [42] are also shown to be necessary below. (The only gap separating the necessary from the sufficient condition is the question of whether or not Ω_2 must have finite perimeter.)

If instead (B) blunt bunching is absent but bunching is present, $\Omega_1^0 = \emptyset \neq \Omega_1$, Theorem 1.5 asserts that $\Omega_1 = \Omega_1^+ \cap \Omega_1^-$ splits into two connected components

$$\Omega_1^\pm := \{(x_1, x_2) \in \Omega_1 \setminus \Omega_1^0 : \pm(x_1 - x_2) > 0\}, \tag{7}$$

placed symmetrically below and above the diagonal. The region Ω_1^- and its reflection Ω_1^+ below the diagonal are foliated by isochoice segments making continuously varying angles θ with the horizontal. The limit of these segments is a segment of length $R_0 > 0$ lying on the boundary of the convex set Ω_0, having endpoint (a, h_0) and making angle $\theta_0 \in [-\pi/4, 0)$ with the horizontal.

Fix any closed convex neighbourhood Ω_0 of (a, a) in $\overline{\Omega}$ which is reflection symmetric around the diagonal and contains such a segment in its boundary. We describe the solution $u = u_1^-$ in Ω_1^- using an Euler-Lagrange equation (10) from [42], rederived more simply in Section 8 below. Index each isochoice segment in Ω_1^- by its angle $\theta \in (-\frac{\pi}{4}, 0)$; (angles which are less than $-\pi/4$ or non-negative are ruled out in the proof of Theorem 1.5). Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to this boundary point $(a, h(\theta))$. Along the hypothesized length $R(\theta)$ of this segment assume u increases linearly with slope $m(\theta)$ and offset $b(\theta)$:

$$u_1^-\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta). \tag{8}$$

Given the initial (angle, height, length) triple $(\theta_0, h_0, R_0) \in [-\pi/4, 0) \times (a, a+1) \times (0, \sqrt{2}/2)$ corresponding to the segment in $\Omega \cap \partial\Omega_0$, and $R : [\theta_0, 0] \longrightarrow [0, 1)$ piecewise Lipschitz with $R(\theta_0) = R_0$, solve

$$m(\theta_0) = 0, \qquad m'(\theta_0) = 0 \qquad \text{such that} \tag{9}$$

(10)
$$(m''(\theta) + m(\theta) - 2R(\theta))(m'(\theta)\sin\theta - m(\theta)\cos\theta + a) = \frac{3}{2}R^2(\theta)\cos\theta\ ;$$

then set

$$h(\theta) = h_0 + \frac{1}{3}\int_{\theta_0}^{\theta}(m''(\vartheta) + m(\vartheta) - 2R(\vartheta))\frac{d\vartheta}{\cos\vartheta}, \tag{11}$$

$$b(\theta) = \int_{\theta_0}^{\theta}(m'(\vartheta)\cos\vartheta + m(\vartheta)\sin\vartheta)h'(\vartheta)d\vartheta. \tag{12}$$

Given (θ_0, h_0, R_0) and $R(\cdot)$ as above, the triple (m, b, h) satisfying (10)–(12) exists and is unique on the interval where $R(\cdot) > 0$. Thus the shape of Ω_1^- and the value of u_1^- on it will be uniquely determined by Ω_0 and $R : [\theta_0, 0] \longrightarrow [0, 1)$. We henceforth restrict our attention to choices of Ω_0 and $R(\cdot)$ for which the resulting set Ω_1^- lies above the diagonal. In this case Ω_1^+ and the value of $u = u_1^+$ on Ω_1^+ are determined by reflection symmetry $x_1 \leftrightarrow x_2$ across the diagonal. This defines $u = u_1$ on Ω_1 and provides the boundary data on $\partial\Omega_1 \cap \partial\Omega_2$ needed for the mixed Dirichlet / Neumann boundary value problem for Poisson's equation,

$$\begin{cases} \Delta u_2 = 3, & \text{on } \Omega_2, \\ (Du_2(x) - x)\cdot \mathbf{n} = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ u_2 - u_1 = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega_1, \\ u_2 = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega_0, \end{cases} \tag{13}$$

which determines $u = u_2$ on $\Omega_2 := \Omega \setminus (\Omega_0 \cup \Omega_1)$. The duality discovered in [42], implies that for at most one choice of Ω_0 and $R(\cdot)$ Lipschitz can convex u (pieced together from u_1, u_2 as above with $u_0 := 0$ on Ω_0) satisfy the supplemental Neumann conditions

$$D(u_2 - u_1) \cdot \hat{\mathbf{n}} = 0, \text{ on } \partial\Omega_2 \cap \partial\Omega_1 \tag{14}$$

$$Du_2 \cdot \hat{\mathbf{n}} = 0, \text{ on } \partial\Omega_2 \cap \partial\Omega_0 \tag{15}$$

required on the free boundaries (since $u \in C^1(\overline{\Omega})$ [16][51]); here $\hat{\mathbf{n}}$ denotes the outer unit normal to Ω_2 at $x \in \partial\Omega_2$. In the course of proving Theorem 1.5 in Section 8 below we complete this circle of ideas — apart from the piecewise Lipschitz hypothesis which Theorem 1.3 falls just short of proving — by showing at least one such choice exists; this choice uniquely solves (1) on the square in case (B). In case (C), Theorem 1.5 shows at least (and therefore exactly [42]) one choice exists satisfying the free boundary problem from [41] in the analogous sense.

We conclude this introduction by outlining the structure of the paper. Section 2 contains preliminaries: the variational inequality associated to (1), some background on Alexandrov second derivatives, and localization results of Rochet–Choné. In Section 3 we prove Theorem 1.1 using perturbation techniques previously used to study the Monge–Ampère equation. In Sections 4 and 5 and we prove some technical results which facilitate our later work in Sections 6 and 7. First, in Section 4, a boundary $C^{1,1}$ result which is new for this problem and extends the interior regularity result of Caffarelli and Lions [10] [39]. Next, in Section 5, propositions quantifying how at points of nonstrict convexity the Neumann boundary condition fails to be satisfied. Section 6 and Section 7 establish Theorem 1.3 using techniques from the study of the obstacle problem. Here we indicate a new connection to the classical obstacle problem. Namely, that the Monopolist's problem gives rise to an obstacle problem where the obstacle is the minimal convex extension of the function defined on Ω_1. The proof of Theorem 1.5 is completed in Section 8 using a case by case analysis based on a careful choice of coordinates to deduce monotonicity of $(Du(x) - x) \cdot \mathbf{n}$ along the upper left boundary $\Omega_W \cap \Omega_N$ of the square. It confirms the economic intuition that the degree to which product selection (hence bunching) is influenced by the market presence of competing consumers decreases as we move away from the exclusion region, i.e. from the lower left toward the upper right region of the square, while on the other hand, increasing as we move the entire square of consumer types away from the outside option by increasing $a \geq 0$. We conclude with an appendix containing some relevant background results. Table 1 contains a list of notation.

Notation	Meaning
Ω	A bounded open convex subset of $\mathbf{R}^n$.
$\overline{\Omega}$	The closure of Ω.
$\operatorname{int}\overline{\Omega}$	The interior of $\overline{\Omega}$.
Ω^c	Set complement $\Omega^c := \mathbf{R}^n \setminus \Omega$ of Ω.
$\mathbf{n}$	Outer unit normal at a point where $\partial\Omega$ is differentiable.
$\tilde{x}$	Bunch $\tilde{x} := \{z \in \overline{\Omega} \ ; \ Du(z) = Du(x)\}$.
$\text{r.i.}(\tilde{x})$	The relative interior of the convex set $\tilde{x}$.
Ω_i	Subset (5) of $\overline{\Omega}$ foliated by $(n-i)$-dimensional bunches.
$\subset\subset$	Compact containment.
v_+	The positive part of a function, $v_+(x) := \max\{v(x), 0\}$.
μ_+	The positive part of a measure μ.
$\operatorname{spt} f$	The support of f, i.e. $\operatorname{spt} f = \text{closure}\{x \ ; \ f(x) \neq 0\}$.
$\mathcal{P}(\Omega)$	The set of Borel probability measures on Ω.
$\llcorner$	Measure restriction: $(\mu \llcorner A)(B) = \mu(A \cap B)$.
$\mathcal{H}^d$	d-dimensional Hausdorff measure.
$dx := d\mathcal{H}^n(x)$	n-dimensional Lebesgue measure, i.e. volume measure.
$dS := d\mathcal{H}^{n-1}$	Surface area measure (or arclength in special case $n = 2$).

Table 1. Table of notation.

2. Variational inequalities and Alexandrov second derivatives

2.1. Variational inequalities

Our basic tools for studying the unique minimizer of the functional

$$L[u] = \int_\Omega \left(\frac{1}{2} |Du - x|^2 + u \right) dx, \tag{16}$$

over

$$\mathcal{U} = \{u : \overline{\Omega} \to \mathbf{R} \ ; \ u \text{ is nonnegative and convex}\},$$

are the variational inequalities stated in the following lemma.

Lemma 2.1 (Variational inequalities)**.** *Let u solve* (1)*. Let $\bar{u} \in \mathcal{U}$ be Lipschitz and $w = \bar{u} - u$. Then each of the following inequalities hold:*

$$0 \leq L'_u(w) := \int_\Omega (n + 1 - \Delta u)\, w dx + \int_{\partial\Omega} (Du - x) \cdot \mathbf{n}\, w dS, \tag{17}$$

$$0 \leq L'_u(\bar{u}) = \int_\Omega (n + 1 - \Delta u)\bar{u}\, dx + \int_{\partial\Omega} (Du - x) \cdot \mathbf{n}\, \bar{u} dS. \tag{18}$$

Moreover if $Du \not\equiv D\bar{u}$ on a set of positive $\mathcal{H}^n$ measure, then

$$0 < L'_{\bar{u}}(w) = \int_\Omega (n + 1 - \Delta\bar{u})\, w dx + \int_{\partial\Omega} (D\bar{u} - x) \cdot \mathbf{n}\, w dS, \tag{19}$$

where $\Delta\bar{u}$ is interpreted as a Radon measure [27, Ch. 6] *and $D\bar{u} \cdot \mathbf{n}$ as the one-sided derivative $\lim_{t \downarrow 0} (\bar{u}(x) - \bar{u}(x - t\mathbf{n}))/t$ which exists by convexity of $\bar{u}$.*

Proof. We begin with (17). Let u be the minimizer and observe $\mathcal{U}$ is convex. Thus for any $\bar{u} \in \mathcal{U}$, $w = \bar{u} - u$ and $t \in [0,1]$ we have

$$L[u] \leq L[u + tw],$$

so, in particular,

$$\begin{aligned} 0 &\leq \frac{d}{dt}\Big|_{t=0} L[u+tw] \\ &= \int_\Omega (Du - x) \cdot Dw + w\,dx. \end{aligned} \tag{20}$$

Note $u \in C^1(\overline{\Omega}) \cap C^{1,1}_{\text{loc}}(\Omega)$ with $\partial^2_{ii} u \geq 0$ so we may apply the divergence theorem and obtain

$$0 \leq \int_\Omega (n + 1 - \Delta u)\, w dx + \int_{\partial\Omega} (Du - x) \cdot \mathbf{n}\, w dS.$$

where $\mathbf{n}$ is the outer unit normal to Ω which exists $\mathcal{H}^{n-1}$-a.e. for the convex domain Ω.

Inequality (18) follows by performing the same argument with $u+\bar{u}$ in place of $\bar{u}$. For (19) we perform similar calculations but use that $h(t) := L[\bar{u} - tw]$ is strictly convex with a minimum at 1 so $h'(0) < 0$. For the divergence theorem we take the one-sided directional derivatives and use that Du is of bounded variation [27, Ch. 6]. ■

Remark 2.2. *(1) It is straightforward to see, again by arguing using a perturbation, that inequality* (18) *holds not just for* $\bar{u} \in \mathcal{U}$ *but for any convex* $\bar{u}$ *with* $\operatorname{spt} \bar{u}_-$ *(the support of the negative part of* $\bar{u}$*) disjoint from the set* $\{u = 0\}$. *The key observation is that for sufficiently small* t, $u + t(\bar{u} - u) \in \mathcal{U}$.
(2) In any neighbourhood where u *is* C^2 *and uniformly convex, that is* u *satisfies an estimate* $D^2 u \geq \lambda I > 0$, *one may perturb — as is standard in the calculus of variations — by smooth compactly supported functions and obtain* $\Delta u = 3$ *in the interior and* $(Du - x) \cdot \mathbf{n} = 0$ *on the fixed boundary of* Ω. *Without a local uniform convexity estimate, even for smooth functions* $\bar{u}$, *it may be that there is no* $t > 0$ *small enough to ensure* $u + t\bar{u}$ *is convex.*

Inequality (19) is useful when one chooses $\bar{u}$ as paraboloid with prescribed Laplacian. We give an example now — the result we prove is required in subsequent sections. It is interesting to contrast the following result with the one-dimensional case, in which minimizers on domains $\Omega \subset [0, \infty)$ satisfy $u'(x) \leq x$.

Proposition 2.3 (Normal distortion is not inward)**.** *Let* u *solve* (1) *where* $\Omega \subset \mathbf{R}^n$ *is bounded, open, and convex. Then for any* $x_0 \in \partial\Omega$ *where the outer normal is defined,*

$$(Du(x_0) - x_0) \cdot \mathbf{n} \geq 0.$$

Proof. By approximation it suffices to prove the result for smooth *strictly* convex domains. Indeed, [31, Corollary 4.7] and its proof imply if $\Omega^{(k)} \supset \Omega$ is a sequence of smooth strictly convex approximating domains and $u^{(k)}$ is the solution of (1) on $\Omega^{(k)}$ then $Du^{(k)}(x) \longrightarrow Du(x)$ at every x where u and each $u^{(k)}$ is differentiable. The $C^1(\overline{\Omega})$ result of [51, 16] implies this is every $x \in \overline{\Omega}$.

Thus we take Ω to be smooth and strictly convex. Up to a choice of coordinates we assume $x_0 = 0$ and $\mathbf{n} = e_1$ (see Figure 2). Recall $p_{x_0}(x) := u(x_0) + Du(x_0) \cdot (x - x_0)$ is the affine support at x_0. For $t > 0$ sufficiently small and $x = (x^1, \dots, x^n) \in \mathbf{R}^n$ we consider the family of admissible perturbations (see Figure 2)

$$\begin{aligned} \hat{u}_t(x) &:= \frac{n+1}{2}\big([x^1 + t]_+\big)^2 + p_{x_0}(x), \\ \bar{u}_t(x) &:= \max\{u(x), \hat{u}_t(x)\}, \\ \Omega_t &:= \{x \in \overline{\Omega}\ ;\ \bar{u}_t(x) > u(x)\}. \end{aligned}$$

Note Ω_t has positive measure since at $x_0 = 0$, $\hat{u}_t(0) = u(0) + (n+1)t^2/2 > u(0)$. Moreover if $x^1 \leq -t$ then $\bar{u}_t(x) \leq u(x)$ and thus $\Omega_t \subset \{x \in \overline{\Omega}\ ;\ x^1 \geq -t\}$. Thus, strict convexity of Ω implies $\overline{\Omega}_t \longrightarrow \{0\}$ in Hausdorff distance [53, §1.8] as $t \longrightarrow 0$. Clearly $\Delta \bar{u}_t = n + 1$ on Ω_t. To derive a contradiction assume $(Du(x_0) - x_0) \cdot \mathbf{n} < 0$. Then for t sufficiently small we also have

$$(D\bar{u}_t(x) - x) \cdot \mathbf{n} < 0 \quad \text{on} \quad \Omega_t \cap \partial\Omega, \tag{21}$$

which holds by the $C^1(\overline{\Omega})$ continuity of u and because $|D\bar{u}_t(x_0) - Du(x_0)| = O(t)$. But (19) with $\Delta \bar{u} = n + 1$ implies

$$0 < \int_{\Omega_t \cap \partial\Omega} (\bar{u}_t - u)(D\bar{u}_t(x) - x) \cdot \mathbf{n}\, dS$$

which contradicts (21) to conclude the proof. ■

An essential tool is that the variational inequality (18) holds not only on Ω but restricted to the contact set $\tilde{x}$ — at least for $\mathcal{H}^n$ almost every x. This novel and powerful technique was pioneered in this context by Rochet and Chonè [51] who exploited the sweeping theory of measures in convex order. In this section we recall the statement of their localization result; for completeness we include a proof in Appendix A.

We introduce the following notation for the variational derivative $\sigma := \delta L/\delta u$ of our objective $L(u)$:

$$d\sigma(x) = (n + 1 - \Delta u) d\mathcal{H}^2 \llcorner \Omega + (Du - x) \cdot \mathbf{n} d\mathcal{H}^1 \llcorner \partial\Omega, \tag{22}$$

which turns out to be a measure with finite total variation. The equivalence relation induced by Du, namely $x_1 \sim x_2$ if and only if $Du(x_1) = Du(x_2)$ yields the partitioning of each Ω_i into leaves. We let $\tilde{x}$ denote the equivalence class of x and can disintegrate σ by conditioning on this equivalence relation. Let the conditional measures $\sigma_{\tilde{x}} = \sigma^+_{\tilde{x}} - \sigma^-_{\tilde{x}}$ be defined by disintegrating separately the positive and negative parts of σ with respect to the given equivalence relation, we recall how

$$0 \leq \int_{\tilde{x}} v(z) d\sigma_{\tilde{x}}(z)$$

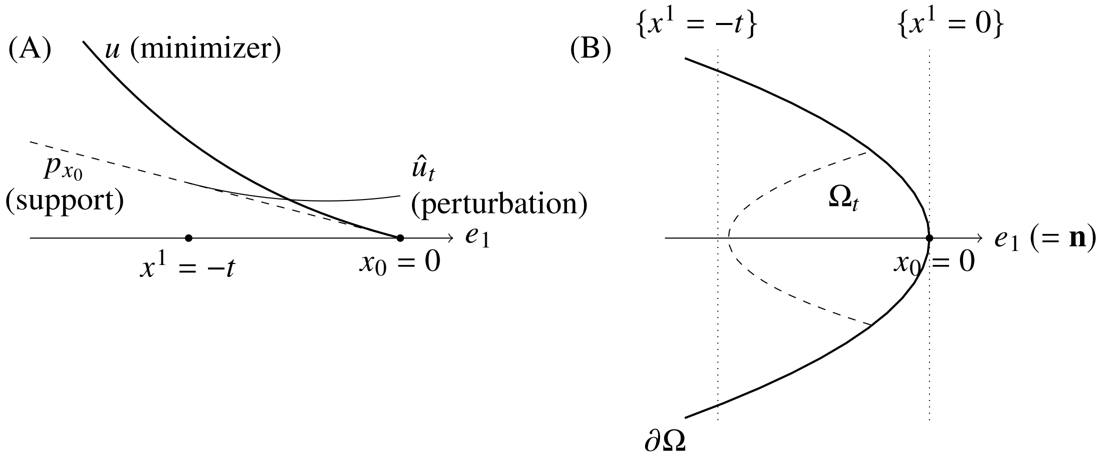


FIGURE 2. Illustrates the constructions in the proof of Proposition 2.3. Subfigure (A) shows a cross-section of the minimizer u, its support p_{x_0}, and the perturbation $\hat{u}_t$. Subfigure (B) illustrates that because $\Omega_t \subset \{x^1 \geq -t\}$ and Ω is strictly convex with outer normal $\mathbf{n} = e_1$ at 0 we have $\overline{\Omega_t} \to \{0\}$ in the Hausdorff distance.

for all convex functions v in Corollary A.9 of Appendix A below.

2.2. Legendre transforms and Alexandrov second derivatives

Recall if $u : \Omega \to \mathbf{R}$ is a convex function, then its Legendre transform is defined by

$$v(y) = \sup_{x\in\Omega} x \cdot y - u(x). \tag{23}$$

A function is called Alexandrov second differentiable at x_0, with Alexandrov Hessian $D^2u(x_0)$ (an $n \times n$ matrix), provided as $x \to x_0$ that

$$u(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T D^2u(x_0)(x - x_0) + o(|x - x_0|^2).$$

Alexandrov proved convex functions are twice differentiable in this sense $\mathcal{H}^n$ almost everywhere.

It's well known that if a differentiable convex function u is Alexandrov differentiable at x_0 and its Legendre transform is Alexandrov second differentiable at $y_0 := Du(x_0)$ then

$$D^2v(y_0) = \left[D^2u(x_0)\right]^{-1}.$$

We have an analogous result even when Alexandrov second differentiability is not assumed.

Lemma 2.4 (Legendre transform of Hessian bounds)**.** *Assume $u : \Omega \to \mathbf{R}$ is a convex function with Legendre transform v, that $x_0 \in \Omega$ and M is an invertible symmetric positive definite matrix. Assume $y_0 \in \partial u(x_0)$. Then*

$$u(x_0 + \delta x) \geq u(x_0) + y_0 \cdot \delta x + \delta x^T M \delta x/2 + o(|\delta x|^2) \quad \text{as } \delta x \to 0,$$

if and only if

$$v(y_0 + \delta y) \leq v(y_0) + x_0 \cdot \delta y + \delta y^T M^{-1} \delta y/2 + o(|\delta y|^2) \quad \text{as } \delta y \to 0.$$

Proof. We prove the "only if" statement; the "if" statement is proved similarly. Up to a choice of coordinates and subtracting an affine support we may assume $x_0, y_0 = 0$ and $u(x_0) = 0$. Whereby we're assuming

$$u(x) \geq x^T Mx/2 + o(|x|^2). \tag{24}$$

It is straightforward to see that (24) holds if and only if for every $\varepsilon > 0$ there is a neighbourhood $\mathcal{N}_\varepsilon$ of 0 on which

$$u(x) \geq (1-\varepsilon) x^T Mx/2. \tag{25}$$

Now, for $y \in \partial u(\mathcal{N}_\varepsilon)$ we have

$$\begin{aligned} v(y) &= \sup_{x \in \Omega} x \cdot y - u(x) \\ &= \sup_{x \in \mathcal{N}_\varepsilon} x \cdot y - u(x) \\ &\leq \sup_{x \in \mathcal{N}_\varepsilon} x \cdot y - (1-\varepsilon) x^T Mx/2. \end{aligned}$$

Provided y lies in the possibly smaller neighbourhood $Y_\varepsilon := [(1-\varepsilon)M\mathcal{N}_\varepsilon] \cap \partial u(\mathcal{N}_\varepsilon)$ the supremum is obtained at $x = [(1-\varepsilon)M]^{-1} y$ and

$$v(y) \leq y^T [(1-\varepsilon)M]^{-1} y/2.$$

The aforementioned equivalence between (25) and (24) gives the desired result. ∎

3. Partition into foliations by leaves that intersect fixed boundary

In this section we present the proof of Theorem 1.1. We use localization (Corollary A.9) to obtain the vanishing Neumann condition throughout $\Omega_n \cap \partial\Omega$, but apart from that the only technique we use is energy comparison, and the variational inequalities (17) – (18) coupled with careful choice of the comparison functions, many of which are inspired by those used to study the Monge–Ampère equation [29, 37]. We denote a subsection to each point of Theorem 1.1.

3.1. Point 1: $\Omega_0 = \{u = 0\}$ if nonempty.

We shall only prove $\Omega_0 \subset \{u = 0\}$; equality follows easily if Ω_0 is nonempty, which is known to be true on strictly convex domains [2].

Proof of Theorem 1.1 (1). For a contradiction we assume there is $x_0 \in \Omega_0$ with $u(x_0) > 0$. Applying Rochet–Choně's localization with $v = u$ we have

$$0 \leq \int_{\tilde{x_0}} (n+1-\Delta u) u \, dx + \int_{\tilde{x_0} \cap \partial\Omega} u(Du - x) \cdot \mathbf{n} \, dS. \tag{26}$$

We plan to show equality holds. By assumption $u(x) = p_{x_0}(x) = u(x_0) + Du(x_0) \cdot (x - x_0)$ on $\tilde{x_0}$. Since $-p_{x_0}(x)$ is convex and $x_0 \in \Omega \setminus \{u = 0\}$ we may apply Corollary A.9 with $v = p_{x_0}$ and obtain

$$0 \geq \int_{\tilde{x_0}} (n + 1 - \Delta u)u\, dx + \int_{\tilde{x_0} \cap \partial\Omega} u(Du - x) \cdot \mathbf{n}\, dS. \tag{27}$$

Using $\Delta u = 0$ on $\tilde{x_0}$, (26) and (27) imply

$$0 = (n + 1) \int_{\tilde{x_0}} u\, dx + \int_{\tilde{x_0} \cap \partial\Omega} u(Du - x) \cdot \mathbf{n}\, dS. \tag{28}$$

We know $u \geq 0$ and $(Du - x) \cdot \mathbf{n} \geq 0$ on $\partial\Omega$ (Proposition 2.3) and $\mathcal{H}^n(\tilde{x}_0) > 0$. Therefore (28) implies $u = 0$ on $\tilde{x_0}$ and this contradiction completes the proof. ∎

3.2. Point 3 of Theorem 1.1

Now we present point 3: that $\Delta u = n + 1$ in Ω_n and that $\Omega_n \cap \Omega$ is open. One can immediately obtain $\Delta u = n + 1$ a.e. in Ω_n via Rochet–Choné's localization (Corollary A.9). However for point 3 of Theorem 1.1 we require in addition, an inequality for $\Delta u(x_0)$ at all points where $Du(x_0) \in \mathrm{int} Du(\Omega)$. Thus we prove the following lemma directly using perturbations.

Lemma 3.1 (Sub- and super-Poisson for interior vs. customized consumption)**.** *Assume $u : \Omega \to \mathbf{R}$ solves* (1) *and $x_0 \in \Omega$ is a point of Alexandrov second differentiability satisfying $Du(x_0) \in \mathrm{int}(Du(\Omega))$. Then*

(1) *There holds $\Delta u(x_0) \geq n + 1$.*
(2) *If, in addition u is strictly convex at x_0, then $\Delta u(x_0) \leq n + 1$.*

Proof. (1) We take $x_0 \in \Omega$ with $Du(x_0) \in \mathrm{int}\, Du(\Omega)$ assumed to be a point of Alexandrov second differentiability. For convenience translate and subtract the affine support at x_0 so that $x_0, u(x_0)$ and $Du(x_0)$ all vanish.

For a contradiction assume $\Delta u(x_0) < n + 1$ and take $\varepsilon > 0$ satisfying $\Delta u(x_0) + n\varepsilon < n + 1$. There is a neighborhood of $x_0 = 0$ on which $u(x) < x^T(D^2u(x_0) + \varepsilon I)x/2$.

Let v denote the Legendre transform (23) of u and set $\tilde{v} = y^T[D^2u(0) + \varepsilon I]^{-1}y/2$. Lemma 2.4 implies $\tilde{v} < v$ in a punctured neighborhood of the origin. Thus as $h \to 0$ the connected component of $\{x\ ;\ v < \tilde{v} + h\}$ containing the origin, which we denote Ω_h^*, converges to $\{0\}$ in the Hausdorff distance. Set

$$v_h(y) = \begin{cases} \tilde{v}(y) + h & y \in \Omega_h^* \\ v(y) & y \notin \Omega_h^*. \end{cases} \tag{29}$$

Let u_h be the Legendre transform of v_h. Note $u_h \leq u$ and this inequality is strict at x_0. Moreover because $D^2v_h \geq (D^2u(0) + \varepsilon I)^{-1}$ at each $y \in \Omega_h^*$ Lemma 2.4 implies $\Delta u_h \leq \Delta u(x_0) + n\varepsilon < n + 1$ on the set $\{u_h < u\}$. This contradicts inequality (19) to establish (1), where $\partial\Omega \subset \{u_h = u\}$ for h small enough has been used.

(2) Now suppose, in addition, x_0 is a point of strict convexity for u and, for a contradiction, that $\Delta u(x_0) > n+1$. Set $\bar{u}(x) = (1-\varepsilon)x^T D^2u(0)x/2$ with $\varepsilon > 0$ chosen so small that $\Delta\bar{u} = (1-\varepsilon)\Delta u(x_0) > n+1$. Note that $\bar{u} < u$ in a punctured neighborhood of 0; (this relies on the strict convexity of u at 0 in the case $D^2u(0)$ has a zero eigenvalue). Thus for sufficiently small $h > 0$ the connected component of $\{x\, ;\, u(x) < \bar{u}(x) + h\}$ containing $x_0 = 0$, which we call $\Omega^{(h)}$, converges to $\{0\}$ in the Hausdorff distance. Set

$$\bar{u}_h = \begin{cases} \bar{u}(x) + h & x \in \Omega^{(h)}, \\ u(x) & x \notin \Omega^{(h)}. \end{cases}$$

Then $\bar{u}_h$ is an admissible interior perturbation of u with $\Delta\bar{u}_h > n+1$ on $\{\bar{u}_h > u\}$. Once again we contradict inequality (19). ■

At any interior point of strict convexity, $x \in \Omega_n \cap \Omega$, we have $Du(x) \in \operatorname{int} Du(\Omega)$ since ∂u is closed. It's now immediate that $\Delta u = n+1$ at each point of Alexandrov second differentiability in Ω_n. It remains to show $\Omega_n \cap \Omega$ is open. We prove in the next subsection that $Du(\bigcup_{i=0}^{n-1}\Omega_i) \subset \partial Du(\Omega)$, that is if $x \in \bigcup_{i=0}^{n-1}\Omega_i$ then $Du(x)$ is in the boundary of the set of gradients. Combined with the $C^{1,1}_{\mathrm{loc}}$ regularity of u we obtain if $x \in \Omega_n \cap \Omega$ the same is true for all sufficiently close $\bar{x}$ (this is because $Du(\bar{x})$ is also in $\operatorname{int} Du(\Omega)$) and this is a sufficient condition for $\bar{x} \in \Omega_n \cap \Omega$. We conclude $\Omega_n \cap \Omega$ is open.

Since we now know u is a $W^{2,\infty}_{\mathrm{loc}}$ (equivalently, $C^{1,1}_{\mathrm{loc}}$) solution of $\Delta u = n+1$ almost everywhere on the open set $\Omega_n \cap \Omega$ the elliptic regularity [32, Theorem 9.19] implies $u \in C^\infty(\Omega_n)$.

3.3. Point 2 of Theorem 1.1

To conclude the proof of Theorem 1.1 we show if $x \in \Omega_i$ for $i = 0, \dots, n-1$ then $\tilde{x}$ extends to the boundary.

For a contradiction assume otherwise. Because u is convex, then there exists x_0 with $\{x_0\} \neq \tilde{x_0} \subset\subset \Omega$ and $y_0 := Du(x_0) \in \operatorname{int} Du(\Omega)$ since ∂u is closed. Because u is $C^{1,1}$ and the sections

$$\Omega^{(h)} := \{x \in \overline{\Omega}\, ;\, u(x) \le u_h(x) := u(x_0) + Du(x_0)\cdot(x - x_0) + h\}$$

converge to $\tilde{x_0}$ in the Hausdorff distance as $h \to 0$, we obtain $Du(\Omega^{(h)}) \subset\subset Du(\Omega)$ for h sufficiently small. In particular by Lemma 3.1 we have $\Delta u \ge n+1$ in $\Omega^{(h)}$. Using $\hat{u} = \max\{u, u_h\}$ as a perturbation function in inequality (17) we see $\Delta u = n+1$ almost everywhere in $\Omega^{(h)}$ (if $\Delta u > n+1$ on a subset of $\{u_h > u\}$ with positive measure, inequality (17) is violated). As in the previous subsection the elliptic regularity implies $u \in C^\infty(\Omega^{(h)})$ is a classical solution of $\Delta u = n+1$ in $\Omega^{(h)}$. Differentiating this PDE twice implies the second derivatives of u are harmonic (and nonnegative by convexity). The strong maximum principle for harmonic functions says in fact $\partial^2_{jj}u > 0$ in $\Omega^{(h)}$ for all $j = 1, 2, ..., n$, so u cannot be affine on $\tilde{x}$. This contradiction completes the proof. Note our use of the strong maximum principle requires $\partial^2_{jj}u > 0$ at some point in $\Omega^{(h)}$. This, however, follows by considering $v = u - u_h$. If $\partial^2_{jj}u = 0$ throughout Ω^h then $Dv(x_0) = 0$, $v(x_0) = -h$ and v is independent of x_j, hence $\Omega^h \cap \partial\Omega \neq \emptyset$, which would contradict $\Omega^h \to \tilde{x_0} \subset\subset \Omega$ as $h \downarrow 0$.

In the course of the above proof we've proved the following lemma which we record here since we require it again and again.

Lemma 3.2 (Interior regularity and strong maximum principle)**.** *Assume $u : \Omega \to \mathbf{R}$ is a $C^{1,1}_{loc}$ convex function. Let $C \in \mathbf{R}$. Assume, in the sense of Alexandrov second derivatives, $\Delta u = C$ almost everywhere in $B_\varepsilon(x_0) \subset \Omega$. Then $u \in C^\infty(B_\varepsilon(x_0))$ and satisfies that for each unit vector ξ either $u_{\xi\xi} \equiv 0$ throughout $B_\varepsilon(x_0)$ or $u_{\xi\xi} > 0$ throughout $B_\varepsilon(x_0)$.*

4. Product selection remains Lipschitz up to the fixed boundary

One of our key techniques for studying the planar Monopolist's problem is the introduction of a new coordinate system defined in terms of the rays which foliate Ω_{n-1}. These new coordinates are a powerful tool for studying the behaviour of the minimizer u but their justification requires two significant technicalities. The first is a boundary regularity result in arbitrary dimensions, namely that in convex polyhedral domains u is $C^{1,1}$ up to the boundary (away from the nondifferentiabilities of the boundary); furthermore, in smooth convex domains u is $C^{1,1}$ on the set of rays having only one end on the boundary (Theorem 4.1). The second required technicality, proved in Section 5, is an equivalence between the Neumann condition and strict convexity stated more precisely in Propositions 5.3 and 5.4. Readers who are interested primarily in the consequences of these technicalities rather than their proof may proceed directly to Section 6.

Boundary regularity beyond $u \in C^1(\overline{\Omega})$ [16][51] for the Monopolist's problem is new. Previously only an interior $C^{1,1}$ result was known [10, 39] and $C^{1,1}$ regularity is known to be sharp.

Theorem 4.1 (Boundary $C^{1,1}$ regularity on convex polyhedral domains)**.** *Let u minimize* (1) *where $\Omega \subset\subset \mathbf{R}^n$ is open, bounded, and convex.*

(1) *There is C depending only on Ω such that if $x_0 \in \Omega_{n-1} \cup \Omega_n$ is a point of Alexandrov second differentiability and $\tilde{x_0} \cap \partial\Omega$ a singleton or empty then*

$$\Delta u(x_0) \leq C.$$

(2) *Assume, in addition, Ω is a convex polyhedron (i.e. an intersection of finitely many half spaces). Let Ω_ε be Ω excluding an ε-ball about each point where $\partial\Omega$ is not smooth. Then there is C depending only on ε and Ω such that*

$$\|u\|_{C^{1,1}(\Omega_\varepsilon)} \leq C.$$

Proof. The key energy comparison ideas are inspired by Caffarelli and Lions's proof of interior regularity [10] and its generalization [39]. However new ideas are required for perturbation near the boundary. We prove there exists C depending only on ε and Ω such that for all $x_0 \in \Omega_\varepsilon$ there holds

$$\limsup_{r\to 0} \frac{\sup_{B_r(x_0)} |u - p_{x_0}|}{r^2} \leq C. \tag{30}$$

Equivalently, there is some r_0 such that for all $r < r_0$ there holds $\sup_{B_r}(u - p_{x_0}) \leq Cr^2$. We emphasize that r_0 will be chosen small depending on quantities which the constant C in (30) is not permitted to depend on, however this does not affect the $C^{1,1}$ estimate[1].

We begin by explaining the proof for part (2), that is, when Ω is a polyhedron and then explain the changes required for part (1) namely, points on rays having an end in the interior of Ω. Note the result for $x \in \Omega_n$ is a straightforward consequence of the convexity of u and $\Delta u = n + 1$ in Ω_n.

Step 1. (Construction of section and comparison function on polyhedrons) We fix $x_0 \in \Omega_\varepsilon$ and translate and subtract a support plane after which we may assume $x_0, u(x_0)$ and $Du(x_0)$ all vanish and thus, $u \geq 0$[2]. Now, after a rotation we may assume the face closest to x_0 is

$$P_{-d} = \partial\Omega \cap \{x = (x^1, \dots, x^n) \ ; x^1 = -d\}.$$

We assume $d < \varepsilon$. (If $d > \varepsilon$ then we already have a $C^{1,1}$ estimate; Caffarelli and Lions's estimate is $|D^2u(x)| \leq C(n)\text{dist}(x, \partial\Omega)^{-1} \sup |Du|$.) Note that x_0 may be close to a single face of the polyhedron but, because we work in Ω_ε, satisfies

$$\text{dist}(x_0, \partial\Omega \setminus P_{-d}) =: \delta \geq C(\varepsilon, \Omega) \tag{31}$$

for a positive constant $C(\varepsilon, \Omega)$ depending only on ε and Ω.

For $r > 0$ to be chosen sufficiently small, but initially $r < d$, set

$$h = \sup_{B_r(0)} u = u(r\xi),$$

where the latter equality defines the unit vector ξ as the direction in which the supremum is obtained. The section

$$S := \{x \in \Omega \ ; u(x) < p(x)\},$$
$$\text{where} \quad p(x) := \frac{h}{2r}(x \cdot \xi + r),$$

satisfies the slab containment condition

$$S \subset \{x \in \Omega \ ; -r < x \cdot \xi < r\} =: S_{\xi,r}. \tag{32}$$

---

[1]This is analogous to an estimate $\left|\frac{f(x+h)-f(x)}{h}\right| \leq C$ for all sufficiently small h implying $|f'(x)| \leq C$ regardless of what dictates our small choice of h. It is interesting to note such an approach would not work for boundary Hölder or $C^{1,\alpha}$ estimates with $0 < \alpha < 1$.

[2]It is worth noting that L is not translation invariant, so after this transformation we should work with $\bar{L}[u] = \int_\Omega \frac{1}{2}|Du|^2 + u - (x + x_0) \cdot Du \, dx$. Inspection of the proof reveals such a change is inconsequential.

The lower estimate is because $p(x) < 0$ when $x \cdot \xi < -r$ and $u \geq 0$. For the upper estimate note

$$Du(r\xi) - Dp(r\xi),$$

is the outer unit normal to S at $r\xi$. However, because u attains its maximum over the boundary of the ball at $r\xi$, Du has zero tangential component and so, by convexity, $Du(r\xi) = \kappa\xi$ for some $\kappa \geq h/r$ meaning the outer normal is

$$Du(r\xi) - Dp_h(r\xi) = \kappa\xi - \frac{h}{2r}\xi.$$

Step 2. (Tilting and shifting at the boundary on polyhedrons) The possibility that S intersects $\partial\Omega$ complicates the boundary estimate. The existing interior estimates use a bound $\mathcal{H}^{n-1}(\partial S \cap \partial\Omega) \leq \frac{C}{\mathrm{dist}(x_0, \partial S \cap \partial\Omega)}\mathcal{H}^n(S)$ which does not suffice near the boundary. Thus we must tilt the affine support to ensure points where ∂S intersects $\partial\Omega$ lie sufficiently far (distance greater than $C(\varepsilon, \Omega)$) from x_0.

We consider the modified plane and section (see Figure 3)

$$\bar{S} = \{x \in \Omega \,;\, u(x) < \bar{p}(x)\},$$

$$\text{where} \quad \bar{p}(x) = \frac{h}{2r}(x \cdot \xi + r) - 2\frac{h}{d}(\mathbf{n} \cdot x) + s, \tag{33}$$

where $\mathbf{n} = -e_1$ is the outer unit normal to Ω along P_{-d} and s is a small positive or negative shift to be specified. The key idea is that provided $r \ll d$, $h/2r \gg 2h/d$ so $D\bar{p}$ is a small perturbation of Dp (in both direction and magnitude). Observe that slab containment, (32), implies $p - u < h$ and on $P_{-d} \subset \{x \,;\, x^1 = -d\}$, we have $\bar{p} = p - 2h + s$, so provided $s < h$ (which we enforce below), $\partial\bar{S}$ is disjoint from P_{-d}. We claim if r is initially chosen sufficiently small, then

$$\bar{S} \subset \{x \in \Omega \,;\, -2r < x \cdot \xi < 2r\} = S_{\xi,2r} \tag{34}$$

where the fact that Ω is bounded has been used.

First we prove the lower bound $\bar{S} \subset \{x \in \Omega \,;\, -2r < x \cdot \xi\}$. This follows because a choice of r sufficiently small ensures the plane $\{\bar{p} = 0\}$ makes an arbitrarily small angle with the original plane $\{p = 0\} = \{x \,;\, x \cdot \xi = -r\}$. Moreover we can ensure $\bar{p}(t\xi) = 0$ for some $0 > t > -3r/2$. Indeed, the plane $\{p = 0\}$, which we used as our original lower bound for the slab S, is necessarily orthogonal to $Dp(x) = \frac{h}{2r}\xi$. Similarly, the plane $\{\bar{p} = 0\}$, which we use as our lower bound for $\bar{S}$, is orthogonal to

$$D\bar{p}(x) = \frac{h}{2r}\xi - \frac{2h}{d}\mathbf{n},$$

provided $r \ll d$ the vectors $D\bar{p}$ and Dp make arbitrarily small angle. Thus the planes $\{\bar{p} = 0\}$ and $\{p = 0\}$ make arbitrarily small angle. Since $\bar{p}(0) = \frac{h}{2} + s$ and $\bar{p}(-3r\xi/2) = -\frac{h}{4} + s + 3\frac{hr}{d}\mathbf{n} \cdot \xi$, provided $|s| < h/8$ and $r < d/24$ there is a point on $\{t\xi \,;\, 0 > t > -3r/2\}$ where $\bar{p} = 0$.

Next we prove the upper bound $S \subset \{x \in \Omega\, ;\, x \cdot \xi < 2r\}$. This is where we choose our vertical shift. Recall

$$u(r\xi) = h \quad \text{and} \quad Du(r\xi) = \kappa\xi,$$

where $\kappa \geq \frac{h}{r}$. Note that

$$\bar{p}(\xi r) = h - 2\frac{h}{d}\mathbf{n} \cdot (r\xi) + s.$$

Because $r < d/24$ the choice $s = 2hr\mathbf{n} \cdot \xi/d$ gives $|s| < h/12$ and we have equality of $\bar{p}$ and u at $r\xi$, i.e. $\bar{p}(r\xi) = u(r\xi) = h$. Then, as before, $Du(r\xi) - D\bar{p}(r\xi)$ is a normal to a support of the convex set $\bar{S}$. However

$$Du(r\xi) - D\bar{p}(r\xi) = \kappa\xi - \frac{h}{2r}\xi + \frac{2h}{d}\mathbf{n}.$$

Recalling $\kappa \geq \frac{h}{r}$ we see again for r sufficiently small this vector makes arbitrarily small angle with ξ. This yields the upper containment in (34).

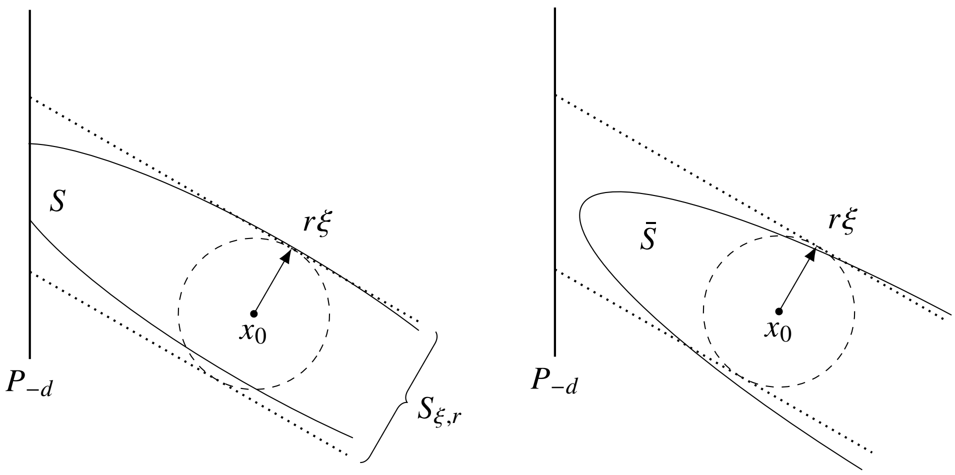


FIGURE 3. An example of the original section and the tilted section. The tilted section is now disjoint from boundary portion P_{-d}. The trade off is it may leave the slab $S_{\xi,r}$. Nevertheless $\bar{S}$ is contained in the slightly larger slab $S_{\xi,2r}$.

Our choice of tilted support $\bar{p}$ implies $\bar{S}$ is disjoint from P_{-d}. Moreover we have

$$\bar{p}(0) - u(0) \geq h/4 \quad \text{and} \quad \sup_{\bar{S}} \bar{p} - u \leq 4h, \tag{35}$$

where the second inequality is because each point in the containment slab $S_{\xi,4r}$ is of distance less than $4r$ from the plane $\{\bar{p} = 0\}$ and $|D\bar{p}| \leq h/r$. These properties are enough for us to employ the dilation argument used by the authors in [39]. For completeness we include full details, but first explain how to obtain the section $\bar S$ in the other setting of the theorem: in arbitrary convex domains at rays with one endpoint on the boundary.

Step 3. (Rays with one endpoint on the boundary in convex domains) Now we explain the choice of a suitable perturbation for case (1) of Theorem 4.1, namely when Ω is merely open, bounded, and convex and $x_0 \in \Omega_{n-1}$ satisfies that $\tilde{x_0}$ has only one point on the boundary. In this case a similar tilting procedure yields a section $\bar S$ which is in fact strictly contained in Ω. Assume, $\tilde{x_0} = \{x_0 + t\zeta\ ;\ -a \leq t \leq b\}$ where $x_0 - a\zeta \in \partial\Omega$. Note, if there is $c_0 > 0$ such that for all r sufficiently small, $|\xi \cdot \zeta| > c_0$ (i.e. the angle between ξ and ζ is bounded away from $\pi/2$) then the slab containment (32) yields $S \subset\subset \Omega$. On the other hand, if $\xi \cdot \zeta \to 0$ (i.e. ξ approaches the orthogonal direction to ζ) we consider the new perturbation

$$\bar p(x) = \frac{h}{2r}(x \cdot \xi + r) + \frac{4h}{a} x \cdot \zeta + s.$$

Provided $|s| < h$, which we can enforce as above, we have for r sufficiently small $\bar S \subset \{x\ ;\ x \cdot \zeta \geq -3a/4\}$. Since, in addition, as $h \to 0$, $\sup_{x \in \bar S} \operatorname{dist}(x, \tilde{x_0}) \to 0$ we have that for h sufficiently small (obtained by an initial choice of r sufficiently small) $\bar S$ will be strictly contained in Ω and satisfy both the slab containment condition (34) and the height estimates (35). Note in the steps that follow the argument is simpler for this case. Indeed, $\bar S \cap \partial\Omega = \emptyset$ implies we do not need to consider the boundary terms in what follows or the dilation argument in Step 5.

Step 4. Initial estimates. Now we use the minimality of the function u for the functional L (defined in (1)) to derive the desired inequality $h \leq Cr^2$. Set $\bar u = \max\{\bar p, u\}$ where $\bar p$ is defined in (33). Minimality implies

$$\begin{aligned} 0 &\leq L[\bar u] - L[u] \\ &= \int_{\bar S} \frac{1}{2}(|D\bar u|^2 - |Du|^2) + (\bar u - u) - (x \cdot D\bar u - x \cdot Du)\, dx \\ &\leq \int_{\bar S} \frac{1}{2}(|D\bar u|^2 - |Du|^2) - x \cdot (D\bar u - Du)\, dx + 4h\mathcal{H}^n(\bar S). \end{aligned} \tag{36}$$

The divergence theorem implies

$$\begin{aligned} -\int_{\bar S} x \cdot (D\bar u - Du)\, dx &= -\int_{\partial\bar S \cap \partial\Omega} (\bar u - u) x \cdot \mathbf{n}\, d\mathcal{H}^{n-1} + n\int_{\bar S} (\bar u - u)\, dx \\ &\leq Ch\mathcal{H}^{n-1}(\partial\bar S \cap \partial\Omega) + 4nh\mathcal{H}^n(\bar S). \end{aligned} \tag{37}$$

Next, using that $\bar{u}$ is linear in $\bar{S}$ (in particular, $\Delta\bar{u} = 0$), we compute

$$
\begin{aligned}
\int_{\bar{S}} |D\bar{u}|^2 - |Du|^2 \, dx &= \int_{\bar{S}} \langle D\bar{u} + Du, D\bar{u} - Du \rangle \, dx \\
&= \int_{\bar{S}} \operatorname{div}\big((D\bar{u} + Du)(\bar{u} - u)\big) - \Delta u(\bar{u} - u) \, dx \\
&= \int_{\bar{S}} \operatorname{div}\big((D\bar{u} + Du)(\bar{u} - u)\big) \\
&\qquad + \operatorname{div}\big((D\bar{u} - Du)(\bar{u} - u)\big) - |D\bar{u} - Du|^2 \, dx \\
&\leq -\int_{\bar{S}} |D\bar{u} - Du|^2 \, dx + 2\int_{\partial\bar{S}\cap\partial\Omega} (\bar{u} - u) D\bar{u} \cdot \mathbf{n} \, d\mathcal{H}^{n-1}. \qquad (38)
\end{aligned}
$$

Substituting (37) and (38) into (36) we have for C depending on $\sup |D\bar{u}|$

$$
\int_{\bar{S}} |D\bar{u} - Du|^2 \leq C(h\mathcal{H}^n(\bar{S}) + h\mathcal{H}^{n-1}(\partial\bar{S} \cap \partial\Omega)). \qquad (39)
$$

Step 5. (Final estimates) To complete the proof we prove for $C > 0$, which in the case of polyhedra depends in particular on ε, there holds

$$
\mathcal{H}^{n-1}(\partial\bar{S} \cap \partial\Omega) \leq C\mathcal{H}^n(\bar{S}) \qquad (40)
$$

$$
\text{and} \quad \int_{\bar{S}} |D\bar{u} - Du|^2 \geq C\frac{h^2}{r^2}\mathcal{H}^n(\bar{S}). \qquad (41)
$$

For the first, in the case of polyhedral domains, recall $\partial\bar{S} \cap P_{-d}$ is empty so $\partial\bar{S} \cap \partial\Omega$ is of distance $C(\varepsilon, \Omega)$ from $x_0 = 0$. Thus the estimate (40), for C depending on ε, is standard in convex geometry and may be proved either as in the work of Chen [23], or the authors [39]. In case (1) of the theorem the estimate is trivial because $\partial\bar{S} \cap \partial\Omega = \emptyset$.

Now we obtain (41). We let $\bar{S}/K$ denote the dilation of $\bar{S}$ by a factor of $1/K$ with respect to x_0. What is again crucial is that $\partial\bar{S} \cap P_{-d} = \emptyset$ so for $D_{x_0,\Omega\setminus P_{-d}}$ defined as in (31), $\partial\bar{S} \cap B_{\delta/2}(x_0)$ consists of interior points of Ω on which $\bar{u} - u = 0$. It is helpful now to choose coordinates such that $\xi = e_1$. For $x = (x^1, x')$, let $P(x) := (0, x')$ be the projection onto $\{x \,;\, x^1 = 0\}$. For each $(0, x') \in P(\bar{S}/K)$ the set $(P^{-1}(0, x') \cap \bar{S}) \setminus (\bar{S}/K)$ is two disjoint line segments. We let $l_{x'}$ be the line segment with greater x_1 component and write $l_{x'} = [a_{x'}, b_{x'}] \times \{x'\}$ where $b_{x'} > a_{x'}$.

Choose $K = \max\{2\text{diam}(\Omega)/\delta, 2\}$ in case (2) which is bounded below by a positive constant depending on ε and Ω. Case (1) is simpler as this dilation is not required. Note that each line segment $l_{x'}$ for $(0, x') \in P(\bar{S}/K)$ has $\bar{u} - u = 0$ at the upper endpoint. This is because from the slab containment condition the upper endpoint lies distance less than $4r \ll \delta$ from $B_{\varepsilon/2}(0) \cap \{x^1 = 0\}$ whereas $\partial\bar{S} \cap \partial\Omega$ lies distance at least δ from $x_0 = 0$. Clearly on $\partial\bar{S} \cap \Omega$ we have $\bar{u} - u = 0$.

We claim each of the following

$$u((b_{x'},x')) - \bar{u}((b_{x'},x')) = 0, \tag{42}$$

$$u((a_{x'},x')) - \bar{u}((a_{x'},x')) \le -\frac{K-1}{K}\frac{h}{4}, \tag{43}$$

$$d_{x'} := b_{x'} - a_{x'} \le 4r. \tag{44}$$

As noted above (42) is because $\bar{u} - u = 0$ on $\partial\bar{S} \cap B_\varepsilon(x_0)$. Then (43) is by convexity of $u - \bar{u}$ along a line segment joining the origin, where $u - \bar{u} \le -h/4$, to $(Ka_{x'}, Kx') \in \partial\bar{S}$, where $u - \bar{u} \le 0$. Finally, (44) is by the modified slab containment condition (34).

Thus, by an application of Jensen's inequality we have

$$\begin{aligned}
\int_{a_{x'}}^{b_{x'}} &[D_{x^1}\bar{u}((t,x')) - D_{x^1}u((t,x'))]^2 dt \\
&\ge \frac{1}{d_{x'}}\left(\int_{a_{x'}}^{b_{x'}} D_{x^1}\bar{u}((t,x')) - D_{x^1}u((t,x'))dt\right)^2 \\
&\ge \frac{1}{d_{x'}}\left(\frac{K-1}{K}\right)^2\frac{h^2}{16} \ge Ch^2/r. \qquad (45)
\end{aligned}$$

To conclude we integrate along all lines $l_{x'}$ for $x' \in P(\bar{S}/K)$. Indeed

$$\begin{aligned}
\int_{\bar{S}} |D\bar{u} - Du|^2\, dx &\ge \int_{P(\bar{S}/K)}\int_{a_{x'}}^{b_{x'}} |D_{x^1}\bar{u}((t,x')) - D_{x^1}u((t,x'))|^2\, dt\, dx' \\
&\ge \int_{P(\bar{S}/K)} Ch^2/r\, dx' \\
&= C\frac{h^2}{r^2}(r|P(\bar{S}/K)|).
\end{aligned}$$

Finally, the convexity of $\bar{S}$ and slab containment (34) implies

$$\int_{\bar{S}} |D\bar{u} - Du|^2\, dx \ge C\frac{h^2}{r^2}|\bar{S}|. \tag{46}$$

Having obtained (41) , substituting inequalities (40) and (41) yields (39) and completes the proof. ■

5. Strict convexity implies the Neumann condition

In this section we continue establishing technical conditions required for the coordinates introduced in Section 6. For planar domains we prove the equivalence between the Neumann condition and strict convexity stated precisely in Propositions 5.3 and 5.4. We begin with two lemmas concerning convex functions in $\mathbf{R}^n$. The first states the upper semicontinuity of the function $x \mapsto \operatorname{diam}(\tilde{x})$ and the second yields the convexity of $\partial^2_{ii}u$ when restricted to a contact set $\tilde{x}$.

Lemma 5.1 (Upper semicontinuity of leaf diameter)**.** *Let Ω be a bounded open convex subset of $\mathbf{R}^n$ and $u \in C^1(\overline{\Omega})$ a convex function. Then the function $x \in \overline{\Omega} \mapsto diam(\tilde{x})$ is upper semicontinuous.*

Proof. We fix a sequence $(x_k)_{k\geq 1}$ converging to some $x_\infty \in \overline{\Omega}$ and note it suffices to prove that

$$\limsup_{k\to\infty} \operatorname{diam}(\tilde{x_k}) \leq \operatorname{diam}(\tilde{x_\infty}).$$

To this end, let $p_k = Du(x_k)$ and take $x_k^{(1)}, x_k^{(2)}$ in $\overline{\Omega}$ realizing

$$|x_k^{(1)} - x_k^{(2)}| = \operatorname{diam}(\tilde{x_k})$$
$$\text{and} \quad \lim_{k\to\infty} |x_k^{(1)} - x_k^{(2)}| = \limsup_{k\to\infty} \operatorname{diam}(\tilde{x_k}).$$

The convergence properties of the subdifferential of a convex function imply $Du(x_k) \to Du(x_\infty)$ and we may assume that, up to a subsequence, $x_k^{(i)} \to x_\infty^{(i)} \in \overline{\Omega}$ for $i = 1, 2$. Thus we may send $k \to \infty$ in the identity

$$u(x_k^{(i)}) = u(x_k) + Du(x_k) \cdot (x_k^{(i)} - x_k) \tag{47}$$

to obtain that for $i = 1, 2$ we have $x_\infty^{(i)} \in \tilde{x_\infty}$ and thus $\tilde{x}$ has diameter greater than or equal to

$$|x_\infty^{(1)} - x_\infty^{(2)}| = \lim_{k\to\infty} |x_k^{(1)} - x_k^{(2)}| = \limsup_{k\to\infty} \operatorname{diam}(\tilde{x_k}).$$

■

Let r.i.$(\tilde{x})$ denote the relative interior of the convex set $\tilde{x}$.

Lemma 5.2 (Existence a.e. of D^2u on a leaf implies convexity of $\partial^2_{ii}u$)**.** *Let $u : \Omega \to \mathbf{R}$ be a differentiable convex function defined on an open convex subset $\Omega \subset \mathbf{R}^n$. Fix any $x \in \Omega$. If $\mathcal{H}^{\dim \tilde{x}}(\tilde{x} \setminus \operatorname{dom} D^2u) = 0$, where* $\operatorname{dom} D^2 u \subset \Omega$ *denotes the set of second differentiability of u, then $u|_{\text{r.i.}(\tilde{x})} \in C^2_{loc}(\text{r.i.}(\tilde{x}))$ and $\partial^2_{ii}u|_{\tilde{x}}$ is a convex function for each $i = 1, \dots, n$.*

Proof. Fix x satisfying $\mathcal{H}^{\dim \tilde{x}}(\tilde{x} \setminus \operatorname{dom} D^2u) = 0$. After subtracting the support at x, we may assume $u(x) = 0, Du(x) = 0$ and that $\tilde{x}$ is not a singleton (since otherwise the result holds trivially). We will show $\partial^2_{ii}u$ is convex along any of those line segments contained in $\tilde{x}$ for which Alexandrov second differentiability holds a.e. . To this end fix $x_0, x_1 \in \tilde{x} \cap \operatorname{dom} D^2u$ along such a segment. Choose orthonormal coordinates such that $x_1 = x_0 + Te_1$ for some $T > 0$. Then since u is affine on $\tilde{x}$ and $\{x_0 + te_1 \ ; 0 \leq t \leq T\} \subset \tilde{x}$ we have $\partial^2_{11}u = 0$ a.e. on $\{x_0 + te_1 \ ; 0 \leq t \leq T\}$.

Next, for $i = 2, \dots, n$ and any $t \in (0, 1)$, convexity of u implies for $r > 0$ sufficiently small

$$u((1-t)x_0 + tx_1 + re_i) \leq (1-t)u(x_0 + re_i) + tu(x_1 + re_i). \tag{48}$$

Here, by r sufficiently small we mean small enough to ensure the above arguments of u are contained in Ω. The definition of Alexandrov second differentiability along with $u, Du = 0$ on $\tilde{x}$ implies

$$u((1-t)x_0 + tx_1 + re_i) = r^2 \partial^2_{ii} u((1-t)x_0 + tx_1)/2 + o(r^2),$$

for a.e. $t \in (0,1)$ and similarly at $x_0 + re_1$ and $x_1 + re_1$. Thus (48) becomes

$$r^2\partial^2_{ii}u((1-t)x_0+tx_1)/2+o(r^2) \le (1-t)r^2\partial^2_{ii}u(x_0)/2+tr^2\partial^2_{ii}u(x_1)/2+o(r^2).$$

Dividing by r^2 and sending $r \to 0$ yields that $\partial^2_{ii}u$ is the restriction of a convex function to the segment $[x_0, x_1]$ — hence continuous on $[x_0, x_1]$. The polarization identity implies the continuity of mixed second order partial derivatives. It follows that $u|_{\mathrm{r.i.}(\tilde{x})} \in C^2_{\mathrm{loc}}$. ■

Proposition 5.3 (No normal distortion nearby implies strict convexity)**.** *Let u minimize* (1) *where $\Omega \subset\subset \mathbf{R}^2$ is open and convex. Let $x_0 \in \partial\Omega$ be a point where $u(x_0) > 0$ and $\tilde{x_0} \cap \partial\Omega = \{x_0\}$. Assume there is $\varepsilon > 0$ with*

$$(Du(x) - x) \cdot \mathbf{n} = 0 \ \text{on} \ B_\varepsilon(x_0) \cap \partial\Omega.$$

Then $\tilde{x_0} = \{x_0\}$, that is u is strictly convex at x_0.

Proof. Because $u \in C^1(\overline{\Omega})$ and Lemma 5.1 implies the upper semicontinuity of R, we may find a possibly smaller $\varepsilon > 0$ such that $u(x) > 0$ and $\tilde{x} \cap \partial\Omega = \{x\}$ for each $x \in \mathcal{N} := B_\varepsilon(x_0) \cap \partial\Omega$. Rochet and Choné's localization (Corollary A.9) with 0 boundary term implies $\Delta u = 3$ almost everywhere on

$$\tilde{\mathcal{N}} := \{x' \in \tilde{x} \,;\, x \in \mathcal{N}\}.$$

To see this, note, by Lemma 5.2, Δu restricted to any $\tilde{x}$ for which $\mathcal{H}^{\dim \tilde{x}}(\tilde{x} \setminus \operatorname{dom} D^2u) = 0$ is a convex function. Thus $v = -(3 - \Delta u)$ is a permissible test function in the localization Corollary A.9 from which we obtain

$$0 \le -\int_{\tilde{x}} (3 - \Delta u)^2 \, (dx)_{\tilde{x}}, \tag{49}$$

where $(dx)_{\tilde{x}}$ denotes the disintegration of the Lebesgue measure with respect to the contact sets (an explanation of disintegration is provided in Appendix A). Inequality (49) implies that $\mathcal{H}^{\dim \tilde{x}}$ almost everywhere on $\tilde{x}$ there holds $(3 - \Delta u)^2 = 0$ and thus the same equality holds on $\tilde{\mathcal{N}}$.

Let x_1 be the interior endpoint of $\tilde{x_0}$ (if $\tilde{x_0} = \{x_0\}$ then we are already done). In a sufficiently small ball $B_\delta(x_1)$, we have just shown $\Delta u = 3$ a.e. in $B_\delta(x_1) \cap \tilde{\mathcal{N}}$. Moreover, Theorem 1.1 implies $\Delta u = 3$ in $B_\delta(x_1) \setminus \tilde{\mathcal{N}} = B_\delta(x_1) \cap \Omega_2$, *which is nonempty*. Our usual maximum principle argument, Lemma 3.2, implies u is strictly convex inside $B_\delta(x_1)$, contradicting that x_1 is the endpoint of a ray. ■

The next proof requires Lemma A.6 which gives the pushforward $Du_\#(\sigma) = \delta_0$ of the variational derivative — and is proved in Appendix A by combining the neutrality implied by localization away from the excluded region $\{u = 0\}$ with the fact that our objective responds proportionately to a uniform increase in indirect utility. We use these to estimate the following:

Proposition 5.4 (One-ended ray lengths bound normal distortion)**.** *Let u solve* (1) *where $\Omega \subset\subset \mathbf{R}^2$ is open and convex. Let $\{x_0\} = \tilde{x_0} \cap \partial\Omega$ with $\partial\Omega$ smooth in a neighbourhood of x_0 and $u(x_0) > 0$. Set $R(x_0) = diam(\tilde{x_0})$. Then*

$$0 \leq (Du(x_0) - x_0) \cdot \boldsymbol{n} \leq CR(x_0), \tag{50}$$

where C depends only on a $C^{1,1}$ bound for u in a neighbourhood of $\tilde{x_0}$.

Proof. The lower bound (50) was established in Proposition 2.3. We first prove (50) assuming $\tilde{x_0}$ nontrivial and at the conclusion of the proof explain why it holds for all x_0 in the theorem. Let $x_0 \in \partial\Omega$ and let $\partial\Omega$ be locally represented by a smooth curve with an arc length parametrization $\gamma : (-\varepsilon, \varepsilon) \to \mathbf{R}^2$ traversing $\partial\Omega$ in the anticlockwise direction with $x_0 = \gamma(0)$ and without loss of generality $\dot{\gamma}(0) = e_2$.

The upper semicontinuity of R from Lemma 5.1 implies

$$\limsup_{x \to x_0} R(x) \leq R(x_0).$$

On the other hand we know from Lemma 3.2 that no subinterval of $\tilde{x_0}$ can be exposed to Ω_2 by which we mean there is no $x \in \tilde{x_0}$ and $\delta > 0$ with $B_\delta(x) \setminus \tilde{x_0} = \Omega_2$. Note in two-dimensions, if $(x_n)_{n\geq 1} \subset \partial\Omega$ satisfies $x_n \to x_0$ and $R(x_n) \to R(x_0)$ then $\tilde{x_n} \to \tilde{x_0}$ in the Hausdorff distance (and such sequences can be found).

As a result there exists sufficiently small $\alpha, \beta > 0$ such that

1. The leaves $\widetilde{\gamma(-\alpha)}$ and $\widetilde{\gamma(\beta)}$ have length at least $3R(x_0)/4$.
2. The leaves $\widetilde{\gamma(-\alpha)}$ and $\widetilde{\gamma(\beta)}$ can be chosen to make fixed but arbitrarily small angle with $\tilde{x_0}$, by e.g. [12, Lemma 16].
3. All leaves intersecting the boundary in $\gamma([-2\alpha, 2\beta])$ have length less than $9R(x_0)/8$ (this holds by the upper semicontinuity of R).

Moreover α, β can be taken as close to 0 as desired. With the smoothness of γ, our two-dimensional setting, and the fact that leaves cannot intersect, this significantly constrains the geometry of

$$A := (Du)^{-1}\big(Du(\gamma([-\alpha, \beta]))\big) = \bigcup_{t \in [-\alpha,\beta]} \widetilde{\gamma(t)}.$$

The set A is strictly contained in a set with left edge $\gamma([-2\alpha, 2\beta])$ and a *vertical right edge of lengths bounded by* $2(\beta + \alpha)$, and top and bottom side lengths bounded by $5R(x_0)/4$ (see Figure 4). Finally we note we can choose sequences α_k, β_k satisfying the above requirements and $\alpha_k, \beta_k \to 0$.

Now, by Lemma A.6, $\sigma(A) = ((Du)_{\#}\sigma)\big(Du(\gamma([-\alpha, \beta]))\big) = 0$. That is,

$$0 = \int_A (n + 1 - \Delta u)\, dx + \int_{\gamma(-\alpha,\beta)} (Du - x) \cdot \mathbf{n}\, d\mathcal{H}^1. \tag{51}$$

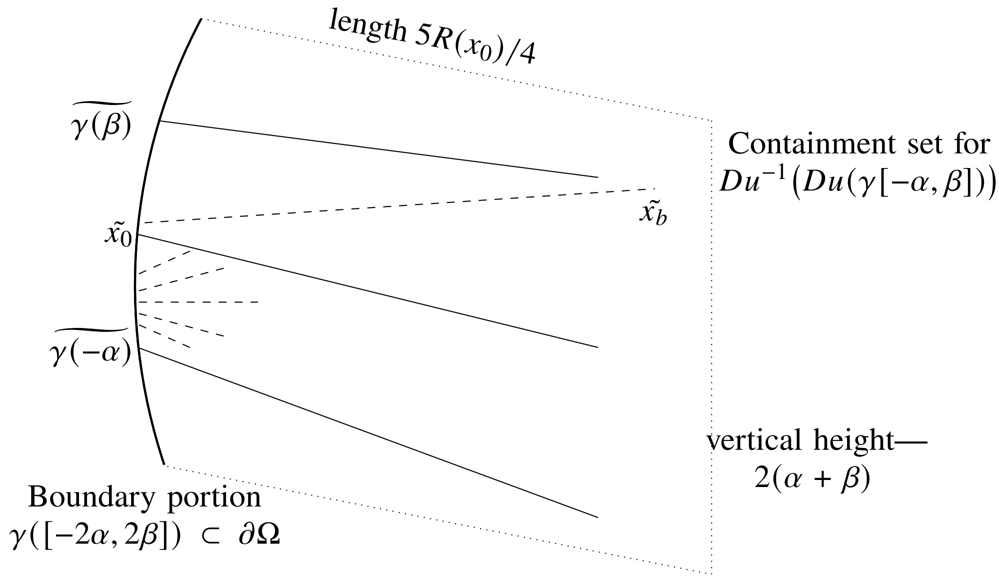


FIGURE 4. Geometry of the constructed set A. Note apriori (though not expected) there may be errant leaves such as $\tilde{x_b}$ or those between $\widetilde{\gamma(-\alpha)}$ and $\tilde{x_0}$. However we have constrained the length of such leaves as less than $9R(x_0)/8$ and, when long, their angles are constrained by the outer leaves $\widetilde{\gamma(-\alpha)}$ and $\widetilde{\gamma(\beta)}$ (which other leaves may not intersect). Thus we obtain the (crude) containment estimate indicated by dotted lines.

Using the boundary $C^{1,1}$ estimate from Theorem 4.1 (proved in Section 4) near $\tilde{x_0}$[3], the constrained geometry of A, and nonnegativity of $(Du - x)\cdot \mathbf{n}$ already established, we see (51) implies

$$\begin{aligned} 0 &\le \int_{\gamma(-\alpha,\beta)} (Du - x)\cdot \mathbf{n}\, d\mathcal{H}^1 \\ &\le \sup_A |n+1-\Delta u| \mathcal{H}^2(A) \\ &\le CR(x_0)(\alpha+\beta) \end{aligned}$$

which is precisely the desired estimate. Indeed, employing this estimate with α_k, β_k in place of α, β, dividing by $\alpha_k + \beta_k$, and sending $\alpha, \beta \longrightarrow 0$ we obtain (50) (after dividing we have an average and $(Du(x_0) - x_0)\cdot \mathbf{n}$ is continuous).

Now, we explain how to obtain the estimate when $\tilde{x}$ is trivial. If $x \in \partial\Omega$ is such that $\tilde{x}$ is trivial and there is a sequence of nontrivial leaves $\partial\Omega \ni x_k \longrightarrow x$ then the estimate follows by the upper semicontinuity of R. If there is no

[3]Because $\tilde{x_0}$ doesn't intersect $\partial\Omega$ at both endpoints and R is upper semicontinuous the same is true for all sufficiently close leaves.

such sequence then $x \in \partial\Omega$ lies in a relatively open subset of the boundary $\mathcal{N} = B_{\varepsilon_0}(x) \cap \partial\Omega$ on which u is strictly convex. Using the nonnegativity of $(Du-x)\cdot\mathbf{n}$ and Corollary A.9 with $v = \pm 1$ applied to $\mathcal{N}$ yields $(Du-x)\cdot\mathbf{n} = 0$ on $\mathcal{N}$. ∎

6. Leafwise coordinates parameterizing bunches in the plane

In this section and the next we study the behavior of the minimizer on Ω_1 and the free boundary $\Gamma = \partial\Omega_1 \cap \partial\Omega_2 \cap \Omega$ in two-dimensions. We introduce one of our main tools: a coordinate system to study the problem on Ω_1 which is flexible enough to include the coordinates proposed earlier by the first and third authors [42], and for which we are finally able to provide a rigorous foundation by proving biLipschitz equivalence to Cartesian coordinates. Moreover, by combining these coordinates with Rochet and Choné's localization technique (Corollary A.9) we are able to provide a radically simpler derivation of the Euler-Lagrange equations (10)–(11) first expressed in [42]; c.f. (68)–(69) and (83)–(84) below,

Let $\gamma : [-a, b] \to \mathbf{R}^2$ be a curve parameterizing $\partial\Omega$ in the clockwise direction with $\dot\gamma(t) \neq 0$ and write $\gamma(t) = (\gamma^1(t), \gamma^2(t))$.

First we give conditions to ensure a neighbourhood of a ray is foliated by rays.

Lemma 6.1 (Local foliation around each tame ray)**.** *Let $\partial\Omega$ be smooth in a neighbourhood of $x_0 \in \mathbf{R}^2$ satisfying $(Du(x_0) - x_0)\cdot\mathbf{n} \neq 0$, $u(x_0) \neq 0$ and $\{x_0\} = \tilde{x_0} \cap \partial\Omega$. Then there exist $\varepsilon, r_0 > 0$ such that diam$(\tilde{x}) \geq r_0$ and $\tilde{x} \cap \partial\Omega = \{x\}$ for all $x \in \partial\Omega \cap B_\varepsilon(x_0)$.*

Proof. We assume without loss of generality that $\gamma(0) = x_0$ and thus there is $\varepsilon > 0$ such that $\gamma : (-\varepsilon, \varepsilon) \to \partial\Omega$ is a smooth curve. Lemma 5.4 implies $\tilde{x_0}$ is nontrivial. For a possibly smaller $\varepsilon > 0$ and all $t \in (-\varepsilon, \varepsilon)$, the $C^1(\overline{\Omega})$ regularity of u implies $\widetilde{\gamma(t)}$ is nontrivial, with length bounded below by some r_0 determined by (50). Furthermore, the upper semicontinuity of $x \mapsto \mathrm{diam}(\tilde{x})$ (Lemma 5.1) implies for a possibly smaller ε, $\widetilde{\gamma(t)} \cap \partial\Omega = \{\gamma(t)\}$ for each $t \in (-\varepsilon, \varepsilon)$. ∎

Recall $x_1 \in \Gamma$ is a tame point of the free boundary $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \cap \Omega$ provided $\tilde{x_1} \cap \partial\Omega = \{x_0\}$ for an x_0 satisfying the hypothesis of Lemma 6.1.

We define $\xi(t) = (\xi^1(t), \xi^2(t))$ as the unit direction vector of the leaf $\widetilde{\gamma(t)}$ pointing into Ω. This means, with $R(t) := \mathrm{diam}(\widetilde{\gamma(t)})$,

$$\widetilde{\gamma(t)} = \{\gamma(t) + r\xi(t)\ ;\ 0 \leq r \leq R(t)\},$$

and subsequently we can write a subset of the connected component of Ω_1 containing x_0 as

$$\mathcal{N} = \mathcal{N} \cap \Omega_1 = \bigcup_{t \in (-\varepsilon,\varepsilon)} \widetilde{\gamma(t)} \tag{52}$$

$$= \{x(r,t) = \gamma(t) + r\xi(t)\ ;\ -\varepsilon < t < \varepsilon, 0 \leq r \leq R(t)\}, \tag{53}$$

and we take (r, t) as new coordinates for $\mathcal{N}$. Because each ray is a contact set along which u is affine there exists functions $b, m : (-\varepsilon, \varepsilon) \to \mathbf{R}$ such that

$$u(x(r,t)) = b(t) + rm(t), \tag{54}$$

and $Du(x(r,t))$ is independent of t.

Our goal is to derive the Euler–Lagrange equations of Lemma 6.3 below which describe the equations the minimizer satisfies in terms of R, ξ, m and b . First, we record the key structural equalities for the new coordinates in the following lemma, which holds under the biLipschitz hypothesis we eventually establish in Corollary 6.6. The quantities (56)–(58) from this lemma also yield a formula for the Laplacian of $u \in C^{1,1}(\mathcal{N})$:

$$\Delta u = \frac{\xi \times w'}{J(r,t)} =: \frac{\delta(t)}{J(r,t)}. \tag{55}$$

Lemma 6.2 (Gradient and Hessian of u in coordinates along tame rays)**.** *Suppose u solves* (1) *where $\Omega \subset \mathbf{R}^2$ is bounded, open and convex. Let γ, ξ, R, m, b on $\mathcal{N}$ be as above* (53)*. If the transformation $x(r,t) = \gamma(t) + r\xi(t)$ is biLipschitz on $\mathcal{N}$, then its Jacobian determinant is positive and given by*

$$0 < J(r,t) = \det\left(\frac{\partial(x^1, x^2)}{\partial(r,t)}\right) = \xi \times \dot\gamma + r\xi \times \dot\xi =: j(t) + r f_\xi(t) \tag{56}$$

where $\xi \times \dot\gamma = \xi^1\dot\gamma^2 - \xi^2\dot\gamma^1$ and similarly $\xi \times \dot\xi$ are evaluated at t. In addition the following formulas for the gradient and entries of the Hessian of (54) *hold $\mathcal{H}^2$-a.e.:*

$$Du(x(r,t)) = \begin{pmatrix} D_1 u \\ D_2 u \end{pmatrix} = \frac{1}{\xi \times \dot\gamma}\begin{pmatrix} \dot\gamma^2 & -\xi^2 \\ -\dot\gamma^1 & \xi^1 \end{pmatrix}\begin{pmatrix} m(t) \\ b'(t) \end{pmatrix} =: \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}, \tag{57}$$

$$D^2u(x(r,t)) = \begin{pmatrix} \partial^2_{11}u & \partial^2_{12}u \\ \partial^2_{21}u & \partial^2_{22}u \end{pmatrix} = \frac{1}{J(r,t)}\begin{pmatrix} -\xi^2(t)w_1'(t) & \xi^1(t)w_1'(t) \\ -\xi^2(t)w_2'(t) & \xi^1(t)w_2'(t) \end{pmatrix}. \tag{58}$$

Proof. Where the transformation $x(r,t) = \gamma(t) + r\xi(t)$ and its inverse are Lipschitz, their Jacobian derivatives are easily compute to be:

$$\frac{\partial(x^1,x^2)}{\partial(r,t)} = \begin{pmatrix} \xi^1(t) & \dot\gamma^1(t) + r\dot\xi^1(t) \\ \xi^2(t) & \dot\gamma^2(t) + r\dot\xi^2(t) \end{pmatrix} \tag{59}$$

$$\frac{\partial(r,t)}{\partial(x^1,x^2)} = \frac{1}{J(r,t)}\begin{pmatrix} \dot\gamma^2(t) + r\dot\xi^2(t) & -\dot\gamma^1(t) - r\dot\xi^1(t) \\ -\xi^2(t) & \xi^1(t) \end{pmatrix}, \tag{60}$$

with $J(r,t)$ from (56). Next, to obtain the gradient expressions we differentiate equation (54) with respect to r to obtain

$$m(t) = \xi^1(t)u_1(x(r,t)) + \xi^2(t)u_2(x(r,t)) = \langle \xi, Du \rangle. \tag{61}$$

Similarly, differentiating (54) with respect to t and equating coefficients of r yields

$$m'(t) = \dot\xi^1(t)D_1u(x(r,t)) + \dot\xi^2(t)u_2(x(r,t)) = \langle \dot\xi, Du\rangle, \tag{62}$$

$$b'(t) = \dot\gamma^1(t)D_1u(x(r,t)) + \dot\gamma^2(t)D_2u(x(r,t)) = \langle \dot\gamma, Du\rangle, \tag{63}$$

where $D_iu = \frac{\partial u}{\partial x^i}$ and that $Du(x(r,t))$ is independent from r has been used. We solve (61) and (63) for D_1u, D_2u and obtain (57). Note the functions w_1 and w_2 in (57) are Lipschitz because $u \in C^{1,1}(\mathcal{N})$. Thus, differentiating the expressions for D_1u and D_2u given by (57) with respect to, respectively, x_1 and x_2 and using the Jacobian (60) gives the formula (58).

We note two facts about the functions j and f_ξ which determine the Jacobian determinant. First

$$j(t) := J(0,t) = \xi \times \dot\gamma > 0, \tag{64}$$

where the nonnegativity is by our chosen orientation: γ traverses $\partial\Omega$ in a clockwise direction and ξ points into the convex domain Ω. Inequality (64) is strict because $\widetilde{\gamma(t)}$ is nontrivial and has only one endpoint on $\partial\Omega$ for each $t \in (-\varepsilon, \varepsilon)$. Next, because ξ is a unit vector, whence $\dot\xi$ is orthogonal to ξ, we have

$$f_\xi(t) = \xi(t) \times \dot\xi(t) = \pm|\dot\xi|, \tag{65}$$

where the value of $\pm$ is determined by the sign of $f_\xi(t)$. From our biLipschitz hypothesis (or nonnegativity of the Laplacian (55)) the sign of $J(r,t)$ is independent of r. Combined with (64) we obtain $J(r,t) > 0$ for $t \in (-\varepsilon, \varepsilon)$ and $0 \le r \le R(t)$. ■

Now we combine our raywise coordinates (r,t) with Rochet–Choné's localization (Corollary A.9). We obtain the following Euler–Lagrange equations which are central to the remainder of our work.

Lemma 6.3 (Poisson data along tame rays)**.** *Let u solve* (1) *where $\Omega \subset \mathbf{R}^2$ is bounded, open and convex. Let the coordinates r, t and functions γ, ξ, R, m, b be as in Lemma 6.2. Then the minimality of u, more precisely Rochet–Chone's localization, implies the Euler–Lagrange equation relating the fixed and free boundaries*

$$R^2(t)|\dot\xi(t)| = 2|\dot\gamma(t)|(Du - x)\cdot \mathbf{n} > 0, \tag{66}$$

and the Euler–Lagrange equation for u

$$3 - \Delta u = \frac{3j(t) + 3r|\dot\xi| - \delta(t)}{J(r,t)} = \frac{3r - 2R(t)}{r + j(t)/|\dot\xi(t)|}. \tag{67}$$

Proof. We compute the disintegration (116) and obtain for $\mathcal{H}^1$-a.e. $t \in (-\varepsilon, \varepsilon)$ that

$$0 \le v(Du - x)\cdot \mathbf{n}|\dot\gamma(t)| + \int_0^{R(t)} \left(3 - \frac{\delta(t)}{J(r,t)}\right) J(r,t)\, v dr.$$

The terms outside the integral are evaluated at $x = x(0,t)$. We consider four choices of test functions, $v|_{\widetilde{\gamma(t)}} = \pm 1$ and $v|_{\widetilde{\gamma(t)}} = \pm r$. Using these in the localization formula we obtain the equalities

$$\left((3j(t) - \delta(t))R(t) + \frac{3}{2}R^2(t)f_\xi(t)\right) + |\dot{\gamma}(t)|(Du - x)\cdot \mathbf{n} = 0 \tag{68}$$

$$(3j(t) - \delta(t))\frac{R(t)^2}{2} + R^3(t)f_\xi(t) = 0. \tag{69}$$

These combine to imply

$$R^2(t)f_\xi(t) = 2|\dot{\gamma}(t)|(Du - x)\cdot \mathbf{n} > 0. \tag{70}$$

We recall (65) and note (70) determines the sign $f_\xi(t) = |\dot{\xi}|$. Thus the Euler–Lagrange equation (66) for the free boundary holds. Equation (66) implies $|\dot{\xi}|$ is bounded away from zero and from above depending only on estimates for the continuous function $(Du - x)\cdot \mathbf{n}$ and our previously given estimate $R(t) > r_0$ (Lemma 6.1).

Combining (55) with (69) we obtain (67). ∎

Remark 6.4 (Tame rays must spread as they leave the boundary)**.** *Comparing* (56) *to* (67) *we recover* $f_\xi(t) = \xi \times \dot{\xi} > 0$*, which asserts that the Jacobian* $J(r,t)$ *is an increasing function of* r*: that is, tame rays spread out as they move away from the boundary. This implies all the rays may be extended into* Ω_2 *without intersecting each other.*

Recalling that $\Delta u = 3$ in Ω_2, we see (67) quantifies how Poisson's equation fails to be satisfied along leaves (with the equation only satisfied at the point $r = 2R(t)/3$). It also implies that when we move from Ω_1 into Ω_2 the Laplacian jumps discontinuously across their common boundary.

In Section 7 we show this discontinuity yields quadratic separation of u from its contact sets and exploit this to obtain estimates on the Hausdorff dimension of $\Gamma \cap \mathcal{N}$.

Let us conclude by justifying the aforementioned Lipschitz continuity of ξ. Note both (67) and (66) yield Lipschitz estimates. However, since their derivation assumed ξ was Lipschitz we need to redo these calculations with a perturbed, Lipschitz, ξ_δ and obtain uniform estimates as the perturbation parameter δ approaches 0. Notice the Lipschitz constant from the following lemma does not depend on the Neumann values $\|\log((Du-x)\cdot\mathbf{n})\|_{L^\infty(\mathcal{N}\cap\partial\Omega)}$.

Lemma 6.5 (Tame ray directions are Lipschitz on the fixed boundary)**.** *With* ξ *defined as above, the function* $t \mapsto \xi(t)$ *is Lipschitz on the interval* $(-\varepsilon,\varepsilon)$ *provided by Lemma 6.1, with Lipschitz constant depending only on an upper bound for* $\sup_{\mathcal{N}} \Delta u$ *and a lower bound on* $R(t)$*.*

Proof. Assume a collection of non-intersecting rays foliate an open set in $\mathbf{R}^2$ and pierce a smooth curve $\gamma(t)$. Provided the intersection of each ray with the curve occurs some fixed distance d from either endpoint of the ray, the assertion of Caffarelli, Feldman, and McCann [12, Lemma 16] says the directions $\xi(t)$ (of the ray passing through $\gamma(t)$) is a locally Lipschitz function with Lipschitz constant depending on γ and d.

Thus, in our setting, if we could extend each ray by length δ outside the domain, $\xi(t)$ would be locally Lipschitz with a constant depending on δ. Then, once we've obtained (67) the Lipschitz constant of ξ is independent of δ. Apriori such an extension outside the domain may not be possible. Thus our strategy below will be to translate the boundary distance δ and use the translated boundary to redefine the $r = 0$ axis of the (r, t) coordinates. In this setting the corresponding direction vector ξ_δ is locally Lipschitz, the above calculations are justified, and sending $\delta \to 0$ gives the Lipschitz estimate on ξ.

Thus, with γ as above let $\mathbf{n}(0)$ be the outer unit normal at $\gamma(0)$ and set

$$\eta(t) = \gamma(t) - \delta\mathbf{n}(0).$$

Because the length of rays intersecting $\gamma(-\varepsilon, \varepsilon)$ is bounded below by r_0, up to a smaller choice of ε, δ we may assume for each $t \in (-\varepsilon, \varepsilon)$ that $\eta(t) \in \Omega_1$ and lies distance at least $\delta/2$ from each endpoint of $\widetilde{\eta(t)}$. Let $\xi_\delta(t)$ denote the unit vector parallel to $\widetilde{\eta(t)}$, where by [12, Lemma 16] ξ_δ is a locally Lipschitz function of t. We may redefine the (r, t) coordinates and write a connected component of Ω_1 containing $\eta(0)$ as

$$\mathcal{N} = \{x(r, t) = \eta(t) + r\xi_\delta(t) \ ; \ t \in (-\varepsilon, \varepsilon) \text{ and } -R_0(t) \le r \le R_1(t)\},$$

where $R_0(t)$ is defined so that $\eta(t) + R_0(t)\xi_\delta(t)$ is the point where the ray $\widetilde{\eta(t)}$ intersects $\partial\Omega$. The function $R_0(t)$ is locally Lipschitz because the curves γ, η are smooth and ξ_δ is Lipschitz (though we don't assert that the Lipschitz constant of R_0 is independent of δ). Thus all our earlier computations may be repeated in these new coordinates. The computations leading to (67) now yield the equation

$$3 - \Delta u = |\dot{\xi}_\delta(t)| \frac{3r - 2R_1 + R_0}{\xi_\delta \times \dot{\eta} + r|\dot{\xi}_\delta|}, \tag{71}$$

which we note satisfies $3 = \Delta u$ when $r = (2R_1 - R_0)/3$ and thus agrees with our earlier coordinate system (in our modified coordinates $r = (2R_1 - R_0)/3$ is the point of distance $2(R_1 + R_0)/3$ from the endpoint of the ray).

Evaluating (71) at $r = 0$ we obtain a Lipschitz estimate on ξ_δ which is independent of δ (using, crucially, the Laplacian estimates of Theorem 4.1). The pointwise convergence of ξ_δ to ξ ensures ξ is Lipschitz. ∎

Corollary 6.6 (Raywise coordinates (53) are biLipschitz)**.** *Let $\mathcal{N}$ denote the subset of Ω_1 defined in* (52)*, (x^1, x^2) Euclidean coordinates, and (r, t) the coordinates defined in* (53)*. Then the change of variables from (x^1, x^2) to (r, t) is biLipschitz with Lipschitz constant depending only on* $\sup |Du|$, $\sup_{t\in(-\varepsilon,\varepsilon)} |\dot{\gamma}|$, $\inf_{t\in(-\varepsilon,\varepsilon)} R(t)$*, and* $\inf_{t\in(-\varepsilon,\varepsilon)} \xi \times \dot{\gamma}$ *(i.e. the transversality of the intersections of rays with the fixed boundary $\{\gamma(t)\}_{t\in(-\varepsilon,\varepsilon)}$).*

Proof. It suffices to estimate each of the entries in the Jacobians $\frac{\partial(x^1,x^2)}{\partial(r,t)}$ and $\frac{\partial(r,t)}{\partial(x^1,x^2)}$ computed earlier in (59) and (60). From (59) it's clear that the entries of this Jacobian permit an estimate from above in terms of $\sup_{t\in(-\varepsilon,\varepsilon)}|\dot\gamma|$ and $\sup_{t\in(-\varepsilon,\varepsilon)}|\dot\xi|$, where the latter may, in turn, be estimated in terms of $\sup|Du|$ and $\inf_{t\in(-\varepsilon,\varepsilon)}R(t)$ thanks to (66). The only additional term we must estimate for the second Jacobian, that is (60), is $1/J(r,t)$. Since $1/J(r,t)$ is decreasing in r by Remark 6.4, it suffices to estimate $J(0,t)$ and this is an immediate consequence of (56) which we recall says $j(t)=\xi\times\dot\gamma>0$. ■

7. On the regularity of the free boundary

In this section we study local properties of the free boundary by transformation to an obstacle problem and prove Theorem 1.3. If u_1 denotes the minimal convex extension of $u|_{\mathcal{N}}$ we show that $v:=u-u_1$ solves an obstacle problem with the same free boundary as our original problem. Standard results for the obstacle problem then imply the free boundary $\Gamma\cap\mathcal{N}$ has Lebesgue measure 0 and, the stronger result, $\Gamma\cap\mathcal{N}$ has Hausdorff dimension strictly less than n.

We use these estimates to establish that the function $t\mapsto R(t)$ from Section 6 is continuous on $(-\varepsilon,\varepsilon)$, Theorem 7.3. In Proposition 7.6 we describe a bootstrapping procedure which shows that if the free boundary — or rather the function $t\mapsto R(t)$ — is Lipschitz, then it is C^∞. In Theorem 7.8 and its corollary, we show R has a Lipschitz graph away from its local maxima.

7.1. Transformation to the classical obstacle problem

Let $x_1\in\Omega$ denote the endpoint of a tame ray and $\tilde{x_1}\cap\partial\Omega=\{x_0\}$ as in Lemma 6.1. Using the coordinates from the previous section we consider a subset of Ω_1,

$$\begin{aligned}\mathcal{N}=\mathcal{N}\cap\Omega_1&=\{\widetilde{\gamma(t)}\ ;\ t\in(-\varepsilon,\varepsilon)\},\\&=\{x(r,t)=\gamma(t)+r\xi(t)\ ;\ t\in(-\varepsilon,\varepsilon)\text{ and }0\le r\le R(t)\}.\end{aligned}$$

In Remark 6.4 we observed that rays spread out as they leave the boundary. Thus the coordinates (r,t) remain well-defined on an extension of $\mathcal{N}$. We denote this extension by $\mathcal{N}_{\text{ext}}$:

$$\mathcal{N}_{\text{ext}}=\{x(r,t)=\gamma(t)+r\xi(t)\ ;\ t\in(-\varepsilon,\varepsilon)\text{ and }0\le r<+\infty\}.$$

On $\mathcal{N}_{\text{ext}}$ we define the minimal convex extension of $u|_{\mathcal{N}}$ by the formula (54)

$$u_1(x)=b(t)+rm(t)\text{ for }x=x(r,t)\in\mathcal{N}_{\text{ext}}.$$

Note there is some $\alpha>0$ such that $B_\alpha(x_1)\subset\subset\mathcal{N}_{\text{ext}}\cap\Omega$. Moreover on $\mathcal{N}_{\text{ext}}\setminus\Omega_1$ we have, by (67), $\Delta u_1\le 3-c_0$ for $c_0>0$ depending only on a lower bound for $R(t)$ and $(Du-x)\cdot\mathbf{n}$. Let $v=u-u_1$ and let $1_{\{v>0\}}$ denote the characteristic function of $\{v > 0\}$. Then $\Delta u = 3$ in Ω_2 implies that

$$\Delta v = f(x)1_{\{v>0\}} \geq c_0 1_{\{v>0\}} \text{ in } B_\alpha(x_1), \tag{72}$$

$$v \geq 0 \text{ in } B_\alpha(x_1), \tag{73}$$

$$v = 0 \text{ in } B_\alpha(x_1) \cap \Omega_1 \text{ and } v > 0 \text{ in } B_\alpha(x_1) \cap \Omega_2 \tag{74}$$

where $f(x) = 3 - \Delta u_1$. Thus v solves the classical obstacle problem in $B_\alpha(x_1)$ with the same free boundary as our original problem. The regularity theory for the obstacle problem yields estimates for the measure of the free boundary. What prevents us from using higher regularity theory for the obstacle problem is that on $B_\alpha(x_1) \cap \{v > 0\}$,

$$\Delta v = f(x) = \frac{3r - 2R(t)}{r + \xi \times \dot{\gamma}/|\dot{\xi}|},$$

may not be Hölder continuous — which is the minimum regularity required to apply regularity results for the obstacle problem. If, R — and hence the free boundary — were Lipschitz, Δv would also be Lipschitz in which case one can bootstrap to C^∞ regularity of R; see Proposition 7.6. As a partial result in this direction we prove that $t \mapsto R(t)$ is continuous. We begin with Hausdorff dimension estimates for the free boundary.

Lemma 7.1 (Hausdorff dimension estimate for the free boundary)**.** *Let x_0, x_1, α be as given above, so that $x_1 \in \Omega$ is the endpoint of a tame ray; c.f. Lemma 6.1. Then the Hausdorff dimension of $B_{\alpha/2}(x_1) \cap \Gamma$ equals $2 - \delta_1$ for some δ_1 depending only on α,* $\inf_{B_\alpha(x_0)\cap\partial\Omega}(Du - x) \cdot \mathbf{n}$ *and* $\inf_{t\in(-\varepsilon,\varepsilon)} R(t)$.

Proof. This is a standard result for the obstacle problem once one notes that f in (72) satisfies $0 < c_0 \leq f = \Delta v \leq 3$ on $\{v > 0\}$ for c_0 depending only on $\inf_{B_\alpha(x_0)}(Du - x) \cdot \mathbf{n}$ and $\inf_{t\in(-\varepsilon,\varepsilon)} R(t)$. We follow the clear exposition of Petrosyan, Shahgholian, and Uraltseva [50, §3.1, 3.2] to establish first quadratic detachment, then porosity.

Step 1. (Quadratic detachment at free boundary points) We claim if $x_2 \in B_{\alpha/2}(x_1) \cap \Gamma$ then

$$\sup_{B_\rho(x_2)} v \geq \frac{c_0}{2}\rho^2. \tag{75}$$

Fix such an x_2 and $\overline{x} \in \Omega_2 \cap B_{\alpha/2}(x_1)$; we will eventually take $\overline{x} \to x_2$. Set $w(x) = v(x) - c_0|x - \overline{x}|^2/2$. On the set $\{v > 0\}$ we have $\Delta w > 0$. Thus, the maximum principle implies

$$0 \leq v(\overline{x}) = w(\overline{x}) \leq \sup_{B_\rho(\overline{x})\cap\{v>0\}} w = \sup_{\partial[B_\rho(\overline{x})\cap\{v>0\}]} w.$$

Because $w < 0$ on $\partial\{v > 0\}$ the supremum is attained at some x on $\partial(B_\rho(\overline{x})) \cap \{v > 0\}$. Because $|x - \overline{x}| = \rho$ we obtain

$$0 \leq v(x) - \frac{c_0}{2}\rho^2,$$

for some $x \in \partial B_\rho(\overline{x})$. We send $\overline{x} \to x_2$ to establish (75).

Step 2. (Nondegeneracy implies porosity) We recall a measurable set $E \subset \mathbf{R}^n$ is called porous with porosity constant δ if for all $x \in \mathbf{R}^n$ and $\rho > 0$ there is $y \in B_\rho(x)$ with

$$B_{\delta\rho}(y) \subset B_\rho(x) \cap E^c.$$

We prove that nondegeneracy, i.e. (75), and Caffarelli–Lions's $C^{1,1}_{\text{loc}}$ implies $\Gamma \cap B_{\alpha/2}(x_1)$ is porous. Take $x_2 \in \Gamma \cap B_{\alpha/2}(x_1)$. Note (75) implies $\sup_{B_\rho(x_2)} |Dv| \geq c_0\rho/2$. Indeed, with $\overline{x} \in B_\rho(x_2)$ such that $v(\overline{x}) \geq c_0\rho^2/2$ we have

$$c_0\rho^2/2 \leq v(\overline{x})-v(x_2) = \int_0^1 Dv(x_2+\tau(\overline{x}-x_2))\cdot(\overline{x}-x_2)\,d\tau \leq \rho \sup_{B_r(x_2)} |Dv|.$$

Now, redefine $\overline{x}$ as a point in $B_{\rho/2}(x_2)$ where $|Dv(\overline{x})| \geq c_0\rho/4$. Using that $\|v\|_{C^{1,1}_{\text{loc}}} \leq M$ (where $\Delta u = 3$, $\Delta u_1 \leq 3$ gives the obvious estimate $M = 6$), we have if $x \in B_{\delta\rho}(\overline{x})$ for $\delta = c_0/8M$ then

$$|Dv(x)| \geq |Dv(\overline{x})| - |Dv(x) - Dv(\overline{x})| \geq c_0\rho/4 - M\delta\rho = c_0\rho/8.$$

Since $Dv \equiv 0$ along Γ this proves $B_{\delta\rho}(\overline{x})$ lies in Γ^c. Thus we've established the porosity condition for balls centered on $\Gamma \cap B_{\alpha/2}(x_1)$. To establish the porosity condition for any ball in $\mathbf{R}^n$ we argue as follows. Let $B_\rho(x) \subset \mathbf{R}^n$. We take $\overline{x} \in B_{\rho/2}(x) \cap \Gamma \cap B_{\alpha/2}(x_1)$, noting if no such $\overline{x}$ exists we're done. Our porosity result applied on $B_{\rho/2}(\overline{x}) \subset B_\rho(x)$ gives porosity of $B_\rho(x)$.

Step 3. (Conclusion) As noted in [50, §3.2.2] by the work of [52] a porous set in $\mathbf{R}^n$ has Hausdorff dimension less than n. ■

For our next result we shall need the following maximum principle on an unbounded strip, which is directly implied by [8, Theorem 1.4, Remark 2.1].

Lemma 7.2 (Maximum principle on a strip [8])**.** *Let $U = \mathbf{R} \times [-1, 0] \subset \mathbf{R}^2$. Let $u \in W^{2,n}(U)$ be a bounded solution of*

$$\begin{cases} \Delta u & \geq 0 \quad \text{in } U, \\ u & = 0 \quad \text{on } \partial U. \end{cases} \tag{76}$$

Then $u \leq 0$ in U.

We now aim to establish continuity of R. Later, when we study the problem on $\Omega = [a, a+1]^2$ we use this result to rule out nontrivial rays entirely contained in $\partial\Omega$. For this reason we state our continuity result in the desired form, but prove a less natural looking lemma from which the continuity result follows immediately.

Theorem 7.3 (Continuity of the tame free boundary)**.** *Taking ε as in Lemma 6.1, the function $R : (-\varepsilon, \varepsilon) \to \mathbf{R}$ defined by $R(t) = diam(\widetilde{\gamma(t)})$ is continuous on $(-\varepsilon, \varepsilon)$.*

The assumptions in the following lemma express precisely the requirements that (1) rays —- apart from possibly the endpoint rays $\widetilde{\gamma(\pm\varepsilon)}$ — have a single endpoint on the boundary and, (2) there are no nearby rays with endpoint elsewhere on the boundary.

Lemma 7.4 (Tame ray lengths vary continuously)**.** *Assume a smooth portion of* $\partial\Omega_1 \cap \partial\Omega$ *is parametrized by* $\{\gamma(t)\}_{t\in(-\varepsilon,\varepsilon)}$ *as in Lemma 6.1. Assume* $\mathcal{N} := \bigcup_{t\in[-\varepsilon,\varepsilon]} \widetilde{\gamma(t)}$ *satisfies*

$$\textit{there is } \delta > 0 \textit{ such that} \quad \{x \in \Omega \,;\, dist(x, \mathcal{N}) < \delta\} \setminus \mathcal{N} \subset \Omega_2,$$

$$\textit{and, for all } t \in (-\varepsilon,\varepsilon) \quad \widetilde{\gamma(t)} \cap \partial\Omega = \gamma(t).$$

Then each $t_\infty \in [-\varepsilon,\varepsilon]$ *satisfies* $R(t_\infty) = \lim_{t\to t_\infty} R(t)$.

Proof. Up to redefining the interval or its orientation, it suffices to prove the result at $t_\infty = \varepsilon$. (This shows a one sided-limit exists. In the interior of $(-\varepsilon,\varepsilon)$ it can be upgraded to a two-sided limit by recalling the upper semicontinuity of R from Lemma 5.1. The possibility

$$2\Delta := R(t_\infty) - \liminf_{t\to t_\infty} R(t) > 0$$

can then be excluded by using the strong maximum principle Lemma 3.2 after setting the center $x_\infty = x(t_\infty, R(t_\infty) - \Delta)$ and reflecting the strictly convex function $u(x) - u(x_\infty) - Du(x_\infty)(x - x_\infty)$ on sufficiently small half-disc in Ω_2 across the ray $\widetilde{x_\infty}$ where it vanishes. At $t_\infty = \pm\varepsilon$ the same argument show the limit coincides with $R(t_\infty) = \operatorname{diam}(\tilde{x}_\infty)$.)

For a contradiction we assume otherwise. This implies there exists a sequence $t_k \to \varepsilon^-$ with

$$\underline{R} := \lim_{k\to\infty} R(t_k) < \limsup_{t\to t_\infty} R(t) =: \overline{R}. \tag{77}$$

The proof is three steps: First we show each $x = x(r, t_\infty)$ for $r \in [\underline{R}, \overline{R}]$ lies in the free boundary. Then we perform a blow-up. Finally, we conclude the blow-up violates the maximum principle.

Step 1. To show for each $r \in [\underline{R}, \overline{R}]$, $x(r, t_\infty) \in \partial\Omega_1$ we suppose otherwise. Then since no such point $x(r, t_\infty)$ can be interior to Ω_2 , the only remaining possibility is that there is $r \in (\underline{R}, \overline{R})$ such that $x(r, t_\infty)$ is an interior point of Ω_1.

Upper semicontinuity of R implies rays sufficiently close to $\widetilde{x(r, t_\infty)}$ have intersection with the boundary close to $x(0, t_\infty) = \partial\Omega \cap \widetilde{x(r, t_\infty)}$. More precisely for every $\varepsilon_0 > 0$ there is $\delta_0 > 0$ such that

$$\{\tilde{x} \,;\, x \in B_{\delta_0}(x(r, t_\infty))\} \cap \partial\Omega \setminus \widetilde{x(r, t_\infty)} \subset B_{\varepsilon_0}(x(0, t_\infty)).$$

Thus, our planar foliation implies that because $x(r, t_\infty)$ is an interior point of Ω_1 then $x(\rho, t_\infty)$ is also an interior point of Ω_1 for each $\rho \le r$ which contradicts that, by (77), $x(\underline{R}, t_\infty) \in \partial\Omega_1$ and completes Step (1).

Step 2. (Blow-up analysis) Now we choose $r_0 \in (\underline{R}, \overline{R})$ satisfying $r_0 \ge 4\overline{R}/5$ and set $x_0 = x(r_0, t_\infty)$. We perform a blow-up analysis and consider the behaviour of

$$u_r(x) := \frac{u(x_0 + rx) - u_1(x_0 + rx)}{r^2},$$

where we recall u_1 denotes the minimal convex extension of $u|_N$. It will be helpful to choose coordinates such that x_0 is the origin,

$$\tilde{x_0} = \{te_1 \ ; t \in [-\alpha, \beta] \text{ for some } \alpha, \beta > 0\},$$

and the positive e_2 direction is orthogonal to $\tilde{x_0}$ and satisfies $\gamma'(t_\infty) \cdot e_2 > 0$. At the outset we fix some half ball $B_\delta^-(x_0) = B_\delta(x_0) \cap \{x^2 \leq 0\}$ with

$$\delta \leq \operatorname{dist}(x_0, \partial\Omega \cap \{x^2 < 0\}).$$

Note that u_r is defined on, at least, the half ball $B_{\delta/r}^-(0)$ and equals 0 along $B_{\delta/r}^-(0) \cap \{x^2 = 0\}$ because $u = u_1$ along rays.

There exists a sequence of r_k with $x_k = -r_k e_2 \in \Omega_1$ and $\operatorname{diam}(\tilde{x_k}) \to \overline{R}$. Consider the sequence of functions $\{u_{r_k}\}$. We will establish that, up to a subsequence, these functions converge to a limit u_∞ defined on $\mathbf{R}^2_- := \{(x_1, x_2) \ ; x_2 \leq 0\}$ satisfying

(1) $|D^2 u_\infty| \in L^\infty(\mathbf{R}^2_-)$ with convergence $u_{r_k} \to u_\infty$ in C^1_{loc},
(2) $u_\infty \equiv 0$ along the lines $l_0 = \{x^2 = 0\}$ and $l_{-1} = \{x^2 = -1\}$,
(3) there is $x = (x^1, x^2)$ between l_0 and l_{-1}, i.e. satisfying $-1 < x^2 < 0$, with $u_\infty(x) > 0$,
(4) $\Delta u_\infty \geq 0$.

All of which combine to contradict the maximum principle (Step 3).

(1) Because $r_0 \geq 4\bar{R}/5$, u and u_1 satisfy a $C^{1,1}$ estimate in $B_\delta^-(x_0)$: In the nearby portion of Ω_1, we have $\Delta u, \Delta u_1 \leq 3$ by (67), and in Ω_2, $\Delta u = 3$, $\Delta u_1 \leq 3$. Thus, Arzela–Ascoli implies for any fixed $B_N^- \subset \mathbf{R}^2$ there is $M \in \mathbf{N}$ sufficiently large that the family $\{u_{r_k}\}_{k \geq M}$ is precompact in $C^1(\overline{B_N^-})$. Hence, up to a subsequence, we obtain $u_\infty : \mathbf{R}^2_- \to \mathbf{R}$ satisfying

$$u_\infty(x) := \lim_{k \to \infty} u_{r_k}(x).$$

Moreover $u_k \to u_\infty$ in $C^1(\Omega')$ for every compact $\Omega' \subset \mathbf{R}^2_-$ and $\|D^2 u_\infty\|_{L^\infty(\mathbf{R}^2_-)} \leq C$ as in [28],[30].

(2) Clearly, $u_\infty = 0$ along $\{x^2 = 0\}$, since $u - u_1$ equals 0 along rays. Moreover we've chosen r_k such that

$$\tilde{x_k} = \{x_k + t(\cos\theta_k, \sin\theta_k) \ ; t \in [-\alpha_k, \beta_k] \text{for some } \alpha_k, \beta_k > 0\},$$

where $\theta_k \to 0$ as $k \to \infty$ and $u - u_1 = 0$ on $\tilde{x_k}$. Note that because $u \equiv 0$ along $\tilde{x_k}$, u_{r_k} is equal to 0 along the line

$$\{-e_2 + t(\cos\theta_k, \sin\theta_k) \ ; t \in [-\alpha_k/r_k, \beta_k/r_k]\}.$$

Because $\theta_k \to 0$ and $u_{r_k} \to u_\infty$ locally uniformly we have $u_\infty = 0$ on $\{x^2 = -1\}$, thereby completing the proof of *(2)*.

(3) This follows as a consequence of the quadratic seperation argument in Lemma 7.1. Indeed, that argument gives is a sequence of z_k on $\partial B_{1/2}^-$ with $|z_k| = 1/2$ such that $u(x_0 + r_k z_k) - u_1(x_0 + r_k z_k) \geq c_0 r_k^2$ for a $c_0 > 0$ independent of k. Uniform convergence implies a limiting $x \in \partial B_{1/2}^-$ where $u_\infty(x) > c_0$

(4) At points of second differentiability for u and u_1 in Ω_1 we have $\Delta u = \Delta u_1$. On the other hand at points of second differentiability in Ω_2 we have $\Delta u_1 < 3$ by (67), whereas here $\Delta u = 3$, establishing (4).

Step 3. (u_∞ contradicts the maximum principle) Conclusions (1)–(4) show u_∞ violates Lemma 7.2, the desired contradiction. ■

Remark 7.5 (From Dini continuity to smoothness of the free boundary)**.** *We have only proved the continuity of R about tame rays. If R were Dini continuous the regularity theory of the obstacle problem, detailed below, should improve the Dini continuity of R to Lipschitz regularity of the free boundary in the neighbourhood of any regular point (defined after* (80)*). Next we show prove the fourth point of Theorem 1.3: that if the function R is Lipschitz then one can bootstrap to a smooth free boundary and minimizer on Ω_1.*

Proposition 7.6 (Tame part of free boundary is smooth where Lipschitz)**.** *Let u solve* (1)*. Let $x_1 \in \Gamma \subset \mathbf{R}^2$ be a tame point of the free boundary and $x_0 = \partial\Omega \cap \tilde{x_1}$. Assume $x \mapsto diam(\tilde{x})$ is Lipschitz on some $B_\varepsilon(x_0) \cap \partial\Omega$. Then there is $\delta > 0$ such that $B_\delta(x_1) \cap \Gamma$ is a smooth curve. On the portion $\mathcal{N}$ of Ω_1 consisting of rays which intersect $B_\delta(x_1)$, the transformation $(r, t) \to x(r, t)$ of* (53) *is a smooth diffeomorphism and $u \in C^\infty(\mathcal{N} \cap \operatorname{int}\Omega)$.*

Proof. We prove by induction on $k = 0, 1, 2, \ldots$ that there is some neighbourhood on which the curve $B_\delta(x_0) \cap \partial\Omega_1$ and the function R are both $C^{k,\alpha}$ for some $0 < \alpha < 1$, while the coordinate transformations and u are $C^{k+1,\alpha}$ in $B_\delta(x_1) \cap \Omega_1$. For $k = 0$ our assumption is that $t \mapsto R(t)$ is Lipschitz, and from Corollary 6.6 the coordinate transformations are biLipschitz. From the formula (66), reproduced here for the readers convenience

$$R^2(t)|\dot{\xi}(t)| = 2|\dot{\gamma}(t)|(Du - x) \cdot \mathbf{n}, \tag{78}$$

and Caffarelli and Lions $u \in C^{1,1}_{\text{loc}}$ (or Theorem 4.1) it follows that $t \mapsto \dot{\xi}(t)$ is also Lipschitz, hence the coordinate transformations improve to $C^{1,1}$ by the Jacobian expressions (59)–(60). From (67), i.e.

$$3 - \Delta u = \left|\frac{\dot{\xi}(t)}{J(r,t)}\right|(3r - 2R), \tag{79}$$

we see $\Delta u \in C^{0,\alpha}$, hence the regularity theory for Poisson's equation implies $u \in C^{k+2,\alpha}$ when $k = 0$ (one derivative more than needed).

Now assume the inductive hypothesis for some fixed k. From (78)–(79) we again deduce u has $C^{k,\alpha}$ Laplacian in Ω_1. Thus the regularity theory for Poisson's equation implies $u \in C^{k+2,\alpha}$. The regularity theory for the obstacle problem (where the obstacle has a $C^{k,\alpha}$ Laplacian; (due to Caffarelli [11, 9], and Kinderlehrer [34] with Nirenberg [33], though the clearest statement we've found is by Blank [6, 5]) implies the free boundary is $C^{k+1,\alpha}$. Note to apply the classical regularity theory for the obstacle problem we are using that the Lipschitz regularity of R implies the set $\overline{\Omega}_1 = \{v = 0\}$ has positive density at each boundary point; here $v = u - u_1$ as in (72) . Now that R is

$C^{k+1,\alpha}$ the same is again true for $\dot{\xi}$ by equation (78) since the smoothness $Du \in C^{k+1,\alpha}$ established in $\Omega_1 \cap B_\delta(x_1)$ propagates down the rays from the free to the fixed boundary using the $C^{k+1,\alpha}$ coordinate transformations; these transformations then improve to $C^{k+2,\alpha}$ by equations (59)–(60) so the induction is established and the proof is complete. ∎

Theorem 1.3(1) and (4) are obtained by combining Lemmas 7.1 and 7.6; parts (2) and (3) will be established in Theorem 7.8 of the next section.

7.2. Criteria for the tame part of the free boundary to be Lipschitz

Having deduced regularity of the free boundary when it is Lipschitz we now turn our attention to the question of characterising the set on which the free boundary is Lipschitz. We will rely on the well known Caffarelli dichotomy for the blow-up of solutions to the obstacle problem. We recall that blowing-up at the edge of the contact region in the classical obstacle problem (without convexity constraints) led Caffarelli to formulate his celebrated alternative [11, 9]: If $w \in C^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ satisfies

$$\Delta w(x) = 1_{\{w>0\}}(x) \quad a.e. \text{ on } \mathbf{R}^n \tag{80}$$

then w is convex and either a quadratic polynomial or a rotated translate of the *half-parabola solution*

$$w(x_1, \ldots, x_n) = \begin{cases} \frac{1}{2}x_1^2 & \text{if } x_1 > 0 \\ 0 & \text{else.} \end{cases}$$

At each point in the free boundary, the density of the contact region is therefore either 0 (called *singular*) or $\frac{1}{2}$ (called *regular*); it cannot equal 1 because of quadratic detachment (as in e.g. the proof of Lemma 7.1). Furthermore, the dichotomy holds for blowups of solution to equations of the form $\Delta u(x) = f(x)1_{\{u>0\}}(x)$ in a domain Ω where f is continuous in the following sense: Take $x_0 \in \Omega \cap \partial\{u = 0\}$ and a sequence $r_k \to 0$. Note that up to taking a sequence the limit

$$u_0(x) := \lim_{k\to\infty} \frac{u\big(r_k(x - x_0)\big)}{r_k^2},$$

exists and is a globally defined solution of $\Delta u(x) = f(x_0)1_{\{u>0\}}(x)$ so that Caffarelli's dichotomy applies to the function u_0. Unfortunately, in our setting we only know $f \in L^\infty_{\mathrm{loc}}$ and not the Hölder continuity required for higher regularity [34].

A real-valued function S on an interval $J \subset \mathbf{R}$ is called *unimodal* if it is monotone, or else if it attains its maximum on a (possibly degenerate interval) $I \subset J$, with S being non-decreasing throughout the connected component of $J \setminus I$ to the left of I, and non-increasing through the connected component to the right of I. The following lemma shows lower semicontinuous functions are unimodal away from their local minima.

Lemma 7.7 (Lower semicontinuous unimodality away from local minima)**.** *Let* $S : E \longrightarrow [-\infty, \infty)$ *be lower semicontinuous on an interval* $E \subset \mathbf{R}$*. Let* T *denote the subset of* E *consisting of local minima for* S*, and* $\overline{T}$ *its closure. Then* S *is unimodal on each connected component* J *of* $E \setminus \overline{T}$*.*

Proof. Fix any open interval $J \subset E \setminus \overline{T}$. We claim S is unimodal on J. Since S is lower semicontinuous but has no local minima on J, for each $c \in \mathbf{R}$ it follows that $J(c) := \{t \in J \mid S(t) > c\} = \cup_i (a_i, b_i)$ is a countable union of open intervals on which $S > c$ with $S \leq c$ on $J \setminus J(c)$. If there were more than one open interval in this union, say (a_1, b_1) and (a_2, b_2) with $b_1 \leq a_2$, then S would attain a local minimum on the compact set $[b_1, a_2] \subset J$, contradicting the fact $J \subset E \setminus \overline{T}$. Thus the set $J(c)$ consists of at most one open interval, which is monotone nonincreasing with $c \in \mathbf{R}$. Let c_0 denote the infimum of $c \in \mathbf{R}$ for which $J(c)$ is empty, and set $I = \cap_{c<c_0} J(c)$. Then S is non-decreasing to the left of I, non-increasing to the right of I, and — if I is nonempty — attains its maximum value on I. ■

We apply this lemma to the diameter $R = -S$ of the rays along the tame part of the free boundary to deduce the free boundary is Lipschitz away from its local maxima.

Theorem 7.8 (Tame free boundary is Lipschitz away from local maxima)**.** *Let* $\gamma : E \longrightarrow \partial\Omega$ *with* $\dot{\gamma}(t) \neq 0$ *for* $t \in E := (-\varepsilon, \varepsilon)$ *smoothly parameterize a fixed boundary interval throughout which the Neumann condition* (6) *is violated. Let* T *denote the subset of* E *consisting of local maxima for* $R(t) := \mathrm{diam}(\widetilde{\gamma(t)})$*, and* J *any connected component of* $E \setminus \overline{T}$*, where* $\overline{T}$ *is the closure of* T*. Then* R *extends continuously to* $\overline{J}$ *and its graph is a Lipschitz submanifold of* $\overline{J} \times \mathbf{R}$*. Similarly, the graph of* $F(t) := \gamma(t) + R(t)\xi(t)$ *over* $\overline{J}$ *is a Lipschitz submanifold of* $\overline{\Omega}$ *(except perhaps at* $t = \pm\varepsilon$*), and* F *is continuous on* $\overline{J}$*.*

Proof. Corollary 6.6 shows the coordinates $x(t, r) = \gamma(t) + r\xi(t)$ are locally biLipschitz on $E \times (0, \infty)$, so the final sentence follows from showing R has a continuous extension to $\overline{J}$ whose graph is a Lipschitz submanifold.

Proposition 5.4 asserts R is upper semicontinuous on $\overline{J}$. Unless R is monotone on J, Lemma 7.7 shows J decomposes into two subintervals on which $-R$ is monotone and they overlap at least at one point t'. Although a monotone function need not be Lipschitz — or even continuous — its graph has Lipschitz constant at most 1. A discontinuity in R on the closure of either of these subintervals can be ruled out as in the proof of Lemma 7.4 (or by Theorem 7.3 in the interior). Thus R is continuous on $\overline{J}$. The graph of R on $\bar{J}$ is obviously Lipschitz, except perhaps when the minimum value of R is uniquely attained at some $t' \in J$. Since t' is a local minimum, R is continuous at t' hence Caffarelli's alternative holds for the blow-up at $F(t')$: the Lebesgue density of Ω_1 at $F(t')$ cannot be zero since $R(t')$ is a local minimum, so it must be exactly $1/2$ [9, 11]. The blow-up limits of $u_2 - u_1$ at $F(t')$ all coincide with the same half-parabola, and R is differentiable at t'.

The Lipschitz graph of R to the left of t' shares the same tangent at t' as the Lipschitz graph of R to the right of t', which completes the proof. ∎

Our next corollary shows that the tame part of the free boundary can only fail to be locally Lipschitz when oscillations with unbounded frequency cause local maxima of R to accumulate, or when an isolated local maximum forms a cusp. In the latter case, the tame free boundary is locally piecewise Lipschitz and the perimeter of Ω_1 is locally finite in this region.

Corollary 7.9 (Is the tame free boundary piecewise Lipschitz?). *Assume T has only finitely many connected components in Theorem 7.8 and R is constant on each of them — as when R has only finitely many local maxima on E. Then the graph of $F(t) := \gamma(t) + R(t)\xi(t)$ is a piecewise Lipschitz submanifold of $(-\varepsilon + \delta, \varepsilon - \delta) \times (0, \infty)$ for each $\delta > 0$. Moreover, if the graph of F fails to be Lipschitz at $F(t')$ for some $t' \in E$, then R has an isolated local maximum at t' and Ω_1 has Lebesgue density zero at $F(t')$.*

Proof. Under the hypotheses of Theorem 7.8, assume T has only finitely many connected components and R is constant on each of them. Then these components must be intervals which are relatively closed in E: otherwise the upper semicontinuous function R has a jump increase at the end of one of them, which leads to a segment in the graph of u_2 — producing the same contradiction to Lemma 3.2 as in the proof of Lemma 7.4. Thus $\overline{T} = T$. For each of the open intervals J comprising $E \setminus \overline{T}$, Theorem 7.8 already asserts that R is continuous and has Lipschitz graph on $\overline{J}$; the only question is whether the graph extends past each endpoint of $\overline{J}$ in E in a Lipschitz fashion. If the endpoint of $\overline{J}$ belongs to a nondegenerate interval in T this is obvious. When the endpoint of $\overline{J}$ is an isolated point t' in T, then Lemma 7.7 shows R nearby is monotone on either side hence must be continuous at t' to avoid an affine segment in the graph of u_2 as before. Now Caffarelli's alternative applies, so the density of Ω_1 at $F(t')$ must be either 0 or $1/2$. In the latter case R has a Lipschitz graph in a neighbourhood of t', as in the proof of Theorem 7.8, hence the corollary is established. ∎

Remark 7.10 (A partial converse). *If Ω_1 fails to have Lebesgue density $1/2$ at some tame point $x' = F(t') \in \Omega \cap \partial\Omega_1$, then there is no neigbourhood of x' whose intersection with Ω_1 is a Lipschitz domain. This follows from the Caffarelli alternative, which requires Ω_1 to have Lebesgue density 0 at x'* [9, 11].

It remains to see whether accumulation points of local maxima of $R(t)$ and/or cusps might be ruled out by combining quadratic detachment shown in Lemma 7.1 with estimates in the spirit of the following lemma.

Lemma 7.11 (A variant on Clarke's implicit function theorem). *Let E be a Lipschitz manifold whose topology is metrized by d_E. Fix a Lipschitz function $f : E \times [a, b] \longrightarrow \mathbf{R}$ such that for each $r \in [a, b]$, $t \mapsto f(t, r)$ has Lipschitz constant L on E. Assume there exists a set $T \subset E$, and nonempty interval*

$(\alpha,\beta) \subset \mathbf{R}$ *such that for each* $t \in T$ *there exists* $R(t) \in [a,b]$ *satisfying*

$$f(t,r) - f(t,R(t)) \geq (r - R(t))\beta \text{ for all } r \in (R(t),b) \tag{81}$$

$$\text{and } f(t,R(t)) - f(t,r) \leq (R(t) - r)\alpha \text{ for all } r \in (a,R(t)). \tag{82}$$

Then the restriction of R *to* T *has Lipschitz constant* $2L/(\beta - \alpha)$.

Proof. Let $t_0, t_1 \in T$ and set $r_i := R(t_i)$ for $i = 0, 1$. Relabel if necessary so that $\delta R := r_1 - r_0 \geq 0$. Conditions (81)–(82) and the Lipschitz continuity of f give

$$\begin{aligned} & f(t_0,r_1) - f(t_0,r_0) \geq \beta\delta R \\ \text{and} \quad & f(t_1,r_1) - f(t_1,r_0) \leq \alpha\delta R. \end{aligned}$$

Subtracting yields

$$2Ld_E(t_0,t_1) \geq (\beta - \alpha)\delta R.$$

This shows R has the asserted Lipschitz constant on T. ■

8. Bifurcations to bunching in the family of square examples

In this section we apply our results and techniques to a concrete example and completely describe the solution on the domain $\Omega = (a, a+1)^2$. We prove Theorem 1.5. For $a \geq \frac{7}{2} - \sqrt{2}$ the solution is as hypothesized in the earlier work of McCann and Zhang [42]. A key strategy involves showing first unimodality and then monotonicity of the normal distortion $(x - Du(x)) \cdot \mathbf{n}$ along each edge of the square. From this we deduce the leaves Ω_1^0 with more than one endpoint on the boundary can only have one endpoint on Ω_W and the other on Ω_S; on these leaves the solution is given explicitly in [51, 42]. We let Ω_1^- denote the set of leaves with one endpoint in Ω and the other on Ω_W, and, finally, let Ω_1^+ denote the set of leaves with one endpoint in Ω and the other on Ω_S. Because, in the course of our proof, we prove Ω_1 does not intersect Ω_N or Ω_E we have $\Omega_1 = \Omega_1^0 \cup \Omega_1^- \cup \Omega_1^+$.

Our main tool to study the minimizer is the coordinates introduced in Section 6. Let us consider a component of Ω_1^- consisting of leaves with one endpoint on the boundary $\Omega_W = \{a\} \times [a, a+1]$ and the other interior to Ω. The argument is similar on each side. We may take the angle θ made by leaves with the horizontal (that is, with the vector $(1,0)$), as the parametrization coordinate of our boundary (i.e. $t = \theta$ and $\xi(t) = (\cos\theta, \sin\theta)$). Then $\gamma(\theta) = (a, h(\theta))$ and (r,θ) satisfy

$$x(r,\theta) = (a + r\cos\theta, h(\theta) + r\sin\theta),$$

where $h(\theta)$ is the height at which the leaf that makes angle θ with the horizontal intersects Ω_W. We work in a connected subset of Ω_1

$$\mathcal{N} = \mathcal{N} \cap \Omega_1^- = \{(r,\theta) \; ; \underline{\theta} \leq \theta \leq \overline{\theta} \text{ and } 0 \leq r \leq R(\theta)\}.$$

In this setting (68) and (69) become

$$0 = (3h' \cos\theta - m'' - m)R(\theta) + \frac{3}{2}R^2(\theta) + h'(\theta)(Du(x) - x) \cdot \mathbf{n} \tag{83}$$

$$0 = (3h' \cos\theta - m'' - m)\frac{R^2(\theta)}{2} + R^3(\theta) \tag{84}$$

where we use the prime notation for derivatives of roman characters as opposed to the dot notation for derivatives of greek characters, and equation (66) becomes

$$R^2(\theta) = 2h'(\theta)(Du - x) \cdot \mathbf{n}. \tag{85}$$

Note when we parameterize with respect to θ, $|\dot{\xi}| = 1$ so m' is Lipschitz by (62). We've used that $h'(\theta) > 0$ as is easily seen by first working with the parametrization $\gamma(t) = (a, t)$ and the angles $\xi(t) = (\cos\theta(t), \sin\theta(t))$, for which the identity $\xi \times \dot{\xi} > 0$ derived in Section 6 implies $\theta'(t) > 0$. Equations (83) – (85) yield a new, expedited, proof of the Euler–Lagrange equations in $\Omega_1^{\pm}$ originally derived by the first and third author [42] via a complicated perturbation argument. Solving (61) and (62) gives

$$u_1(x) = m(\theta)\cos\theta - m'(\theta)\sin\theta \tag{86}$$

$$\text{and} \quad u_2(x) = m(\theta)\sin\theta + m'(\theta)\cos\theta. \tag{87}$$

Thus, along Ω_W

$$(Du(x_0) - x_0) \cdot \mathbf{n} = a - u_1(x_0) = a + m'(\theta)\sin(\theta) - m(\theta)\cos\theta$$

Substituting into (85) we obtain

$$R^2(\theta) = 2h'(\theta)(a + m'\sin(\theta) - m(\theta)\cos\theta).$$

After multiplying by $\cos\theta$ and solving (84) for $h'\cos\theta$ we obtain (10), which coincides precisely with equation (4.22) of [42].

We obtain Theorem 1.5 as a combination of Lemmas. As required by the theorem, we henceforth make the tacit assumption $a \geq 0$.

Lemma 8.1 (Exclusion includes right-angled triangle in lower left corner)**.** *Let u minimize* (1) *with* $\Omega = (a, a+1)^2$. *Then* $\mathcal{H}^2(\Omega_0) > 0$ *and* $\Omega_0 \subset [a, c]^2$ *for some* $c > a$ *satisfying* $\Omega_0 \cap \partial\Omega = [a, c]^2 \cap \partial\Omega$.

Proof. Whenever Ω is a subset of the first quadrant, symmetry shows the minimizer satisfies $D_i u \geq 0$. Since the inclusion $\Omega_0 \subset \{u = 0\}$ of Theorem 1.1 becomes an equality if Ω_0 is nonempty, monotonicity of convex gradients implies $\Omega_0 = \{u = 0\} \subset [a, c]^2$ for some $c > a$ such that $\Omega_0 \cap \partial\Omega = [a, c]^2 \cap \partial\Omega$ in this case; here symmetry across the diagonal and the fact that $\{u = 0\}$ is closed have been used. Armstrong has proved that Ω_0 has positive measure whenever Ω is strictly convex [2] and this result has been extended to general benefit functions by Figalli, Kim, and McCann [31]. It is straightforward to adapt their proof to our setting. Indeed, convexity of Ω_0 and symmetry across the diagonal means $\mathcal{H}^1(\partial\Omega_0 \cap \partial\Omega) > 0$ implies $\mathcal{H}^2(\Omega_0) > 0$ and this implication is the only place strict convexity is used in [31, Theorem 4.8]. ■

Now we establish that $\Omega_N \cup \Omega_E \subset \Omega_2$.

Proposition 8.2 (No normal distortion along top right boundaries)**.** *Let u solve* (1) *with $\Omega = (a, a+1)^2$. Then $(Du - x) \cdot \mathbf{n} = 0$ throughout Ω_N and Ω_E. Consequently, $\Omega_E \cup \Omega_N \subset \Omega_2$.*

Proof. For a contradiction we assume (without loss of generality, by Proposition 2.3) there is $x_0 = (a+1, t_0) \in \Omega_E$, at which $(Du(x_0) - x_0) \cdot \mathbf{n} > 0$; when $x_0 \in \Omega_E$ is a vertex of Ω we interpret $\mathbf{n} = (1, 0)$. With this interpretation the continuity of Du implies we may in fact assume, without loss of generality, that x_0 lies in the relative interior of Ω_E. Thus $\tilde{x_0}$ has positive diameter and the same is true in a relatively open portion of the boundary, by Proposition 5.4. Working on Ω_E, it is convenient to let θ denote the clockwise angle a ray makes with the inward normal $(-1, 0)$. Thus $\theta > 0$ corresponds to a ray with nonpositive slope. Parametrizing the boundary as $\gamma(\theta) = (a+1, h(\theta))$ using this θ, the derivation of the equation (85) is unchanged along Ω_E. In particular $h'(\theta) > 0$, which is most easily seen by beginning with the clockwise oriented parametrization $\gamma(t) = (a+1, a+1-t)$ and $\xi(t) = (-\cos\theta(t), \sin\theta(t))$ in the coordinate arguments of Section 6 and recalling $\xi \times \dot{\xi} > 0$.

Clearly $\theta|_{\tilde{x_0}} \geq 0$ or $\theta|_{\tilde{x_0}} < 0$; we will derive a contradiction in either case.

Case 1. (Nonpositively sloped leaf). First let's assume the leaf has nonpositive slope (i.e. $\theta \geq 0$). The inequality $h'(\theta) > 0$ implies leaves above, but in the same connected component of Ω_1, as x_0 with one endpoint on $\{a+1\} \times [t_0, a+1]$ are also nonpositively sloped.

At the endpoint of each leaf the Neumann condition is not satisfied, that is $(Du(x) - x) \cdot \mathbf{n} > 0$, equivalently $D_1 u(x) > a+1$ (by the sign condition on the Neumann value). On the boundary portion where leaves have nonnegative slope, $t \mapsto u_1(a+1, t)$ is a nondecreasing function (see Figure 5(A)). Thus $D_1 u(a+1, t) > a+1$ for all $t \in [t_0, a+1]$ and, by Theorem 1.1 and Proposition 5.4, each $x \in \{a+1\} \times [t_0, a+1]$ is the endpoint of a nontrivial leaf of nonpositive slope; (such leaves cannot accumulate onto the convex set $\Omega_0 = \{u = 0\}$ in view of Lemma 8.1). We consider the following dichotomy and derive a contradiction in either case: there is a sequence of leaves approaching $(a+1, a+1)$ with one endpoint on Ω_E and the other in Ω or else there is not, in which case all sequences of leaves approaching $(a+1, a+1)$ have one end on Ω_E and the other on Ω_N.

Case 1a. (All leaves approaching the vertex have one end in the interior). In the first case take a sequence $(x_k)_{k \geq 1} \subset \Omega_E$ with $x_k = (x_k^1, x_k^2) = (a+1, x_k^2)$ increasing along Ω_E to a limit x_∞ with $(a+1, a+1) \in \tilde{x}_\infty$. Provided no ray is contained entirely in the boundary, which we prove subsequently in Lemma 8.7, we obtain $\tilde{x}_\infty = \{(a+1, a+1)\}$ in view of Lemma 8.1. Moreover, we can take $\tilde{x}_k$ to contain points of Alexandrov second differentiability of u since the leaves occupy positive area by Corollary 6.6 and Fubini's theorem. Let the corresponding angles be θ_k. Because the leaves don't intersect other sides of the square, symmetry and the sign of the angle yield $R(\theta_k) \to 0$.

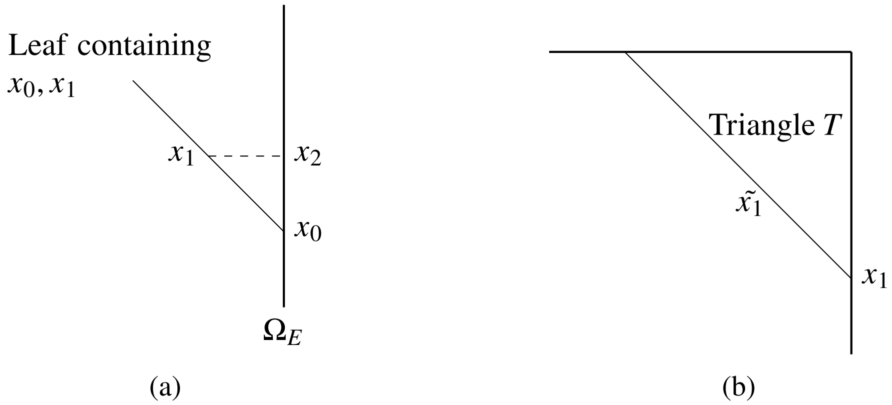


FIGURE 5. (A) Explanation of why $t \mapsto u_1(a+1, t)$ is monotone nondecreasing when leaves make positive angle with the horizontal (i.e. have nonpositive slope). Because $x_1 \in \tilde{x_0}$, $Du(x_0) = Du(x_1)$. Then monotonicity of the gradient of a convex function implies $D_1u(x_2) \geq D_1u(x_1) = D_1u(x_0)$. Thus $t \mapsto D_1u(a+1, t)$ is nondecreasing.
(B) Since Du is constant along $\tilde{x_1}$, monotonicity of the gradient implies $D_1u(x) \geq D_1u(x_1)$ and $D_2u(x) \geq D_2u(x_1)$ for all $x \in T$.

Theorem 4.1(1) provides a $C^{1,1}$ estimate along leaves with one endpoint on the boundary. Thus from (67),

$$\Delta u - 3 = \frac{2R - 3r}{h' \cos\theta + r},$$

evaluated at $r = 0$ we obtain an estimate

$$\frac{R(\theta_k)}{h'(\theta_k)} \leq C.$$

Combined with (85), i.e. $R^2(\theta) = 2h'(\theta)(Du - x) \cdot \mathbf{n}$, we contradict that $R(\theta) \to 0$ but $(Du(x) - x) \cdot \mathbf{n}$ is positive and increasing.

Case 1b. (There exists a leaf crossing the domain). In the second case we pick any leaf with one endpoint (call it x_1) on Ω_E and the other on Ω_N. Then $|\theta| = \pi/4$ by symmetry. Note $D_1u(x_1), D_2u(x_1) > a+1$ (by the Neumann inequality on Ω_E and Ω_N). Also this leaf bounds a right triangle T with sides $\tilde{x_1}$, and segments of Ω_N, Ω_E (Figure 5(B)). Define

$$\bar{u}(x) := \begin{cases} u(x_1) + Du(x_1) \cdot (x - x_1) & \text{for } x \in T \\ u(x) & x \in \Omega \setminus T \end{cases}. \tag{88}$$

Because $\bar{u}$ is defined by extension of an affine support for u, for all $x \in \operatorname{int} T$, $\bar{u}(x) < u(x)$. Moreover for $x \in T$ we have

$$|D\bar{u}(x) - x| \leq |Du(x) - x|,$$

this is because monotonicity of the gradient and the Neumann condition implies for $x \in T$, $D_i u(x) \geq D_i u(x_1) > a + 1$, whereas $x^i \leq a + 1$. Thus $\bar{u}$ is admissible for (1) and strictly decreases $L[u] = \int_\Omega |Du - x|^2/2 + u\, dx$, a contradiction, given that u minimizes L.

Case 2. (Positively sloped leaf). If our originally chosen leaf has positive slope (i.e. $\theta < 0$) the proof is similar, with slight modifications in the lower right corner. Indeed, $h'(\theta) > 0$ implies all leaves below our chosen leaf also have positive slope and on such leaves $t \mapsto D_1 u(a + 1, t)$ is a decreasing function (by monotonicity of Du, as in Case 1). Thus the Neumann value $(Du - x) \cdot \mathbf{n} = D_1 u(a + 1, t) - a - 1$ increases as we move towards the lower right corner. Proposition 5.4 then implies each $x \in \{a + 1\} \times [a, t_0]$ is the endpoint of a nontrivial leaf with positive slope (since Lemma 8.1 again prevents such rays from accumulating onto the convex set Ω_0, and rays in the boundary are ruled out by Lemma 8.7 below). Consider the same two alternatives as above: there is a sequence of leaves whose endpoints on Ω_E converge to $(a + 1, a)$ and whose other endpoint is interior to Ω, or there is not.

In the first case the contradiction is the same as in Case 1a above. In the second case choose a leaf with endpoint x_1 on Ω_E and other endpoint on Ω_S. By the Neumann inequality, Proposition 2.3, $D_1 u(x_1) > a + 1$ while $D_2 u(x_1) \leq a$. For x in the interior of the right triangle T formed by $\tilde{x_1}$ and segments of Ω_E, Ω_S monotonicity of the gradient implies

$$D_1 u(x) > D_1 u(x_1) > a + 1, \tag{89}$$

$$D_2 u(x) \leq D_2 u(x_1) \leq a. \tag{90}$$

Thus the affine extension as in (88) once again satisfies $|D\bar{u} - x| < |Du - x|$ (because $x \in [a, a + 1]^2$) and $\bar{u} < u$ in T. Thus $L[\bar{u}] < L[u]$ — the same contradiction as in Case 1b above.

Conclusion: $\Omega_E \cup \Omega_N \subset \Omega_2$. It remains to be shown that the corners $(a, a + 1)$, $(a + 1, a)$, and $(a + 1, a + 1)$ are not the endpoints of rays. This follows by the maximum principle, Lemma 3.2, combined with a reflection argument. For example, suppose a ray $x_0 = (a, a + 1) \in \Omega_N$ is the endpoint of a nontrivial ray $\tilde{x_0}$ which has negative slope and thus enters Ω; That this negative slope cannot be infinite holds because no ray can be entirely contained in the boundary, see Lemma 8.7 below. Fix any point x_1 in the relative interior of $\tilde{x_0}$ and let ν be the normal to $\tilde{x_0}$ that has positive components. Then for ε sufficiently small $B_\varepsilon(x) \subset \Omega$ and the half ball

$$B^+_\varepsilon(x) = B_\varepsilon(x) \cap \{x\ ;\ (x - x_1) \cdot \nu > 0\},$$

is contained in Ω_2 (because no rays intersect the relative interior of Ω_N). After subtracting from u its support at x and extending the resulting function to

$$B^-_\varepsilon(x) = B_\varepsilon(x) \cap \{x\ ;\ (x - x_1) \cdot \nu < 0\},$$

via reflection from $B^+_\varepsilon(x)$ we obtain a function which violates Lemma 3.2. We conclude no rays intersect $(a, a+1)$ and, via an identical argument, no rays intersect $(a+1, a)$ and $(a+1, a+1)$. ∎

Remark 8.3 (No ray has positive slope)**.** *A similar argument to the above implies no leaf intersecting Ω_W or Ω_S has positive slope. Indeed, if a leaf on Ω_W has positive slope its other endpoint is interior to Ω (the leaf cannot intersect Ω_N or Ω_E). The same argument as Case 1 above implies as one moves vertically up Ω_W each boundary point remains the endpoint of a nontrivial leaf of nonpositive slope. Lemma 8.7 ensures leaves must have length shrinking to* 0 *as they approach* $(a, a+1)$ *and thus we obtain the same contradiction as in Case 1a above. As a result along $\Omega_W \cap \Omega_1$ the function $\theta \mapsto u_1(x(0,\theta))$ is nondecreasing (equivalently $\theta \mapsto (Du - x) \cdot \mathbf{n}\big|_{x=x(0,\theta)}$ is nonincreasing; see again Figure 5(A)).*

Remark 8.4 (Neumann data: counting stray rays on convex polygons)**.** *The monotonicity argument of the preceding proposition combines with Propositions 2.3, 5.3–5.4 and Remark 6.4 to yield the following more general result. Let u minimize* (1) *for a convex polygon $\Omega \subset [0,\infty)^2$ with vertex set V. On each subinterval $I \subset \partial\Omega$ that is disjoint from $V \cup \Omega_0 \cup \Omega_1^0 \cup \Omega_2$, the function $(x - Du(x)) \cdot \mathbf{n}(x)$ is unimodal (hence non-vanishing on I except perhaps at a single point denoted x_I): its value at x dominates its values throughout the orthogonal projection of $\tilde{x}$ onto the boundary segment of Ω containing x. Here $\Omega_1^0 = \{x \in \Omega_1 : \#(\tilde{x} \cap \partial\Omega) \geq 2\}$. The arguments and conclusions of Lemmas 8.5 and 8.7 below also adapt equally well to this more general geometry. At least one endpoint of the interval I therefore lies in $\Omega_0 \cup \Omega_1^0$. This shows stray rays cannot occur where such intervals I accumulate, hence can only occur on (an at most countable subset of) $\{x_I\}_I$.*

To deal with the remaining case deferred from the proof of Proposition 8.2 — that no ray can be entirely contained in the boundary — we first introduce a suitable perturbation.

Lemma 8.5 (Perturbation of a boundary ray)**.** *Let u minimize the Monopolist's problem* (1) *on $\Omega = (a, a+1)^2$. Assume there is $x_0 \in \partial\Omega$ with nontrivial $\tilde{x_0} \subset \partial\Omega$ and that M satisfies $\Delta u(x_0) < M$. Then there exists a family of perturbations u_h such that:*

1. $-h \leq u_h - u \leq 0$ *and* $\mathcal{N}_h := \{u_h < u\} \to \tilde{x}$ *in the Hausdorff distance as* $h \to 0$,
2. $u_h - u \leq -h/2$ *on a segment of* $\tilde{x_0}$ *with* $\mathcal{H}^1$ *measure greater than or equal to* $\mathcal{H}^1(\tilde{x_0})/4$.
3. $\Delta u_h < M$
4. $(Du_h - x) \cdot \mathbf{n} \geq (Du - x) \cdot \mathbf{n}$ *on* $\mathcal{N}_h \cap \partial\Omega$.

Remark 8.6. *The lemma requires the inequality $\Delta u(x_0) < M$ only in the viscosity sense, namely that a paraboloid satisfying* (91) *exists.*

Proof. We fix an x_0 as in the statement of the lemma. For concreteness we assume $x_0 \in \text{r.i.}(\Omega_N)$ and $\tilde{x_0} \subset \Omega_N$; the reader will see the proof is unchanged regardless of which of the four edges of the square we take. Let $x_0 = (x_0^1, x_0^2) = (x_0^1, a+1)$ and $y_0 = Du(x_0) = (y_0^1, y_0^2)$. By assumption $\Delta u(x_0) < M$ implies there is $c, b > 0$ with $c + b < M$ and

$$u(x) < u(x_0) + y_0 \cdot (x - x_0) + \frac{c}{2}(x^1 - x_0^1)^2 + \frac{b}{2}(x^2 - x_0^2)^2, \tag{91}$$

satisfied in some punctured neighbourhood $B_{\delta_0}(x_0) \cap \bar{\Omega} \setminus \{x_0\}$.

Now, we consider the Legendre transform $v : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$v(y) = \sup_{x \in \bar{\Omega}} x \cdot y - u(x).$$

Note, by (91) and the proof of Lemma 2.4

$$v(y) > q_0(y) := v_0(y_0) + x_0 \cdot (y - y_0) + \frac{1}{2c}(y^1 - y_0^1)^2 + \frac{1}{2b}(y^2 - y_0^2)^2,$$

on a punctured neighbourhood $B_{\delta_1}(y_0) \cap Du(\bar{\Omega}) \setminus \{y_0\}$. Thus the connected component $\mathcal{N}_h^*$ of

$$\{v(y) < q_0(y) + h\} \cap Du(\bar{\Omega}),$$

containing y_0 has diameter going to zero as $h \to 0$.

For h chosen sufficiently small define convex $v_h : Du(\bar{\Omega}) \to \mathbf{R}$ by

$$v_h(y) = \begin{cases} q_0(y) + h & y \in \mathcal{N}_h^*, \\ v(y) & \text{otherwise.} \end{cases}$$

Our desired perturbations are

$$u_h(x) := \sup_{y \in Du(\bar{\Omega})} x \cdot y - v_h(y).$$

We will establish (1)–(4). First, $v_h \geq v$ implies $u_h \leq u$ and $v_h \leq v + h$ implies $u_h \geq u - h$. Moreover, since $\mathcal{N}_h^* \to \{y_0\}$ in Hausdorff distance, $\mathcal{N}_h \to \tilde{x_0}$ in Hausdorff distance, thereby establishing (1).

To establish (2) we note the restriction of $u_h - u$ to $\tilde{x}$ is a convex function. Then (2) is a straightforward consequence of $u_h(x_0) - u(x_0) = -h$, along with $u_h - u \leq 0$ at either endpoint of $\tilde{x_0}$, and the definition of convexity.

To establish (3) note at each point of $\bar{\mathcal{N}}_h^*$, v_h is supported by the paraboloid q_0. Thus, Lemma 2.4 implies $\Delta u_h < M$ at each point of Alexandrov second differentiability in $\mathcal{N}_h$.

Finally, we consider (4) — the Neumann inequality. We need to compute the value of $Du_h(x)$ for $x \in \partial\Omega$, and it is equivalent to ask where a given $x \in \partial\Omega$ supports v_h. Because we are giving the proof assuming $x \in \Omega_N$, we need only consider boundary terms on Ω_N and—if $(a, a+1) \in \tilde{x_0}$—Ω_W. First we consider $x \in \Omega_N$. Note that in $\mathcal{N}_h^*$ our explicit formula for v_h implies $D_2 v_h = a + 1$ precisely throughout the line $\{y \; ; \; y^2 = y_0^2\}$. This implies any $x \in \Omega_N \cap \mathcal{N}_h$ supports v_h at a point with $y^2 = y_0^2$ and subsequently $(Du_h - x) \cdot \mathbf{n} = (Du - x) \cdot \mathbf{n}$. Next, if $(a, a+1) \in \tilde{x_0}$ we consider $x \in \Omega_W$. Note that along the line $\{y \; ; \; y^1 = y_0^1\}$ our formula for v_h implies $D_1 v_h = x_0^1 > a$. Thus, by monotonicity of the gradient of a convex function if $x \in \Omega_W \cap \mathcal{N}_h$, i.e. x has $x^1 = a$, it supports v_h at a point y satisfying $y^1 \leq y_0^1$. Hence, because $\mathbf{n} = -e_1$, along $(D\bar{u} - x) \cdot \mathbf{n} \geq (Du - x) \cdot \mathbf{n}$ is satisfied also along $\Omega_W \cap \mathcal{N}_h$. ∎

Lemma 8.7 (No ray is a subset of the fixed boundary)**.** *Let u minimize the Monopolist's problem* (1) *on $\Omega = (a, a+1)^2$. There is no nontrivial ray contained in $\partial\Omega$*

Proof. For a contradiction assume such a ray exists. Namely that there is nontrivial $\tilde{x_0} \subset \partial\Omega$, where, without loss of generality, x_0 is not a corner (though $\tilde{x_0}$ may contain a corner). Set $L = \mathcal{H}^1(\tilde{x_0})$.

We consider two cases $(Du(x_0) - x_0) \cdot \mathbf{n} > 0$ and $(Du(x_0) - x_0) \cdot \mathbf{n} = 0$. In each case we obtain a contradiction by employing Lemma 8.5 to obtain a perturbation u_h which contradicts (19), which in our setting is

$$0 < \int_\Omega (n + 1 - \Delta u_h)\,(u_h - u)dx + \int_{\partial\Omega} (Du_h - x) \cdot \mathbf{n}\,(u_h - u)dS. \tag{92}$$

Case 1. ($(Du(x_0) - x_0) \cdot \mathbf{n} > 0$) This case is straightforward. One may take any fixed M in Lemma 8.5. Then (92) becomes

$$\int_\Omega (n + 1 - \Delta u_h)\,(u_h - u)dx + \int_{\partial\Omega} (Du_h - x) \cdot \mathbf{n}\,(u_h - u)dS \\ \leq C(M)|\mathcal{N}_h|h - L(Du(x_0) - x_0) \cdot \mathbf{n}h/8,$$

which, since $|\mathcal{N}_h| = o(1)$, contradicts (92) for h sufficiently small.

Case 2. ($(Du(x_0) - x_0) \cdot \mathbf{n} = 0$) In this case we do not have the negative term on the boundary but, we will see, are assured we can choose x_0 such that $\Delta u(x_0) < 3$. We claim, x_0 can be chosen so that there exists $\varepsilon > 0$ with

$$u(x) < u(x_0) + y_0 \cdot (x - x_0) + \frac{\varepsilon}{2}(x^1 - x_0^1)^2 + \frac{(3 - 2\varepsilon)}{2}(x^2 - x_0^2)^2 \tag{93}$$

in some punctured neighbourhood $B_{\delta_0}(x_0) \cap \bar{\Omega} \setminus \{x_0\}$. Indeed, note that because the Neumann condition is 0 along the ray $\tilde{x_0}$, $\tilde{x_0}$ must be a limit of tame rays with endpoint on a different edge to $\tilde{x_0}$. (Tame rays approaching $\tilde{x_0}$ whose endpoint lies in the same edge fall into Case 1 by an argument as in Proposition 8.2 based on the monotonicity exhibited in Figure 5; stray rays approaching $\tilde{x_0}$ can be ruled out by the same monotonocity argument, since Propositions 5.3–5.4 ensure they would be interspersed with tame rays to which it applies; boundary rays which fail to be approximated by other rays can only be adjacent to Ω_2 — a possibility excluded by the strong maximum principle of Lemma 3.2 combined with a reflection across the fixed boundary.)

Using the Laplacian formula (67) at points distance $4R/5$ along these rays and that the length of these rays varies continuously (Lemma 7.4), implies there is a point, which we take as our choice of x_0, such that in a sufficiently small neighbourhood of x_0, we have $\Delta u < 3$ while both $\partial^2_{11}u$ and hence

$\partial^2_{12}u$ are as small as desired. Thus Lemma 8.5 gives a perturbation $u_h \le u$ satisfying $\Delta u_h < 3 - \varepsilon$ and $(Du_h - x) \cdot \mathbf{n} \ge 0$, again contradicting (92). ■

Another key point that it will be helpful to have at our disposal is the following.

Lemma 8.8 (Concave nondecreasing profile of stingray's tail)**.** *Let ω be some connected subset of Ω_1^-. Then $Du(\omega) = \{Du(x) = (y^1, y^2) \,;\, x \in \omega\}$ is such that y^2 is a strictly convex increasing function of y^1 lying above the diagonal whose monotonicity* (94) *and convexity* (95) *are easily quantified below in terms of the parameters $\theta(t) = t$ and $m(t) = m(\theta)$ from* (54)*.*

Proof. Connectivity of $\omega \subset \Omega_1^-$ combines with Theorem 1.1(2) and the definition of Ω_1^- to imply the set $\hat{\omega} = \{\tilde{x} \cap \partial\Omega \,;\, x \in \omega\}$ of fixed boundary endpoints of rays intersecting ω is also connected — hence forms an interval on the left boundary Ω_W of the square; it cannot intersect the relative interior of Ω_N according to Proposition 8.2. On the set of rays whose endpoints lie in the relative interior of this interval $\hat{\omega}$, the Lipschitz regularity of m' from Theorem 4.1 and Corollary 6.6 justifies the following computations.

We wish to consider the convexity of the curve $y(\theta) = (y^1(\theta), y^2(\theta))$ where by (86) and (87)

$$y^1(\theta) = m(\theta)\cos\theta - m'(\theta)\sin\theta$$

$$y^2(\theta) = m(\theta)\sin\theta + m'(\theta)\cos\theta.$$

Using Lipschitz regularity of m we have for $\mathcal{H}^1$ almost every θ

$$\frac{dy_2}{dy_1} = \frac{\dot{y}^2(\theta)}{\dot{y}^1(\theta)} = -\frac{\cos\theta}{\sin\theta} > 0. \tag{94}$$

Here the sign condition comes from Remark 8.3. Note this implies the curve $y(\theta)$ is such that y^2 is an increasing function of y_1. We see $\frac{dy_2}{dy_1}$ is an increasing function of θ, namely

$$\frac{d}{d\theta}\frac{dy_2}{dy_1} = \frac{1}{\sin^2\theta} \tag{95}$$

and subsequently, by Remark 8.3 which implies θ is an increasing function of y^1, $\frac{dy_2}{dy_1}$ is an increasing function $y_1 = u_1$.Thus we have the required monotonicity of the derivative to conclude y_2 is a convex function of y_1. ■

Symmetry implies connected components of $Du(\Omega_1^+)$ are curves with y^2 a concave function of y^1.

Now we can combine all the Lemmas we've just proved and complete the proof of Theorem 1.5.

Proof. (Theorem 1.5). By symmetry about the diagonal and $\Omega_1 \cap \Omega_E = \emptyset$ we can prove each point of the theorem by an analysis of the function $t \mapsto (Du - x) \cdot \mathbf{n}\big|_{x=(a,t)}$. Lemma 8.1 asserts (a, a) is in Ω_0 as is $\{(a, a+t) \,;\, 0 \le t \le \alpha\}$ for some $\alpha = c - a \in (0, 1]$. On $\Omega_0 \cap \Omega_W$ we have $(Du - x) \cdot \mathbf{n} = a$.

whereas on $\Omega_2 \cap \Omega_W$ we have $(Du - x) \cdot \mathbf{n} = 0$. Thus, for $a > 0$, $u \in C^1(\overline{\Omega})$ [16, 51] implies some portion of Ω_1 must abut Ω_0 as one moves up Ω_W.

Now we consider the configuration of Ω_1. Since leaves in Ω_1^0 reach the diagonal, by symmetry they are orthogonal to it and $Du(x) = (b, b)$ on such leaves, i.e. the product $Du(x)$ selected lies on the diagonal.

Step 1. (Configuration of domains) We claim as one moves vertically up Ω_W there is, in order, (i) a closed interval of Ω_0 with positive length, (ii) a half-open interval of Ω_1^0 which is empty for a sufficiently small and nonempty for a sufficiently large, (iii) a nonempty open interval of Ω_1^-, and finally an interval of Ω_2. All we must show is there is at most a single component of Ω_1^0, and it is followed by Ω_1^-. This is because, if Ω_1^0 and Ω_1^- exist their ordering follows from Lemma 8.8. Indeed if a portion of Ω_1^- is preceded by Ω_0 or Ω_1^0 then followed by Ω_1^0 we have the contradiction of a strictly convex curve lying above the diagonal with a start and endpoint on the diagonal in the stingray's profile.

Step 2. (Blunt bunching (i.e. $\Omega_1^0 \neq \emptyset$) for $a \geq \frac{7}{2} - \sqrt{2}$). Recall u cannot be affine on any segment in the closure of Ω_2, by a reflection argument combined with the strong maximum principle of Lemma 3.2. It follows that Ω_1^- is nonempty whenever Ω_1^0 is nonempty; this was previously established by a different approach in [41]. Next we assume Ω_1^0 is empty and show $a < \frac{7}{2} - \sqrt{2}$. Let $(a, \underline{x}_2)$ be the upper endpoint of $\Omega_0 \cap \Omega_W$ and let $(a, \overline{x}_2)$ be the lower endpoint of $\Omega_2 \cap \Omega_W$. The segment $\{a\} \times (\underline{x}_2, \overline{x}_2)$ consists of endpoints of leaves in Ω_1^-. By the Neumann condition we have $D_1 u(a, \underline{x}_2) = 0$ and $D_1 u(a, \overline{x}_2) = a$. Thus,

$$\int_{\underline{x}_2}^{\overline{x}_2} \partial^2_{12} u(a, x_2)\, dx_2 = a.$$

As in (4.17) of [42], using the (r, θ) coordinates we compute

$$\partial^2_{12} u(a, x_2) = -\sin(\theta) \frac{m''(\theta) + m(\theta)}{h'(\theta)}.$$

From (84), which reads $m''(\theta) + m(\theta) - 3h'(\theta)\cos\theta = 2R(\theta)$, we have

$$\begin{aligned} \partial^2_{12} u(a, x_2) &= -\sin(\theta) \frac{2R(\theta)}{h'(\theta)} - 3\sin(\theta)\cos(\theta) \\ &= -2\sin(\theta) R(\theta) \theta'(x_2) - \frac{3}{2}\sin(2\theta). \end{aligned}$$

We've used the inverse function theorem to rewrite $\frac{1}{h'(\theta)} = \theta'(x_2)$. Using $-\sin(\theta) \geq 0$ from Remark 8.3 and $R \leq 1$ and $\theta \geq -\frac{\pi}{4}$ for convexity of Ω_0

we conclude

$$\begin{aligned} a &= \int_{\underline{x}_2}^{\bar{x}_2} \partial_{12}^2 u(a, x_2)\, dx_2 \\ &= -\int_{\underline{x}_2}^{\bar{x}_2} [2\sin(\theta)R(\theta)\theta'(x_2) + \frac{3}{2}\sin(2\theta)]\, dx_2 \\ &< 2\|R\|_\infty[\cos(0) - \cos(-\frac{\pi}{4})] + \frac{3}{2}[\bar{x}^2 - \underline{x}^2] \\ &\leq \frac{7}{2} - \sqrt{2}. \end{aligned}$$

Step 3. (No blunt bunching (i.e. $\Omega_1^0 = \emptyset$) for $a \ll 1$ sufficiently small) Suppose for a contradiction that there is a sequence $a_k \downarrow 0$ such that the minimizer on $\Omega^{(k)} = (a_k, a_k + 1)^2$ has Ω_1^0 nonempty. Let u_k be the minimal convex extension to $\mathbf{R}^n$ of the corresponding minimizer to (1). Let, for example $(\Omega_0^1)^{(k)}$ denote Ω_0^1 for the problem on $(a_k, a_k + 1)^2$, and domains with no superscript denote the corresponding domain for $a = 0$. The convergence result [31, Corollary 4.7] implies $u_k|_{B_\varepsilon} \to u|_{B_\varepsilon}$ locally uniformly for any $B_\varepsilon \subset (0, 1)^2$ where u is the minimizer for $a = 0$.

It is clear that Ω_1 is empty when $a = 0$. Indeed, for $a = 0$ the solution on $[0, 1]^2$ is the restriction of the solution on $[-1, 1]^2$. The solution on $[-1, 1]^2$ satisfies $\Omega_1 = \emptyset$: Theorem 1.1(2) asserts the rays all extend to the boundary but Proposition 8.2, which is valid also on $[-1, 1]^2$ asserts there can be no ray intersecting the boundaries $\{1\} \times [0, 1]$ and $[0, 1] \times \{1\}$. By symmetry there are no rays intersecting anywhere on $\partial[-1, 1]^2$ and hence no rays whatsoever.

Recall (4.18) of [42] asserts $\{u_k = 0\}$ is the triangle $(x^1, x^2) \in [a_k, a_k+1]^2$ defined by $x^1 + x^2 \leq 2a + \frac{2a}{3}(\sqrt{1 + \frac{3}{2a^2}} - 1)$. The limit $\Omega_0 := \{u = 0\}$ from Theorem 1.1 must therefore contain the triangle $x^1 + x^2 \leq \sqrt{2/3}$ of area $\frac{1}{3}$ in $[0, 1]^2$. Nor can Ω_0 be larger than this triangle, since Proposition 2.3 implies both integrands are non-negative in the identity

$$1 = \int_{\Omega_0} 3d\mathcal{H}^2 + \int_{\Omega_0 \cap \partial\Omega} (Du - x) \cdot \mathbf{n} d\mathcal{H}^1$$

asserted by Lemma A.6. But the previous paragraph implies $\Omega_1 = \emptyset$, so outside the triangle Ω_0 the minimizing $u \in C^{1,1}_{\text{loc}}((0, 1)^2)$ is a strictly convex solution to Poisson's equation $\Delta u = 3$. Reflecting this solution across the line $x^1 + x^2 = \sqrt{2/3}$ where it vanishes contradicts the strong maximum principle (Lemma 3.2). This is the desired contradiction which establishes Step 3.

Step 4. (No further Ω_1 components) From Remark 8.3 any leaves intersecting Ω_W have nonpositive slope. Thus, recalling Figure 5(A), the Neumann value $(Du - x) \cdot \mathbf{n}$ is a positive decreasing function of x_2 along $\Omega_1 \cap \Omega_W$ and 0 along $\Omega_2 \cap \Omega_W$. Thus there cannot exist an Ω_1 component above an Ω_2 component on Ω_W. ■

Remark 8.9 (Estimating the bifurcation point)**.** *It is clear that the criterion $a \geq \frac{7}{2} - \sqrt{2}$ for the existence of Ω_1^0 is not sharp. However the presence of a bifurcation reflects the radically different behavior we have shown the model to display for small and large a. We expect there is a single bifurcation value a_0 such that Ω_1^0 is nonempty for $a > a_0$ while Ω_1^0 is empty for $0 < a \leq a_0$. It would be interesting to confirm this expectation, and to find or estimate a_0 more precisely.*

Appendix A. Rochet and Choné's sweeping and localization

We recall as in (18) the minimizer u of (1) satisfies the variational inequality

$$0 \leq \int_\Omega (n+1-\Delta u)v(x)\,dx + \int_{\partial\Omega} v(x)(Du - x)\cdot \mathbf{n}\,d\mathcal{H}^{n-1}, \tag{96}$$

for all convex v with $\operatorname{spt} v_-$ disjoint from Ω_0. Since $u \in C^1(\overline{\Omega})$ [16], we know

$$d\sigma := (n+1-\Delta u)\,d\mathcal{H}^n \llcorner \Omega + (Du - x)\cdot \mathbf{n}\,d\mathcal{H}^{n-1}\llcorner \partial\Omega, \tag{97}$$

is a measure of finite total-variation and we can rewrite (96) as

$$0 \leq \int_{\overline{\Omega}} v(x)\,d\sigma(x). \tag{98}$$

The disintegration theorem, which we state in Section A.3 (see also [1, Theorem 5.3.1], [25, 78-III]) implies we may disintegrate the measure σ (by separately disintegrating its positive and negative parts σ^+ and σ^-) with respect to the map Du (equivalently with respect to the contact sets $\tilde{x} = Du^{-1}(Du(x))$). Our goal in this section is to prove Corollary A.9, namely to show that for $\mathcal{H}^n$ almost every $x \in \overline{\Omega}$ (96) holds for the disintegration on $\tilde{x}$. In fact, provided $x \notin \Omega_0$ we will prove the result for general convex v, and for $x \in \Omega_0$ we will prove the result for v satisfying $u + v \geq 0$. More precisely, the disintegration theorem implies there exists families of measures $\sigma_{+,y}$ and $\sigma_{-,y}$ such that all Borel f satsify the following analog of Bayes' rule for conditional expectation:

$$\int_{\overline{\Omega}} f(x)\,d\sigma_+(x) = \int_{Du(\overline{\Omega})}\int_{Du^{-1}(y)} f(x) d\sigma_{+,y}(x)\,d(Du_\#\sigma_+)(y),$$

$$\int_{\overline{\Omega}} f(x)\,d\sigma_-(x) = \int_{Du(\overline{\Omega})}\int_{Du^{-1}(y)} f(x) d\sigma_{-,y}(x)\,d(Du_\#\sigma_-)(y).$$

We show for $(Du)_\#\sigma_+$ almost every y

$$0 \leq \int_{Du^{-1}(y)} v(x) d(\sigma_{+,y} - \sigma_{-,y})(x), \tag{99}$$

for all convex v with $\operatorname{spt} v_-$ disjoint from $\{u = 0\}$. As we will prove the same result (with $\tilde{x}$ replacing $Du^{-1}(y)$) then holds for $\mathcal{H}^n$ almost every $x \in \Omega$ and $\mathcal{H}^{n-1}$ almost every $x \in \partial\Omega$.

We emphasize that this appendix, whilst included for completeness, merely provides some more details on Rochet and Chonè's proof of this localization property [51].

A.1. Measures in convex order and sweeping operators.

We begin with the following definition which is used in the theory of martingales and clearly related to (98).

Definition A.1 (Convex order). *Let $\Omega \subset \mathbf{R}^n$ be a Borel set. Let $\sigma_1, \sigma_2 \in \mathcal{P}(\Omega)$ be Borel probability measures. We write $\sigma_1 \preceq \sigma_2$ (read as σ_1 precedes σ_2 in convex order) provided for every continuous convex $u : \Omega \to \mathbf{R}$ there holds*

$$\int_\Omega u \, d\sigma_1 \leq \int_\Omega u \, d\sigma_2.$$

The "Sweeping Theorem" characterizes measure in convex order and requires some more definitions. We take this Theorem from the work of Strassen [55, Theorem 2] where it's attributed to "Hardy–Littlewood–Pólya–Blackwell–Stein–Sherman–Cartier–Fell–Meyer" (see also [22, Théorème 1], [44, T51, T53]).

Definition A.2 (Sweeping operators and Markov kernels). *(1) By a Markov kernel on Ω we mean a function $T : \Omega \to \mathcal{P}(\Omega)$ where for each $x \in \Omega$, $T_x := T(x) \in \mathcal{P}(\Omega)$ is a probability measure. As a technicality we require for each Borel E that $x \mapsto T_x(E)$ is Borel measurable.*
(2) A Markov kernel T is called a sweeping operator[4] *provided it satisfies that for each affine $p : \Omega \to \mathbf{R}$,*

$$p(x) = \int_\Omega p(\xi) \, dT_x(\xi) =: (Tp)(x). \tag{100}$$

(3) If $\sigma \in \mathcal{P}(\Omega)$ and T is a Markov kernel on Ω, then we define $T\sigma \in \mathcal{P}(\Omega)$ by

$$(T\sigma)(A) = \int_\Omega T_x(A) \, d\sigma(x). \tag{101}$$

We note two points: First, (100) is equivalent to the requirement

$$\int_\Omega \xi \, dT_x(\xi) = x.$$

Second, for each integrable $f \in L^1(T\sigma)$, from (101) we have

$$\int_\Omega f(x) \, d(T\sigma)(x) = \int_\Omega \int_\Omega f(\xi) \, dT_x(\xi) \, d\sigma(x). \tag{102}$$

The sweeping theorem gives a necessary and sufficient condition for σ_1 to precede σ_2 in convex order.

Theorem A.3 (Sweeping characterization of convex order; see [55]). *The measures $\sigma_1, \sigma_2 \in \mathcal{P}(\Omega)$ satisfy $\sigma_1 \preceq \sigma_2$ if and only if there exists a sweeping operator T such that $\sigma_2 = T\sigma_1$.*

[4]The term sweeping is also sometimes known as a balayage or dilation.

A.2. First characterization; Lagrange multiplier and sweeping

We aim to apply Theorem A.3 to the positive and negative parts of σ. However Theorem A.3 does not apply directly because we do not have the condition $\sigma_- \preceq \sigma_+$ but only the weaker condition obtained by testing against *nonnegative* convex functions. Nevertheless, we obtain the following lemma.

Lemma A.4 (Restoring neutrality)**.** *Let σ be as in* (97)*. There exists a nonnegative measure λ supported on $\{u = 0\}$ and a sweeping operator T on $\overline{\Omega}$ such that*

$$\sigma - \lambda = (\sigma - \lambda)_+ - (\sigma - \lambda)_- \tag{103}$$

$$= T\omega - \omega, \tag{104}$$

for $\omega := (\sigma - \lambda)_-$.

The representation (103) is, of course, trivial. The essential conclusion is that after subtracting the Lagrange multiplier λ (our reason for designating it so will be clear in the proof) $(\sigma - \lambda)_- \preceq (\sigma - \lambda)_+$.

Proof of Lemma A.4. First note for all $t \geq 0$, tu is admissible for the minimization problem. Thus by minimality

$$0 = \frac{d}{dt}\bigg|_{t=1} L[tu] = \int_{\overline{\Omega}} u(x)\, d\sigma. \tag{105}$$

Combined with the minimality condition (98) we have

$$0 = \inf\left\{\int_{\overline{\Omega}} v\, d\sigma\ ;\ v \text{ is convex with } v \geq 0\right\},$$

with $v = u$ realizing the infimum. A classical theorem in the calculus of variations says the constraint $v \geq 0$ may be realized by a Lagrange multiplier (see, for example, [38, Theorem 1,pg. 217]). Thus there is a nonnegative Radon measure λ such that

$$0 = \inf\left\{\int_{\overline{\Omega}} v\, d(\sigma - \lambda)\ ;\ v \text{ is convex}\right\}, \tag{106}$$

with $v = u$ still realizing the infimum. Using that u attains the infimum along with (105) yields

$$\int u\, d\lambda = 0.$$

Since λ is nonnegative we conclude $\mathrm{spt}\lambda \subset \{u = 0\}$. Since (106) implies $(\sigma - \lambda)_- \preceq (\sigma - \lambda)_+$ we see (104) follows by Theorem A.3. Note that $(u - \lambda)_-$ and $(u - \lambda)_+$ may not be probability measures but are of finite and equal mass (finiteness follows from $u \in C^1(\overline{\Omega})$ [16]), equality from testing against $v = \pm 1$) and this justifies our use of Theorem A.3. ■

The following property of the sweeping operator T is essential to our arguments.

Lemma A.5 (Localization to leaves)**.** *Let T be the sweeping operator given by Lemma A.4. Then for ω almost every x the measure T_x is supported on $\tilde{x}$.*

Proof. First, by (102)

$$\int_{\overline{\Omega}} u(x)\,d(T\omega)(x) = \int_{\overline{\Omega}}\int_{\overline{\Omega}} u(\xi)\,dT_x(\xi)\,d\omega(x). \tag{107}$$

For each $x \in \overline{\Omega}$ we have $u(\xi) \geq p_x(\xi) = u(x) + Du(x)\cdot(\xi - x)$ with equality if and only if $\xi \in \tilde{x}$. Thus (107) and the definition of a sweeping operator (100) imply

$$\begin{aligned}\int_{\overline{\Omega}} u(x)\,d(T\omega)(x) &\geq \int_{\overline{\Omega}}\int_{\overline{\Omega}} p_x(\xi)\,dT_x(\xi)\,d\omega(x)\\ &= \int_{\overline{\Omega}} u(x)\,d\omega(x).\end{aligned}$$

However (106) implies we must have equality. Subsequently we obtain

$$\int_{\overline{\Omega}} u(\xi)\,dT_x(\xi) = \int_{\overline{\Omega}} p_x(\xi)\,dT_x(\xi),$$

for ω a.e. x, which can only occur if for a.e. x, T_x is supported on $\tilde{x}$. ■

At this point we have almost everything needed to obtain the localization property. Now we establish the following lemma which was was used earlier in the proof of Proposition 5.4.

Lemma A.6 (Objective responds proportionately to uniform utility increase)**.** *Let σ denote the variational derivative (recall equation* (22)*). Then $(Du)_{\#}\sigma = \delta_0$, that is $(Du)_{\#}\sigma$ is a unit Dirac mass at the origin.*

Proof. That the variational derivative σ assigns measure 1 to $\{u = 0\}$ was observed by Rochet and Chonè [51]: this follows from the leafwise neutrality outside $\{u = 0\}$ implied by Lemmas A.4–A.5, and from $L(u + t) = t + L(u)$ for all $t \in \mathbf{R}$. Thus it suffices to prove $(Du)_{\#}\sigma(A) = 0$ for any $A \subset Du(\overline{\Omega})$ not containing 0 or, equivalently,

$$(T\omega)(Du^{-1}(A)) = \omega(Du^{-1}(A)). \tag{108}$$

Because T_x is supported on $\tilde{x}$ for ω a.e. x we have

$$(T\omega)(Du^{-1}(A)) = \int_{\Omega} T_x(Du^{-1}(A))\,d\omega(x) = \int_{Du^{-1}(A)} T_x(Du^{-1}(A))\,d\omega(x),$$

where we've used $T_x(Du^{-1}(A)) = 0$ for $x \notin Du^{-1}(A)$. Since for $x \in Du^{-1}(A)$ we have (up to normalization) $T_x(Du^{-1}(A)) = 1$ this proves (108). ■

A.3. Disintegration and localization

In this section we complete the proof that the variational inequality (19) holds on almost every contact set. We use the Disintegration Theorem in the following form, which can be viewed as a continuum generalization of Bayes' theorem (see [1, Theorem 5.3.1] and [25, 78-III]).

Theorem A.7 (Disintegration of measure)**.** *Let X, Y be Radon separable metric spaces, $\mu \in \mathcal{P}(X)$, and let $F : X \to Y$ be a Borel map used to define the push forward $\nu = F_{\#}\mu \in \mathcal{P}(Y)$. Then there exists a ν-a.e. uniquely determined Borel family of probability measures $\{\mu_y\}_{y\in Y} \subset \mathcal{P}(X)$ such that μ_y vanishes outside $F^{-1}(y)$ for ν-a.e. y, and*

$$\int_X f(x)\,d\mu(x) = \int_Y \int_{F^{-1}(y)} f(x)\,d\mu_y(x)\,d\nu(y)$$

for every Borel $f :\to [0, +\infty]$.

Theorem A.8 (Leafwise Euler-Lagrange condition)**.** *Let $\sigma_{+,y}$ and $\sigma_{-,y}$ be the conditional measures obtained from* (97) *by applying Theorem A.7 with the projection $F = Du$, $\tilde{x} = Du^{-1}(y)$, and μ equal to σ_+ and σ_- respectively. Put $\sigma_y = \sigma_{+,y} - \sigma_{-,y}$ and let v be a convex function with spt v_- disjoint from $\{u = 0\}$. Then for both $\mathcal{H}^n$ almost every $x \in \Omega$ and $\mathcal{H}^{n-1}$ almost every $x \in \partial\Omega$ there holds for $y = Du(x)$*

$$\int_{\tilde{x}} v(\xi)\,d\sigma_y(\xi) \geq 0. \tag{109}$$

We work away from $\{u = 0\}$ and use $\omega \llcorner \{u > 0\} = \sigma_- \llcorner \{u > 0\}$ without further reference to this restriction. The idea of the proof is that because T_x is supported on $\tilde{x}$ the conditioning of $T\omega$, denoted $(T\omega)_y$, is obtained by sweeping the conditioning of ω, denoted ω_y. More succinctly

$$\sigma_{+,y} = (T\omega)_y = T(\omega_y) = T\big(\sigma_{-,y}\big). \tag{110}$$

Then Theorem A.3 implies (109).

Proof. We apply the Disintegration Theorem a number of times in this proof, each time with $F(x) = Du(x)$. Applying the Disintegration Theorem to the measure $\omega = (\sigma - \lambda)_-$ from Lemma A.4, we obtain a family of measures ω_y such that for any Borel $f : \overline{\Omega} \to [0, +\infty]$

$$\int_{\overline{\Omega}} f(x)\,d\omega(x) = \int_{Du(\overline{\Omega})} \int_{Du^{-1}(y)} f(x)\,d\omega_y(x)\,d\big((Du)_{\#}\omega\big)(y). \tag{111}$$

We consider two ways of expressing the result of disintegrating $T\omega$. First, by a direct application of the disintegration theorem

$$\int_{\overline{\Omega}} f(x)\,d(T\omega)(x) = \int_{Du(\overline{\Omega})} \int_{Du^{-1}(y)} f(x)\,d(T\omega)_y(x)\,d\big((Du)_{\#}(T\omega)\big)(y). \tag{112}$$

On the other hand using the sweeping operator and the disintegration of ω, (111), we obtain

$$\begin{aligned}\int_{\overline{\Omega}} f(x)\,d(T\omega)(x) &= \int_{\overline{\Omega}} \int_{\overline{\Omega}} f(\xi)\,dT_x(\xi)\,d\omega \\ &= \int_{Du(\overline{\Omega})} \int_{Du^{-1}(y)} \int_{\overline{\Omega}} f(\xi)\,dT_x(\xi)\,d\omega_y(x)\,d\big((Du)_{\#}\omega\big)(y). \qquad (113)\end{aligned}$$

Using that T_x is supported on $Du^{-1}(Du(x))$ the inner two integrals in (113) become integration against $T(\omega_y)$. Namely,
(114)

$$\int_{\overline{\Omega}} f(x)\, d(T\omega)(x) = \int_{Du(\overline{\Omega})} \int_{Du^{-1}(y)} f(x)\, d(T(\omega_y))(x)\, d\big((Du)_{\#}\omega\big)(y).$$

To conclude we recall from Lemma A.6 that $(Du)_{\#}(T\omega) = (Du)_{\#}\omega$. Thus comparing (114) and (112) and using the uniqueness a.e of the conditional measures we obtain

$$(T\omega)_y = T(\omega_y).$$

which is (110) so by the sweeping characterization of convex order (Theorem A.3) we obtain (109) for $(Du)_{\#}\sigma_+$ almost every $y \in Du(\overline{\Omega})$.

To finish the proof we must translate back to stating the result in terms of $\mathcal{H}^n$ a.e. $x \in \overline{\Omega}$. Let $B \subset \overline{\Omega}$ be the set of x for which the leafwise localized inequality (109) does not hold. Then $(Du)_{\#}\sigma_+(B) = 0$ and by Lemma A.6 $(Du)_{\#}\sigma_-(B) = 0$. Hence
(115)

$$0 = \int_{Du^{-1}(Du(B))} (n+1-\Delta u)_{\pm}\, dx + \int_{Du^{-1}(Du(B))\cap\partial\Omega} ((Du-x)\cdot \mathbf{n})_{\pm}\, d\mathcal{H}^{n-1}.$$

It follows that

$$0 = \int_{Du^{-1}(Du(B))} |n+1-\Delta u|\, dx + \int_{Du^{-1}(Du(B))\cap\partial\Omega} |(Du-x)\cdot \mathbf{n}|\, d\mathcal{H}^{n-1}.$$

Now on each $\tilde{x}$ with $x \in B$ because the leafwise inequality does not hold we have $n+1-\Delta u \neq 0$ on a positive $\mathcal{H}^{\dim \tilde{x}}$ measure subset of $\tilde{x}$. Indeed, since the leafwise inequality does not hold either $n+1-\Delta u \not\equiv 0$ or $(Du-x)\cdot\mathbf{n} \neq 0$ and in this latter case mass balance, Lemma A.6, implies $n+1-\Delta u \not\equiv 0$. Thus $Du^{-1}(Du(B))$, which clearly contains B, has measure 0. Thus $\mathcal{H}^n(B) = 0$.

Finally to obtain the result for $\mathcal{H}^{n-1}$ almost every $x \in \partial\Omega$, note at boundary points of strict convexity the leafwise (now, pointwise) inequality is satisfied by Lemma A.6. Moreover if the set of $x \in \partial\Omega \cap B$ which are contained in nontrivial contact sets and has $\mathcal{H}^{n-1}(B\cap\partial\Omega) > 0$ then, by Fubini's theorem $\mathcal{H}^n(B) > 0$. We conclude $\mathcal{H}^{n-1}(B\cap\partial\Omega) = 0$. ∎

Corollary A.9 (Rochet and Chonè's leafwise localization)**.** *Every convex* $v : \mathbf{R}^n \longrightarrow \mathbf{R}$ *satisfies*

$$0 \le \int_{\tilde{x}} v(z) d\sigma_{\tilde{x}}(z) \tag{116}$$

for $\mathcal{H}^n$ *almost every* $x \in \overline{\Omega}\backslash\{u=0\}$ *and for* $\mathcal{H}^{n-1}$ *almost every* $x \in \partial\Omega\backslash\{u = 0\}$*. (The same conclusions extend to* $x \in \{u=0\}$ *if also* $u+v \ge 0$*.)*

Proof. To obtain Corollary A.9 from Theorem A.8: for $x \in \overline{\Omega} \setminus \{u=0\}$ apply Theorem A.8 with the convex function $\xi \mapsto v(\xi) + M\mathrm{dist}(\xi,\tilde{x})$ for M chosen sufficiently large; if instead $u(x) = 0$ and $u+v \ge 0$ apply Theorem A.8 to the convex function $v + \varepsilon + M\mathrm{dist}(\cdot,\tilde{x})$ which becomes positive for $M \ge \|u\|_{C^{0,1}(\Omega)}$, and then send $\varepsilon \downarrow 0$. ∎

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