Valuing multiple dimensions of heterogeneity: the monopolist's free boundary problem in the plane

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with Cale Rankin (Monash U) and Kelvin Shuangjian Zhang (Fudan U)

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Outline

Monopolist's problem

Given $X \subset \mathbf{R}^m$ compact convex, $Y \subset \mathbf{R}^n$, and 'direct utility'
$b(x, y)$ = value of product $y \in Y$ to buyer $x \in X$
$c(y)$ = monopolist's cost to produce $y \in Y$
$d\mu(x)$ = relative frequency of buyer $x \in X$ (as compared to $x' \in X$)

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$$\tilde{\Pi}(v) := \int_X [v(y_v(x)) - c(y_v(x))] d\mu(x), \qquad \text{where}$$

Agent x's problem: choose $y_v(x)$ to maximize

$$y_v(x) \in \arg\max_{y \in Y} b(x, y) - v(y)$$

Constraints: v lower semicontinuous, $0 \in Y$ and $v(0) = c(0) = 0$.

Examples of asymmetric information

• airline ticket pricing

• insurance

• educational signaling

• optimal taxation: replace profit maximization with a budget constraint for providing services

Two landmarks (very abridged):

Mirrlees '71, Spence '73 ($n = 1 = m$): $\frac{\partial^2 b}{\partial x \partial y} > 0$ implies $\frac{dy_v}{dx} \geq 0$

Rochet-Choné '98 ($n = m > 1$): $b(x, y) = x \cdot y$ *bilinear* implies $y_v(x) = Dv^*(x)$ convex gradient; bunching

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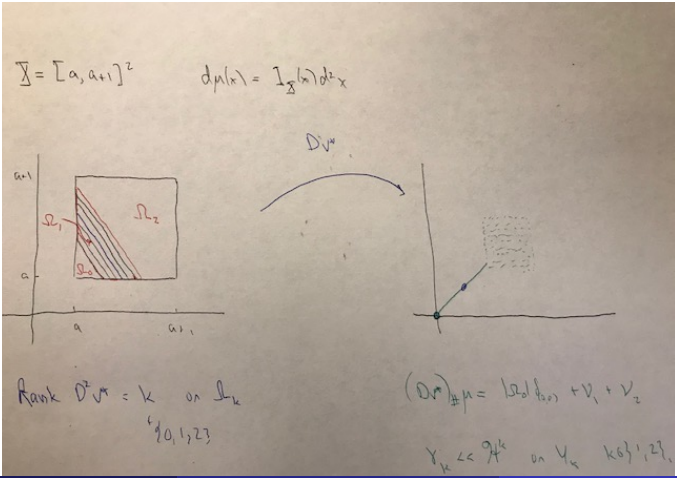
Rochet-Choné '98 ($n = m > 1$): $b(x, y) = x \cdot y$ *bilinear* implies $y_v(x) = Dv^*(x)$ convex gradient; bunching and unique for $c(y) = \frac{1}{2}|y|^2$



Mathematical developments

Carlier–Lachand-Robert '03: b bilinear gives $v^* \in C^1(X)$ where $X = \mathrm{spt}\,\mu$;

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under strengthening of Ma–Trudinger–Wang's '05 fourth order (curvature) conditions on b (and more generally, non-quasilinear preferences), where

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is called the 'indirect utility' to shopper x

Rochet–Choné $b(x, y) = x \cdot y$ in terms of buyers' utilities u

$$u(x) := v^*(x) := \max_{y \in Y}[x \cdot y - v(y)] \tag{1}$$

is attained where the f.o.c.

$$Du(x) = y_v(x)$$

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hold. Therefore maximize

$$\begin{aligned} \tilde{\Pi}(v) &= \int_X (v - c)(Du(x)) d\mu(x) \\ &= \int_X [b(x, y) - u(x) - c(y)]_{y = Du(x)} d\mu(x) =: -L(u) \end{aligned}$$

among u of form (1) (i.e. among convex $u(\cdot) \geq 0$ with $Du \in Y$)

Following Rochet–Choné '98 choose $b(x, y) = x \cdot y$ so profit

$$-L(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$
$$\in \mathcal{U} := \{u : X \longrightarrow [0, \infty] \text{ convex} \mid Du(X) \subset Y\}$$

- henceforth specialize to $c(y) = |y|^2/2$ and $X \subset Y := [0, \infty)^n$
- makes constraint $Du \in Y$ redundant (by unconditional symmetry)

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$$L(u) := \int_X \left(\frac{1}{2} |Du(x) - x|^2 + u - \frac{1}{2}|x|^2 \right) d\mathcal{H}^n(x)$$

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- among $u : X \longrightarrow [0, \infty]$ convex; (without convexity, have obstacle problem!)

Explicit solutions? $\frac{d\mu}{dx} = 1_X$ uniform on cube $X = [a, a+1]^n$

c.f. Mussa-Rosen '78

BUYER'S MARKET on INTERVAL: $a < 1 = n$ optimized by

$$u(x) = \begin{cases} (x - \frac{a+1}{2})^2 & \text{if } x \geq \frac{a+1}{2} \\ 0 & \text{else.} \end{cases}$$

- buyers $x \in (0, \frac{a+1}{2})$ opt out; remaining x get customized products $u'(x)$

SELLER'S MARKET: $a \geq 1 = n$ optimized by

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SELLER'S MARKET: $a \geq 1 = n$ optimized by $u(x) = (x - \frac{a+1}{2})^2 - (\frac{a-1}{2})^2$

- no distortion at top type: $u'(a+1) = a+1$
- downward distortion elsewhere $x - u'(x) = a + 1 - x \geq 0$
- distortion increases with a but decreases with x in $X = [a, a+1]$
- each type x of buyer gets a customized product $u'(x)$

THIS TALK: WHAT HAPPENS IN HIGHER DIMENSIONS $n \geq 2$?

$n \geq 2$: partition X into convex leafs of varied dimension

$$u \in \operatorname*{arg\,min}_{\text{convex } u' \geq 0} L(u')$$

minimizes net loss $L(u') := \int_X \left(\frac{1}{2}|Du'(x) - x|^2 + u' - \frac{1}{2}|x|^2 \right) d\mathcal{H}^n(x)$

(Closed convex) isoproduct bunch (= equivalence class = contact set = leaf)

$$\tilde{x} := (Du)^{-1}(Du(x)) = \{x' \in X \mid Du(x') = Du(x)\} \subset X$$

foliate interior of $\Omega_{n-i} := \{x \in X \mid \dim(\tilde{x}) = i\}$.

Theorem (Leaves reach boundary; any normal distortion is outward)

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Theorem (Leaves reach boundary; any normal distortion is outward)

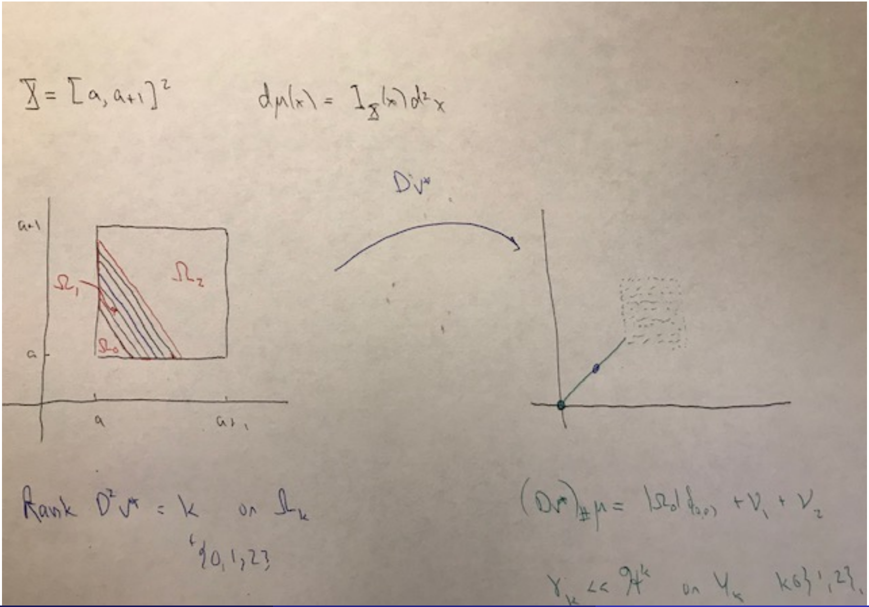
(i) $\Omega_0 = \{x \in X \mid u = 0\}$ *foliated by a single leaf (unless* $\Omega_0 = \emptyset$*.*)*

(ii) if $x \in \Omega_1 \cup \cdots \Omega_{n-1}$ *there exists* $x' \in \tilde{x} \cap \partial X$ *and* $\hat{n}(x') \cdot (Du(x') - x') \geq 0$.

(iii) Ω_n *is relatively open in* X*, foliated by points, i.e.* u *is strictly convex.*

Offers possibility to describe u throughout X using behaviour on ∂X(!)

Rochet-Choné's square example revisited; $c(y) = \frac{1}{2}|y|^2$



Proof (ii): one-sided variations; maximum principle

For $u + \epsilon w \geq 0$ convex,

$$0 \leq L'_u(w) := \frac{d}{d\epsilon}\Big|_{\epsilon=0^+} L(u+\epsilon w) = \int_X w \frac{\delta L}{\delta u}$$

$$= \int_X [n+1-\Delta u]\, w\, d\mathcal{H}^n + \int_{\partial X} (Du - x)\cdot \hat{n}\, w\, d\mathcal{H}^{n-1}$$

where $\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$; (neglecting convexity get $\frac{1}{n+1}\Delta u = 1_{\{u>0\}}$ on X)

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- then $u \in C^\infty(U)$; from $\Delta(\partial^2_{\xi\xi} u) = 0$ strong maximum principle yields either
  - $\partial^2_{\xi\xi} u > 0$ throughout U forcing $\tilde{x} = \{x\}$ or
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□

Characterizing $\Omega_1 \subset \mathbf{R}^2$: obstacle problem plus convexity

Setting $u_i := u$ on $\Omega_i := \{x \in X \mid \mathrm{Dim}(\tilde{x}) = n - i\}$ (now $n = 2$) gives

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$$k(s) = \frac{3}{4}s^2 - as - \log|s - 2a| + const$$

subject to boundary conditions $u_1 = u_0$ and $Du_1 = Du_0$ at lower boundary.

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u_2 = u_1 \quad &\text{on} \quad \partial\Omega_2 \cap \partial\Omega_1 \quad \text{(Dirichlet)}
\end{aligned}
$$

OVERDETERMINED!

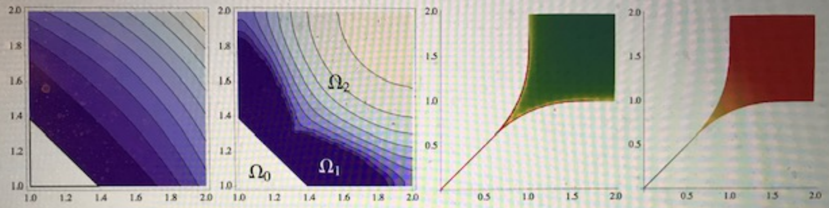

Fig. 1 Numerical approximation U of the solution of the classical Monopolist's problem (1), computed on a 50×50 grid. *Left* level sets of U, with $U = 0$ in *white*. *Center left* level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in *white*); note the degenerate region Ω_1 where $\det(\nabla^2 U) = 0$. *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). *Color scales* on Fig. 10 (color figure online)

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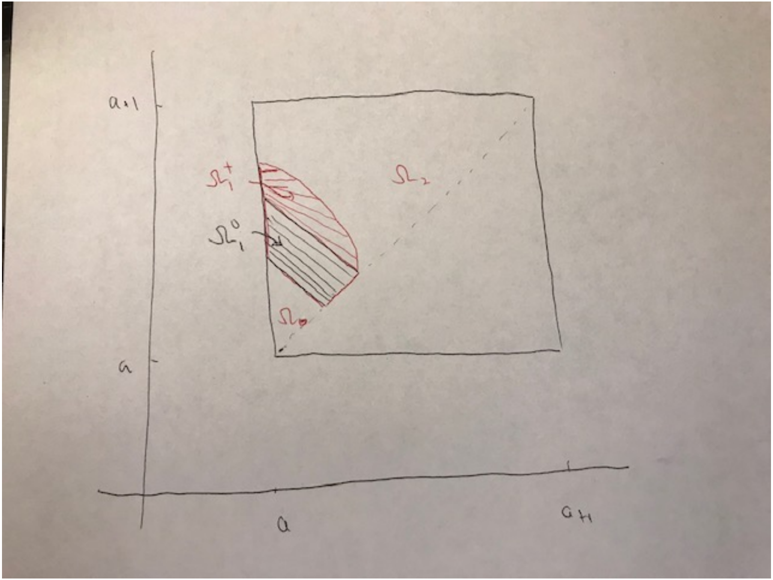
J.-M. Mirebeau (2016)



c.f. M-Z '24; Boerma-Tsyvinski-Zimin '22+ blunt Ω_1^0 vs targeted Ω_1^+ bunching

Free boundary problem

$u = u_i$ on Ω_i where

- on Ω_0 exclusion: $u_0 = 0$
- BLUNT: on Ω_1^0, Rochet-Choné's ODE: $u_1(x_1, x_2) = \frac{1}{2}k(x_1 + x_2)$ where

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- TARGETED: on Ω_1^+, $u_1 = u_1^+$ given by a NEW system of ODE (for height $h(\cdot)$ and length $R(\cdot)$ of isochoice segments together with profile of $u_1^+(\cdot)$ along them), with boundary conditions

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- on Ω_2, PDE: $\Delta u_2 = 3$ with Rochet-Choné's overdetermined conditions

$$\begin{aligned}
(Du_2(x) - x) \cdot \hat{n}_{\Omega_2}(x) = 0 &\quad \text{on} \quad \partial X \cap \bar{\Omega}_2 \text{ and on } \{x_1 = x_2\} \\
(Du_2 - Du_1^+) \cdot \hat{n}_{\Omega_2}(x) = 0 &\quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Neumann)} \\
u_2 = u_1^+ &\quad \text{on} \quad \partial\Omega_2 \cap \partial\Omega_1^+ \quad \text{(Dirichlet)}
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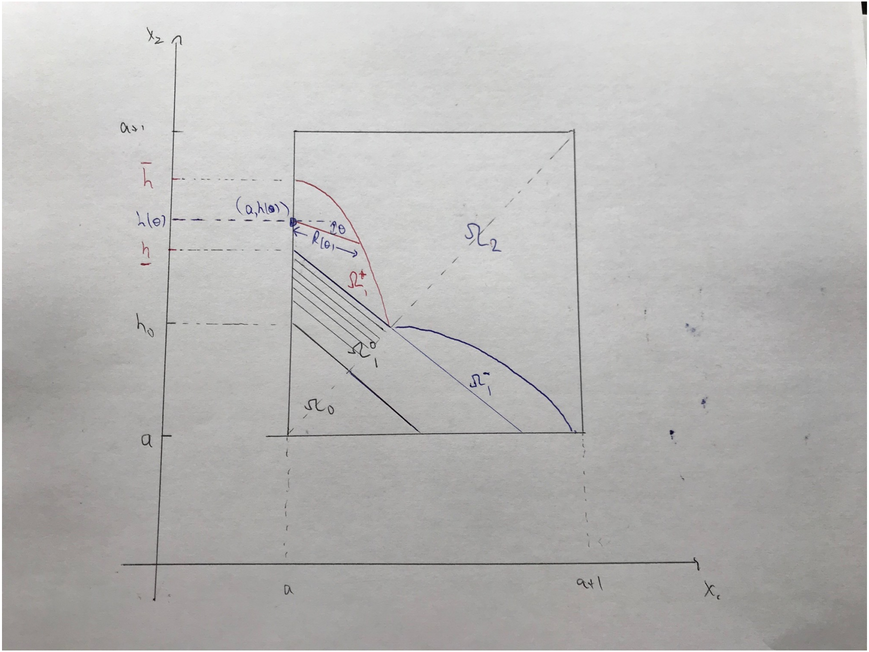
x_2
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$\bar{h}$
$h(\theta)$
$\underline{h}$
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a
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2θ
$R(\theta)$
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a
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Precise Euler-Lagrange equation in the 'missing' region Ω_1^+

Index each isochoice segment in Ω_1^+ by its angle $\theta \geq \theta_0 \in [-\frac{\pi}{4}, 0)$ to horizontal. Let $(a, h(\theta))$ denote its left-hand endpoint and parameterize the segment by distance $r \in [0, R(\theta)]$ to $(a, h(\theta))$. Along this segment of length $R(\theta)$,

$$u_1^+\Big((a, h(\theta)) + r(\cos\theta, \sin\theta)\Big) = m(\theta)r + b(\theta).$$

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$$\frac{3}{2}R^2(\theta)\cos\theta = [m''(\theta) + m(\theta) - 2R(\theta)](m'(\theta)\sin\theta - m(\theta)\cos\theta + a) \qquad (*)$$

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$$h(t) = \underline{h} + \frac{1}{3}\int_{\theta_0}^{t} [m''(\theta) + m(\theta) - 2R(\theta)]\frac{d\theta}{\cos\theta}, \qquad (3)$$

$$b(t) = \frac{1}{2}k(a + \underline{h})1_{-\pi/4}(\theta_0) + \int_{\theta_0}^{t} (m'(\theta)\cos\theta + m(\theta)\sin\theta)h'(\theta)d\theta. \qquad (4)$$

• for $\underline{h} \in [a, a+1]$, $\theta_0 \in [-\frac{\pi}{4}, 0)$, $R : [\theta_0, \frac{\pi}{2}] \longrightarrow [0, 1)$ Lipschitz (say, and $R(\theta_0) = \frac{1}{\sqrt{2}}(\underline{h} - a)$ if $\theta_0 = -\pi/4$) we can solve (*)–(4) to find Ω_1^+ and u_+^1.

• we can then solve the resulting Neumann problem for $\Delta u_2 = 3$ on Ω_2

• M.–Rankin–Zhang 24+ shows some choice of $\underline{h}$ and $R(\cdot)$ (not known to be Lipschitz) also yields $u_2 - u_1 = const$ on $\partial\Omega_2 \setminus \partial X$,

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WHY IS IT NATURAL FOR SUCH A CHOICE TO EXIST?

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• a unique optimizer $\bar{u} \in \mathcal{U}$ is known to exist (Rochet-Choné) and $\bar{u} \in C^{1,1}_{loc}(X^0)$ (Caffarelli-Lions); if the sets Ω_i where its Hessian is rank i are smooth enough, and Ω_1 has the expected 3 components, then (*)–(4) and the overdetermined Poisson problem $\Delta u_2 = 3$ must be satisfied

• but maybe Ω_i are not finite perimeter, or Ω_1 is not (simply) connected and/or has more than three components (some too small for the numerics to resolve); these possibilities excluded by M.–Rankin–Zhang '24+ (and M.–O'Brien '26+).

The obstacle problem (without convexity constraint)

Blowing-up at the edge of the contact region in the obstacle problem led to

Theorem (Caffarelli's alternative; circa 1980)

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$$w(x_1, \ldots, x_n) = \begin{cases} \frac{1}{2}x_1^2 & \text{if } x_1 > 0 \\ 0 & \text{else.} \end{cases}$$

Corollary

At each point in $\mathbf{R}^n$*, the density of the contact region* $\{w = 0\}$ *is either*

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Corollary

At each point in $\mathbf{R}^n$*, the density of the contact region* $\{w = 0\}$ *is either* 0*,* $\frac{1}{2}$*, or* 1*. On the free boundary, only* 0 *(called 'singular') and* $\frac{1}{2}$ *(called 'regular') occur.*

When is our free boundary Lipschitz? Smooth?

Our problem reduces to an obstacle problem for customization u_2; obstacle is minimal convex extension of u_1 from bunching Ω_1 to $\mathbf{R}^2$; $0 < \Delta(u_2 - u_1) \in L^\infty$

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for some $C > 0$, $\quad 0 \le (Du - x) \cdot \hat{n} \le C\, diam(\tilde{x})$;
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- call $\tilde{x}$ a *ray* if $\mathrm{diam}(\tilde{x}) > 0$, and *stray* if also $(Du - x) \cdot \hat{n} = 0$ for $x \in \tilde{x} \cap \partial X$

Theorem (M.–Rankin–Zhang 24+: Free boundary regularity)

For $X \subset \mathbf{R}^2$ *convex (smooth or polyhedral), apart from stray rays*
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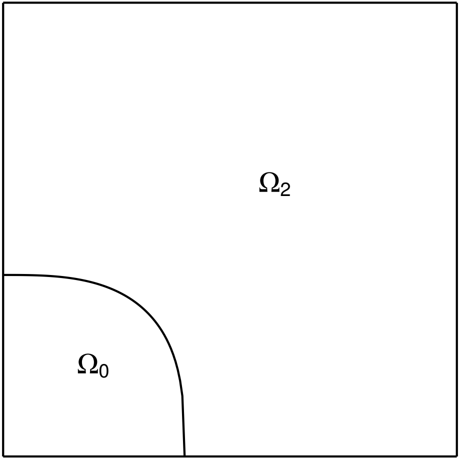
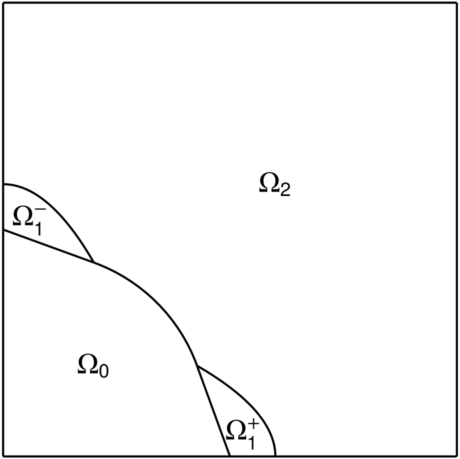
(iii) $\partial\Omega_1 \cap \partial\Omega_2$ *is locally Lipschitz* $\Leftrightarrow$ *each (accumulation point of) local maxima of* $diam(\tilde{x})$ *is regular (not singular) in the Caffarelli alternative*
(iv) $diam(\tilde{x})$ *is smooth wherever it is locally Lipschitz in* $\{(Du - x) \cdot \hat{n} > 0\}$
(v) if $X = [a, a+1]^2$ *then* $diam(\tilde{x})$ *is unimodal, hence there are no stray rays.*

Transition first to targeted and then to blunt bunching

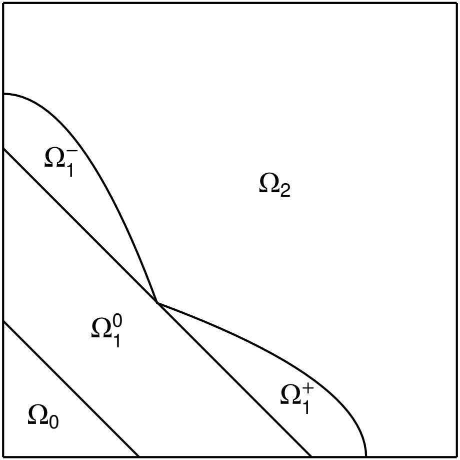
- No bunching (apart from exclusion): if $a = 0$ then $\Omega_1 = \emptyset$



Targeted bunching: if $0 < a \ll 1$ then $\Omega_1^0 = \emptyset \neq \Omega_1^\pm$ (and small)



- **Blunt bunching:** if $a \geq 7/2 - \sqrt{2}$ then $\Omega_1^0 \neq \emptyset \neq \Omega_1^{\pm}$



Ingredients of proof

Recall: Caffarelli-Lion's '06+ assert $u \in C^{1,1}_{loc}(X^0)$.

- we extend this estimate to the edges of square (and corners of $\Omega_1^{\pm}$)
- shows on tame rays, the coordinates $x(r, \theta)$ are biLipschitz
- on customization region Ω_2 have $\Delta u = 3$.
- on Ω_1 return to variational analysis of $\min\{L(u) \mid 0 \le u \text{ convex}\}$ where

$$L(u) = \int_{[a,a+1]^2} \left(\frac{1}{2}|Du - x|^2 + u - \frac{|x|^2}{2} \right) d\mathcal{H}^2(x)$$

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recalling

$$L'_u(w) := \frac{d}{d\epsilon}\bigg|_{\epsilon=0^+} L(u + \epsilon w) = \int_X w \frac{\delta L}{\delta u}$$

i.e. $w \geq 0$ convex implies $\int w d\sigma \geq 0$ for

$$d\sigma = \frac{\delta L}{\delta u} = (3 - \Delta u) d\mathcal{H}^2|_X + (Du - x) \cdot \hat{n} d\mathcal{H}^1|_{\partial X}.$$

Thus positive and negative parts of σ are in convex order: $\sigma^-(w) \leq \sigma^+(w)$;
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Rochet-Choné '98: convex order inherited by $\tilde{\sigma}$-a.e. conditional measure: $\sigma^-_{\tilde{x}}(w) \leq \sigma^+_{\tilde{x}}(w) \forall w$ convex. Thus $\sigma^\pm_{\tilde{x}}$ have the same mass & center of mass; get $\sigma^+_{\tilde{x}}$ from $\sigma^-_{\tilde{x}}$ by sweeping / balayage / mean-preserving spreads / Martingales if $0 \notin \tilde{x}$ (Cartier-Fell-Meyer '56).

- In the blunt region $x \in \Omega^0_1$, this tells uniform negativity of $d\sigma_{\tilde{x}}(r) \sim -dr$ over the segment interior is balanced by positive Dirac masses at the endpoints.
- In the targeted region $x \in \Omega^+_1$, it tells $d\sigma_{\tilde{x}}(r) \sim (3r - 2R)dr$ increases affinely in $0 < r < R(\theta)$, balancing a positive Dirac mass at $r = 0$.

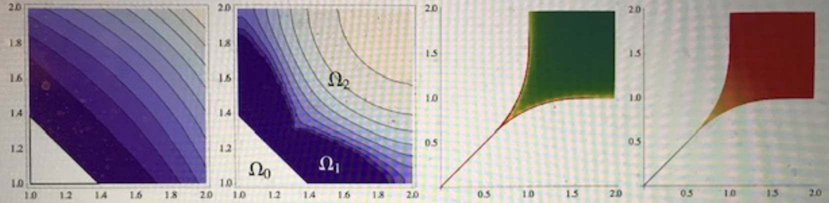

Fig. 1 Numerical approximation U of the solution of the classical Monopolist's problem (1), computed on a 50×50 grid. *Left* level sets of U, with $U = 0$ in *white*. *Center left* level sets of $\det(\nabla^2 U)$ (with again $U = 0$ in *white*); note the degenerate region Ω_1 where $\det(\nabla^2 U) = 0$. *Center right* distribution of products sold by the monopolist. *Right* profit margin of the monopolist for each type of product (margins are low on the one dimensional part of the product line, at the *bottom left*). *Color scales* on Fig. 10 (color figure online)

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THM: Away from corners, (r, θ) are biLipschitz coordinates.

Now $x(r, \theta) = (a, h(\theta)) + r(\cos\theta, \sin\theta)$ and $u_1^+(x) = m(\theta)r + b(\theta)$ yield

$$\text{Jacobians} \qquad d\mathcal{H}^2|_X = |h' \cos\theta + r| dr d\theta$$

$$d\mathcal{H}^1|_{\partial X} = |h'(\theta)| d\theta$$

$$\text{Laplacian} \qquad \Delta u = \frac{m'' + m}{h' \cos\theta + r}$$

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$$\text{so} \qquad -d\sigma = -\frac{\delta L}{\delta u} = (\Delta u - 3) d\mathcal{H}^2|_X - \hat{n}\cdot(Du - x) d\mathcal{H}^1|_{\partial X}.$$

factors into conditional measures (on $\tilde{x}$ with slope $\tan\theta$) given by

$$\mp d\sigma_{\tilde{x}} = [m'' + m - 3(h'\cos\theta + r) - \hat{n}(x)\cdot(Du - x)h'(\theta)\delta_0(r)]dr$$

- the last term represents a point mass where the segment $\tilde{x}$ intersects ∂X

$$\mp \frac{d\sigma_{\tilde{x}}}{dr} = m'' + m - 3(h' \cos\theta + r) - \hat{n}(x) \cdot (Du - x) h'(\theta) \delta_0(r)$$

Since $\sigma_{\tilde{x}}^- \leq \sigma_{\tilde{x}}^+$ in convex order, $\int_0^R w d\sigma_{\tilde{x}} = 0$ for $\pm w(r) \in \{1, r\}$ yields

$$[m'' + m - 3h' \cos\theta]R - \frac{3}{2}R^2 \quad = \quad \hat{n}(x) \cdot (Du - x) h'(\theta) \tag{5}$$

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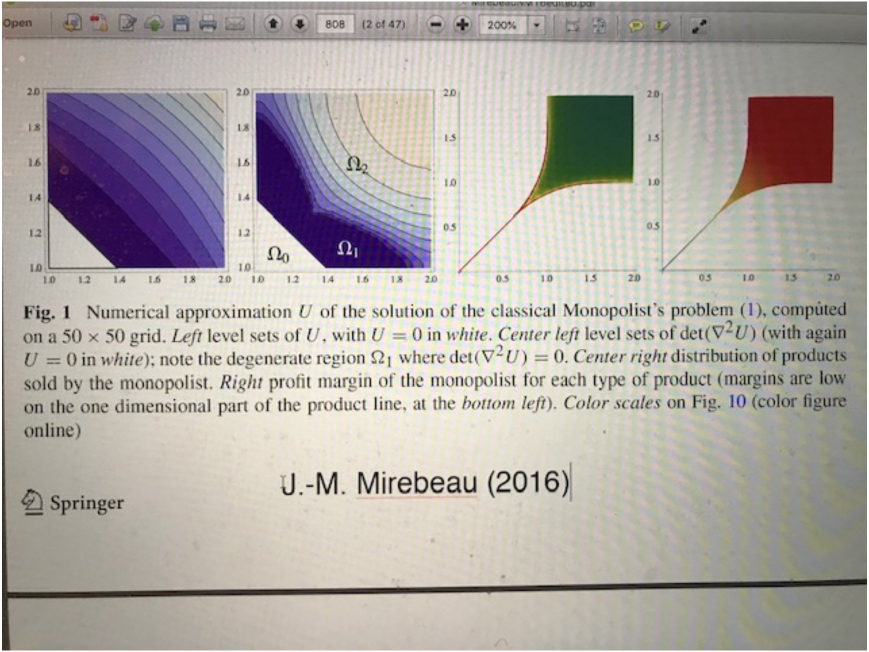
Choosing $w(r)$ strictly convex shows $\sigma^+_{\tilde{x}}$ must be obtained from $\sigma^-_{\tilde{x}}$ by mean-preserving spread; hence the point mass is in $\sigma^+_{\tilde{x}}$ not $\sigma^-_{\tilde{x}}$. From (5)-(6),

$$0 \le \frac{1}{2}R(\theta)^2 = \hat{n}(x)\cdot(Du - x)h'(\theta). \quad (7)$$

Rays spread as they leave the boundary! Hence $\frac{d\mathcal{H}^1|_{\partial X}}{d\theta} = |h'(\theta)| = +h'(\theta) \ge 0$. Also $R > 0$ implies point mass (7)$\neq 0$ hence $0 \neq \Delta u - 3 = \frac{2R-3r}{h'\cos\theta + r}$.



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$$e(\theta) := y_2 = \frac{\partial u}{\partial x_2} = m' \cos\theta + m \sin\theta$$

$$f(\theta) := a - y_1 = \hat{n} \cdot (Du - x) = (m' \sin\theta - m \cos\theta + a).$$

Substituting $h' = \frac{R^2}{2f}$ from (7) in (6) yield our ODE for m in terms of R:

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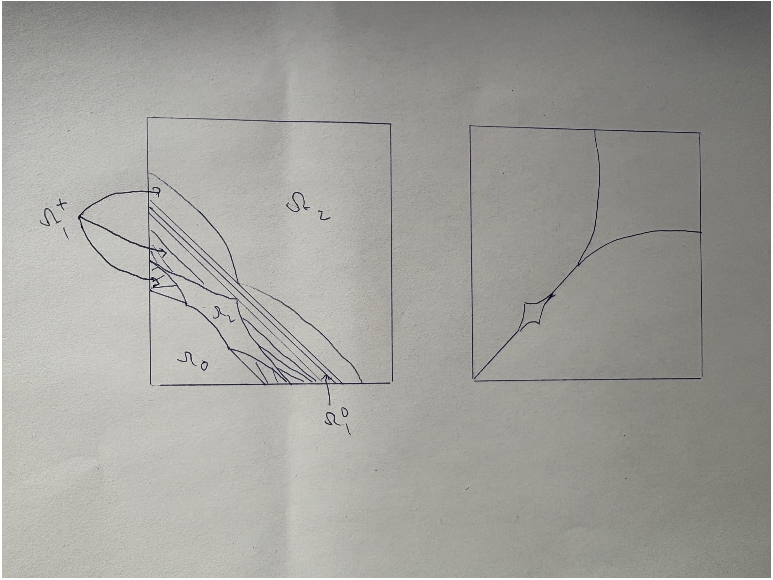
$$\text{Also} \qquad -\frac{dy_1}{dy_2} = \frac{df}{de} = \frac{f'(\theta)}{e'(\theta)} = \tan\theta < 0$$

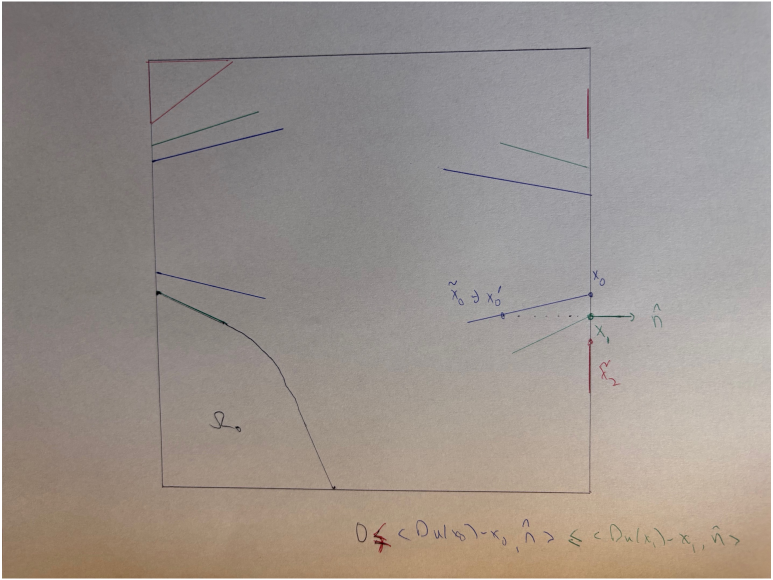
which shows $-\frac{1}{\tan\theta}$ gives the slope of the boundary of the products consumed. This boundary is convex since

$$-\frac{d^2y_1}{dy_2^2} = \frac{d^2f}{de^2} = -\frac{1}{e'(\theta)}\frac{d\tan\theta}{d\theta} = -\frac{1}{(m''+m)\cos^3\theta} < 0.$$

Shows Ω_1^+ must be connected:



$(a, a) \in \Omega_0 \not\ni (a, a+1)$; top and right boundaries $\subset \Omega_2$



rays intersecting top or right boundaries ruled out by

$$\begin{aligned} 0 &\le \Delta x \cdot \Delta y \\ &= (x_1 - x_0') \cdot (y_1 - y_0) \\ &= s \times \hat{n} \cdot (Du(x_1) - Du(x_0)) \end{aligned}$$

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so the length of intersection of Ω_0 with the bottom of the square is $\ell < \sqrt{\frac{2}{3}} < 1$

Beyond this stylized example

- other (convex) domains $X \subset \mathbf{R}^2$

- nonuniform agent densities $d\mu(x) = f(x) d\mathcal{H}^n(x)$ on X;
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- other quasilinear and nonquasilinear agent preferences $b(x, y) \neq x \cdot y$
(assuming fourth-order Ma-Trudinger-Wang type conditions)

- higher dimensions $X \subset Y = [0, \infty)^n$ (starting with $X = [a, a+1]^3$)
(n free boundaries separating $n+1$ regions $\Omega_0, \ldots, \Omega_n$ with complicated Euler-Lagrange PDEs)

Thanks to the audience. . .

Thanks to the audience… and the organizers!

A regularity result: Lipschitz product selection

Theorem (M.-Rankin-Zhang '23+)

If b and $\tilde{b}(y,x) = b(x,y)$ both satisfy (B0-B3), c satisfies (C0-C2) and $d\mu(x) = fdx$ with $\log f \in C^{0,1}$ then $u \in C^{1,1}_{loc}(X^0)$.

- extends Caffarelli-Lions '06+ to b & c non-quadratic
- improves Chen '13 from C^1_{loc} to $C^{1,1}_{loc}$
- sharp: examples for $n = 1 = m$ show $u \notin C^2_{loc}(X^0)$
- idea: use energetic comparison to pinch u between parabolas

Lemma (A geometric lemma)

Given $d > 0$*, there exists* $C_0, C_1, C_2 > 0$ *such that if* $u = u^{\tilde{b}b}$ *is optimal and* $d(x_0, \partial X) > d$ *and* $y_0 = \bar{y}_b(Du(x_0), x_0)$ *then if* $r < C_0$ *and*

$$h = \sup_{x \in B_r(x_0)} u(x) - [u(x_0) + b(x, y_0) - b(x_0, y_0)] > 0$$

then some $A(\cdot) = b(\cdot, y') + a'$ *makes* $S := \{x \in X \mid u < A\}$ *a neighbourhood of* x_0 *with*

$$\sup_{x \in S} A(x) - u(x) \leq h$$

and

$$\frac{1}{|S|} \int_S \Big[c(y) - b(x, y) \Big]_{y=y'}^{y=\bar{y}(Du(x),x)} f(x) dx \geq -C_1 h + C_2 \frac{h^2}{r^2}.$$

Proof:

A new duality for bilinear preferences

Following Rochet-Choné '98 choose $b(x, y) = x \cdot y$ and $X, Y \subset \mathbf{R}^n$ convex so profit

$$-L(u) = \int_X [x \cdot Du - u(x) - c(Du(x))] d\mu(x)$$

with

$$u(x) = v^*(x) := \sup_{y \in Y} x \cdot y - v(y)$$

$$\in \mathcal{U} := \{u : X \longrightarrow [0, \infty] \; convex \mid Du(X) \subset Y\}$$

THM (M.-Zhang, to appear in M3AS) Y a convex cone; c.f. Kolesnikov-Sandomirskiy-Tsyvinski-Zimin 22+ on Beckmann auctions):

$$\max_{u \in \mathcal{U}} -L(u) =$$

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Proof: Rockafellar-Fenchel duality; ($\leq$): $S \in \mathcal{S}$, $u \in \mathcal{U}$ and definition of c^*

$$-L(u) = \langle x \cdot Du(x) - u - c(Du(x)) \rangle_\mu \leq \cdots \leq \langle c^* \circ S \rangle_\mu$$

□

- gives new necessary and sufficient criterion for optima