Ironing, Sweeping, and Multidimensional Screening

Jean-Charles Rochet; Philippe Chone

Econometrica, Vol. 66, No. 4 (Jul., 1998), 783-826.

Econometrica, Vol. 66, No. 4 (July, 1998), 783–826

IRONING, SWEEPING, AND MULTIDIMENSIONAL SCREENING

BY JEAN-CHARLES ROCHET AND PHILIPPE CHONÉ[1]

We provide existence proofs and characterization results for the multidimensional version of the multiproduct monopolist problem of Mussa and Rosen (1978). These results are also directly applicable to the multidimensional nonlinear pricing problems studied by Wilson (1993) and Armstrong (1996). We establish that bunching is robust in these multidimensional screening problems, even with very regular distributions of types. This comes from a strong conflict between participation constraints and second order incentive compatibility conditions. We consequently design a new technique, the sweeping procedure, for dealing with bunching in multidimensional contexts. This technique extends the ironing procedure of Mussa and Rosen (1978) to several dimensions. We illustrate it on two examples: we first solve a linear quadratic version of the bidimensional nonlinear pricing problem, where consumers' types are exponentially distributed. The solution involves pure bundling for consumers with low demands. The second example is the bidimensional version of the Mussa and Rosen problem when consumers' types are uniformly distributed on a square. The solution is such that the seller offers a full differentiation of products in the upper part of the qualities spectrum, but only limited choice for lower qualities. This seems to be a quite general pattern for multidimensional screening problems. The sweeping procedure is potentially applicable to other multidimensional screening problems.

KEYWORDS: Screening, nonlinear pricing, adverse selection, incentives, bundling.

1. INTRODUCTION

THIS ARTICLE OFFERS A SYSTEMATIC ANALYSIS of a particular class of multidimensional screening models, which are natural extensions of the nonlinear pricing models studied by Mussa and Rosen (1978), Roberts (1979), Spence (1980), Maskin and Riley (1984), and Wilson (1993). We show that the solutions differ markedly from their one-dimensional counterparts, essentially because "bunching"[2] cannot be ruled out easily. We design a new technique, the "sweeping" procedure, for dealing with this difficulty. This new technique is potentially applicable to many other multidimensional screening problems.

[1] This is a revision of a manuscript of the first author only (January, 1995) with the same title. A previous version (1992) circulated under the title "Optimal Screening of Agents with Several Characteristics." We benefited from the detailed comments and suggestions of a co-editor and three referees, as well as seminar participants in Chicago, Montréal, Stanford, Toulouse, and a workshop in Bonn, and offer particular thanks to Bruno Jullien, Jean-Jacques Laffont, Jean Tirole, Bob Wilson, and Preston McAfee (who gave us the idea for the title). Patrice Loisel was very helpful in providing several numerical solutions. The usual disclaimer applies.

[2] In the jargon of incentives theory, "bunching" refers to a situation where a group of agents (here, buyers) having different types are treated identically in the optimal solution.

1.1. *A Brief Survey of the Literature*

The analysis of optimal screening of agents with unknown characteristics has been the subject of a large theoretical literature in the last twenty years. This is partly justified by the great variety of contexts to which such an analysis can be applied. Indeed, one of the merits of this theoretical literature has been to show that such diverse questions as nonlinear pricing, product lines design, optimal taxation, regulation of public utilities..., could be handled within the same framework.

Although it is often recognized that agents typically have several characteristics and that principals typically have several instruments, this problem has most of the time been examined under the assumption of a single characteristic and a single instrument. In this case, several qualitative results can be obtained with some generality:

(i) When the single-crossing condition is satisfied, only local (first and second order) incentive compatibility constraints can be binding.

(ii) In most problems, the second order (local) incentive compatibility constraints can be ignored, provided that the distribution of types is not too irregular.

(iii) If it is the case, bunching is ruled out and the optimal solution can be found in two steps:

(a) First compute the expected rent of the agent as a function of the allocation of (nonmonetary) goods.

(b) Second, find the allocation of goods that maximizes the surplus of the principal, net of the expected rent computed above.

Considering the typical multiplicity of characteristics and instruments in most applications, it is important to know if the results above can be extended to multidimensional contexts. This question has already been examined in the signaling literature: Kohleppel (1983) and Wilson (1985) have considered examples of multidimensional signaling equilibria, while Quinzii and Rochet (1985) and Engers (1987) have obtained existence results. In the screening literature, the question has been essentially considered in two polar cases: one instrument—several characteristics (Laffont, Maskin, and Rochet (1987), Lewis and Sappington (1988), Srinagesh (1991)) and one characteristic—several instruments (Matthews and Moore (1987)). In the first case, it is of course impossible to obtain perfect discrimination among agents. The same level of the instrument (for example the quantity produced by a regulated firm) is chosen by many different agents (in Lewis and Sappington (1988) this corresponds to firms having different costs and demands). However Laffont, Maskin, and Rochet (1987) and Lewis and Sappington (1988) show that it is possible to aggregate these types and reason in terms of the average cost function of all firms having chosen the same production level. Once this aggregation has been performed, the problem resembles a one-dimensional problem, in which types have been aggregated as above. Of course the situation is more complex, since this aggregation procedure is endogenous.

The second case, that of several instruments and one characteristic is very different. For instance Matthews and Moore (1987) extend the Mussa and Rosen model by allowing the monopolist to offer different levels of warranties as well as qualities. One of their most striking results is that the allocation of qualities is not necessarily monotonic with respect to types. As a consequence, nonlocal incentive compatibility constraints may be binding at the optimum.

The most interesting (and probably most difficult) case is the one with several instruments and several characteristics. It has been examined in relatively few papers: Seade (1979) has studied the optimal taxation problem for multidimensional consumers and has shown that it was equivalent to a calculus of variations problem with several variables. Rochet (1984) has studied an extension of the Baron and Myerson (1982) regulation problem to a bidimensional context, where both marginal cost and fixed cost are unknown to the regulator. Contrary to Lewis and Sappington, Rochet allows for stochastic mechanisms (as Baron and Myerson did): this provides additional flexibility to the regulator. On a particular example, Rochet (1984) shows that the optimal mechanism can indeed be stochastic, as conjectured by Baron and Myerson. McAfee and McMillan (1988) have provided a decisive step in the study of nonlinear pricing under multidimensional uncertainty. They introduce a "Generalized Single Crossing Condition" under which incentive compatibility constraints can be replaced by the local (first and second order) conditions of the agents' decision problems. Then they also show that the optimal screening mechanism can be obtained as the solution of a calculus of variations problem. Wilson (1993) contains a very original and almost exhaustive treatment of nonlinear pricing models: in particular several multidimensional examples are solved. Finally, Armstrong (1996) exemplifies some of the difficulties involved in the multidimensional nonlinear pricing problem. He shows that the participation constraint typically binds for a set of consumers with positive measure, and gives closed form solutions for several parametric examples. The present article builds directly on Wilson (1993) and Armstrong (1996).

1.2. *Outline of the Paper*

We start in Section 2 by specifying a particular class of multidimensional screening problems which contains two examples of interest: the multidimensional extension of the Mussa and Rosen (1978) article on product line design by a monopolist, and the multiproduct nonlinear pricing problem studied by Wilson (1993) and Armstrong (1996). Section 3 contains a heuristic presentation of our results. We proceed in Section 4 by studying the simpler problem (which we call the "relaxed" problem $\mathscr{P}^*$) in which the second order conditions of consumers' choice are neglected. We give a formal proof of existence and uniqueness of the solution (Theorem 1), and an economic interpretation of its characterization (Theorem 2). As a corollary, we find a simple necessary condition on the distribution of consumers' types for this solution to be admissible in the "complete" problem $\bar{\mathscr{P}}$ (i.e. for the second order conditions of consumers' choice not to be binding). This condition is very restrictive: except for some specific situations (like the ones that Wilson (1993) and Armstrong (1996) have solved explicitly), the second order conditions of the consumers' program will be violated. The economic interpretation is that bunching is present in the solution of most multidimensional screening problems. This comes from a strong conflict between participation constraints and second order incentive compatibility conditions, which are neglected in the relaxed problem. This is explained in Section 5.

We therefore proceed (in Section 6) to the study of the complete problem $\bar{\mathscr{P}}$, where the possibility of bunching is explicitly taken into account. We establish existence and uniqueness of the solution to $\bar{\mathscr{P}}$ (Theorem 1 bis) and give an economic interpretation of its characterization (Theorem 2 bis). This characterization involves a new technique, the "sweeping procedure," which generalizes the ironing procedure, invented by Mussa and Rosen (1978)[3] for dealing with unidimensional bunching. In Section 7 we illustrate the use of the sweeping procedure by solving two examples. These examples give interesting insights on the economic constraints faced by multiproduct firms when designing their product lines and price schedules in an adverse selection context. Section 8 concludes.

2. STATEMENT OF THE PROBLEM

2.1. *The Model*

We consider a multidimensional extension of the Mussa and Rosen (1978) model, in which a multiproduct monopolist sells indivisible goods to a heterogenous population of consumers, each buying at most one unit. Utilities are quasilinear:

(2.1) $$U = u(t,q) - p.$$

Here, t is a K dimensional vector of characteristics of the consumer (his "type"), q is a K dimensional vector of attributes of the good ("qualities"), and p is the price of the good. The monetary equivalent of one unit of the good is represented by $u(t,q)$ with a vector of characteristics q and for a consumer of type t. For simplicity, we assume that the technology exhibits constant returns to scale: the unit cost of producing product q is $C(q)$. Notice that q and t have the same dimensionality, so that perfect screening (a pattern in which consumers with different types always buy different goods) is not ruled out by dimensionality considerations. We assume that the surplus function $S(t,q)$, defined by

(2.2) $$S(t,q) = u(t,q) - C(q),$$

[3] The ironing procedure has been applied to more general unidimensional situations by Guesnerie and Laffont (1984).

is strictly concave, twice continuously differentiable, and has for all t a unique maximum. In that sense, product differentiation is "fundamental": it is not a consequence of monopoly power, since it is present in the first best solution. For technical reasons that we explain below, we will have to assume that the parameterization of preferences is linear in types:

$$u(t,q)=\sum_{k=1}^{K} t_k v_k(q),$$

where $v(\cdot)=(v_1(\cdot),\dots,v_K(\cdot))$ is a one-to-one mapping. Up to a redefinition of the vector of attributes q and the cost function $C(q)$, we can, without further loss of generality, take

$$v_k(q)=q_k \qquad (k=1,\dots,K),$$

which gives us a bilinear specification:

$$u(t,q)=\sum_{k=1}^{K} t_k q_k = t\cdot q. \tag{2.3}$$

Such a parameterization is used by Wilson (1993) and Armstrong (1996) in the different context of nonlinear pricing. For example, q may be a utility vector $(q_1(x),\dots,q_K(x))$ where x is a vector of *quantities* (i.e. a bundle of different goods). Denoting by $x(q)$ the (unique) bundle that provides the utility vector q, the cost function is then

$$C(q)=\sum_{k=1}^{K} C_k x_k(q),$$

where C_k denotes the unit cost of good k, for $k=1,\dots,K$. Another example is the multiproduct linear demand model where customers' benefit functions have the form

$$S(t,q)=t\cdot q-\frac{1}{2}\sum_{k=1}^{K} q_k^2,$$

and the unit costs are normalized to zero. The surplus function therefore coincides with this customer benefit function, and formula (2.2) holds with

$$C(q)=\frac{1}{2}\sum_{k=1}^{K} q_k^2.$$

Although we focus here on the product line interpretation à la Mussa and Rosen, our model is thus compatible with the nonlinear pricing models of Wilson (1993) and Armstrong (1996). This will allow us to compare our results with those obtained by these two authors.

2.2. *The Monopolist's Problem*

The monopolist has to design a product line Q (i.e. a subset of $\mathbb{R}^K$) and a price schedule p (i.e. a mapping from Q to $\mathbb{R}$) that jointly maximize her overall profit,[4] knowing only the statistical distribution of types in the population of potential buyers. We assume that this distribution is continuous on its domain Ω, a convex open bounded subset of $\mathbb{R}_+^K$, with a density function $f(t)$ with respect to the Lebesgue measure on $\mathbb{R}^K$. This density f is $\mathscr{C}^1$ on Ω, and can be extended by continuity to the closure of Ω, which is a convex compact subset of $\mathbb{R}_+^K$, denoted $\overline{\Omega}$. By convention, the total mass of consumers is normalized to 1:

$$(2.4) \qquad \int_\Omega f(t)\, dt = 1.$$

When he buys from the monopolist, a consumer of type t chooses the product $q(t)$ that solves

$$(2.5) \qquad \max_{q \in Q} \{t \cdot q - p(q)\} = U(t).$$

However, consumers of type t can also abstain from buying, in which case they obtain their reservation utility level denoted $U_0(t)$. This reservation utility level is often supposed to be type independent, and normalized to zero.[5] We will deal here with the more general and more interesting case where $U_0(t)$ is associated to an "outside good" (as in Salop (1979)), corresponding to a quality q_0 sold at a price p_0:

$$(2.6) \qquad U_0(t) = t \cdot q_0 - p_0.$$

The total profit of the monopoly is

$$\Pi = \int_{\Omega_1} \{p(q(t)) - C(q(t))\}\mathbf{1}\{U \geq U_0\} f(t)\, dt,$$

where $q(t)$ realizes the maximum in (2.5), and $\mathbf{1}\{U \geq U_0\}$ is the indicator function of the participation set:

$$\{t \in \Omega,\, U(t) \geq U_0(t)\}.^6$$

It is easy to see that the solution $q(t)$ of (2.5) is uniquely defined for a.e. t (see, for instance, Rochet (1987)), so that the value of the integral defining Π is not ambiguous. However the participation set depends on the price schedule in a complex fashion, which may introduce nonconvexities, at least when $p_0 < C(q_0)$

[4] This is only to fix ideas. The case of a regulated monopoly, required to maximize a weighted sum of its profit and consumers' surplus, could be treated in the same fashion (see Wilson (1993) for details).

[5] This is the case, for instance, in Wilson (1993) or Armstrong (1996). A thorough treatment of type dependent participation levels in unidimensional models is, however, to be found in Jullien (1996).

[6] When indifferent, the consumer is supposed to buy from the monopoly. As we will see, this assumption will turn out to be innocuous in our context.

(this happens already in dimension 1; see Jullien (1996) for a discussion). We will therefore consider only the case where $p_0 \geq C(q_0)$, which means that the monopoly does not lose money if he sells the outside good at price p_0. In this case, it is always in the interest of the monopolist to choose $p(\cdot)$ in such a way that all consumers participate, so that

$$(2.7) \qquad U(t) \geq U_0(t) \quad \text{for all } t \text{ in } \Omega.$$

The integral defining Π can then be computed over all Ω:

$$(2.8) \qquad \Pi = \int_{\Omega} \{p(q(t)) - C(q(t))\} f(t)\, dt.$$

The monopolist's problem therefore consists in finding Q and $p(\cdot)$ that maximize Π (given by (2.8)), under the constraints (2.5), (2.6), and (2.7).

2.3. *Two Possible Approaches*

We now discuss the two approaches that can be used for solving this problem. The first approach is due to Wilson (1993): it is direct and very powerful in the one-dimensional case. The second approach, due to Mirrlees (1971), is indirect, more complex, but can be extended easily to multidimensional models.

2.3.1. *The Direct Approach*

If the price schedule p: $Q \to \mathbb{R}$ is "well behaved" (i.e. Q is convex, p is strictly convex and $\mathscr{C}^2$) then the quality assignment $t \to q(t)$ is one-to-one and smooth: $q = q(t) \Leftrightarrow \nabla p(q) = t$, where

$$\nabla p = \left(\frac{\partial p}{\partial q_1}, \ldots \frac{\partial p}{\partial q_K} \right)$$

denotes the gradient of p. Then one can change variables in (2.8) and write

$$(2.9) \qquad \Pi = \int_{Q} [p(q) - C(q)] f(\nabla p(q)) J(q)\, dq,$$

where $J(q) = |D^2 p(q)|$ is the Jacobian of the change of variables. In dimension 1, this gives immediately the solution after integration by parts. Indeed, in this case

$$\Pi = \int_{q_0}^{+\infty} (p(q) - C(q)) f(p'(q)) p''(q)\, dq,$$

which is also equal to

$$(2.10) \qquad \Pi = \int_{q_0}^{+\infty} [p'(q) - C'(q)](1 - F(p'(q)))\, dq,$$

where F is the c.d.f. associated to f. The optimal marginal price $p'(q)$ is therefore obtained by pointwise maximization[7] in (2.10):

$$p'(q) = \arg\max_t (t - C'(q))(1 - F(t)). \tag{2.11}$$

Unfortunately, this direct approach does not generalize easily to multidimensional problems, for two reasons: one cannot get rid of the Jacobian $J(q)$ by integration by parts so that we are stuck with a second order variations calculus problem with a very complex Euler equation; moreover, we will establish below that the optimal price schedule is almost never "well behaved" in the above sense, so that the change of variables is not allowed. Therefore, we have to adopt the dual approach, initiated by Mirrlees (1971) and which we present now.

2.3.2. *The Dual Approach*

It consists in using the indirect utility function $U(t)$ as the instrument chosen by the monopoly. This is justified by the following implementation result:

LEMMA 1 (Rochet (1987)): *Let $U(\cdot)$ and $q(\cdot)$ be defined on Ω, with values respectively in $\mathbb{R}$ and $\mathbb{R}^K$. There exists a product line $Q \subset \mathbb{R}^K$ and a price schedule p: $Q \to \mathbb{R}$ such that $U(t)$ satisfies (2.5) for a.e. t (the maximum being obtained for $q = q(t)$) if and only if:*[8]

$$q(t) = \nabla U(t) \quad \textit{for a.e. } t \textit{ in } \Omega, \tag{2.12}$$

$$U \textit{ is convex continuous on } \Omega. \tag{2.13}$$

$\nabla U(t)$, the gradient of U at t, is defined for a.e. t because of condition (2.13). Thanks to Lemma 1, we can reformulate the monopoly problem: it suffices to replace, in formula (2.8), $q(t)$ by $\nabla U(t)$ and $p(q(t))$ by $t \cdot \nabla U(t) - U(t)$. We obtain a problem of calculus of variations:

$$\max \phi(U) = \int_\Omega \{t \cdot \nabla U(t) - C(\nabla U(t)) - U(t)\} f(t)\, dt$$

under the constraints:

$$U \text{ convex continuous on } \Omega, \tag{2.13}$$

$$U(t) \geq U_0(t) \text{ for a.e. } t \text{ in } \Omega. \tag{2.14}$$

[7] One can check that when $q(\cdot)$ is increasing, the first order condition of this problem gives the familar result:

$$C'(q(t)) = t - \frac{1 - F(t)}{f(t)}.$$

However this condition holds only when there is no bunching, whereas (2.11) is always true.

[8] Without our assumption that $u(t, q)$ is linear in t, such a simple characterization is impossible to obtain (see Rochet (1987) for details), so that a complete solution is out of reach. The generalized single crossing condition of McAfee and McMillan (1988) is slightly more general than linearity, but gives rise to untractable integrability constraints.

If we ignore (2.14), the problem boils down to what is known in physics as an *obstacle problem*, i.e. a problem of calculus of variations with an inequality constraint. This problem, which we call the *relaxed problem* $\mathscr{P}^*$, is studied in Section 4. In Section 5 we show that the solution of this relaxed problem is seldom convex: it typically violates constraint (2.14). The complete problem $\bar{\mathscr{P}}$, which includes constraint (2.14), is much more difficult to handle, and is studied in Section 6. For the moment, we give a heuristic presentation of our results.

3. A HEURISTIC PRESENTATION OF OUR RESULTS

Even if the above problem is very natural from an economic viewpoint, its solution necessitates mathematical tools that are not standard in economics. For the sake of exposition, we have dedicated this section to a heuristic presentation of our results, which are then rigorously established in the rest of the paper. Our starting point is the functional $\phi(U)$ that expresses the monopolist's profit as a function of the buyers' indirect utility function U:

$$\phi(U) = \int_{\Omega} \{t \cdot \nabla U(t) - C(\nabla U(t)) - U(t)\} f(t)\, dt. \tag{3.1}$$

3.1. *The Problem is not Decomposable*

Up to a constant, this profit is also equal to the difference between the expectation of the total surplus $S(t, q(t))$ (with $q(t) = \nabla U(t)$) and the expectation of the buyers' rent $U(t) - U_0(t)$. In dimension one (i.e. when $\Omega = [a, b]$), it is easy to transform the second term, as soon as one knows where the participation constraint is binding. For example in the most familiar situation, $U(a) = U_0(a)$ (i.e. the participation constraint binds "at the bottom") and an integration by parts gives

$$\int_a^b (U(t) - U_0(t)) f(t)\, dt = \int_a^b (q(t) - q_0)(1 - F(t))\, dt. \tag{3.2}$$

The problem is thus decomposable into two subproblems:

(i) Compute expected rents as a function of the product assignment $q(\cdot)$; this is formula (3.2).

(ii) Then choose the products assignment $q(\cdot)$ that maximizes the "virtual" surplus (Myerson (1981)):

$$\int_a^b \{S(t, q(t)) f(t) - (q(t) - q_0)(1 - F(t))\}\, dt.$$

Unfortunately when $K > 1$ (multidimensional setting), this does not work for (at least) two reasons:

(a) Given an arbitrary product assignment $t \to q(t)$, it is not in general possible to find a function $U(t)$ such that $\nabla U = q$; q has to satisfy integrability conditions, which are complex to manage.

(b) Even if q satisfies these integrability conditions, there are an infinity of ways to compute the expected rent. Indeed $[U(t) - U_0(t)]$ can be computed as the integral of $q - q_0$ along *any* path joining Ω_0 (where the rent equals zero) to the point t. Thus there are an infinity of vector fields $t \to \nu(t)$ such that

$$\int_{\Omega} (U(t) - U_0(t)) f(t)\, dt = \int_{\Omega} (q(t) - q_0(t)) \cdot \nu(t)\, dt. \tag{3.3}$$

In more intuitive terms, the problem is that one does not know the direction of the (local) incentive compatibility constraints that will be binding. This is why we cannot (in general)[9] decompose the problem, and have to work directly on $U(\cdot)$ instead of $q(\cdot)$.

3.2. *Characterization of the Solution when there is no "Bunching"*

When there is no bunching (i.e. when different types always get different products) the monopolist's problem reduces to maximizing $\phi(U)$ under the sole constraint $U \geq U_0$. The solution U^* of this (relaxed) problem $\mathscr{P}^*$ is such that $\phi(U^* + \epsilon h) \leq \phi(U^*)$ for any *admissible* function $(U^* + \epsilon h)$. Let the marginal loss in the direction h be defined as

$$L(h) = \lim_{\epsilon \to 0^+} \frac{\phi(U^*) - \phi(U^* + \epsilon h)}{\epsilon} \tag{3.4}$$

(i.e. the opposite of the directional derivative $\phi'(U^*)h$). The first order condition of $\mathscr{P}^*$ expresses that $L(h)$ is nonnegative for any admissible direction h, i.e. for any function h that is nonnegative on the "indifference set" Ω_0, where the consumers' rent equals zero. We establish below (in Section 4) that the marginal loss can be written as the integral of h with respect to some measure μ, supported by $\overline{\Omega}$:

$$L(h) \stackrel{\text{def}}{=} \int_{\overline{\Omega}} h(t)\, d\mu(t).$$

More precisely μ is the sum of a measure on Ω, with a density $\alpha(t)$, and a (singular) measure on $\partial\Omega$, with a density $\beta(t)$ with respect to the Lebesgue measure σ on $\partial\Omega$ (the expressions of α and β are given in Section 4):

$$L(h) = \int_{\Omega} h(t)\alpha(t)\, dt + \int_{\partial\Omega} h(t)\beta(t)\, d\sigma(t). \tag{3.5}$$

The expression of the first order conditions that characterize U^* now becomes very simple (see Theorem 2 in Section 4): the measure μ has to be positive (i.e. $\alpha(t) \geq 0$, $\beta(t) \geq 0$) and supported by Ω_0 (i.e. α and β are equal to zero outside Ω_0). The difficulty is then to find the boundary Γ that separates Ω_0 from its

[9] The only exception of which we are aware is the family of problems solved explicitly by Wilson (1993) and Armstrong (1996). As we explain in Section 5, these examples are not robust.

complement Ω_1. Once Γ has been determined (see Section 4 for details), Ω_0 and Ω_1 are also determined and the value of U^* in Ω_1 is obtained by solving a partial differential equation which expresses two crucial properties of the solution. To interpret them, let us define the marginal distortion vector at t:

$$\nu(t) \overset{\text{def}}{=} \frac{\partial S}{\partial q}(t, q^*(t))f(t). \tag{3.6}$$

(i) *First Property of the Solution*: The boundary of the strict participation set Ω_1 is the union of two sets: Γ, where the participation constraint is binding ($U^* = U_0$) and $\partial\Omega_1 \cap \partial\Omega$ (the "outside" boundary of Ω), where the distortion vector $\nu(t)$ is tangent to $\partial\Omega$. This generalizes the property known as "no distortion at the top" of the unidimensional solution. It can be interpreted as follows: The optimal solution is never distorted in the direction of the outside normal $\vec{n}(t)$ to $\partial\Omega$ because there are no types outside Ω, and therefore there are no binding incentive compatibility constraints in that direction.

(ii) *Second Property of the Solution*: The sum of the marginal variations of ν at t in all directions[10] equals the opposite of $f(t)$:

$$\sum_{i=1}^{K} \frac{\partial \nu_i}{\partial t_i}(t) = -f(t). \tag{3.7}$$

This condition (already discussed in Wilson (1993)), expresses the basic trade-off between rent extraction and surplus formation. In dimension one, this equation can be solved explicitly:

$$\nu(t) = -F(t) + \text{constant},$$

where the constant is determined by the boundary condition discussed above. When $K > 1$, ν can in general only be found numerically.

3.3. *Bunching is Robust in Multiple Dimension*

Armstrong (1996) has shown that when Ω is strictly convex, the indifference set[11] Ω_0 cannot be reduced to a singleton: it has even a nonempty interior, as we show below. We establish that $\mu(\Omega_0)$ equals one (see Proposition 2 in Section 4.3), which immediately implies Armstrong's result. Moreover, we also prove (Proposition 1) that Ω_0 necessarily contains all the boundary points t of $\overline{\Omega}$ that are "*directly exposed to the outside product* q_0" in the sense that the line from $\nabla C(q_0)$ to t does not intersect Ω.[12] The set of such points is denoted by $\partial_-\Omega$ and its convex hull by Ω_-. One of our main results is that whenever Ω_- is large enough, then U^* is not convex and thus differs from the solution $\overline{U}$ of the complete problem $\overline{\mathscr{P}}$, which involves bunching.

[10] The expression $\sum_{i=1}^{K}(\partial\nu_i/\partial t_i)(t)$, known as the divergence of ν at t, is denoted $\operatorname{div}\nu(t)$ and plays a crucial role in the sequel.

[11] While in Armstrong's context, Ω_0 is interpreted as the nonparticipation set, we prefer to interpret it here as the set of types for which all the rent is extracted by the monopoly.

[12] The precise interpretation of this property is given in Section 4.3.

Intuitively, this comes from a conflict between participation constraints and second order incentive compatibility conditions: on the one hand, it is optimal to extract all the surplus from all the types in $\partial_- \Omega$ (who are directly exposed to the outside product); on the other hand, if U^* is convex (as a function) then Ω_0 is also convex (as a set) and therefore has to contain the convex hull Ω_- of $\partial_- \Omega$, which may conflict with the condition $\mu(\Omega_0) = 1$. On a series of examples, we show this is more the rule than the exception, and that the explicit solutions found by Wilson (1993) and Armstrong (1996) (who do not satisfy it) are not robust to a small perturbation of Ω.

3.4. *Characterization of the Solution when there is Bunching*

When there is bunching, the complete characterization of the solution necessitates a complex technique, the sweeping procedure, that we discuss in detail in Section 6. For the moment, we just explore the economic interpretation of the results that we obtain thanks to this technique (Theorem 2 bis). The general pattern of the solution $\overline{U}$ is as follows. Ω is partitioned into three regions:[13]

(i) the indifference region (still denoted Ω_0), where $\overline{U} = U_0$; it is convex, its boundary is now strictly included in $\partial_- \Omega$ (contrary to the case of U^*) but we still have the property $\mu(\Omega_0) = 1$;

(ii) the nonbunching region Ω_1, where $\overline{U}$ is strictly convex and μ is zero; therefore the marginal distortion vector satisfies the two properties described above in part 3.2;

(iii) finally the bunching region Ω_B, itself partitioned into "bunches" $\Omega(q)$ of types who buy the same product q; the measure μ has now a positive part μ_+ and a negative part μ_-. We prove that the restrictions of μ_+ and μ_- to each bunch $\Omega(q)$ have the same mass and the same mean. If one remembers that μ measures the marginal losses associated to variations of consumers' rent, these conditions mean that whenever the specification of product q or its price $P(q)$ are marginally altered around the solution, the total contribution to the firm's profit is zero.

We now explore in details the techniques needed to solve our problem. We start with the relaxed problem $\mathscr{P}^*$.

4. THE RELAXED PROBLEM

In this section, we study the (relaxed) problem where the convexity constraint (2.14) (which can be interpreted as the second order condition of the buyers' program (2.5)) is neglected. This is the traditional way to solve the one-dimensional version of the problem, in which (2.14) boils down to the condition that $t \to q(t)$ is nondecreasing. When there is no "bunching" (i.e., different types always get different products), the relaxed problem and the complete problem

[13] See Figures 6 and 7 in Section 7 for two examples.

have the same solutions. It is therefore natural to try the same procedure in the multidimensional case. This has been done by Wilson (1993) and Armstrong (1996) on several examples. We provide here a systematic treatment by obtaining existence and characterization results for the solution of the relaxed problem.

4.1. *Existence of the Solution*

As already mentioned, the relaxed problem is a problem of calculus of variations with an inequality constraint. Similar problems are called "obstacle problems" in Physics and many existence and regularity results are available (see, for instance, Kinderlehrer-Stampacchia (1980) or Rodrigues (1987)). The appropriate functional space for solving such problems is $H^1(\Omega)$, the space of functions[14] U from Ω to $\mathbb{R}$ such that U and ∇U are square integrable. This space is a Hilbert space for the norm $|U|$ defined by

$$|U|^2 = \int_\Omega (U^2(t) + \|\nabla U(t)\|^2)\, dt. \tag{4.1}$$

The mathematical formulation of our relaxed problem is therefore

$$(\mathscr{P}^*)\begin{cases} \max \phi(U), \\ U \in K^*, \end{cases} \tag{4.2}$$

where

$$\phi(U) = \int_\Omega \{S(t, \nabla U(t)) - U(t)\} f(t)\, dt \tag{4.3}$$

and

$$K^* = \{U \in H^1(\Omega) / U \geq U_0 \text{ a.e.}\}. \tag{4.4}$$

THEOREM 1: *We assume that f is bounded away from zero, that C is $\mathscr{C}^2$ and that its second derivative D^2C has uniformly bounded eigenvalues*:

$$\begin{aligned} &\exists \epsilon > 0 \textit{ and } \exists M > \epsilon \textit{ such that } \forall q\ \forall h \\ &\epsilon \|h\|^2 \leq D^2C(q)(h,h) \leq M\|h\|^2. \end{aligned} \tag{4.5}$$

Then $\mathscr{P}^*$ has a unique solution U^*.

PROOF:[15] See Appendix 1.

[14] Actually, $H^1(\Omega)$ is a set of *equivalence classes* of functions for the relation $U_1 \sim U_2 \Leftrightarrow U_1 = U_2$ almost everywhere on Ω.

[15] Although similar, the obstacle problems encountered in Physics are somewhat simpler, since they typically require $U = U_0$ on the boundary of Ω, which we denote from now on by $\partial\Omega$. This is not the case here, and our Theorem 1 is not a direct corollary of classical results in the calculus of variations. However, the structure of the proof is standard: we first prove that ϕ is concave and continuous on K^*, and that K^* is closed and convex. The difficult part is to prove that ϕ is *coercive*, i.e., that $\phi(U)$ tends to $-\infty$ when $|U|$ tends to $+\infty$. The uniqueness of U^* is easy to establish.

4.2. *The Characterization of U^**

We come now to the characterization of U^*, which relies on the first order condition of $\mathscr{P}^*$. This first order condition expresses that whenever a function h constitutes an *admissible* (marginal) variation of U^*, then it has to generate a marginal loss for the monopoly. The following lemma shows that this marginal loss can be computed as the integral of h for some measure μ on $\overline{\Omega} = \Omega \cup \partial\Omega$. We first need a definition.

DEFINITION 1: The *marginal loss of the monopoly at U^* for a variation h* is the opposite of the directional derivative of the profit function ϕ:

$$(4.6) \qquad L(h) \overset{\text{def}}{=} -\phi'(U^*)h \overset{\text{def}}{=} \lim_{\epsilon \to 0^+} \frac{\phi(U^*) - \phi(U^* + \epsilon h)}{\epsilon}.$$

LEMMA 2: *The marginal loss of the monopoly at U^* for a variation h can be computed as follows*:

$$(4.7) \qquad L(h) = \int_{\Omega} h(t)\alpha(t)\,dt + \int_{\partial\Omega} h(t)\beta(t)\,d\sigma(t),$$

where $d\sigma$ denotes the Lebesgue measure on the boundary $\partial\Omega$, div *denotes the divergence operator,*

$$(4.8) \qquad \alpha(t) = f(t) + \operatorname{div} \nu(t),$$

$$(4.9) \qquad \beta(t) = \nu(t) \cdot \vec{n}(t) f(t).$$

$$\nu(t) = \frac{\partial S}{\partial q}(t, q^*(t)) f(t)$$

is the marginal distortion vector, and $\vec{n}(t)$ is the (outward) normal to $\partial\Omega$.

PROOF: See Appendix 2.

$\alpha(t)$ measures the marginal loss of the seller when the rent of type t buyers increases marginally; formula (4.8) shows that it is the sum of $f(t)$ (direct affect) and $\operatorname{div} \nu(t)$ (indirect effect). The proof of Lemma 2 relies on the *divergence theorem*[16] (see, for instance, Rodrigues (1987)), which asserts that under some regularity conditions

$$(4.10) \qquad \int_{\Omega} \operatorname{div}\{a(t)\}\,dt = \int_{\partial\Omega} a(t) \cdot \vec{n}(t)\,d\sigma(t),$$

where $a(\cdot)$ is a vector field from Ω to $\mathbb{R}^K$, and $\vec{n}(t)$, $\sigma(t)$ are defined as in the above lemma. The last technical tool for characterizing U^* is the following lemma, which relies on the fact that K^* is a convex cone of vertex U_0 (this means that if $U_0 + h$ belongs to K^*, then for any positive number λ, $U_0 + \lambda h$ also belongs to K^*).

[16] The divergence theorem has been used in similar contexts by Mirrlees (1971) and more recently by McAfee and McMillan (1988), Wilson (1993), Armstrong (1996), and Jehiel et al. (1996).

LEMMA 3: *U^* is the maximum of ϕ on K^* if and only if*

$$\forall h \geq 0 \quad \phi'(U^*)h \leq 0,$$
$$U^* - U_0 \geq 0 \quad \text{and} \quad \phi'(U^*)(U^* - U_0) = 0.$$

PROOF: See Appendix 2.

The last condition in Lemma 3 is to be interpreted as a complementarity slackness condition. Using Lemma 3 and formulas (4.7) to (4.9), we can now establish the characterization of U^*.

THEOREM 2: *Under the assumptions of Theorem 1, the unique solution U^* of $\mathscr{P}^*$ (with $q^* = \nabla U^*$) is of class $\mathscr{C}^1$ on $\overline{\Omega}$. It is characterized by the following conditions:*

$$\alpha(t) = f(t) + \operatorname{div} \nu(t) \geq 0,$$
$$a.e. \text{ on } \Omega \text{ (with equality if } U(t) > U_0(t)),$$
$$\beta(t) = -\nu(t)\cdot n(t) \geq 0,$$
$$\sigma - a.e. \text{ on } \partial\Omega \text{ (with equality if } U(t) > U_0(t)),$$

where

$$\nu(t) = \frac{\partial S}{\partial q}(t, q^*(t))\cdot f(t).$$

PROOF: See Appendix 2.

Except for very specific cases (see Section 5) U^* cannot be found explicitly.[17] The main difficulty is to find the shape of the "free boundary:" $\Gamma = \partial\Omega_0 \cap \partial\Omega_1$, which separates the strict-participation region Ω_1 (where $U^* > U_0$) from its complement Ω_0 (where $U = U_0$). In Section 4.3 we will show however that the extremities of Γ can be determined, by using the continuity of $q^*(\cdot)$ up to the boundary of Ω. To go further, numerical solutions are possible, either by discretizing the first order conditions, or by discretizing the economic problem $\mathscr{P}^*$, and solving it by standard (discrete) programming techniques. For example, we have studied the following case: $K = 2$, $\Omega = [a, b]^2$, f uniform, $C(q) = (c/2)(q_1^2 + q_2^2)$, $q_0 = 0$. We have solved numerically the discretized version of $\mathscr{P}_1$, which gives a standard quadratic programming problem, and extended it by linearity to the whole set Ω. We have taken a uniform distribution on a finite $N \times N$ grid. For N large enough (typically, $N \geq 15$), the solution is smooth and satisfies (numerically)[18] the necessary and sufficient condition of Theorem 2. The shape of the solution is represented in Figure 1 for the case $a = 2$, $b = 3$, $c = 1$.

[17] This is not surprising: boundary value problems for partial differential equations almost never have explicit solutions.

[18] A formal proof of convergence of the solution of the discretized problem towards the solution of the continuous problem is given in Choné and Rochet (1997).

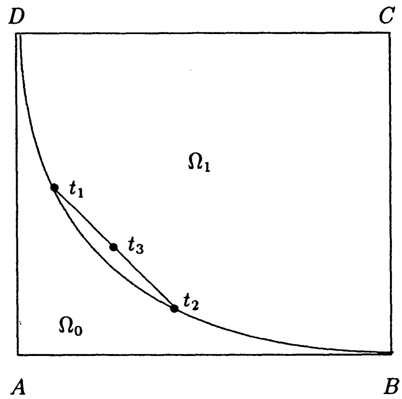

FIGURE 1.—The free boundary of the relaxed problem for a uniform distribution on a square.

We now give an economic interpretation of the conditions found in Theorem 2.

4.3. *The Economic Interpretation of the Characterization of* U^*

The divergence theorem has thus allowed us to compute two functions, α (defined on Ω) and β (defined on $\partial\Omega$) which determine at each point of $\bar{\Omega} = \Omega \cup \partial\Omega$ the local net cost of adverse selection (in terms of foregone profit) for the monopolist. These functions measure the marginal loss for the monopolist due to a local increase of the consumers' informational rent.

Theorem 2 establishes that, for the optimal solution U^*, this marginal loss is always nonnegative, and is zero anywhere the participation constraint does not bind (i.e. on Ω_1). Thus on this participation set Ω_1, U^* satisfies a partial differential equation (the Euler equation of our problem of calculus of variations):

$$\text{(4.11)} \qquad \operatorname{div}\left\{ \frac{\partial S}{\partial q}(t, q^*(t)) f(t) \right\} + f(t) = 0,$$

which expresses the trade-off between surplus maximization and rent extraction. This equation has a unique solution that satisfies also the (mixed) boundary conditions:

$$\text{(4.12)} \qquad \frac{\partial S}{\partial q}(t, q^*(t)) \cdot \vec{n}(t) = 0 \quad \text{on } \Omega_1 \cap \partial\Omega \quad \text{(no distortion at the “top”)}$$

and

$$\text{(4.13)} \qquad U(t) = U_0(t) \quad \text{on } \Gamma$$

(binding participation constraint at the "bottom").

4.3.1. *The Geometry of the Participation Constraint*

Let us consider the indifference set Ω_0. When Ω_0 has a nonempty interior (a robust situation, by Armstrong's (1996) result), we have by construction $U(t) = t \cdot q_0 - p_0$, $q^*(t) = q_0$, for t in Ω_0. For convenience we will denote $\nabla C(q_0)$ by t_0. t_0 corresponds to the type of consumers for which q_0 is the first best choice. The position of t_0 with respect to Ω will play an important role in the determination of $q^*(\cdot)$. On Ω_0 the first order conditions become inequalities:

(4.14) $\quad \forall t \in \Omega_0, \quad \alpha(t) = \operatorname{div}\{(t - t_0)f(t)\} + f(t) \geq 0.$

(4.15) $\quad \forall t \in \partial\Omega_0 \cap \partial\Omega, \quad \beta(t) = (t_0 - t)\cdot \vec{n}(t) \geq 0.$

Condition (4.14) does not necessarily restrict Ω_0, since it can be seen as a regularity condition of the density function f. It is automatically satisfied, for instance, if f is uniform, or close to a uniform distribution in the $\mathscr{C}^1$ topology. It generalizes a previous condition, used by McAfee and McMillan (1988) in their analysis of the "bundling" problem.

On the other hand, condition (4.15) *cannot* be satisfied for all t in $\partial\Omega$, at least when $t_0 \notin \overline{\Omega}$. It is a *geometric* condition, linked to the relative positions of t_0 and Ω. It imposes strong restrictions on the boundary of the nonparticipation region Ω_0. To understand its economic meaning, let us recall that for $t_1 \in \partial\Omega$ the convex set Ω is entirely contained in the half space $(t - t_1)\cdot \vec{n}(t_1) < 0$ (see Figure 2, case 1). Now condition (4.15) means exactly that, when t_1 belongs to the no participation region Ω_0, then necessarily t_0 does *not* belong to this half space: t_0 and Ω are separated by the tangent hyperplane at t_1. In other words, the boundary type t_1 is *directly exposed to the attraction of the outside good* q_0 (materialized in the type space by $t_0 = \nabla C(q_0)$). On the contrary, when condition (4.15) is not satisfied, there are other types in Ω that are more directly exposed to the attraction by q_0 (see Figure 2, case 2).

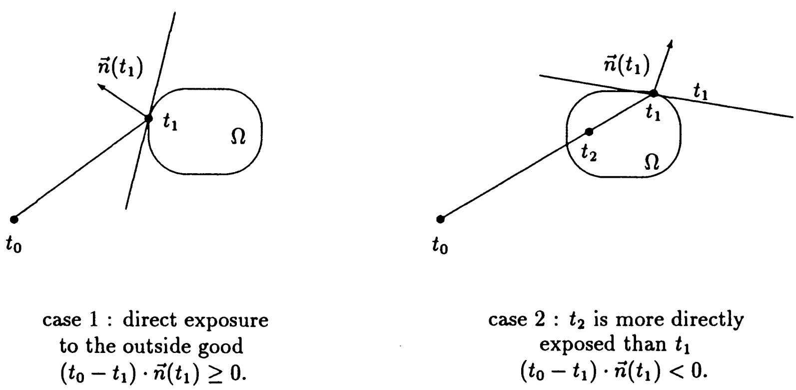

FIGURE 2.—The notion of direct exposure to the outside good.

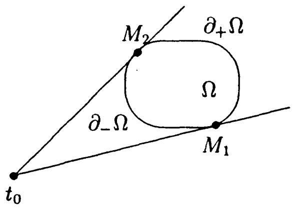

FIGURE 3.—The partition of $\partial\Omega$ when Ω is a smooth and strictly convex subset of $\mathbb{R}^2$.

This motivates the partition of $\partial\Omega$ into two sets:

$$\partial_-\Omega \overset{\text{def}}{=} \{t \in \partial\Omega, (t_0 - t)\cdot\vec{n}(t) \geq 0\},$$

the set of boundary points that are directly exposed to the outside good, and its complement,

$$\partial_+\Omega = \{t \in \partial\Omega, (t_0 - t)\cdot\vec{n}(t) < 0\}.$$

For example, if Ω is a smooth and strictly convex subset of $\mathbb{R}^2$ (and if $t_0 \notin \overline{\Omega}$), there are exactly two points M_1 and M_2 in $\partial\Omega$ for which $(t_0 - t)\cdot\vec{n}(t) = 0$. These two points separate $\partial_-\Omega$ and $\partial_+\Omega$, as in Figure 3.

It can be proved that Γ necessarily passes through M_1 and M_2.[19] More generally, let us define

$$\partial_0\Omega = \partial\Omega \cap \partial\Omega_0,$$

the "outside" boundary of the indifference region, and similarly for Ω_1,

$$\partial_1\Omega = \partial\Omega \cap \partial\Omega_1,$$

which is also the "outside" boundary of Ω_1. We have the following proposition.

PROPOSITION 1: *When Ω is smooth, strictly convex, and $t_0 \notin \overline{\Omega}$, the outside boundary of Ω_0 consists exactly of the boundary points of Ω that are directly exposed to the outside good*:

$$\partial_0\Omega = \partial_-\Omega.$$

As a consequence, we also have that

$$\partial_1\Omega = \partial_+\Omega.$$

PROOF: Condition (4.15) implies that $\partial_0\Omega$ is included in $\partial_-\Omega$. Moreover, Theorem 2 asserts that U^* is $\mathscr{C}^1$ on $\overline{\Omega}$. This implies that $\beta(t) = (\nabla C(q^*(t)) - t)\cdot\vec{n}(t)f(t)$ is continuous on $\partial\Omega$. Now on $\partial_0\Omega$ $\beta(t) = (t_0 - t)\cdot\vec{n}(t)f(t)$ and, by the first order condition, equals zero on the complement of $\partial_1\Omega$.

[19] In the example of Figure 1, Ω is not smooth (because of the corners) but there is no ambiguity: M_1 and M_2 correspond precisely to the corners B and D.

By continuity of $\beta(\cdot)$, it is therefore also equal to zero at all the limit points of $\partial_1 \Omega$ (in our example, M_1 and M_2) which means that $\partial_0 \Omega$ actually coincides with $\partial_- \Omega$ (and that $\partial_1 \Omega = \partial_+ \Omega$). *Q.E.D.*

4.3.2. *The Measure of the Indifference Region*

The last important consequence of Theorem 2 is that Ω_0 cannot be too small. More specifically, we have the following proposition.

PROPOSITION 2: *Let* μ *be the measure associated to* U^* *by* (4.7), (4.8), *and* (4.9), *and let* $\Omega_0 = \{t, U^*(t) = U_0(t)\}$ *be the indifference region. Then we have*

$$\mu(\Omega_0) = 1. \tag{4.16}$$

PROOF: First let us remark that by definition of μ and ϕ we have $\mu(\Omega) = -\phi'(U^*)\mathbf{1} = 1$. Now, Theorem 2 establishes that μ is supported by Ω_0, which proves (4.16). *Q.E.D.*

We will show in Section 5 that condition (4.16) is also satisfied by the solution $\bar{U}$ of the complete problem (using of course the measure μ and the set Ω_0 that correspond to $\bar{U}$). We will also give an interesting interpretation of this condition.

5. BUNCHING IS ROBUST IN MULTIDIMENSIONAL SCREENING PROBLEMS

As we already discussed, Armstrong (1996) has proved that Ω_0 had typically a nonempty interior: except for peculiar cases, there is always a nonnegligible set of consumers who don't gain anything from the presence of the monopoly.[20] They are "bunched" on the outside option q_0. In this section, we establish an equally surprising result: except for specific cases (like those solved explicitly in Wilson (1993) or Armstrong (1996)) there is always a nonnegligible set of consumers outside Ω_0 who are not screened by the monopoly: they receive the same product even though they have different tastes. We call this property "bunching of the second type." This is surprising for at least two reasons: in our set up, the first best solution involves a complete screening of consumers. Our result means therefore that, in multidimensional contexts, product differentiation is somewhat impeded (and not entailed) by exploitation of market power, contrary to that suggested by a naive extrapolation of the Mussa and Rosen (1978) results (where the monopolist's product line is typically much larger than the efficient one). The second reason is technical: in dimension one, bunching can be easily discarded by reasonable assumptions on the distribution of types.

[20] Armstrong considers the case $q_0 = 0$, $p_0 = C(q_0) = 0$, and interprets Ω_0 as a nonparticipation set. In the more general case $q_0 \neq 0$, $p_0 \geq C(q_0)$, it is more natural to interpret Ω_0 as the set where the participation constraint is exactly binding. Armstrong's interpretation is more natural in the case $p_0 < C(q_0) = 0$, which we have excluded, since it gives rise to nonconvexities.

In multidimensional screening problems, it is not possible to do so, which imposes the use of more complex techniques (see Section 6). In this section we start by giving the economic intuition why bunching is robust (subsection 5.1) and then give the precise statement of our result (subsection 5.2). We then discuss in detail the closed form solutions obtained by Wilson (1993) and Armstrong (1996) (subsection 5.3).

5.1. *Economic Intuition of Why Bunching is Robust*

Let us examine the uniform-quadratic specification on the square $\Omega = [a, b]^2$ that we have already used as an illustration. The shape of the solution is given in Figure 1. It is easy to see that U^* cannot be convex. Indeed the indifference region Ω_0 can be described as $\{t \in \Omega, U^*(t) \leq t \cdot q_0 - p_0\}$. This would be a convex set if U^* was a convex function. It is immediate from Figure 1 that Ω_0 is not convex. Intuitively (this is particularly so when a is large), the monopoly would like to serve as many consumers as possible, and choose Γ as close as possible to the lower boundary of Ω. This creates a conflict with second order incentive compatibility conditions, which are neglected in the relaxed problem. For example, in Figure 1, it is impossible to find a quality vector q and a price p that simultaneously give a positive utility to type $t_3 = (t_1 + t_2)/2$ and do not attract t_1 or t_2 (which have zero utility):

$$t_1 \cdot q - p \leq 0 \quad \text{and} \quad t_2 \cdot q - p \leq 0 \quad \text{imply:} \quad t_3 \cdot q - p \leq 0.$$

As we will see in Section 6, the optimal trade-off between surplus extraction on the marginal consumers and second order incentive compatibility conditions will imply giving the same products to types t_1, t_2, and t_3, which will result in Γ becoming a straight line (see Figure 7). This is a robust source of bunching.

5.2. *A Formal Statement of the Result*

Since U^* cannot be computed explicitly in general, it is difficult to prove directly that it is not convex. However, as usual in such problems, the necessary conditions that characterize U^* imply strong properties. For example, we have seen in Proposition 1 that the lower boundary of Ω_0 was equal to the set

$$\partial_- \Omega = \{t \in \partial\Omega, (t_0 - t) \cdot \vec{n}(t) \geq 0\},$$

i.e. the boundary points that are directly exposed to the outside good. If Ω_0 is convex, it contains necessarily the convex hull of $\partial_- \Omega$, which we have denoted Ω_-. Now Proposition 2 establishes that

$$\mu(\Omega_0) = 1. \tag{5.1}$$

Thus a necessary condition for Ω_- to be contained in Ω_0 (and therefore for U^* to be convex) is

$$\mu(\Omega_-) = \int_{\Omega_-} \alpha(t)\, dt + \int_{\partial_- \Omega} \beta(t)\, d\sigma(t) \leq 1.$$

When Ω_0 has a nonempty interior, then $\nu(t)=t-t_0$ on Ω_0 so that $\alpha(t)$ and $\beta(t)$ have an explicit expression. By contraposition, we obtain the following proposition.

PROPOSITION 3: *Under the assumptions of Proposition* 1 *and when* Ω_0 *has a nonempty interior, the following condition implies that* U^* *is not convex, and therefore that bunching occurs*:

$$(5.2)\qquad 1<\int_{\Omega_-}[f(t)+\operatorname{div}\{(t-t_0)f(t)\}]\,dt+\int_{\partial_-\Omega}(t_0-t)\cdot\vec{n}(t)f(t)\,dt$$

where

$$(5.3)\qquad \partial_-\Omega=\{t\in\partial\Omega,(t_0-t)\cdot\vec{n}(t)\geq 0\},$$

and Ω_- *is the convex hull of* $\partial_-\Omega$.

As a first illustration of condition (5.2), let us consider the case where f is uniform and Ω is an ellipse in $\mathbb{R}^2$, of center $(a,0)$: $\Omega=\{t=(t_1,t_2),(t_1-a)^2+(t_2^2/b^2)\leq 1\}$.

For simplicity, we take $t_0=(0,0)$. When $b=1$, Ω is a disk; on the other hand when b tends to zero, Ω shrinks to the line $\{(t_1,0),a-1\leq t_1\leq a+1\}$. In this limit case, the solution of the relaxed problem is always convex and Ω_0 is a singleton whenever $a\geq 3$. However, by Armstrong's result (1996) we know that for $b>0$, Ω_0 has always a nonempty interior. We can therefore apply Proposition 3. It is easy to see that $\Omega_-=\{t\in\overline{\Omega}/t_1\leq a-(1/a)\}$ (see Figure 4) and that

$$\mu(\Omega_-)=3\left[\frac{1}{2}-\frac{1}{\pi}\arcsin\frac{1}{a}-\frac{\sqrt{a^2-1}}{\pi a^2}\right].$$

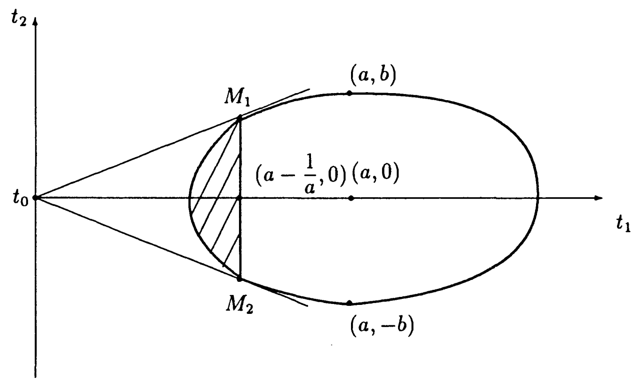

FIGURE 4.—When Ω is an ellipse, Ω_- is represented by the shaded area.

This quantity is less than 1 if and only if $a < a^* \sim 3.77$. Therefore when $a > a^*$, U^* is not convex and bunching of the second type appears. Notice that this condition does not depend on b. However when $b = 0$ and $a \geq 3$ there is no bunching at all: Proposition 3 cannot be applied, since Ω_0 has an empty interior.

By Proposition 1, we know that the free boundary Γ passes through M_1 and M_2. When $a > a^*$, Γ is contained in Ω_- so that U^* cannot be convex.

5.3. *The Closed-form Solutions of Wilson* (1993) *and Armstrong* (1996)

Wilson (1993, Chapter 13) and Armstrong (1996) were able to obtain closed-form solutions for several problems closely related to $\mathscr{P}^*$. These solutions adapt very naturally to our context. For example, Wilson finds the explicit solution of (the analogous to) $\mathscr{P}^*$ when $C(q) = (1/2)\|q\|^2$, $q_0 = 0$, and t is uniformly distributed on the "north east" quarter of the unit disk:

$$\Omega = \{t \in \mathbb{R}^2_+, t_1^2 + t_2^2 \leq 1\}.$$

The solution is characterized by

$$q^*(t) = \frac{1}{2}\max\left(0, 3 - \frac{1}{\|t\|^2}\right)t. \tag{5.4}$$

It is illustrated by Figure 5a.

It is easily checked that q^* satisfies the necessary conditions of Theorem 2. In particular the distortion vector $t - \nabla C(q^*(t))$ is colinear to t, and vanishes on BC (since $\|t\|^2 = 1$). Also, q^* equals zero on $\Omega_0 = \{t \in \mathbb{R}^2_+, t_1^2 + t_2^2 \leq (1/3)\}$ and is continuous on Ω. The most interesting property is that the function $t \to (t - t_0) \cdot \vec{n}(t)$ is identically zero on the whole lower boundary $CA \cup AB$. This comes from the joint facts that $t_0 = A$ and that the lower boundary of Ω coincides with the coordinate axes. This causes a certain degeneracy of the problem: the points M_1 and M_2 of Figure 3 are not well defined, and $\partial_- \Omega = CA \cup AB$ does not

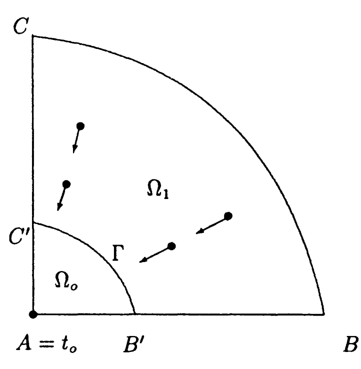

FIGURE 5a.—The explicit solution of Wilson (1993): arrows represent the directions of binding incentive constraints (they are always radial).

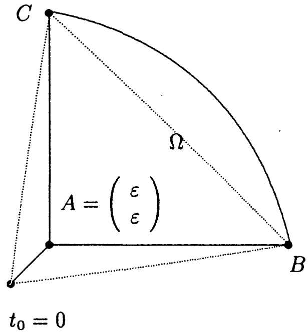

FIGURE 5b.

coincide with $\partial_0 \Omega = C'A \cup AB'$. However, condition (5.1) is satisfied, since

$$\int_{\Omega_0'} [f(t) + \operatorname{div}\{(t - t_0) f(t)\}]\, dt = 3\int_{\Omega_0} f(t)\, dt = 1,$$

while

$$\int_{\partial_- \Omega} (t_0 - t) \cdot \vec{n}(t) f(t)\, d\sigma(t) = 0.$$

Armstrong (1996) also finds closed form solutions for cases where $t_0 = 0$ and $\Omega = \mathbb{R}^n_+$. Therefore these examples also satisfy the property that $(t - t_0) \cdot \vec{n}(t)$ is identically zero. This property is not robust: it is lost after an arbitrarily small perturbation of Ω (or t_0). For example consider a small perturbation of Figure 5 (Figure 5b), whereby Ω is shifted by a vector

$$\begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix}.$$

It is easy to see, in this example, that condition (5.2) (which implies bunching) is satisfied. Indeed $\partial_- \Omega$ now equals the lower boundary $CA \cup AB$, which means that Ω_- is the triangle CAB. The distribution of t being uniform ($f(t) \equiv (4/\pi)$), the first integral in the right-hand side of (5.2) is easily computed:

$$\int_{\Omega_-} [f(t) + \operatorname{div}(t f(t))]\, dt = 3\int_{\Omega_-} f(t)\, dt = \frac{6}{\pi} > 1.$$

The second integral being necessarily nonnegative, condition (5.2) is satisfied. This means that an arbitrarily small perturbation of the problem solved explicitly above gives rise to bunching.[21]

[21] This also indicates that the free boundary Γ is not a continuous function of the support Ω of the types' distribution. This comes from the shape of Ω, and more specifically from the discontinuity of $t \to \vec{n}(t)$ at B and C.

6. THE COMPLETE PROBLEM AND SWEEPING OPERATORS

We are now back to our initial problem, which can be written, in conformity with our previous notation,

$$(\overline{\mathscr{P}})\begin{cases}\max \phi(U),\\ U\in \overline{K},\end{cases}$$

where

$$\phi(U)=\int_{\Omega}\{t\cdot \nabla U(t)-C(\nabla U(t))-U(t)\}f(t)\,dt,$$

and

$$\overline{K}=\{U\in H^1(\Omega),U\geq U_0 \text{ and } U \text{ is convex}\}.$$

It is immediately seen that, like K^*, $\overline{K}$ is a (closed) convex cone of vertex U_0: if (U_0+h) belongs to $\overline{K}$ (which means exactly that h is convex nonnegative, since $U_0(t)=t\cdot q_0-p_0$ is linear) then for any positive constant λ, $(U_0+\lambda h)$ also belongs to $\overline{K}$. Thus our first two results on the relaxed problem $\mathscr{P}^*$ can immediately be adapted to the complete problem $\overline{\mathscr{P}}$.

6.1. *Existence and First Characterization of the Solution*

THEOREM 1′: *Under the assumptions of Theorem* 1, *the complete problem* $\overline{\mathscr{P}}$ *has a unique solution* U.

PROOF: See Appendix 1.

The second result is the variational characterization of $\overline{U}$.

LEMMA 3′: *The solution* $\overline{U}$ *of the complete problem* $\overline{\mathscr{P}}$ *is characterized by the properties*:

(6.1) *For all convex* $h\geq 0$, $\phi'(\overline{U})h\leq 0$;

(6.2) $\overline{U}-U_0\geq 0$, *convex, and* $\phi'(\overline{U})(\overline{U}-U_0)=0$.

PROOF: Identical to that of Lemma 3. *Q.E.D.*

The difference with Lemma 3 is that the first condition is less restrictive, since $\phi'(\overline{U})h$ has to be ≤ 0 only when h is convex, but the second condition is more restrictive, since $\overline{U}$ (or equivalently $\overline{U}-U_0$) is now required to be convex. In particular $\overline{U}=U^*$ if and only if U^* is convex, which is very seldom the case (by our Proposition 3). We have seen in Section 4 that the directional derivative of ϕ could be written as a measure

$$\phi'(\overline{U})h=-\int h(t)\,d\mu(t),$$

where μ is the sum of a measure on Ω, with a density[22] $\alpha(t)=f(t)+\operatorname{div}\{(t-\nabla C(\bar{q}(t)))f(t)\}$, and a (singular) measure supported by $\partial\Omega$, of density $\beta(t)=(\nabla C(\bar{q}(t))-t)\cdot\vec{n}(t)f(t)$. Contrarily to the case of problem $\mathscr{P}^*$, the first order conditions of $\overline{\mathscr{P}}$ do not imply that μ is a positive measure, since (6.1) is only required when h is *convex* nonnegative. Therefore it is natural to decompose μ into its positive and its negative parts:

$$\int h(t)\,d\mu(t)=\int h(t)\,d\mu_+(t)-\int h(t)\,d\mu_-(t). \tag{6.3}$$

The next step is to introduce the important notion of *sweeping operator*.

6.2. *The Notion of Sweeping Operator*

DEFINITION 2: A *transition probability* T on $\overline{\Omega}$ is a family $T(t,\cdot)$ of probability measures on Ω, defined for a.e. t in $\overline{\Omega}$, and measurable in t. To any measure μ on $\overline{\Omega}$ one can associate the measure $T\mu$ defined by

$$\forall h\in\mathscr{C}(\overline{\Omega},\mathbb{R})\quad (T\mu)h=\iint h(s)T(t,ds)\,d\mu(t). \tag{6.4}$$

DEFINITION 3: If the transition probability T satisfies

$$\int sT(t,ds)=t\quad\text{for a.e. } t \text{ in } \overline{\Omega}, \tag{6.5}$$

then T is called a *sweeping operator*.

This notion of a sweeping operator is an important tool in potential theory (see Meyer (1966)). The word "sweeping" is a literal translation of the French word "balayage" coined by Meyer. It refers to the fact that the measure $T\mu$ is obtained from μ by "sweeping mass" around each point t, while preserving the center of gravity (condition (6.5)). It is the exact generalization of the notion of a mean preserving spread (see Rothschild and Stiglitz (1970)), defined for probability distributions on the real line. It is easy to see from (6.5) and Jensen's lemma that for any convex function h and any t in Ω

$$\int h(s)T(t,ds)\geq h(t).$$

Therefore if μ is a positive measure, (6.4) implies

$$(T\mu)h\geq\int h(t)\,d\mu(t)=\mu h. \tag{6.6}$$

[22] Again, we assume that $\bar{q}(\cdot)$ is almost everywhere differentiable (otherwise $\alpha(t)$ is not defined everywhere), which will be checked ex post.

It turns out that Cartier's theorem (see Meyer (1966)) asserts the converse property: if μ_1 and μ_2 are two positive measures such that $\mu_2 h \geq \mu_1 h$ for all convex functions h, then μ_2 can be obtained from μ_1 by a sweeping operator.[23] This is in fact a generalization of the property of "increasing risk," defined and characterized by Rothschild and Stiglitz (1970) for real-valued random variables.[24]

Our condition (6.1) is a little more complex: using the decomposition (6.3), it amounts to saying that $\mu_+ h \geq \mu_- h$ for any convex *nonnegative* function h. By an adaptation of Cartier's theorem, we obtain the following result.

PROPOSITION 4: *The integral* $\int h\, d\mu$ *is nonnegative on all convex continuous nonnegative functions h if and only if there exists a sweeping operator T and two nonnegative measures* λ *and* ν *such that*

$$\mu = \lambda + T\nu - \nu. \tag{6.7}$$

PROOF: See Appendix 2.

The next (and final) step in the characterization of $\bar{U}$ is the transformation of condition (6.2), the second part of the first order conditions obtained in Lemma 3′. This is what we do in the next subsection.

6.3. *The Characterization of "Bunches"*

What we call a "bunch" is a (nonsingleton) set $\Omega(q)$ of types t who choose the same quality q:

$$\Omega(q) = \{t \in \bar{\Omega}, \bar{q}(t) = q\} = \{t \in \Omega, \bar{U}(t) = t \cdot q - p(q)\}. \tag{6.8}$$

Note that, by definition of $\bar{U}$, we always have $\forall t \in \bar{\Omega}, \bar{U}(t) \geq t \cdot q - p(q)$, so that $\Omega(q)$ can also be defined by $\Omega(q) = \{t \in \Omega, \bar{U}(t) \leq t \cdot q - p(q)\}$.

By convexity of $\bar{U}$, $\Omega(q)$ is therefore a convex subset of Ω (or an empty set). By continuity of $\bar{q}(\cdot)$, it is also a closed set. Notice that $\Omega(q_0)$ coincides with the indifference set Ω_0. When $\bar{U}$ is strictly convex on $\Omega_1 = \Omega \setminus \Omega_0$, Ω_0 is the only bunch. As we have seen, this is seldom the case. On the contrary, when there is nontrivial bunching, $\bar{U}$ is affine on all bunches (by (6.8)) and conversely, if $s \notin \Omega(q)$ and $t \in \Omega(q)$, then $U(s) > U(t) + (t - s) \cdot q$.

Therefore, the incentive compatibility constraint between s and t is binding if and only if s and t belong to the same bunch $\Omega(q)$. This defines a.e. an equivalence

[23] A simple example is when μ_1 is a uniform distribution on a line $[t_1, t_2]$, and μ_2 equals $(M/2)(\delta_{t_1} + \delta_{t_2})$, where δ_t denotes the Dirac measure in t and M is the total mass of μ_1. This is an extreme case of sweeping: all the mass of μ_1 is shifted to the boundary. However extreme, this situation will appear in one of the examples we solve explicitly in Section 7.

[24] Indeed, according to the definition of Rothschild and Stiglitz, a distribution μ_2 on $\mathbb{R}$ is riskier than another distribution μ_1 if and only if for all *concave* functions v, we have $\mu_2 v \leq \mu_1 v$. This is clearly equivalent to our property.

relation between types.[25] The measures λ and $T(t, \cdot)$ have to be interpreted as the Lagrange multipliers associated respectively to the participation constraint $U(t) \geq U_0(t)$, and to the incentive compatibility constraints $U(s) \geq U(t) + q(t) \cdot (s - t)$.

To see this, it is enough to prove that these measures vanish outside the corresponding "bunches."

PROPOSITION 5: *The measures* λ, ν, *and* $T(t, \cdot)$ (*obtained from Proposition* 4) *satisfy*

$$\operatorname{supp} \lambda \subset \Omega_0,$$
$$\operatorname{supp} T(t, \cdot) \subset \Omega(q(t)) \quad \text{for } \nu \text{ a.e. } t.$$

PROOF: See Appendix 3.

6.4. *The Quality Does Not Jump*

Another important property is stated in the next proposition.

PROPOSITION 6: *The mapping* $t \to \bar{q}(t)$ *is continuous on* Ω.

In the unidimensional case, the continuity of $\bar{q}(\cdot)$ had been established by Mussa and Rosen (1978). It has important consequences: for example, all bunches $\Omega(q)$ are closed sets, as preimages of a singleton $\{q\}$ by a continuous mapping. Also, the consumers' program has for all t a *unique* maximizer: if it were not so, the indirect utility function $\bar{U}$ would not be $\mathscr{C}^1$ (not even differentiable) at t. This gives another (ex-post) justification for using the dual approach: the optimal indirect utility function $\bar{U}$ is $\mathscr{C}^1$, whereas the optimal price schedule $\bar{P}$ is typically not differentiable. Finally, the continuity of $\bar{q}(\cdot)$ has a surprising consequence in the multidimensional context: since Armstrong (1996) has shown that the nonparticipation region Ω_0 had most of the time a nonempty interior, q_0 belongs to the set Q of products offered to the consumers. The mapping $\bar{q}(\cdot)$ being continuous, this implies that Q is connected, which means that the monopoly will typically find it optimal to sell products which are arbitrarily close to the "outside product" q_0. An example of such a product line is given in Figure 8.

6.5. *The Complete Characterization of the Solution*

It is given in the following theorem.

THEOREM 2′: *Under the assumptions of Theorem* 1, *the solution* $\bar{U}$ *of* $\bar{\mathscr{P}}$ *is* $\mathscr{C}^1$ *on* $\bar{\Omega}$. *It is characterized by a partition of* $\bar{\Omega}$ *into three regions* Ω_0, Ω_B, *and* Ω_1,

[25] Notice the important difference with discrete models, where this relation defines instead a partial ordering.

with the following properties:

(i) *In the indifference region* $\Omega_0, \bar{U}(t) = U_0(t)$. *Moreover* $\mu(\Omega_0) = 1$ *and the outside boundary of* Ω_0 *is (strictly) included in* $\partial_- \Omega$, *the set of types that are directly exposed to the outside product.*

(ii) *In the nonbunching region* Ω_1, $\bar{U}$ *is strictly convex and satisfies the Euler equation*

$$\alpha(t) = 0 \quad on \quad \Omega_1 \cap \Omega \quad and \quad \beta(t) = 0 \quad on \quad \Omega_1 \cap \partial\Omega.$$

(iii) *Finally the bunching region* Ω_B *is partitioned into bunches* $\Omega(q)$. *On each of these bunches* $\bar{U}$ *is affine and the restriction of* μ *satisfies Cartier's property*:

$$\mu_+^{\Omega(q)} = T\mu_-^{\Omega(q)}.$$

Notice that the characterization of $\bar{U}$ is more complex than that of U^*, obtained in Theorem 2. For example, $\alpha(\cdot)$ (the density of the regular part of μ) can be negative. However, it must be the case that the negative parts of α can be "swept" towards the boundary, while respecting the bunches. Consider for instance a bunch $\Omega(q)$. Theorem 2′ states that

$$\mu_+^{\Omega(q)} = T\mu_-^{\Omega(q)}.$$

This implies in particular that the restriction of μ to $\Omega(q)$ satisfies two conditions:

$$\int_{\Omega(q)} d\mu(t) = 0, \tag{6.9}$$

and

$$\int_{\Omega(q)} t\, d\mu(t) = 0. \tag{6.10}$$

The interpretation of condition (6.9) is very natural if one remembers two basic formulas:

$$\bar{U}(t) = \max_{q \in Q} \{tq - p(q)\}, \tag{6.11}$$

$$\phi'(\bar{U})h = -\int h(t)\, d\mu(t). \tag{6.12}$$

Consider now what happens if the monopolist replaces her price schedule $p(\cdot)$ by a small variation $(p + \epsilon k)(\cdot)$. By the envelope formula applied to (6.11) and the chain differentiation rule applied to (6.12) we obtain the marginal profit change:

$$\Delta\Pi \sim \epsilon \int_{\Omega} k(\bar{q}(t))\, d\mu(t).$$

This is zero for any variation $k(\cdot)$ if and only if condition (6.9) is satisfied for all q. This is analogous to a conditional expectations condition: the total (marginal) contribution to the firm's profit of the types who are bunched together in $\Omega(q)$ is zero for all q. Similarly, condition (6.10) corresponds to an analogous extremality condition with respect to variations of the product line Q. Notice however that (6.9) and (6.10) do not exhaust the "sweeping conditions," which also imply (roughly speaking) that the positive part of μ is more concentrated towards the boundary. Typically, these conditions will be satisfied if $\alpha(t) \leq 0$ and $\beta(t) \geq 0$, as we will see on a series of examples in Section 7.

Finally in the nonbunching region, the Euler equation is satisfied:

$$\alpha(t) = \operatorname{div}\{(t - \nabla C(\bar{q}(t)))f(t)\} + f(t) = 0.$$

This can be expressed in terms of the distortion vector $\nu(t)$:

$$\nu(t) = (t - \nabla C(\bar{q}(t)))f(t).$$

This vector satisfies the partial differential equation:

$$\operatorname{div}(\nu(t)) = -f(t),$$

(which expresses the optimal trade-off between distortion and rent), and the boundary condition:

$$\forall t \in \Omega_1 \cap \partial\Omega \quad \nu(t) \cdot \vec{n}(t) = 0 \quad \text{(no distortion "at the boundary")}.$$

As we will see in the examples solved in Section 7, Ω_1 typically contains portions of the *lower* boundary of Ω. Therefore, extrapolating the expression "no distortion at the top" (correct in the one-dimensional case) would be misleading, even if the participation constraint binds in the lower regions of Ω. The correct formulation is that, anywhere outside the bunching regions Ω_0 and Ω_B, there is no distortion "*at the boundary.*"

To conclude this section, let us remark that the condition (5.1) is also satisfied by the solution $\bar{U}$ of the complete problem. Indeed, by Theorem 2′,

$$\mu(\Omega_0) = \int_{\Omega_0} \alpha(t)\,dt + \int_{\partial\Omega_0} \beta(t)\,d\sigma(t) = 1,$$

which gives exactly condition (5.1). This condition has a nice interpretation, that we now explain. We first need two notations, for $\epsilon > 0$:

$$\Omega^{\epsilon} = \left\{t \in \Omega, \bar{U}(t) - U_0(t) \geq \epsilon\right\},$$

and $B(\epsilon)$ is the profit obtained by uniformly raising prices by ϵ (starting from the optimal solution $\bar{U}$). We have therefore

$$B(\epsilon) = \int_{\Omega^{\epsilon}} \{t \cdot \nabla\bar{U}(t) - C(\nabla\bar{U}(t)) - \bar{U}(t) + \epsilon\} f(t)\,dt.$$

It is possible to establish[26] that B has a right derivative in 0 equal to $1 - \mu(\Omega_0)$. Therefore, condition (5.1) becomes transparent. It describes the optimal trade-off

[26] The proof is technical and has not been included. It is available from the authors upon request.

between market share and the general level of prices: uniformly increasing the level of prices implies gaining more profit from participating customers but losing the marginal ones. The optimal trade off is given by formula (5.1).

Finally, let us remark that Armstrong's result (that Ω_0 cannot have zero measure when Ω is strictly convex) is immediately deduced from (5.1). Indeed, if it were the case, Ω_0 would necessarily be a singleton (remember that it is convex). Then $\mu(\Omega_0)$ would be 0, and (5.1) would be violated.

7. THE USE OF THE SWEEPING CONDITIONS: SEVERAL EXAMPLES

As is unfortunately often the case, the characterization result obtained in Theorem 2′ is not constructive. It does not tell how to find the solution $\overline{U}$. However if we have a candidate solution (obtained, for example, by numerical solution of a discretized problem), Theorem 2′ will allow us to check whether it is indeed the true solution. In this section, we study several examples along these lines.

7.1. *The Unidimensional Case*: *Ironing and Sweeping*

PROPOSITION 7: *In the unidimensional case, the sweeping conditions coincide with the ironing conditions of Mussa and Rosen.*

PROOF: See Appendix 4.

7.2. *Nonlinear Pricing with an Exponential Distribution of Types*

We consider here a two-dimensional nonlinear pricing problem of the simplest form, as in Wilson (1993). The gross utility obtained by a consumer of type $t=(t_1,t_2)$ when he consumes a bundle $q=(q_1,q_2)$ is

$$S(t,q)=t_1q_1+t_2q_2-\frac{1}{2}q_1^2-\frac{1}{2}q_2^2.$$

Marginal costs are normalized to zero. We assume that t is distributed on $[a,+\infty]^2$, with an exponential density

$$f(t)=\exp(2a-t_1-t_2).$$

We assume that $a>1$. Notice that t_1 and t_2 are independently distributed across the population of consumers, while $S(t,q)$ is separable in q_1 and q_2. Therefore a natural candidate for the optimal price schedule is

$$p(q)=p_1(q_1)+p_2(q_2),$$

where $p_1(\cdot)$ and $p_2(\cdot)$ are the solutions of the unidimensional pricing problems for q_1 and q_2 *separately*. Because of the exponential distribution of types, these solutions are easy to determine explicitly. Indeed we have

$$q_i(t_i)=t_i-\frac{1-F_i(t_i)}{f_i(t_i)}=t_i-1,$$

so that

$$p_i'(q_i) = t_i - q_i(t_i) \equiv 1.$$

Since $a > 1$, all consumers participate, and $p_i(0)$ is determined in such a way that $U_i(a) = 0$. We obtain finally

$$p_i(q_i) = q_i + \frac{1}{2}(a-1)^2 \quad \text{and} \quad p(q_1, q_2) = q_1 + q_2 + (a-1)^2.$$

The corresponding indirect utility function is

$$U(t) = \frac{1}{2}(t_1 - 1)^2 + \frac{1}{2}(t_2 - 1)^2 - (a-1)^2.$$

However, this *cannot* be the solution, since $\{t, U(t) = 0\}$ is reduced to the singleton $\{(a, a)\}$, which is never optimal. As shown by Theorem 2′, the solution $\overline{U}$ is in fact characterized by a partition of $\Omega = \mathbb{R}_+^2$ into three regions:

(i) the nonparticipation region Ω_0, on which $\overline{U}(t) \equiv 0$;
(ii) the bunching region Ω_B, where $\overline{U}$ only depends on $\tau = t_1 + t_2$;
(iii) the nonbunching region Ω_1, where $\overline{U}$ is strictly convex.

PROPOSITION 8: *The three regions are separated by two parallel lines*:

$$\Gamma_0 = \{t, t_1 + t_2 = \tau_0\},$$

the boundary between Ω_0 *and* Ω_B, *and*

$$\Gamma_1 = \{t, t_1 + t_2 = \tau_1\},$$

the boundary between Ω_B *and* Ω_1. *The values of* τ_0 *and* τ_1 *are given in Appendix 4. In the region* Ω_B, *there is "commodity bundling": consumers are restricted to consume the same quantity of the two goods.*

The shape of the solution is represented in Figure 6. The proof of Proposition 8, as well as the precise features of the solution are given in Appendix 4.

7.3. *The Mussa-Rosen Problem when Types are Uniformly Distributed on a Square*

As a final illustration, we come back to the Mussa-Rosen problem, in the case of a uniform distribution of types. For simplicity we specify the parameters as follows: $t_0 = 0$, $C(q) = (c/2)\|q\|^2$, and $\Omega = [a, b]^2$, with $b = a + 1$. In this case, which we have solved numerically, $\overline{U}$ has a different shape in the three different regions that partition Ω. These regions are, as before:

(i) the indifference region Ω_0, on which $\overline{U}(t) \equiv 0$;
(ii) the bunching region Ω_B, where $\overline{U}$ only depends on $t_1 + t_2$;
(iii) the nonbunching region Ω_1, where $\overline{U}$ is strictly convex.

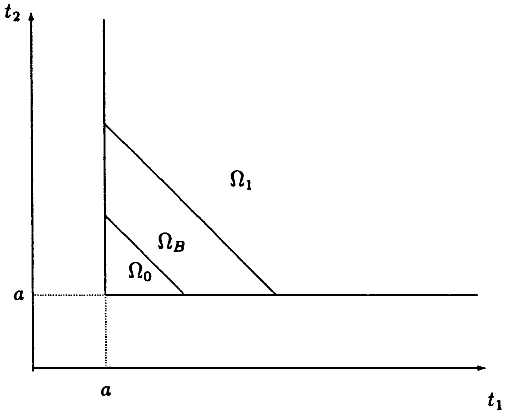

FIGURE 6.—The partition of Ω in the nonlinear pricing problem with linear demands and exponential distribution of types.

The shape of the solution is represented in Figure 7. The quality assignment $\bar{q}(t)$ is characterized by different formulas in these three regions:

(i) On Ω_0, $\bar{q}(t) \equiv 0$ so that

$$\begin{cases} \alpha(t) = \mathrm{div}(tf(t)) + f(t) = 3, \\ \beta(t) = (t_0 - t)\cdot \vec{n}(t) = a \quad \text{on} \quad D_0 A \cup AB_0. \end{cases}$$

The frontier $D_0 B_0$ between Ω_0 and Ω_B is the line $t_1 + t_2 = \tau_0$, where τ_0 is determined by condition (5.1):

$$1 = \int_{\Omega_0} \alpha(t)\, dt + \int_{\partial_0 \Omega} \beta(t)\, d\sigma(t).$$

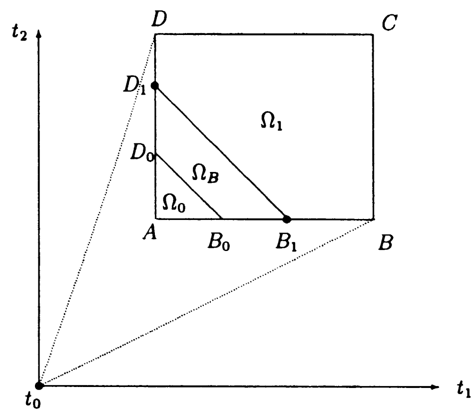

FIGURE 7.—The different regions: indifference Ω_0, bunching Ω_B, nonbunching Ω_1, for the case of a uniform distribution on a square.

We obtain $1=(3/2)(\tau_0-a)^2+2a(\tau_0-a)$, which gives

$$\tau_0=\frac{4a+\sqrt{4a^2+6}}{3}.$$

(ii) On Ω_B, $\bar{q}_1(t)=\bar{q}_2(t)=q_B(\tau)$, with $\tau=t_1+t_2$. Thus

$$\begin{cases}\alpha(t)=3-c\operatorname{div}\bar{q}(t)=3-2cq_B'(\tau) & \text{on} \quad \Omega_B,\\ \beta(t)=(c\bar{q}(t)-t)\cdot\vec{n}(t)=a-cq_B(\tau) & \text{on} \quad \partial\Omega_B.\end{cases}$$

The sweeping conditions are satisfied if $3-2cq_B'(\tau)\leq 0$, $a-cq_B(\tau)\geq 0$, and on each bunch

$$\int_a^{\tau-a}\alpha(t_1,\tau-t_1)\,dt_t+\beta(a,\tau-a)+\beta(\tau-a,a)=0.$$

This gives a differential equation in $q_B(\cdot)$:

$$(3-2cq_B'(\tau))(\tau-2a)+2(a-cq_B(\tau))=0.$$

This equation is easily solved:

$$cq_B(\tau)=\frac{3}{4}\tau-\frac{a}{2}+\frac{k}{2(\tau-2a)}.$$

The constant k is determined by the smooth pasting condition $q(\tau_0)=0$, which gives $k=-1$.

The frontier between Ω_1 and Ω_B is again a straight line D_1B_1, of equation $t_1+t_2=\tau_1$, where τ_1 is determined by the continuity condition on $\partial\Omega$, which gives $cq_B(\tau_1)=a$.

(iii) Finally on Ω_1, the solution is determined by the Euler equation:[27]

$$\alpha(t)=3-c\Delta\bar{U}(t)=0 \quad \text{on} \quad \Omega_1\cap\Omega,$$

together with two boundary conditions,

$$\beta(t)=(c\bar{q}(t)-t)\cdot\vec{n}(t)=0 \quad \text{on the upper boundary of } \Omega_1, \quad \text{and}$$

$$c\bar{q}(t)=\begin{pmatrix}a\\a\end{pmatrix} \quad \text{on } D_1B_1 \quad \text{(smooth pasting again)}.$$

It is interesting to determine the set of qualities Q that are actually sold by the monopolist in this example.

PROPOSITION 9: *The set $Q=\{\bar{q}(t),t\in\Omega\}$ consists of the efficient product line plus a set of "basic" qualities constituted by the line OA (see Figure 8).*

[27] The convexity of $\bar{U}$ on Ω_1 has been checked numerically.

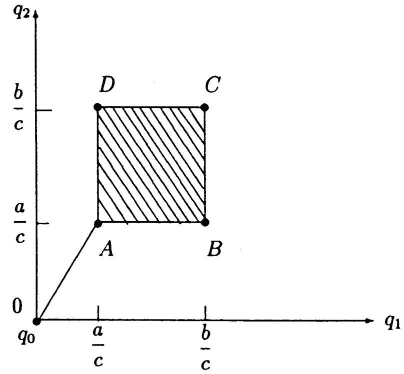

FIGURE 8.—The product line sold by the monopoly consists of the efficient product line (the square *ABCD*) plus the line OA.

PROOF: Although we have not solved explicitly for $\bar{q}$ in the region Ω_1, the boundary conditions give us a lot of information:

(i) The lower boundary D_1B_1 of Ω_1 corresponds in fact to the last bunch of the region Ω_B:

$$\forall t \in D_1B_1 \quad \bar{q}(t) = \begin{pmatrix} a/c \\ a/c \end{pmatrix}.$$

(ii) On the north-east boundary D_1D, $\bar{q}_1(t)$ equals a/c, while $\bar{q}_1(t)$ increases continuously from a/c to b/c (the situation is symmetric on B_1B).

(iii) On the north boundary DC, $\bar{q}_2(t)$ equals b/c, while $\bar{q}_1$ increases continuously from a/c to b/c (the situation is symmetric on BC).

Therefore the boundary of Ω_1 is mapped by $\bar{q}$ into the boundary of the efficient set (i.e. the square $[a/c, b/c]^2$). Since $\bar{q}$ is the gradient of a strictly convex function $\bar{U}$ (on Ω_1), there cannot be any "hole" in $\bar{q}(\Omega_1)$, which means that $\bar{q}(\Omega_1)$ equals exactly the square $[a/c, b/c]^2$. Now it is easy to see that the image of Ω_B by $\bar{q}$ is just the straight line from

$$q_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a/c \\ a/c \end{pmatrix}.$$

The qualitative pattern of Figure 8 is also present in several examples that we have solved numerically: except in specific cases, there is always a set of types (close to those who don't participate) who are bunched together on the same "basic" qualities. As we already explained, this comes from a conflict between participation constraints and transverse incentive compatibility conditions. As a consequence (see Figure 8) the monopolist offers less choice to the "lower median" consumers (at least when the outside quality is low). This pattern is confirmed by casual empiricism: when they have some market power, firms selling differentiated products (like cars or computers) seem indeed to offer less variety at the lower end of the spectrum of qualities.

8. CONCLUSION

The main results of this paper are the following:

(i) We have proved that, under multidimensional adverse selection, the multiproduct monopolist's problem has a unique solution,[28] both if we neglect (relaxed problem) or incorporate (complete problem) the second order conditions of the consumers' program.

(ii) We have characterized these solutions by computing two functions, α (defined on the interior Ω of the set of types) and β (defined on the boundary $\partial\Omega$) which measure the local net cost (in terms of foregone profit) of the informational rents left to the consumers. These two functions naturally define a measure $d\mu = \alpha dt + \beta d\sigma$ on $\overline{\Omega}$.

(iii) At the solution U^* of the relaxed problem $\mathscr{P}^*$, both functions have to be equal to zero except in the indifference region Ω_0, where rents are equal to zero and both functions can be positive. We establish a necessary condition (involving simultaneously the geometry of Ω, the distribution of types and the position of the outside option q_0) for U^* to be admissible for the complete problem.

(iv) On a series of examples, we show that this condition is seldom satisfied and that the closed form solutions obtained by Wilson (1993) and Armstrong (1996) correspond to very peculiar patterns. In other words, the second order conditions are typically binding, which suggests that bunching is robust in multidimensional adverse selection problems.

(v) We then proceed to study the complete problem $\overline{\mathscr{P}}$, and characterize its solution $(\overline{U}, \bar{q} = \nabla \overline{U})$ by adapting the notion of sweeping operator used in potential theory. A sweeping operator is a transition probability that preserves the expectations: in other words it transforms positive measures on Ω by evenly "sweeping" mass towards the boundary of Ω. The characterization of $\overline{U}$ is given in terms of the measure μ and of the "bunches" $\Omega(q)$ (which are defined as the sets of types who choose the same quality q). We prove that $\overline{U}$ is the solution if and only if, on any bunch $\Omega(q)$ the positive part of μ (denoted μ_+) is obtained from its negative part (denoted μ_-) by a sweeping operator supported by $\Omega(q)$. A typical situation is when α is nonpositive on Ω, β is nonnegative on $\partial\Omega$, and the restriction of μ to each bunch $\Omega(q)$ has a zero mass. We give economic interpretations for these conditions.

The final section is dedicated to the presentation of three examples which illustrate these new techniques.

[28] As far as we know, such a formal existence result is novel in the literature on the topic.

(i) In subsection 7.1 we show that in the unidimensional case, the sweeping procedure is equivalent to the ironing procedure invented by Mussa and Rosen (1978) for dealing with unidimensional bunching.

(ii) In subsection 7.2 we solve the bidimensional nonlinear pricing problem when preferences are quadratic and types are exponentially distributed. We find that the solution involves some "commodity bundling": the consumers who do not want to buy more than some threshold quantity are forced to buy the same amount of the two goods.

(iii) Finally in subsection 7.3 we solve the bidimensional Mussa-Rosen problem when types are uniformly distributed on a square. We find that the seller offers a full differentiation of products in the upper part of the qualities spectrum, but only "limited choice" (which generates bunching) for lower qualities. We conjecture[29] that this is a general pattern of multiproduct lines under imperfect competition.

Even if we don't provide a general algorithm for solving multidimensional screening problems, our results give a simple, operational form of the first order conditions that characterize the solutions to these problems. Combined with the numerical techniques that we develop elsewhere (Choné and Rochet (1997)), this provides a powerful instrument, applicable to a wide range of such problems, where similar patterns are likely to emerge. For example, Biais et al. (1996) study a mechanism design problem under bidimensional adverse selection, motivated by initial price offerings. The solution involves complete bunching in one direction, a pattern that may appear also in other contexts. Among many other potential applications one can cite auctions with externalities (Jehiel et al. (1996)) or the analysis of collusion under asymmetric information (Laffont and Martimort (1997)).

IDEI-GREMAQ and Institut Universitaire de France, Université de Toulouse I, Place Anatole France, 31042 Toulouse Cedex, France; rochet@cict.fr

and

ENSAE, 3, avenue Pierre Larousse, 92241 Malakoff Cedex, France; CHONE@ENSAE.FR; http://www.ENSAE.FR

Manuscript received March, 1995; final revision received June, 1997.

APPENDIX 1

PROOF OF THEOREM 1: The functional ϕ is concave and continuous on K^* (which is closed and convex). Thus we have only to check that ϕ is coercive on K^* (see, e.g., Kinderlehrer and Stampacchia (1980, Chapter 1)), i.e. that $\phi(U)$ tends to $-\infty$ when $|U|_{H^1}$ tends to $+\infty$. For all U in $H^1(\Omega)$, we denote by $\underline{U}$ the mean value of U in Ω:

$$\underline{U} = \frac{1}{|\Omega|}\int_{\Omega} U(t)\, dt.$$

[29] This conjecture was suggested to the first author by Bob Wilson.

We recall Poincaré's inequality (see, e.g., Kinderlehrer and Stampacchia (1980, Chapter 1)). It asserts that there exists a constant $M(\Omega)$ such that, for all U in $H^1(\Omega)$, $|U-\underline{U}|_{L^2} \leq M(\Omega)|\nabla U|_{L^2}$.

Using the decomposition $U=\underline{U}+U-\underline{U}$, we obtain

$$\text{(A.1)} \qquad |U|_{L^2}^2 = \underline{U}^2 + |U-\underline{U}|_{L^2}^2 \leq \underline{U}^2 + M(\Omega)^2 |\nabla U|_{L^2}^2,$$

which leads to

$$\text{(A.2)} \qquad |U|_{H^1} \to +\infty \Leftrightarrow \underline{U}^2 \to +\infty \quad \text{or} \quad |\nabla U|_L^2 \to +\infty.$$

Remark that

$$\text{(A.3)} \qquad C(q) = \int_0^1 \nabla C(tq)\cdot q\,dt = \int_0^1 \left[\nabla C(0)q + \int_0^1 D^2C(stq)(q,q)\,ds\right] dt$$
$$\geq \nabla C(0)\cdot q + \epsilon|q|^2.$$

By assumption, f is bounded away from zero: $f \geq \alpha > 0$ on Ω. It follows from (A.1) and (A.3) that

$$\text{(A.4)} \qquad \phi(U) = \int (t\cdot\nabla U - C(\nabla U) - U) f(d)\,dt$$
$$\leq -\int \{\alpha\epsilon|\nabla U|^2 + \nabla C(0)\cdot\nabla U f(t)\}\,dt + \int\{t\cdot\nabla U - (U-\underline{U})\}f(t)\,dt - \underline{U}$$
$$\leq -\alpha\epsilon|\nabla U|_{L^2}^2 + M(\Omega)|\nabla U|_{L^2} - \underline{U}.$$

The coerciveness of ϕ follows from (A.2) and (A.4). Let us now prove the uniqueness of the solution. Suppose by contradiction that U_1 and U_2 are two distinct solutions of the relaxed problem. The first order conditions imply

$$\int (t\cdot\nabla(U_2-U_1) - \nabla C(\nabla U_1)\cdot\nabla(U_2-U_1) - (U_2-U_1))f(t)\,dt \leq 0,$$

$$\int (t\cdot\nabla(U_1-U_2) - \nabla C(\nabla U_2)\cdot\nabla(U_1-U_2) - (U_1-U_2))f(t)\,dt \leq 0,$$

which leads to

$$\int (\nabla C(\nabla U_1) - \nabla C(\nabla U_2))\cdot\nabla(U_1-U_2)f(t)\,dt \leq 0.$$

We conclude (using the strict convexity of C) that $\int|\nabla(U_1-U_2)|^2=0$, which means that U_1-U_2 is a constant. Given the form of the functional ϕ, $\phi(U_1)$ and $\phi(U_2)$ differ by the same constant, which contradicts the fact that U_1 and U_2 are both solutions of the relaxed problem. *Q.E.D.*

PROOF OF THEOREM 1′: The proof is exactly identical to that of Theorem 1. The only thing which needs to be proved is that $\overline{K}$ is closed. In other words if a sequence (U_n) of convex functions converges (for the H^1-norm) to $\overline{U}$, is it true that $\overline{U}$ is convex? In fact, this property results immediately from the much stronger result of Dudley (1977): if a sequence (U_n) of convex functions converges to a generalized function $T \in \mathscr{D}'(\Omega)$ (see, for instance, Kinderlehrer and Stampacchia (1980) for a definition of the space $\mathscr{D}'(\Omega)$ of generalized functions on Ω), then T is a convex function. Since the topology on $\mathscr{D}'(\Omega)$ is weaker than the norm-topology on H^1, this implies the desired property that $\overline{K}$ is closed. *Q.E.D.*

APPENDIX 2

PROOF OF LEMMA 2: It consists of two steps:

Step 1: Show that, for all h in H^1,

$$L(h) = \int_\Omega \{h(t) - t \cdot \nabla h(t)\} f(t)\, dt + \int_\Omega \nabla C(U^*(t)) \cdot h(t) f(t)\, dt.$$

The first integral corresponds to the linear part of ϕ and requires no proof. The second integral is obtained by remarking that

$$\lim_{\epsilon \to 0^+} \frac{C(\nabla U^*(t) + \epsilon \nabla h(t)) - C(\nabla U^*(t))}{\epsilon} = \nabla C(\nabla U^*(t)) \cdot \nabla h(t),$$

the difference being bounded by $(1/2)M\|\nabla h(t)\|^2$, thanks to assumption (4.5).

Step 2: Transform the above formula by using the divergence theorem. We obtain

$$\int_\Omega (\nabla C(U^*(t)) - t) \cdot \nabla h(t) f(t)\, dt = \int_{\partial\Omega} \beta(t)\, dt + \int_\Omega \operatorname{div}[(t - \nabla C(U^*(t))) f(t)] h(t)\, dt,$$

and the result is established.

PROOF OF LEMMA 3: ϕ is concave and K^* is convex; therefore the maximum U^* of ϕ on K^* is characterized by the first order condition

(A.5) $\{U^* \in K^* \text{ and } \forall U \in K^* \phi'(U^*)(U - U^*) \leq 0\}.$

Let us apply this condition in turn to $U = U_0$ and $U = (1/2)(U_0 + U^*)$, which both belong to K^*. We obtain: $\phi'(U^*)(U_0 - U^*) \leq 0$ and $(1/2)\phi'(U^*)(U^* - U_0) \leq 0$.

This implies $\phi'(U^*)(U^* - U_0) = 0$, and the second property is established. Using this, and the fact that $U - U^* = (U - U_0) - (U^* - U_0)$, we see that (A.5) is then equivalent to the first property. The converse is immediate. *Q.E.D.*

PROOF OF THEOREM 2: The properties of $\alpha(t)$ and $\beta(t)$ result directly from the decomposition of $\phi'(U^*)h$ (relation (3.9)) and Lemma 2. It remains to establish that U^* is $\mathscr{C}^1$ on $\bar{\Omega}$, or equivalently that q^* is continuous on $\bar{\Omega}$. This property is immediate in the interiors of Ω_0 (since $q^*(t) \equiv q_0$) and Ω_1 (since q^* satisfies the partial differential equation $\alpha(t) = 0$). The only difficulty is to prove that $q^*(t)$ tends to q_0 when t tends to the free boundary Γ. This "smooth pasting" condition is in fact standard in obstacle problems and is established by a direct application of the penalization methods of Kinderlehrer and Stampacchia (1980). A complete proof is available from the authors upon request.

PROOF OF PROPOSITION 5: It follows directly from Cartier's theorem and Theorem 1 of Luenberger (1969, Chapter 1), which guarantees the existence of a Lagrange multiplier λ for the constraint $h \geq 0$. Indeed, by the assumption of Proposition 2 we have

$$0 = \inf\left\{ \int h\, d\mu, h \text{ convex}, h \geq 0 \right\}.$$

Taking $X = Z = \mathscr{C}(\bar{\Omega})$, $\mathscr{C}$ the set of convex functions on $\bar{\Omega}$ and G: $u \to -u$, we can apply Theorem 1 of Luenberger. Thus there exists a positive measure λ such that

$$0 = \inf\left\{ \int h\, d\mu - \int h\, d\lambda, h \text{ convex} \right\}.$$

In other words, the measure $\mu - \lambda$ is nonnegative on $\mathscr{C}$. By Cartier's theorem (see Meyer (1966)), there exists a sweeping operator T such that $(\mu - \lambda)_+ = T(\mu - \lambda)_-$.

Let $\nu = (\mu - \lambda)_-$. We have thus $\mu = \lambda + T\nu - \nu$, which was to be proven. *Q.E.D.*

APPENDIX 3

PROOF OF PROPOSITION 6: It results from the application of the characterization obtained in Theorem 2 to the condition (6.2):

$$\int \{\overline{U}(t) - U_0(t)\}\, d(\lambda + T\nu - \nu)(t) = 0. \tag{A.6}$$

Since $\overline{U} - U_0$ is nonnegative and λ is a positive measure, we have

$$\int \{\overline{U}(t) - U_0(t)\}\, d\lambda(t) \geq 0. \tag{A.7}$$

Moreover $\overline{U} - U_0$ is convex, T is a sweeping operator and ν is a positive measure. Therefore

$$\int \{\overline{U}(t) - U_0(t)\}\, d(T\nu - \nu) \geq 0. \tag{A.8}$$

Putting together relations (A.6), (A.7), and (A.8), we see that relations (A.8) and (A.9) are in fact equalities. In particular,

$$\int \{\overline{U}(t) - U_0(t)\}\, d\lambda(t) = 0,$$

which means exactly that supp $\lambda \subset \Omega_0$. We also have, for the same reason,

$$\int \{\overline{U}(t) - U_0(t)\}\, d(T\nu)(t) = \int \{\overline{U}(t) - U_0(t)\}\, d\nu(t). \tag{A.10}$$

Recall now the incentive compatibility conditions,

$$\forall s, t \quad \overline{U}(s) \geq \overline{U}(t) + \overline{q}(t) \cdot (s - t),$$

with equality if and only if s and t belong to the same bunch. We also have, naturally,

$$\forall s, t \quad U_0(s) = U_0(t) + q_0 \cdot (s - t).$$

By definition of a sweeping operator (property (6.5)), these two properties imply

$$\forall t \int \{\overline{U}(s) - U_0(s)\} T(ds, t) \geq U(t) - U_0(t). \tag{A.11}$$

Integrating this relation with respect to the positive measure ν, we obtain

$$\int \{\overline{U}(t) - U_0(t)\}\, d(T\nu)(t) \geq \int \{\overline{U}(t) - U_0(t)\}\, d\nu(t).$$

Comparing with (A.10) we see that we have in fact equality: this can only be true if (A.11) is an equality for ν a.e. t, which in turn implies that the support of $T(t, \cdot)$ is included in the "bunch" $\Omega(q(t))$ for ν a.e. t. *Q.E.D.*

PROOF OF THEOREM 2′: Let us start by proving the properties of $\alpha(\cdot)$ and $\beta(\cdot)$. They result from Proposition 3. For example in the nonbunching region Ω_1, λ is zero and T is trivial so that $\mu_+ = \mu_-$ or equivalently $\mu = 0$. Therefore for all t in $\Omega_1 \cap \Omega$, $\alpha(t) = 0$ and for all t in $\Omega_1 \cap \partial\Omega$, $\beta(t) = 0$.

Similarly for all t belonging to the bunch $\Omega(q)$, the second part of Proposition 3 implies that $T(t,\cdot)$ is supported by $\Omega(q)$. Therefore the restriction of T to $\Omega(q)$ is a sweeping operator on $\Omega(q)$ and the restriction of μ to $\Omega(q)$ satisfies Cartier's property:

$$\mu_+^{\Omega(q)} = T\mu_-^{\Omega(q)}.$$

Finally in the nonparticipation region Ω_0, we only have that $\mu = \lambda + T\nu - \nu$, where T is a sweeping operator on Ω_0, and λ, ν are positive measures supported by Ω_0. This tells us only two things:

(i) $\mu(\Omega_0) = 1$ (this comes from the joint facts that $\mu(\Omega) = -\phi'(\overline{U})\mathbf{1} = 1$, and $\mu(\Omega_B) = 0$, given that T preserves the mass).

(ii) If t is an extreme point of Ω_0, we have $\beta(t) \geq 0$, since a sweeping operator T is necessarily trivial on the set $\mathscr{E}_0$ of such extreme points. Therefore on Ω_0: $\mu = \lambda \geq 0$.

Finally, the proof that $\bar{q}$ is continuous (Proposition 7) is technical: we will skip it for the sake of conciseness. It is available from the authors upon request.

APPENDIX 4

Proof of Proposition 8: In the unidimensional case ($K = 1$, $\Omega = [a, b]$), recall that, for $t \in]a, b[$,

$$\alpha(t) = \frac{d}{dt}[(t - C'(\bar{q}(t)))f(t)] + f(t), \quad \text{and}$$

$$\begin{cases} \beta(a) = -(C'(\bar{q}(a)) - a)f(a), \\ \beta(b) = +(C'(\bar{q}(b)) - b)f(b). \end{cases}$$

As is well known, there is never any bunching "at the top," so that $\beta(b) = 0$. If there is no bunching either "at the bottom," i.e. when $\Omega_0 = \{a\}$, condition (5.8) means exactly. that λ is concentrated in a. Therefore $\lambda = \beta(a)\delta_a$, where δ_a denotes the Dirac measure in a. Moreover, since λ has unit mass, $\beta(a)$ equals one, which means $C'(\bar{q}(a)) = a - (1/f(a))$. Similarly, at any interior point t where there is no bunching, we have $\alpha(t) = 0$, which by continuity of $\bar{q}(\cdot)$ implies

$$C'(\bar{q}(t)) = t - \frac{1 - F(t)}{f(t)}.$$

Suppose now that $\Omega(q) = [t_1, t_2]$ is an interior bunch. Then Proposition 3 means that $\alpha(t)$ changes sign on $[t_1, t_2]$, with, moreover, $\int_{t_1}^{t_2} \alpha(t)\,dt = \int_{t_1}^{t_2} t\alpha(t)\,dt = 0$. Now by definition, for all t in $[t_1, t_2]$,

$$\alpha(t) = \frac{d}{dt}[(t - C'(q))f(t)] + f(t).$$

This is the derivative of $A(t) = [t - C'(q)]f(t) + F(t) - 1$. Therefore the first condition means

$$[t_1 - C'(q)]f(t_1) + F(t_1) = [t_2 - C'(q)]f(t_2) + F(t_2).$$

By continuity of $\bar{q}(\cdot)$ at t_1 and t_2, this quantity necessarily equals 1, so that

$$C'(q) = t_1 - \frac{1 - F(t_1)}{f(t_1)} = t_2 - \frac{1 - F(t_2)}{f(t_2)}.$$

The second condition can be transformed by integrating by parts:

$$0 = \int_{t_1}^{t_2} tA'(t)\,dt = t_2 A(t_2) - t_1 A(t_1) - \int_{t_1}^{t_2} A(t)\,dt.$$

Since $A(t_2) = A(t_1) = 0$, this condition is satisfied if the integral of $A(t)$ on $[t_1, t_2]$ equals 0. Now, we also have that

$$\forall t \in \Omega(q): \quad C'(q) = t - \frac{1 - F(t)}{f(t)} - \frac{A(t)}{f(t)},$$

where

$$A(t_1) = A(t_2) = 0,$$

and

$$\int_{t_1}^{t_2} A(t)\, dt = 0.$$

The last condition to be checked is that $\alpha(t)$ is indeed obtained by a sweeping operation. This means that for any convex function $h: [t_1, t_2] \to \mathbb{R}$ one has

$$\int_{t_1}^{t_2} h(t)\, \alpha(t)\, dt \geq 0.$$

Since $\alpha(t) = A'(t)$ and $A(t_1) = A(t_2) = 0$, an integration by parts shows that this is equivalent to

$$\int_{t_1}^{t_2} k(t) A(t)\, dt \leq 0,$$

for all nondecreasing function $k = h'$ (the derivative of a convex function). As is well known, this is satisfied if and only if

$$\forall x \in [t_1, t_2] \quad \int_{t_1}^{x} A(t)\, dt \leq 0.$$

It is easy to see that the above conditions correspond exactly to the "ironing" conditions obtained by Mussa and Rosen (1978) for dealing with bunching in dimension one. Therefore, our "sweeping" procedure is the natural extension of "the ironing procedure" to multidimensional contexts. *Q.E.D.*

PROOF OF PROPOSITION 9: We start by computing the measure μ. For t in Ω we have

$$\alpha(t) = f(t) + \operatorname{div}((t - q(t)) f(t))$$
$$= (3 - \tau + q_1(t) + q_2(t) - \operatorname{div} q(t)) \exp(2a - \tau).$$

$\partial\Omega$ consists of two types of points:

(i) $M_1(\tau) \stackrel{\text{def}}{=} (a, \tau - a)$ where $\vec{n}(M_1(\tau)) = -(1,0)$ and $\beta(M_1(\tau)) = (a - q_1(a, \tau - a))\exp(2a - \tau)$.
(ii) $M_2(\tau) \stackrel{\text{def}}{=} (\tau - a, a)$ where $\vec{n}(M_2(\tau)) = -(0,1)$, and $\beta(M_2(\tau)) = (a - q_2(\tau - a, a))\exp(2a - \tau)$.

Let us check the conditions obtained in Theorem 2′. We start by Ω_0, where we have

$$q(t) \equiv 0, \quad \text{so that} \quad \alpha(t) = (3 - \tau)\exp(2a - \tau) \quad \text{and}$$
$$\beta(M_1) = \beta(M_2) = a \exp(2a - \tau).$$

The first free boundary Γ_0 is determined by condition (5.1): $\mu(\Omega_0) = 1$. After easy computations, we find that this is equivalent to the equation: $(\tau_0 - 2a)(\tau_0 - 1) = 1$. The relevant solution (i.e. such that $\tau_0 > 2a$) is

$$\tau_0 = \frac{1}{2} + a + \sqrt{\left(a - \frac{1}{2}\right)^2 + 1}\,.$$

Notice that $\beta \geq 0$ on $\partial_0 \Omega$ but that α can be positive or negative. In the nonbunching region Ω_1, $\overline{U}$ satisfies the Euler equation,

$$\Delta\overline{U} - \frac{\partial \overline{U}}{\partial t_1} - \frac{\partial \overline{U}}{\partial t_2} = 3 - t_1 - t_2.$$

Together with the boundary condition

$$\begin{cases} \dfrac{\partial \overline{U}}{\partial t_1}(t) = a & \text{if } t_1 = a \text{ or } t_1 + t_2 = \tau_1, \\ \dfrac{\partial \overline{U}}{\partial t_2}(t) = a & \text{if } t_2 = a \text{ or } t_1 + t_2 = \tau_1. \end{cases}$$

This equation has solutions (which only differ by a constant term) if and only if the following compatibility condition is satisfied:

$$\int_{\Omega_1} f(t)\,dt = \int_{\Gamma_1} (t - q(t))\cdot \vec{n}(t) f(t)\,d\sigma(t).$$

This is easily shown to be equivalent to the following equation, which determines the second free boundary $\Gamma = \{t, t_1 + t_2 = \tau_1\}$: $(\tau_1 - 2a)^2 = (\tau_1 - 2a) + 1$. The relevant solution (i.e. such that $\tau_1 > 2a$) is

$$\tau_1 = 2a + \frac{1 + \sqrt{5}}{2}.$$

Finally, in the bunching region $\Omega_B = \{t, \tau_0 < t_1 + t_2 < \tau_1\}$, $\overline{U}$ only depends on $t_1 + t_2$, so that

$$q_1(t) = \frac{\partial \overline{U}}{\partial t_1}(t) \quad \text{and} \quad q_2(t) = \frac{\partial \overline{U}}{\partial t_2}(t)$$

are equal. In other words, all consumers such that $t \in \Omega_B$ consume the same quantity of the two goods. We denote this quantity by $q_B(\tau)$. On each bunch $\Omega(q) = \{t, t_1 + t_2 = \tau$, with $q_B(\tau) = q\}$ we have

$$\alpha(t) = \exp(2a - \tau)(3 - \tau - 2\dot{q}_B(\tau) + 2q_B(\tau)),$$

and

$$\beta(M_1(\tau)) = \beta(M_2(\tau)) = \exp(2a - \tau)(a - q_B(\tau)).$$

Notice that α is constant on each bunch and that β is symmetric. Therefore the sweeping conditions reduce to $\mu(\Omega(q)) = 0$ (total mass is preserved), and $\beta(M_1) = \beta(M_2) \geq 0$ (positive part is on the boundary).

The first condition gives a differential equation in q_B:

$$(3 - \tau - 2q_B' + 2q_B)(\tau - 2a) - (a - q_B) = 0.$$

Its solutions have the following form:

$$q_B(\tau) = k\frac{\exp(\tau - 2a)}{\tau - 2a} + \frac{1}{2}\left(\tau - 1 - \frac{1}{\tau - 2a}\right),$$

where k is an arbitrary constant. The continuity of $\overline{q}$ ("smooth pasting") implies two conditions:

$$q_B(\tau_0) = 0 \quad \text{and} \quad q_B(\tau_1) = a.$$

There is apparently an overdetermination of k, but it turns out that $k=0$ leads to a $q_B(\cdot)$ that satisfies both conditions. Therefore the expression of q_B in the bunching region Ω_B is actually very simple:

$$q_B(\tau)=\frac{1}{2}\left[\tau-1-\frac{1}{\tau-2a}\right].$$

Notice that $q_B(\cdot)$ is increasing (so that $\bar{U}$ is indeed convex on Ω_B),[30] and that the sign condition $\beta \geq 0$ (i.e., $q_B(\tau)\leq a$) is satisfied. Clearly, the optimal solution involves offering a limited choice $q_1=q_2=q_B$ for all quantities below a (this is usually referred to as "commodity bundling"). All the consumers who only want to consume small quantities of the goods are restricted to choose the bundle (q_B, q_B) that maximizes their net utility:

$$\bar{U}(t)=\max_{q_B}\{(t_1+t_2)q_B-q_B^2-P_B(q_B)\}.$$

The first order condition gives $\tau=2q_B(\tau)+P_B'(q_B(\tau))$. Comparing with the expression of $q_B(\tau)$ given above, we find $P_B'(q_B)=\tau(q_B)-2q_B$, where $\tau(q_B)$ is the greatest solution of the equation

$$\tau-1-\frac{1}{\tau-2a}=2q_B.$$

This gives

$$\tau(q_B)=a+q_B+\frac{1}{2}+\sqrt{\left(a-q_B-\frac{1}{2}\right)^2+1},$$

and finally

$$P'(q_B)=a-q_B+\frac{1}{2}+\sqrt{\left(a-q_B-\frac{1}{2}\right)^2+1}.$$

Notice that $P'(0)=\tau_0>2a$ and that $P'(a)=\tau_1-2a$.

REFERENCES

ARMSTRONG, M. (1996): "Multiproduct Nonlinear Pricing," *Econometrica*, 64, 51–75.

BARON, D. P., AND R. B. MYERSON (1982): "Regulating a Monopolist with Unknown Costs," *Econometrica*, 50, 911–930.

BIAIS, B., P. BOSSAERTS, AND J. C. ROCHET (1996): "An Optimal Mechanism for IPOs," Mimeo, GREMAQ-IDEI, University of Toulouse.

CHONÉ, P., AND J. C. ROCHET (1997): "Computation and Estimation of Optimal Nonlinear Prices," Mimeo, GREMAQ-IDEI, University of Toulouse.

DUDLEY, R. M. (1977): "On Second Derivatives of Convex Functions," *Mathematical Scandinavian*, 41, 159–174.

ENGERS, M. (1987): "Signaling with Many Signals," *Econometrica*, 55, 663–674.

GUESNERIE, R., AND J. J. LAFFONT (1984): "A Complete Solution to a Class of Principal Agent Problems with an Application to the Control of a Self-Managed Firm," *Journal of Public Economics*, 25, 329–369.

JEHIEL, P., B. MOLDOVANU, AND E. STACCHETTI (1996): "Multidimensional Mechanism Design for Auctions with Externalities," Mimeo, University of Mannheim.

JULLIEN, B. (1996): "Participation, Efficiency and Optimal Contracts under Adverse Selection," Mimeo, GREMAQ-IDEI, University of Toulouse.

[30] The convexity of $\bar{U}$ on Ω_1 has been checked numerically.

KINDERLEHRER, D., AND G. STAMPACCHIA (1980): *An Introduction to Variational Inequalities*. Boston: Academic Press.

KOHLEPPEL, L. (1983): "Multidimensional Market Signalling," Mimeo, Bonn University.

LAFFONT, J. J., AND D. MARTIMORT (1997): "Collusion under Asymmetric Information," *Econometrica*, 65, 875–912.

LAFFONT, J. J., E. MASKIN, AND J. C. ROCHET (1987): "Optimal Non Linear Pricing with Two Dimensional Characteristics," in *Information, Incentives and Economic Mechanisms*, ed. by T. Groves, R. Radner, and S. Reiter. Minneapolis: Univ. of Minnesota Press, pp. 256–266.

LEWIS, T. R., AND D. M. SAPPINGTON (1988): "Regulating a Monopolist with Unknown Demand and Cost Functions," *Rand Journal of Economics*, 19, 438–457.

LUENBERGER, D. (1969): *Optimization by Vector Space Methods*. New York: John Wiley and Sons.

MASKIN, E., AND J. RILEY (1984): "Monopoly with Incomplete Information," *The Rand Journal of Economics*, 15, 171–196.

MATTHEWS, S., AND J. MOORE (1987): "Monopoly Provision of Quality and Warranties: An Exploration of the Theory of Multidimensional Screening," *Econometrica*, 55, 441–467.

MCAFEE, R. P., AND J. MCMILLAN (1988): "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 46, 335–354.

MEYER, P. A. (1966): *Probabilities and Potentials*. Waltham, Massachusetts: Blaisdell Publ. Co.

MIRRLEES, J. A. (1971): "An Exploration of the Theory of Optimum Income Taxation," *Review of Economic Studies*, 38, 175–208.

MUSSA, M., AND S. ROSEN (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, 18, 301–317.

MYERSON, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–63.

QUINZII, M., AND J. C. ROCHET (1985): "Multidimensional Signalling," *Journal of Mathematical Economics*, 4, 113–128.

ROBERTS, K. W. (1979): "Welfare Considerations on Nonlinear Pricing," *Economic Journal*, 89, 66–83.

ROCHET, J. C. (1984): "Monopoly Regulation with Two Dimensional Uncertainty," Mimeo, Université Paris 9.

——— (1987): "A Necessary and Sufficient Condition for Rationalizability in a Quasi Context," *Journal of Mathematical Economics*, 16, 191–200.

RODRIGUES, E. (1987): *Obstacle Problems in Mathematical Physics*. Amsterdam: North Holland.

ROTHSCHILD, M., AND J. STIGLITZ (1970): "Increasing Risk I: a Definition," *Journal of Economic Theory*, 2, 225–243.

SALOP, S. (1979): "Monopolistic Competition with Outside Goods," *Bell Journal of Economics*, 10, 141–156.

SEADE, J. (1979): "Non Linear Prices with Multidimensional Consumers," Mimeo, Bellcore, Livingston, N.J.

SPENCE, A. M. (1980): "Multi-product Quantity-dependent Prices and Profitability Constraints," *Review of Economic Studies*, 47, 821–842.

SRINAGESH, P. (1991): "Non Linear Prices with Multidimensional Consumers," Bellcore, Livingston, N.J., USA.

WILSON, R. (1985): "Multidimensional Signaling," *Economics Letters*, 19, 17–21.

——— (1993): *Non Linear Pricing*. Oxford: Oxford University Press.