# Trading linearity for ellipticity: a nonsmooth approach to Einstein's theory of gravity and Lorentzian splitting theorems

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*www.math.toronto.edu/mccann/Talk3.pdf*

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# What is a splitting theorem? (dimension reduction...)

**Example (convex functions, not necessarily smooth)**

If the graph of a convex function $u : \mathbf{R}^n \longrightarrow \mathbf{R}$ contains a full line, say $u(t, 0, \ldots, 0) = 0$ for all $t \in \mathbf{R}$, then $u(x) = U(x_2, \ldots, x_n)$ for all $x \in \mathbf{R}^n$

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## Example (smooth Riemannian manifolds; Cheeger-Gromoll '71)

If a connected complete Ricci nonnegative Riemannian manifold $(M^n, g_{ij})$ contains a length minimizing line, then $M$ is a geometric product of $(\mathbf{R}, dr^2)$ with a submanifold $(\Sigma^{n-1}, h_{ij} = g_{ij}|_\Sigma)$: i.e.

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Gigli '13+: nonsmooth version for infinitesimally Hilbertian metric-measure spaces $(M, d, m)$ satisfying curvature-dimension condition $RCD(0, N)$ à la Sturm '06, Lott & Villani '09, (. . . M. '94)

This talk: *Lorentzian* analogs relevant to Einstein's theory of gravity

# Schematic proof of Cheeger-Gromoll splitting theorem:

Let $\gamma : \mathbf{R} \longrightarrow M^n$ be the doubly-infinite minimizing geodesic line.

Busemann '32: $b_r(x) := d(x, \gamma(r)) - d(\gamma(0), \gamma(r))$ and $\pm b^{\pm} := \lim_{r \to \pm\infty} b_r$

- note $b_r$ is 1-Lipschitz and $|\nabla b_r| = 1 = |\nabla b^{\pm}|$ a.e.; for $r > 0$,
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- along $\Sigma$, metric splits into tangent $g_{ij} dy^i dy^j$ and normal components $dr^2$
- $(r, y) \in \mathbf{R} \times \Sigma \mapsto \exp_y r \nabla b(y)$ is surjective hence a global isometry

# General relativity: Einstein's gravity and field equation

• Einstein's gravity is formulated on smooth Lorentzian manifolds, but often predicts such manifolds are geodesically incomplete (nonsmooth)

Gravity not a force, but rather a manifestation of curvature of spacetime
"Spacetime tells matter how to move" (along timelike/null geodesics... )

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| | | |
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• understanding Ricci and Einstein (à la Kip Thorne) $i, j \in \{0, 1, 2, 3\}$

• just integrate this local conservation law for $T_{ij}(x)$ to find $g_{ij}$...

What if matter distribution is unknown?

Can look at initial value problem (nonlinear wave equation); linearization produces gravity waves; no smoothing; singularities propagate...

# Elliptic vs hyperbolic geometry

ELLIPTIC: $\mathbf{R}^n$ equipped with Euclidean norm $|v|_E := (\sum v_i^2)^{1/2}$

- $|v+w|_E \leq |v|_E + |w|_E$

HYPERBOLIC: Minkowski space $\mathbf{R}^n$ equipped with the *hyperbolic 'norm'*

$$|v|_F := \begin{cases} (v_1^2 - \sum_{i\geq 2} v_i^2)^{1/2} & v \in F := \left\{ v \in \mathbf{R}^n \mid v_1 \geq (\sum_{i\geq 2} v_i^2)^{1/2} \right\} \\ -\infty & \textit{else} \end{cases}$$

- $|v+w|_F \geq |v|_F + |w|_F$, but terribly asymmetric

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the future $F \subset \mathbf{R}^n$ is a convex cone; $v \in F$ called *causal* or *future-directed*

- $v$ is *timelike* if $v \in F \setminus \partial F$
- $v$ is *lightlike (or null)* if $v \in \partial F \setminus \{0\}$

(• $v$ is *spacelike* iff $\pm v \notin F$ and *past-directed* if $-v \in F$)

- smooth curves are called *timelike (etc.)* if all tangents are timelike (etc.)

# A crash course in differential geometry: action principles

Manifold $M^n$ with symmetric nondegenerate $C^m$-smooth tensor $g_{ij} = g_{ji}$

RIEMANNIAN: $(g_{ij}) > 0$ defines Euclidean norm on each tangent space

- its geometry is also encoded in the (symmetric) distance function

$$d(x,y) := \inf_{\sigma(0)=x,\sigma(1)=y} \left( \int_0^1 |\dot\sigma_t|^q_{E_g} dt \right)^{1/q} \qquad q > 1$$

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$$\ell(x,y) := \sup_{\sigma(0)=x,\sigma(1)=y} \left( \int_0^1 |\dot\sigma_t|^q_{F_g} dt \right)^{1/q} \qquad 0 \neq q < 1$$

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- $(-\infty)^q := -\infty$ so $\ell(x,y) = -\infty$ unless a causal curve links $x$ to $y$
- extremizers are independent of $q$; they are called *geodesics*
- $\ell(x,z) \geq \ell(x,y) + \ell(y,z)$ (analog of the triangle inequality $d$ satisfies)

# The Riemann curvature tensor

Given (timelike) *geodesics* $(\sigma_s)_{s\in[0,1]}$ and $(\tau_t)_{t\in[0,1]}$ with $\sigma_0 = \tau_0$ and $\dot\tau_0, \dot\sigma_0, \dot\tau_0 - \dot\sigma_0 \in F \setminus \partial F$, if $g_{ij} \in C^3$

$$\ell(\sigma_s, \tau_t)^2 = |t\dot\tau_0 - s\dot\sigma_0|^2_{F_g} - \frac{Sec}{6} s^2 t^2 + O((|s| + |t|)^5)$$

where sectional curvature $Sec = R(\dot\sigma_0, \dot\tau_0, \dot\sigma_0, \dot\tau_0)$ is quadratic in $\dot\sigma_0 \wedge \dot\tau_0$ and measures the leading order correction to Pythagoras

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- its trace $\mathrm{Ric}_{ik} = g^{jl} R_{ijkl}$ yields the *Ricci* tensor; $\mathrm{Ric}(v, v)$ measures the correction to Pythagoras averaged over all triangles including side $v$
- second trace $R = g^{ik}\mathrm{Ric}_{ik}$ yields the *scalar curvature*; in the elliptic case it gives leading order correction to the area of a sphere of radius $r$ (and to the volume of a ball of radius $r$)
- $d\mathrm{vol}_g(x) = \sqrt{|\det(g)|}\, d^n x$ in coordinates; (in the Riemannian case it coincides with the $n$-dimensional Hausdorff measure associated to $d$)

# Energy conditions and singularity theorems

WEC (weak energy condition): $T(v, v) \geq 0$ for all future $v \in F$ (physical)
SEC (strong energy condition): $\mathrm{Ric}(v, v) \geq 0$ for all future $v \in F$ (less ")
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[Cosmological constant (dark matter): $\geq (n-1)Kg(v,v)$]

THM: Hawking '66 (big bang) singularities are generic:
SEC + mean curvature bound $H_\Sigma \geq h > 0$ on a suitable hypersurface $\Sigma$ implies finite-time singularities along all timelike geodesics through $\Sigma$

Cavalletti-Mondino '20+: also in $TCD(0,N)$ metric-measure spacetimes

THM: Penrose '65 (stellar collapse) singularities are generic
NEC + trapped codimension-2 compact surface $S$ + suitable noncompact hypersurface $\Sigma$ imply finite-time singularity along some null geodesic

Graf '20 holds for $g_{ij} \in C^1$;

Open: version for $TCD(0,N)$ metric-measure spacetimes?

# Smooth Lorentzian splitting theorems

- *'spacetime'*: a connected Lorentzian manifold $(M^n, g_{ij})$ which admits a continuous choice of $F_g$ (distinguishing future from past).
- *'strong energy condition' SEC:* $g(v,v) > 0$ implies $\mathrm{Ric}(v,v) \geq 0$
- *'timelike geodesically complete'*: all (unit speed) timelike geodesics admit doubly-infinite extensions (maximizing locally but not necessarily globally)

**Theorem (conjectured by Yau '82; proved by Newman '90)**

*Let $(M^n, g_{ij})$ be a SEC spacetime containing a maximizing timelike line. If $M$ is (a) timelike geodesically complete, then $M$ is a geometric product of* $\mathbf{R}$ *with a ($\mathrm{Ric} \geq 0$, complete) Riemannian submanifold $\Sigma^{n-1}$*

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Galloway '84: proved assuming compact Cauchy surfaces exist

Beem, Ehrlich, Galloway, Markvorsen '85: proved under timelike sectional curvature nonnegativity assuming (b) global hyperbolicity;

Beran-Ohanyan-Rott-Solis '23: extension to g.h. Lorentzian length spaces

Eschenburg '88: proved under (a) + (b)

Galloway '89: proved under (b) without (a) using Bartnik '88

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**Theorem (BBCGMORS 24+ 'nonsmooth $p$-d'Alembert comparison')**

*For $p < 1$, the operator $\Box_p u := -\nabla \cdot (|\nabla u|_F^{p-2} \nabla u)$ is nonuniformly elliptic and (SEC) implies $\Box_p b_r^+ \leq \frac{n-1}{\ell(\cdot,\gamma(r))}$ distributionally, i.e. $\forall\ 0 \leq \phi \in C_c^1(M)$*

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$$\int_M g\left(\nabla \phi, \frac{\nabla b_r^+}{|\nabla b_r^+|_F^{2-p}}\right) d\mathrm{vol}_g \leq (n-1) \int_M \frac{\phi(\cdot) d\mathrm{vol}_g(\cdot)}{\ell(\cdot, \gamma(r))}.$$

- for the distributional limit $r \to \infty$, need $\nabla b_r^+ \longrightarrow \nabla b^+$ strongly
- need uniform ellipticity; must bound $\{\nabla b_r^+\}_{r \geq R}$ away from lightcone

# Convex $p$-energy: trading linearity for ellipticity

Additional conditions may ensure $\ell \neq +\infty$ and extremizers exist
• complete or *proper* (boundedly compact) in the Riemannian case
• (b) *globally hyperbolic* in the Lorentzian case (i.e. compact diamonds, future $F$ varies continuously over $M$, no closed future-directed curves)

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• note $L = H^*$ jumps from 0 to $+\infty$ across future cone boundary $\partial F$ (but $H$ diverges continuously at the boundary of the dual cone $F^*$)
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Beran Braun Calisti Gigli M. Ohanyan Rott Sämann (octet): extremizers of $p$-Dirichlet energy $u \mapsto \int_M H(du) d\mathrm{vol}_g$ rel. to compactly supported perturbations satisfy a new nonuniformly elliptic nonlinear PDE
- trade linearity of d'Alembertian for ellipticity of $p$-d'Alembertian!

# Nondivergence expression of (nonuniform) ellipticity

$$\Box_p b = \nabla_i \left( \frac{\partial H}{\partial w_i}\Big|_{db} \right) = H^{ij} \nabla_i \nabla_j b$$

$$H(w) = -\frac{1}{p} |w|^p_{F^*}$$

$$H^i := \frac{\partial H}{\partial w_i} = -|w|^{p-2} g^{ik} w_k$$

$$H^{ij} := \frac{\partial^2 H}{\partial w_i \partial w_j} = |w|^{p-2} \left[ (2-p) g^{ik} g^{jl} \frac{w_k w_l}{|w|^2} - g^{ij} \right]$$

$$\sim |w|^{p-2} \begin{bmatrix} 2-p-1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \quad \text{if } p < 1$$

choosing normal coordinates around $\gamma(0)$ in which $w = db$ is the time axis

Nomizu-Ozeki '61 give a complete Riemannian metric $\tilde{g}$ on $(M, g)$.

**Theorem (Eschenburg '88 ... Galloway-Horta '96)**

*Under (a) and/or (b), $\gamma(0)$ admits a neighbourhood $X$ and constants $R, C$ such that if $r \geq R$ then (i) a maximizing geodesic $\sigma$ connects each $x \in X$ to $\gamma(r)$; (ii) each such geodesic satisfies $\tilde{g}(\sigma'(0), \sigma'(0)) \leq C g(\sigma'(0), \sigma'(0))$ hence $\{b_r^+\}$ is timelike and uniformly equiLipschitz on $X$.*

- intersecting ellipsoid and hyperboloid uniformize ellipticity on $X$

**Lemma (BGMOS: equi-semiconcavity '24/-superdifferentiability '25+)**

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**Lemma (BGMOS: equi-semiconcavity '24/-superdifferentiability '25+)**

*$g \in C^{1,1}$ yields $\tilde{C}$ so all $u \in \{b_r^+\}_{r \geq R}$ and $(v, x) \in TX$ satisfy*

$$\lim_{t \to 0} \frac{u(\exp_x^{\tilde{g}} tv) + u(\exp_x^{\tilde{g}} -tv) - 2u(x)}{\tilde{g}(v, v)} \leq \tilde{C}$$

(• $p$-d'Alembert comparison then follows from smooth calculations)

- gives $db_r^+ \to db^+$ pointwise a.e., hence $|db^\pm|_{F^*} = 1$ a.e. and
- $\pm b^\pm$ are distributionally $p$-superharmonic $\Box_p b^+ \leq 0 \leq \Box_p b^-$

• now strong maximum principle improves $b^+ \geq b^-$ to $b^+ = b^- \in C^{1,1}(X)$ if $g \in C^{1,1}$ (or $b^\pm = b \in C^1(X)$ if $g_{ij} \in C^1(M)$)

Homogeneity $2p - 2 < 0$ variant on Bochner Ohta '14, Mondino-Suhr '23: $|du|_{F^*} = 1$ and $\Box_p u = 0$ imply (by elliptic regularity $b \in W^{2,2}_{loc}(X)$ if $g \in C^1(M)$), and

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\begin{aligned}
0 =& \nabla_i (H^{ij}|_{du} \nabla_j (H|_{du})) - H^i \nabla_i (\nabla_j (H^j|_{du})) \\
=& H^{ij} u_{jk} H^{kl} u_{li} + R_{ij} H^i H^j \\
=& \mathrm{Tr}\left[ \left( \sqrt{D^2 H} \nabla^2 u \sqrt{D^2 H} \right)^2 \right] + \mathrm{Ric}(DH, DH)
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- allow Bakry-Émery '85 weights, à la Case '10 and Woolgar-Wylie '18

## Theorem (BGMOS 25+: $C^1$ Eschenburg splitting with weight $V$)

*Let $(M^n, g_{ij})$ with $V, g_{ij} \in C^1$ be an (a) timelike geodesically complete, (b) globally hyperbolic spacetime and (c) timelike nonbranching, containing a maximizing timelike line. If $N \in (n, \infty]$ and*

$$\mathrm{Ric}^{(N,V)} := \mathrm{Ric} + \mathrm{Hess}\, V - \frac{1}{N-n} dV \otimes dV \geq 0$$

*timelike distributionally* then $M^n$ is ($C^2$-isometric to) a geometric product of $(\mathbf{R}, dr^2)$ with a (complete, $\mathrm{Ric}^{(N-1,V)} \geq 0$ distributionally) Riemannian submanifold $\Sigma^{n-1}$. Also $V(x(r,y))$ is independent of $r \in \mathbf{R}$.*

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Proof: Set $\nabla^{(g,V)} \cdot := (\nabla - dV)\cdot$ and $\Box_p^{(g,V)} u := -\nabla^{(g,V)} \cdot (|\nabla u|^{p-2} \nabla u)$, and show Busemann functions satisfy $\pm \Box_p^{(g,V)} b^{\pm} \leq 0$ hence agree and are weighted $p$-harmonic. Use weighted Bochner-Ohta identity to deduce $\mathrm{Hess}\, b^+ = 0$ and argue as before. □

THANK YOU VERY MUCH!