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ASSORTATIVE MATCHING AND SEARCH[1]

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In Becker's (1973) neoclassical marriage market model, matching is positively assortative if types are complements: i.e., match output $f(x, y)$ is *supermodular* in x and y. We reprise this famous result assuming time-intensive partner search and transferable output. We prove existence of a search equilibrium with a continuum of types, and then characterize matching. After showing that Becker's conditions on match output no longer suffice for assortative matching, we find sufficient conditions valid for any search frictions and type distribution: supermodularity not only of output f, but also of $\log f_x$ and $\log f_{xy}$. Symmetric submodularity conditions imply negatively assortative matching. Examples show these conditions are necessary.

KEYWORDS: Search frictions, matching, assignment.

1. INTRODUCTION

THIS PAPER REEXAMINES a classic insight of the assignment literature—when matching is assortative—in an environment with search frictions. We assume a continuum of heterogeneous agents who can produce only in pairs. If two agents form a match, they generate a flow of divisible output. We depart from the neoclassical assignment literature (e.g., Becker (1973)) in assuming that match creation is time consuming: each unmatched agent faces a Poisson arrival of potential mates (Diamond (1982), Mortensen (1982), Pissarides (1990)). As matching precludes further search, agents must weigh the opportunity cost of ceasing to search for better options, against the benefit of producing immediately.

Individuals' behavior is described by their acceptance sets, which specify with whom they are willing to match; only mutually acceptable matches are consummated. When agents match, they evenly divide the match surplus, i.e., their output flow in the match less their values while searching, as in the Nash bargaining solution. Equilibrium requires that everyone's acceptance set maxi-

[1]This paper answers questions stemming from a 1994 version of our mimeo "Matching, Search, and Heterogeneity." The current version of that paper solves a constrained social planner's problem and focuses on the relationship between equilibrium and socially optimal matching patterns.

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mizes her expected payoff, and the distribution of unmatched agents is in steady state. We provide what we believe to be the first general existence theorem for search models with ex ante heterogeneous agents. The proof is complicated by the observation that agents' acceptance sets affect the steady state unmatched distribution, and thus the agents' matching opportunities and their willingness to accept matches.

We then turn to assortative matching. Becker's (1973) sufficient condition in the frictionless model is well-known. Assume that types x and y in $[0,1]$ produce $f(x,y)$ when matched and nothing otherwise. With complementarity ($f_{xy}>0$), the marginal product of a higher partner rises in one's type; therefore, in a core allocation, matching is positively assortative—matched partners are identical. The easy derivation of this famous result offers hope that it naturally extends to a model with search frictions: Agents match with an interval of types around their own. Examples in Figure 3 disprove this. First, for the complementary function $f(x,y)=(x+y-1)^2$, type $\frac{1}{2}$ produces nothing in the core allocation. With search frictions, nearby agents will match with $\frac{1}{2}$. When type $\frac{1}{2}$ agents meet each other, they then prefer to wait for a profitable match, as matching produces nothing, but precludes further search. By continuity, this argument extends to types near $\frac{1}{2}$. These agents match with higher and lower types, but not among themselves.

That $(\frac{1}{2},\frac{1}{2})$ minimizes f is inessential to this critique. Let $f(x,y)=(x+y)^2$ and suppose "high" types are willing to match with "middle" types. This opportunity is wonderful for middle types, and so if two of them should meet, they prefer to continue to search for high types. On the other hand, if they meet a "low" type without such a valuable option, matching is mutually agreeable.

Despite these setbacks, we find restrictions on the production function alone that ensure assortative matching for any search frictions or type distribution. By this we formally mean that any two matches can be severed, and the greater two and lesser two types agreeably rematched. Observe that our motivational failures of assortative matching in Figure 3 featured individuals with nonconvex matching sets—some agents only willing to match with higher and lower types. In fact, we show that matching set convexity is logically necessary for assortative matching, and—along with simple conditions that orient matching sets—suffices as well. Convexity, in turn, follows if all agents' preferences over partners' types are single-peaked.

This suggests an indirect attack on assortative matching. We show that single-peaked preferences, and hence assortative matching, ask that not only the production function be supermodular,[3] but also its log first- and cross-partial derivatives, $\log f_x$ and $\log f_{xy}$. Supermodularity of f ensures that any high enough type's utility rises in her partner's type; supermodularity of $\log f_x$ yields single-peaked preferences for low types; and supermodularity of $\log f_{xy}$ provides a single-crossing property which allows us to classify every type as either low or high. Finally, we prove that negatively assortative matching—matching with opposite types—obtains under symmetric submodularity conditions.

[3] We define supermodularity in Assumption A1-Sup. See Topkis (1998) for details.

We are aware of two other papers that consider "transferable utility" (i.e., shared output) search models with ex ante heterogeneity.[4] Sattinger (1995) does not explore the link with models in the frictionless assignment literature. Lu and McAfee (1996) establishes assortative matching when $f(x, y) \equiv xy$, a production function that satisfies our sufficient conditions. However, they do not consider other production functions, and so do not touch on the necessary and sufficient conditions that are central to our paper. In addition, Lu and McAfee's (1996) existence proof sidesteps the endogeneity of the unmatched distribution. While Sattinger (1995) endogenizes the unmatched distribution, he does not prove existence of a search equilibrium.

By way of overview, Section 2 summarizes Becker's two frictionless results, and then introduces our search model. We define and characterize search equilibria in Section 3, and establish their existence in Section 4. Section 5 first defines assortative matching and shows that it requires convex matching sets. We then prove that convex matching sets and Becker's condition ensure assortative matching. Finally, we prove that our three supermodularity condition imply convex matching sets, and address necessity with counterexamples. Less intuitive proofs are appendicized.

2. THE MODEL

There is an atomless continuum of agents, each indexed by her exogenously given and publicly observable productivity type $x \in [0, 1]$. Normalize the mass of agents to unity, and let $L : [0, 1] \mapsto [0, 1]$ be the type distribution. Associated with this is a type density function l. For the existence of an equilibrium, we require that l be positive and boundedly finite: $0 < \underline{l} < l(x) < \bar{l} < \infty$ for all x. Our language in the paper, as well as the interpretation we lend to it, implicitly assumes a continuum of every type of agent. Agents then belong to the graph $\{(x, i) | x \in [0, 1],\ 0 \leq i \leq l(x)\}$ in $\mathbb{R}^2$ with Lebesque measure, where i is an index number of the type x agent.

Without loss of generality, normalize the flow output of an unmatched agent to 0. When agents (types) x and y are matched together, their flow output depends on their types, $f : [0, 1]^2 \mapsto \mathbb{R}$. We later refer to a basic set of assumptions:

A0 (REGULARITY CONDITIONS): *The production function $f(x, y)$ is nonnegative, symmetric ($f(x, y) \equiv f(y, x)$), continuous, and twice differentiable, with uniformly bounded first partial derivatives on* $[0, 1] \times [0, 1]$.

2.1. *The Frictionless Matching Benchmark*

In the core allocation, prices allocate the scarce resource, high productivity agents. When is there *positively assortative matching* (PAM), where each agent

[4] Morgan (1995), Burdett and Coles (1997), and Smith (1998) study heterogeneous-agent search models with nontransferable utility (NTU), i.e., with exogenous output sharing rules.

matches with another of the same type? A sufficient condition is that types be complementary:

A1-SUP (STRICT SUPERMODULARITY): *The production function f is strictly supermodular. That is, the own marginal product of any* $x>0$ *is strictly increasing in her partner's type; or if* $x>x'$ *and* $y>y'$, *then* $f(x,y)+f(x',y')>f(x,y')+f(x',y)$.

If A1-Sup obtains, all agents have higher marginal products when they match with high productivity agents. In the core allocation there must be PAM. The proof of this well-known result of Becker is simple: Any allocation in which some type x agents match with type $x'\neq x$ agents admits a Pareto-improvement. Output rises if all such agents rematch with another of their own type, since A1-Sup implies $f(x,x)+f(x',x')>2f(x,x')$ whenever $x\neq x'$. Thus the unique output-maximizing allocation entails PAM, and hence so must the core.

A1-SUB (STRICT SUBMODULARITY): *The production function f is strictly submodular: if* $x>x'$ *and* $y>y'$, *then* $f(x,y)+f(x',y')<f(x,y')+f(x',y)$.

Under A1-Sub, the unique core allocation entails *negatively assortative matching* (NAM): Each agent x matches with her "opposite" type $y(x)$, where $L(x)+L(y(x))\equiv 1$. For A1-Sub implies that if there are four agents, $z_1<z_2\leq z_3<z_4$, the allocation in which z_1 and z_4 are matched and z_2 and z_3 are matched Pareto dominates the two other possible allocations in which these four agents match in pairs.

Throughout, we maintain A1-Sup or A1-Sub. This excludes production functions with nonmonotonic marginal products (like $f(x,y)\equiv\max\langle x^2y,xy^2\rangle$, in Kremer and Maskin (1995)), for which matching patterns are not easily characterized.

2.2. *Matching with Search*

We now develop a continuous time, infinite horizon matching model with search frictions, in which meeting other agents is time-consuming and haphazard.

Action Sets: At any instant in continuous time, an agent is either *matched* or *unmatched.* Only the unmatched engage in (costless) search for a new partner. When two unmatched agents meet, they immediately observe each other's type. Either may veto the proposed match; it is only consummated if both accept. Since in a steady state environment, a match that is profitable to accept is profitable to sustain, we simplify our notation by ignoring the possibility of quits.

To maintain a steady state population of unmatched agents, we assume exogenous match dissolutions. Thus, nature randomly destroys any match with a constant flow probability (Poisson rate) $\delta>0$, i.e., it lasts an elapse time of t

with chance $e^{-\delta t}$. At the moment the match is destroyed, both agents re-enter the pool of searchers.

Preferences: Each agent maximizes her expected present value of payoffs, discounted at the interest rate $r > 0$. As in Becker, we assume that match output $f(x, y)$ is shared. Thus, x earns an endogenous flow payoff $\pi(x|y)$ when matched with y. Because payoffs exhaust match output, $\pi(x|y) + \pi(y|x) \equiv f(x, y)$.

Unmatched Agents and Search: Let $u \leq l$ denote the *unmatched density function*, i.e., $\int_X u(x)\,dx$ is the mass of unmatched agents with types $x \in X \subseteq [0, 1]$.

Search frictions capture the following story. Were it possible, an unmatched individual would meet a random unmatched or matched agent at the flow rate $\rho > 0$. However, it is infeasible to meet someone who is already matched—she is engaged, and so misses any meeting. Thus, one simply meets any $y \in Y \subseteq [0, 1]$ at a rate proportional to the mass of those unmatched in Y: $\rho \int_Y u(y)\,dy$. Our conclusion underscores that our descriptive theory extends well beyond this search technology, but we use this assumption in our equilibrium existence proof.

Strategies: A steady state (pure) strategy for agent of type x is a time-invariant[5] Borel measurable set $A(x)$ of agents with whom x is willing to match. (That agents of the same type use the same strategy is not a restriction, as will follow from §3.1.) The strategy depends only on the unmatched density function u, the sole payoff-relevant state variable.

Next, agent *x's matching set* $\mathcal{M}(x) \equiv A(x) \cap \{y | x \in A(y)\}$ is the set of acceptable types y who are willing to match with her. Call a match (x, y) *mutually agreeable* if $y \in \mathcal{M}(x)$. By construction, matching sets are symmetric, $y \in \mathcal{M}(x)$ if and only if $x \in \mathcal{M}(y)$. We shall sometimes consider the matching correspondence, $\mathcal{M}$: $[0, 1] \rightrightarrows [0, 1]$. Finally, for use in our existence result, Proposition 1, we define a match indicator function α: $\alpha(x, y) = 1$ if $y \in \mathcal{M}(x)$ and 0 otherwise.

Steady State: In steady state, the flow creation and flow destruction of matches for every type of agent must exactly balance. The density of matched agents $x \in [0, 1]$ is $l(x) - u(x)$; these agents' matches exogenously dissolve with flow probability δ. The flow of matches created by unmatched agents of type x is $\rho u(x) \int_{\mathcal{M}(x)} u(y)\,dy$. Putting this together, in steady state for all types $x \in [0, 1]$,

$$(1) \qquad \delta(l(x) - u(x)) = \rho u(x) \int_{\mathcal{M}(x)} u(y)\,dy = \rho u(x) \int_0^1 \alpha(x, y) u(y)\,dy.$$

[5]In a stationary world, assuming stationary acceptance sets is without loss of generality: As the strategy of no single agent affects the future state of the economy, if an acceptance set is optimal at time s, it remains so at time $t > s$.

3. SEARCH EQUILIBRIUM

In a *steady state search equilibrium* (SE), (i) everyone maximizes her expected payoff, taking all other strategies as given; (ii) if matching weakly increases both agents' payoffs, then they both accept the match;[6] and (iii) all unmatched rates are in steady-state. This section formalizes (i) and (ii) in a compact, recursive way.

3.1. *Analytic Description of Search Equilibrium*

The Two Bellman Equations: Let $W(x)$ denote the expected value of an unmatched agent x. Similarly, let $W(x|y)$ be the present value for x while matched with y, and thus $S(x|y) \equiv W(x|y) - W(x)$ is her "personal" surplus when matched. We begin by providing the Bellman equations solved by these values.

While unmatched, x earns nothing, but at flow rate $\rho\int_{\mathcal{M}(x)} u(y)\,dy$, she meets and matches with some $y \in \mathcal{M}(x)$, enjoying a capital gain $S(x|y)$. Summarizing:

$$rW(x) = \rho\int_{\mathcal{M}(x)} S(x|y)u(y)\,dy. \tag{2}$$

Similarly, x gets an endogenous flow payoff $\pi(x|y)$ when matched with y. With Poisson rate δ, her match is destroyed, and she suffers a capital loss $S(x|y)$. Hence,

$$rW(x|y) = \pi(x|y) - \delta S(x|y). \tag{3}$$

Match Surplus Division: Search frictions create temporary bilateral rents, since an agreeable match now is generically strictly preferred to waiting for a better future match. This shifts the determination of the flow payoffs π into the realm of bargaining theory. We follow a number of authors, e.g., Pissarides (1990), closing the model with the Nash bargaining solution: namely, $S(x|y) \equiv S(y|x)$ for all (x, y). Using this, equation (3), and the resource constraint $\pi(x|y) + \pi(y|x) \equiv f(x, y)$, we have

$$S(x|y) = \frac{f(x,y) - rW(x) - rW(y)}{2(r+\delta)}. \tag{4}$$

Personal surplus is half the excess of flow match output over both flow unmatched values. Discounting accounts both for impatience and match impermanence.

Matching Sets: In a SE, an agent's strategy is to accept any match that weakly exceeds her expected present unmatched value: $S(x|y) \geq 0$ if and only if

[6] In our 1996 working paper (available on request), we allow that agents may reject a match if they are just indifferent. This generalization does not affect our conclusions. We omit it here, as it complicates our analysis throughout the paper.

$y \in A(x)$. Since $S(x|y) \equiv S(y|x)$, this implies $y \in A(x)$ if and only if $x \in A(y)$, and so $\mathcal{M}(x) \equiv A(x)$. Thus by (4), the mutual optimality condition is

$$(5) \qquad S(x,y) \equiv f(x,y) - w(x) - w(y) \geq 0 \Leftrightarrow y \in \mathcal{M}(x)$$

where $w(x) \equiv rW(x)$ is the *average* present value of an unmatched agent, her "reservation wage," and $S(x,y)$ is the *(flow) match surplus*.

The Value Equation: Substituting (4) into (2) yields an implicit value equation

$$(6) \qquad w(x) = \theta \int_{\mathcal{M}(x)} (f(x,y) - w(x) - w(y)) u(y)\, dy$$

$$\equiv \theta \int_{\mathcal{M}(x)} s(x,y) u(y)\, dy$$

where $\theta \equiv \rho/2(r+\delta)$. An agent's unmatched value is proportional to her share of match surplus.

Summary: A SE is fully described by specifying: (i) who is matched with whom (matching sets $\mathcal{M}$); (ii) the measure of types searching (unmatched density u); and (iii) how much everyone's time is worth (unmatched value w).

DEFINITION (SE CHARACTERIZATION): A SE can be represented as a triple $(w, \mathcal{M}, u)$ where: w solves the implicit system (6), given $(\mathcal{M}, u)$; $\mathcal{M}$ is optimal given w, i.e., it obeys (5); and u solves the steady state equation (1) given $\mathcal{M}$.

Figure 1 graphically depicts the matching sets for the production function $f(x,y) = xy$ as well as a particular choice of search frictions, impatience, and type distribution.[7] Since this production function satisfies A1-Sup, in the frictionless benchmark agents are only willing to match with their own type. With search frictions, agents match with an interval of types, including their frictionless partner.

3.2. *Properties of Value Functions and Matching Sets*

We now summarize the critical properties of the value function.

LEMMA 1: *Given A0, the value function w satisfies the value inequality*

$$(7) \qquad w(x) \geq \theta \int_M (f(x,y) - w(x) - w(y)) u(y)\, dy$$

for arbitrary $M \subseteq [0,1]$. In particular, w is nonnegative. Also, w is Lipschitz,

[7]Figures in this paper represent numerical approximations of the equilibrium matching sets. To create them, we divided the type space into 500 discrete types. We posited a matching set, calculated the associated steady state unmatched rates using (1), calculated the value function using (6), and then calculated a new matching set using (5). This tâtonnement process converged, and thus by definition, to a SE. The program is available upon request.

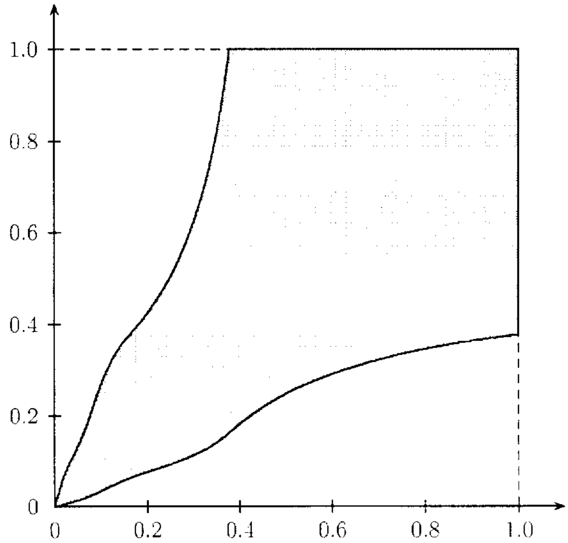

FIGURE 1.—Equilibrium Matching Sets: This depicts the matching sets for $f(x,y)=xy$, with $\delta=r$, $\rho=100r$, and a uniform distribution of agents, $L(x)=x$ for $x\in[0,1]$. If $x\in\mathcal{M}(y)$ (so $y\in\mathcal{M}(x)$ also), then the points (x,y) and (y,x) are shaded in the graph. The graph is therefore symmetric in (x,y). This is the same production function that Lu and McAfee (1996) use; therefore, this picture is similar to their Figure 1 (p. 129).

continuous, and a.e. differentiable in a SE. When differentiable, its derivative is

$$(8)\qquad w'(x)=\frac{\theta\int_{\mathcal{M}(x)}f_x(x,y)u(y)\,dy}{1+\theta\int_{\mathcal{M}(x)}u(y)\,dy}.$$

Here is an intuitive overview of the proof (found in Appendix A). First, the value inequality follows because matches are agreeable if and only if they produce nonnegative match surplus. For Lipschitz and continuity, anyone can do almost as well as nearby types simply by imitating their matching pattern, since the production function is continuous. If they optimize, they will do better still, so the value function cannot jump. Finally, when the matching set is suitably differentiable in x, $w'(x)$ is found by application of the Fundamental Theorem of Calculus. Surplus vanishes along the boundary of the matching set; therefore, we can ignore the effect of changes in the matching set, and simply differentiate (6) under the integral sign.

Regularity Assumption A0 also imposes some restrictions on matching sets.

LEMMA 2: *Posit A0. All matching sets $\mathcal{M}(x)$ are nonempty and closed and the matching correspondence $\mathcal{M}$ is upper hemicontinuous (u.h.c.).*

These conclusions are established in the proof of Lemma 1 (Part 3, Step 1).

4. EXISTENCE OF A SEARCH EQUILIBRIUM

PROPOSITION 1 (SE EXISTENCE): *Given A0 and A1-Sup or A1-Sub, a SE exists.*

Since values w play an analogous role to prices in Walrasian models, we look for a fixed point in an appropriate map from the space of players' value functions into itself. Even though a SE is a triple, this program works because values encode all the information needed to recover matching sets $\mathcal{M}$ and unmatched densities u.

The proof demonstrates the continuity of three maps: condition (5) maps value functions w into matching sets $\mathcal{M}$ (Lemma 3); equation (1) maps matching sets $\mathcal{M}$ into steady state unmatched densities u (Lemma 4); and equation (6) is the composite map of values and induced matching sets and unmatched densities into new values (Proposition 1). SE's are fixed points of these mappings.

Previous existence theorems for heterogeneous agent search models exploit an a priori known threshold structure of NTU matching sets to prove existence by construction (Morgan (1995), Burdett and Coles (1997)). Some have also assumed that the unmatched density u does not depend on agents' matching sets, an interaction that we believe is of significant economic interest.

We consider value functions w as elements of the space $C[0,1]$ of continuous maps on $[0,1]$ with the sup norm: $\|w\|_\infty = \sup_{x \in [0,1]}|w(x)|$. Instead of matching sets $\mathcal{M}$, for Lemma 3, we work with the associated match indicator functions α, and the norm $\|\alpha\|_1 = \int_0^1 \int_0^1 |\alpha(x,y)|\, dx\, dy < \infty$, for $\alpha \in \mathscr{L}^1([0,1]^2)$. However, we restrict focus to the convex set $\mathscr{A}$ of α's with range in $[0,1]$. Finally, to satisfy Lemma 4, unmatched densities u must be given the norm $\|u\|_1 = \int_0^1 |u(x)|\, dx$.

LEMMA 3: *Posit A0 and A1-Sup or A1-Sub. Any Borel measurable map $w \mapsto \alpha_w$ from value functions to match indicator functions in $\mathscr{A}$ solving* (5) *is continuous.*

A1-Sup or A1-Sub rule out an atom of zero surplus matches, a possibility that would invalidate Lemma 3 and considerably complicate the proof of Proposition 1.

LEMMA 4: *The map $\alpha \mapsto u_\alpha$ from match indicator functions in $\mathscr{A}$ to the unmatched density implied by the steady-state equation* (1) *is both well-defined and continuous.*

PROOF: *Step* 1: *$\alpha \mapsto u_\alpha$ is well defined, so there exists a unique solution u_α to* (1). The critical idea is a log transformation of the unmatched density. Let us for convenience define $\hat{\rho} \equiv \rho/\delta$. Let Γ be the set of measurable maps v of $[0,1]$ into $[\log \underline{l} - \log(1 + \hat{\rho}\bar{l}), \log \bar{l}]$. For all $x \in [0,1]$ and $v \in \Gamma$, define

$$\Phi_\alpha v(x) \equiv \log\left(\frac{l(x)}{1 + \hat{\rho}\int_0^1 \alpha(x,y) e^{v(y)}\, dy} \right).$$

Here, $u \equiv e^{v}$ solves the steady-state condition (1) if and only if $\Phi_\alpha v = v$. To prove the steady state unmatched density is unique, we show that Φ_α has a unique fixed point.

As $0 \le \alpha(x, y) \le 1$ and $\underline{l} < l(x) < \bar{l}$, one can verify that Φ_α maps Γ into itself. In this step alone, and not the continuity proof, we will use the sup-norm: $\|v\|_\infty = \sup_{x \in [0,1]} |v(x)|$, so that v belongs to the complete space $\mathscr{L}^\infty([0,1])$ of essentially bounded functions. By the Contraction Mapping Theorem, Φ_α has a unique fixed point if there is a $\chi \in (0,1)$ with $\|\Phi_\alpha v^1 - \Phi_\alpha v^2\|_\infty < \chi \|v^1 - v^2\|_\infty$ for any $v^1, v^2 \in \Gamma$.

Use the definition of Φ_α for arbitrary $x \in [0,1]$ and $v^1, v^2 \in \Gamma$:

$$\Phi_\alpha v^2(x) - \Phi_\alpha v^1(x) = \log\left(\frac{1 + \hat{\rho}\int_0^1 \alpha(x,y) e^{v^1(y)}\,dy}{1 + \hat{\rho}\int_0^1 \alpha(x,y) e^{v^2(y)}\,dy} \right)$$

$$\le \log\left(\frac{1 + \hat{\rho} e^{\|v^1 - v^2\|_\infty} \int_0^1 \alpha(x,y) e^{v^2(y)}\,dy}{1 + \hat{\rho}\int_0^1 \alpha(x,y) e^{v^2(y)}\,dy} \right)$$

$$\le \log\left(\frac{1 + \hat{\rho}\bar{l} e^{\|v^1 - v^2\|_\infty}}{1 + \hat{\rho}\bar{l}} \right).$$

The first inequality uses $e^{v^1(y)} \le e^{\|v^1 - v^2\|_\infty} e^{v^2(y)}$ for all y. Since $e^{\|v^1 - v^2\|_\infty} > 1$, the resulting fraction is increasing in the integral. That the integral is less than $\bar{l}$ yields the second inequality. To bound this final expression, observe

$$\frac{\log\left(1 + \hat{\rho}\bar{l} e^{\|v^1 - v^2\|_\infty}\right) - \log(1 + \hat{\rho}\bar{l})}{\|v^1 - v^2\|_\infty}$$

$$\le \frac{\log\left(\underline{l} + \hat{\rho}\bar{l}^2(1 + \hat{\rho}\bar{l})\right) - \log\left(\underline{l}(1 + \hat{\rho}\bar{l})\right)}{\log \bar{l} - \log \underline{l} + \log(1 + \hat{\rho}\bar{l})} \equiv \chi \in (0,1)$$

as the left hand side rises in $\|v^1 - v^2\|_\infty \le \log \bar{l} - \log \underline{l} + \log(1 + \hat{\rho}\bar{l})$, given $v^1, v^2 \in \Gamma$.

Reversing the roles of v^1 and v^2 proves that $|\Phi_\alpha v^2(x) - \Phi_\alpha v^1(x)| \le \chi \|v^1 - v^2\|_\infty$. Since this holds for all x, we have proven that $\|\Phi_\alpha v^2 - \Phi_\alpha v^1\|_\infty \le \chi \|v^1 - v^2\|_\infty$. Everywhere uniqueness of the solution to $v = \Phi_\alpha v$ follows.

Step 2: $\alpha \mapsto u_\alpha$ is continuous on the space $\mathscr{A}$. Intuitively, (1) forms a system of equations $G(u_\alpha, \alpha) = 0$, with G continuous. By some implicit function theorem, at points β near α, the unique solution to this equation u_β must lie near u_α. This logic requires that the derivative of G with respect to u be invertible and continuous, which we address in Appendix B. *Q.E.D.*

PROOF OF PROPOSITION 1: Given values w in $C[0,1]$, we follow the Schauder Fixed Point Theorem program in §17.4 of Stokey and Lucas (1989).

• STEP 1: THE BEST RESPONSE VALUE: Consider the map $T: C[0,1] \mapsto C[0,1]$

$$Tw(x) \equiv \frac{\theta \int_0^1 \max\langle f(x,y) - w(y), w(x)\rangle u^w(y)\,dy}{1+\theta \bar{u}^w} \tag{9}$$

where $u^w \equiv u_{\alpha_w}$ is the unmatched density implied by the value function w (by Lemmas 3 and 4), and $\bar{u}^w \equiv \int_0^1 u^w(z)\,dz$ is the implied mass of unmatched agents. By definition, a fixed point of the mapping $Tw = w$ is a SE.

• STEP 2: THE FAMILY $\mathscr{G}$: To establish the existence of a fixed point of the operator T, we need a nonempty, closed, bounded, and convex domain space $\mathscr{G} \subseteq C[0,1]$ such that (i) $T: \mathscr{G} \to \mathscr{G}$; (ii) $T(\mathscr{G})$ is an equicontinuous family; and (iii) T is a continuous operator. Let $\mathscr{G}$ be the space of Lipschitz functions w on $[0,1]$ satisfying $0 \le w(x) \le \sup_y f(x,y)$ for all x and $|w(x_2) - w(x_1)| \le \kappa |x_2 - x_1|$ for all x_1, x_2, where $\kappa \equiv \sup_{x,y} |f_x(x,y)|$, as in the proof of Lemma 1. This subset of $C[0,1]$ is clearly nonempty, closed, bounded, and convex.

• STEP 3: $T: \mathscr{G} \to \mathscr{G}$ IS CONTINUOUS AND $T(\mathscr{G})$ IS EQUICONTINUOUS: Quite easily, if $w \in [0, \sup_y f(\cdot, y)]$, then so is Tw. Next, $|Tw(x_2) - Tw(x_1)|$ is at most

$$\frac{\theta}{1+\theta\bar{u}^w} \int_0^1 |\max\langle f(x_2,y) - w(y), w(x_2)\rangle$$
$$- \max\langle f(x_1,y) - w(y), w(x_1)\rangle| u^w(y)\,dy$$
$$\le \frac{\theta \int_0^1 |\max\langle f(x_2,y) - f(x_1,y), w(x_2) - w(x_1)\rangle| u^w(y)\,dy}{1+\theta\bar{u}^w}.$$

Since $f(x,y)$ and $w(x)$ are each Lipschitz in x with modulus κ, Tw is Lipschitz with modulus $\kappa\theta\bar{u}^w/(1+\theta\bar{u}^w) \le \kappa$. A family of Lipschitz functions of the same modulus is equicontinuous. Finally, Appendix B proves continuity of T algebraically. *Q.E.D.*

5. DESCRIPTIVE THEORY

5.1. *Assortative Matching*

In the frictionless world, supermodularity (A1-Sup) ensures PAM—any type x only matches with another type x. In a frictional setting, individuals are generally willing to match with sets of agents, and so mismatch is the rule. Our first step is to formulate a sensible generalization of assortative matching to this environment.

DEFINITION: Take $x_1 < x_2$ and $y_1 < y_2$. There is PAM if the matching sets form a *lattice* in $\mathbb{R}^2$: $y_1 \in \mathscr{M}(x_1)$ and $y_2 \in \mathscr{M}(x_2)$ whenever $y_1 \in \mathscr{M}(x_2)$ and $y_2 \in \mathscr{M}(x_1)$. There is NAM if $y_1 \in \mathscr{M}(x_2)$ and $y_2 \in \mathscr{M}(x_1)$ whenever $y_1 \in \mathscr{M}(x_1)$

and $y_2 \in \mathscr{M}(x_2)$. This definition[8] generalizes Becker's frictionless one in Section 2.1, $x \in M(x)$ for all x. The presumed contrary matches do not exist with singleton matching sets, and so the implications are true. Moreover, with PAM and symmetric matching sets, an agent who is willing to match with someone, will match with her own type: $y \in \mathscr{M}(x)$ implies $x \in \mathscr{M}(y)$ by symmetry, so $x \in \mathscr{M}(x)$ and $y \in \mathscr{M}(y)$ by PAM.

The definition also captures the intuition that PAM (NAM) describes a preference for matching with similar (opposite) types. To understand why, we must explore the links between assortative matching and matching set convexity. First, we have the following proposition.

PROPOSITION 2 (ASSORTATIVE MATCHING ⇒ CONVEXITY): *Given PAM or NAM, if all matching sets are nonempty, then they are convex as well.*

PROOF: Since the two cases are symmetric, assume PAM. Take any $x_1 < x_2 < x_3$ and $y_2 \in [0,1]$, with x_1 and x_3 in $\mathscr{M}(y_2)$. By assumption, there exists $y' \in \mathscr{M}(x_2)$. If $y' > y_2$, like y_3 in Figure 2, then $x_2 \in \mathscr{M}(y')$ and $x_3 \in \mathscr{M}(y_2)$ imply $x_2 \in \mathscr{M}(y_2)$, using PAM. If $y' < y_2$, like y_1 in Figure 2, then $x_2 \in \mathscr{M}(y')$ and

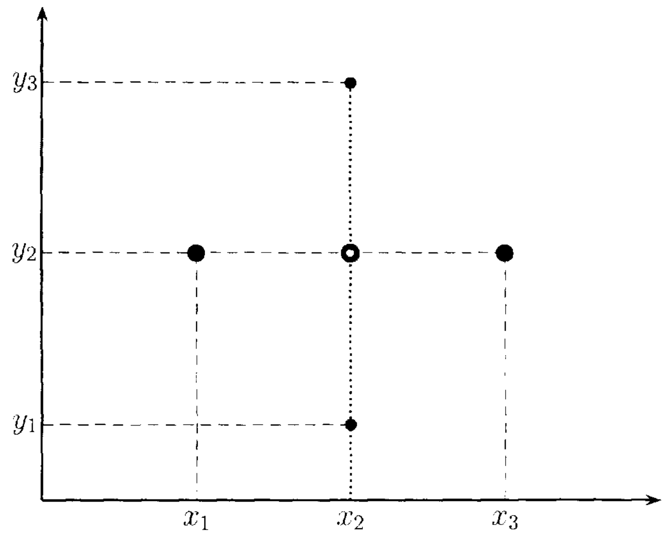

FIGURE 2.—Proposition 2 Illustrated. If low and high types (x_1 and x_3) match with some agent y_2, then so must middle types (x_2), given PAM or NAM.

[8]An equivalent formulation, valid in higher dimensions, is simply to say that the matching indicator function α is *affiliated*: $\alpha(\vec{\beta} \wedge \vec{\beta}')\alpha(\vec{\beta} \vee \vec{\beta}') \geq \alpha(\vec{\beta})\alpha(\vec{\beta}')$ for any two matches $\vec{\beta} = (x, y)$ and $\vec{\beta}' = (x', y')$. As usual, $\vee$ and $\wedge$ denote componentwise vector maxima and minima. Matching is negatively assortative if the reverse inequality obtains (negative affiliation). This also extends our formulation to probabilistic acceptance decisions, as would be necessary if there were atoms in the type distribution; the probability-of-matching function must be affiliated. Milgrom and Weber (1982) is the classic (auction-theory) economic application of affiliation.

$x_1 \in \mathcal{M}(y_2)$ imply $x_2 \in \mathcal{M}(y_2)$. Finally, if $y' = y_2$, then obviously $x_2 \in \mathcal{M}(y_2)$. *Q.E.D.*

Next, suppose matching sets are convex, closed, and nonempty; the last two assumptions follow in particular from Assumption A0, by Lemma 2. Then they are fully described by *lower* and *upper bound functions*: $a(x) \equiv \min\{y | y \in \mathcal{M}(x)\}$ and $b(x) \equiv \max\{y | y \in \mathcal{M}(x)\}$. These easily-visualized functions provide an intuitive characterization of assortative matching:

PROPOSITION 3 (MATCHING SET BOUND FUNCTIONS): *Assume that matching sets are closed and nonempty. Then there is PAM (NAM) if and only if matching sets are convex and the bound functions a and b are nondecreasing (nonincreasing).*

(The proof is in Appendix C.) This confirms higher types have higher (lower) matching sets under PAM (NAM). For example, higher types have higher mean and median partners under PAM.

Establishing PAM or NAM requires a comparison of a 4-tuple of values, a complex task that prevents us from directly finding conditions that ensure assortative matching. Instead, we assault the problem indirectly. We first show that convex matching sets and conditions that orient matching sets are sufficient for assortative matching, essentially the converse of Proposition 2. In Section 5.2, we complete the argument by finding primitive sufficient conditions for convexity.

PROPOSITION 4 (SUFFICIENT CONDITIONS FOR ASSOCIATIVE MATCHING): *Assume symmetric, convex, and nonempty matching sets $\mathcal{M}(x)$ for all x, and an upper hemicontinuous matching correspondence $\mathcal{M}$. There is PAM if and only if $0 \in \mathcal{M}(0)$ and $1 \in \mathcal{M}(1)$. There is NAM if and only if $0 \in \mathcal{M}(1)$ and $1 \in \mathcal{M}(0)$.*

PROOF: As the two cases are identical, we concentrate on PAM. We have already argued that PAM and symmetric, nonempty matching sets imply $x \in \mathcal{M}(x)$ for all x. Hence, $0 \in \mathcal{M}(0)$ and $1 \in \mathcal{M}(1)$.

To prove sufficiency of these conditions, take $x_1 < x_2$ and $y_1 < y_2$ with $x_2 \in \mathcal{M}(y_1)$ and $x_1 \in \mathcal{M}(y_2)$. We first prove $x_1 \in \mathcal{M}(y_1)$. Our 'Intermediate Value Theorem' in Claim 1 of Appendix C then applies, since the matching correspondence is nonempty- and convex-valued and u.h.c. Consequently, $0 \in \mathcal{M}(0)$ and $y_2 \in \mathcal{M}(x_1)$ (by symmetry) imply that there is $x_0 \in [0, x_1]$ with $y_1 \in \mathcal{M}(x_0)$, and thus $x_0 \in \mathcal{M}(y_1)$, again by symmetry. If $x_0 = x_1$, we are done. Otherwise, with $x_0 < x_1 < x_2$, convexity of $\mathcal{M}(y_1)$, $x_0 \in \mathcal{M}(y_1)$, and $x_2 \in \mathcal{M}(y_1)$, ensure that $x_1 \in \mathcal{M}(y_1)$. A parallel construction uses $1 \in \mathcal{M}(1)$ to prove that $x_2 \in \mathcal{M}(x_2)$, establishing PAM. *Q.E.D.*

5.2. *Sufficient Conditions for Convexity*

Since we have shown that convex matching sets are necessary and, with conditions that orient matching patterns, sufficient for assortative matching, it is logical to attack assortative matching indirectly by finding conditions under which matching sets are convex. As matches occur at points where the match surplus function s is nonnegative valued, according to condition (5), we look for conditions under which s is quasi-concave in each argument—i.e., each agent has single-peaked preferences.

The logic of the frictionless model of Section 2.1 suggests that Assumption A1 might suffice. For assume PAM, so identical types match. Competition ensures that output is evenly split: x earns $w^0(x)=f(x,x)/2$, where w^0 is the 'zero search frictions' value function, analogous to w. Define a frictionless surplus function, $s^0(x,y)\equiv f(x,y)-w^0(x)-w^0(y)$. If $s^0(\cdot)\leq 0$ for all matches, then the wages w^0 decentralize this allocation. For given x, an increase in her partner's type yields marginal surplus

$$s_y^0(x,y)=f_y(x,y)-f_x(y,y)/2-f_y(y,y)/2=f_y(x,y)-f_y(y,y)$$

since f is symmetric. If A1-Sup obtains, then $f_y(x,y)\gtrless f_y(y,y)$ as $x\gtrless y$. So for a given x, the surplus function is increasing or decreasing in her partner's type y as $x\gtrless y$. Thus, $s^0(x,\cdot)$ is quasiconcave and maximized at x, with $s^0(x,x)=0$. Parallel results obtain given NAM and A1-Sub.

If frictions reduce everyone's value equally, the surplus function would shift up, but the set of points y with $s(x,y)\geq 0$ would remain convex, the result we desire.

Examples disprove this conjecture. The production function $f(x,y)=(x+y-1)^2$ in the top panel of Figure 3 obeys A1-Sup, and yet equilibrium matching sets are not convex, given appropriate search frictions. For any x close enough to $\frac{1}{2}$ won't match with another x, but will match with both higher and lower types. The intuition for this example is clear—$\frac{1}{2}$ produces nothing when matched with her own type.

The bottom panel of Figure 3 uses the function $f(x,y)=(x+y)^2$ to illustrate that the nonconvexity is quite general, and does not require f nonmonotonic. With enough search frictions, types near 1 are willing to match with intermediate types like 0.2. This opportunity offers a windfall for 0.2: When two of them meet, they prefer to continue to search for types near 1. On the other hand, if 0.2 meets a sufficiently low type, with no such valuable outside option, match surplus is once again positive, and the match is mutually agreeable.

We now introduce additional conditions on the production function f that ensure convex matching sets. We hope the serious interplay of three forms of supermodularity makes for an interesting application of the ongoing research program here.

A2-SUP: *The first partial derivative of the production function is log-supermodular:*[9] *for all* $x_1\leq x_2$ *and* $y_1\leq y_2$, $f_x(x_1,y_1)f_x(x_2,y_2)\geq f_x(x_1,y_2)f_x(x_2,y_1)$.

A3-SUP: *The cross partial derivative of the production function is log-supermodular: for all* $x_1\leq x_2$, $y_1\leq y_2$, $f_{xy}(x_1,y_1)f_{xy}(x_2,y_2)\geq f_{xy}(x_1,y_2)f_{xy}(x_2,y_1)$.

[9] A positive function is log-supermodular if its log is supermodular. These definitions extend this notion to possibly negative functions, in the way needed in this paper.

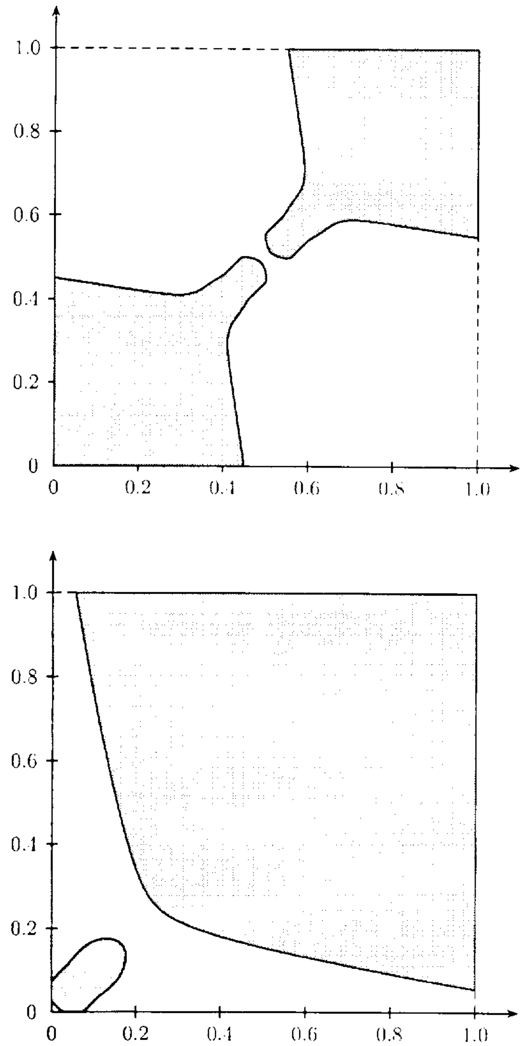

FIGURE 3.—Non-Convex Matching. The top panel depicts matching sets for $f(x, y) = (x + y - 1)^2$, $\delta = r$, $\rho = 100r$, and $L(x) = x$ on $[0, 1]$. The bottom panel depicts matching sets for $f(x, y) = (x + y)^2$, $\delta = r$, $\rho = 35r$, and $L(x) = x$ on $[0, 1]$.

A2-SUB: *The first partial derivative of the production function is log-submodular: for all* $x_1 \leq x_2$ *and* $y_1 \leq y_2$, $f_x(x_1, y_1) f_x(x_2, y_2) \leq f_x(x_1, y_2) f_x(x_2, y_1)$.

A3-SUB: *The cross partial derivative of the production function is log-submodular: for all* $x_1 \leq x_2$ *and* $y_1 \leq y_2$, $f_{xy}(x_1, y_1) f_{xy}(x_2, y_2) \leq f_{xy}(x_1, y_2) f_{xy}(x_2, y_1)$.

While Assumptions A1, A2, A3-Sup are independent, as are A1, A2, A3-Sub, they jointly are interrelated. For example, A3-Sup and A3-Sub simultaneously hold for production functions of the form $f(x,y) \equiv c_1 + c_2(g(x)+g(y)) + c_3 h(x)h(y)$. Under the additional restriction $g \equiv h$, A2-Sup and A2-Sub obtain as well. Thus, A2-Sup and A2-Sub jointly imply A3-Sup and A3-Sub, but not conversely.

To better understand these assumptions, consider the constant elasticity of substitution (CES) production function $f(x,y) = (\frac{1}{2}(x^a + y^a))^{1/a}$, with Cobb-Douglas limit $f(x,y) = (xy)^{1/2}$ when $a = 0$. This satisfies A1-Sup when the elasticity of substitution is negative, $a < 1$. For $a < 0$—an elasticity of substitution between -1 and 0—A2-Sup and A3-Sup are satisfied as well. When inputs are more easily substituted, $a > 0$, A2-Sup is violated. Finally, A3-Sup obtains when $a \leq 1/2$.

PROPOSITION 5 (CONVEX MATCHING): *Posit A0. Given A1-Sup, A2-Sup, and A3-Sup, or A1-Sub, A2-Sub, and A3-Sub, all matching sets are convex.*

This is proven in Section 5.3. The importance of A1 is clear from the frictionless benchmark. The production functions in Figure 3 satisfy A1-Sup, A2-Sub, and both A3-Sup and A3-Sub, proving A2 necessary. We postpone the subtle issue of the necessity of A3.

As an important robustness check, both the primary premises of Proposition 5 —A1, A2, A3—and the convexity conclusion in $\mathbb{R}$, are scale- and order-independent. Consider the cardinal specification of the type distribution. For instance, relabel each agent x by her type's percentile $L(x)$, and let $\hat{f}(L(x), L(y)) \equiv f(x,y)$. Then f satisfies the assumptions in Proposition 5 if and only if $\hat{f}$ does. Likewise, we may reverse agents' ordering by relabeling each x as $1-x$, and letting $\hat{f}(1-x, 1-y) \equiv f(x,y)$. This observation will play a key role in the proof of Lemma 5.

Our main result follows from Propositions 4 and 5.

PROPOSITION 6 (ASSORTATIVE MATCHING CHARACTERIZATION): *Posit A0. Then A1-Sup, A2-Sup, A3-Sup and $f_y(0,y) \leq 0 \leq f_y(1,y)$ for all y imply PAM; A1-Sub, A2-Sub, A3-Sub and $f_y(0,y) \geq 0$ for all y or $0 \geq f_y(1,y)$ for all y imply NAM.*

We use A1, A2, and A3 to establish matching set convexity through Proposition 5. These assumptions do not guarantee assortative matching, as Figure 4 attests. The function $f(x,y) = x + y + xy$ obeys all three supermodularity assumptions, but matching sets violate the additional requirement from Proposition 4 that $0 \in \mathcal{M}(0)$. With search frictions, some agents $y > 0$ match with 0, since such matches are productive. Therefore, $w(0) > 0$ by (6), and the match surplus of $(0,0)$ is negative. By (5), $0 \notin \mathcal{M}(0)$. We preempt such difficulties via the boundary conditions on the marginal product of f.

PROOF OF PROPOSITION 6: By construction, matching sets are symmetric. They are convex by Proposition 5, and nonempty, with a u.h.c. correspondence by

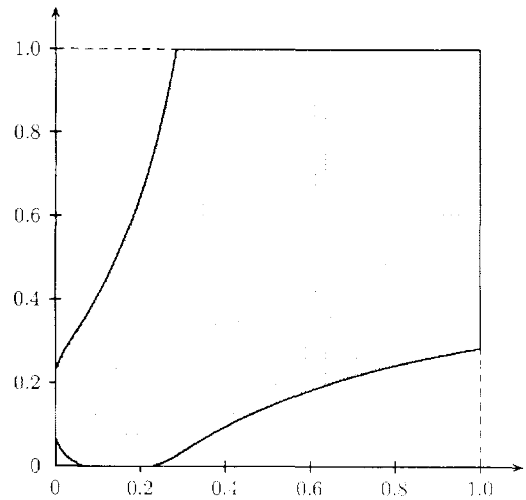

FIGURE 4.—A Failure of PAM. This depicts matching sets for $f(x,y) = x + y + xy$, $\rho = 750r$, $\delta = r$, and $L(x) = x$ on $[0,1]$. Although f satisfies A1-Sup, A2-Sup, and A3-Sup, PAM does not arise. Indeed, that $f(0,0) = 0$ but $f_y(0,y) > 0$ forces $0 \notin \mathscr{M}(0)$.

Lemma 2. As all the assumptions of Proposition 4 are satisfied, there is PAM if $0 \in \mathscr{M}(0)$ and $1 \in \mathscr{M}(1)$; and there is NAM if $0 \in \mathscr{M}(1)$ and $1 \in \mathscr{M}(0)$.

We first show that $0 \in \mathscr{M}(0)$ under the supermodularity and associated boundary assumptions. (That $1 \in \mathscr{M}(1)$ is similar.) For y with $w'(y) \geq 0$, $f_y(0,y) \leq 0$ trivially implies $s_y(0,y) \leq 0$. For all other y, by (8):

$$w'(y) = \frac{\theta \int_{\mathscr{M}(y)} f_y(x,y) u(x)\, dx}{1 + \theta \int_{\mathscr{M}(y)} u(x)\, dx} > \frac{\int_{\mathscr{M}(y)} f_y(x,y) u(x)\, dx}{\int_{\mathscr{M}(y)} u(x)\, dx} \geq f_y(0,y)$$

where the first inequality follows from the negative numerator (as $w'(y) < 0$), and the second from A1-Sup. In either case, $s_y(0,y) \leq 0$, so that type 0 prefers matching with lower types. Finally, $0 \in \mathscr{M}(0)$, as matching sets are nonempty (Lemma 2).

Next take the submodularity assumptions, with $f_y(1,y) \leq 0$. If $w'(y) \geq 0$, then we trivially have $s_y(1,y) \leq 0$. Otherwise, use A1-Sub in the above supermodularity argument to prove $w'(y) \geq f_y(1,y)$. Hence, 1 prefers to match with lower types; thus, $0 \in \mathscr{M}(1)$ by nonemptiness, and $1 \in \mathscr{M}(0)$ by symmetry. The proof with $f_y(0,y) \geq 0$ is similar. *Q.E.D.*

5.3. *Convexity Argument*: *Proof of Proposition* 5

We prove that match surplus is quasiconcave in each argument.

LEMMA 5 (QUASICONCAVITY): *Posit A*0 *and fix z. Given A*1, *A*2, *A*3-*Sup or A*1, *A*2, *A*3-*Sub, the match surplus function* $s(z,y)$ *is quasiconcave in* y.

By condition (5), Proposition 5 follows immediately from Lemma 5.

Each of the three assumptions plays a distinct role in our proof. In the frictionless model, A1-Sup ensures that the highest type has an increasing surplus function, $s_y^0(1, y) \geq 0$. Search frictions depress and *flatten* all value functions w. As a result, the highest type's match surplus is strictly increasing in her partner's type. By continuity, this argument extends to other high types.

For the lowest type, A1-Sup implies that the frictionless surplus function is decreasing. Flattening the value function changes this in complex ways. A2-Sup imposes that the percentage decline in productivity from 'mismatch' is smaller for high types than for low types. As a result, low types suffer a larger percentage decline in value. This ensures that low types continue to prefer to match with relatively cheap low types. Finally, A3-Sup provides a single crossing property (SCP), Lemma 6 below, which allows us to treat everyone as either a high type or a low type.

The formal proof of Lemma 5 rules out local minima in any z's surplus function, showing that if $s(z, \cdot)$ is falling at y_1, it is falling at $y_2 > y_1$. Here let us assume $w'(y_1) > 0$. Let $\tilde{z}$ solve $f_y(\tilde{z}, y_1) \equiv w'(y_1)$. By A1-Sup, $f_y(z, y_1) > f_y(\tilde{z}, y_1)$ for $z > \tilde{z}$, so that $s(z, \cdot)$ is increasing at y_1.

Dealing with $z < \tilde{z}$ uses the other assumptions. By A2-Sup, $f_y(\tilde{z}, y_1) f_y(z, y_2) \leq f_y(z, y_1) f_y(\tilde{z}, y_2)$ for any $y_2 > y_1$. By the definition of $\tilde{z}$, the left-hand side is equal to $w'(y_1) f_y(z, y_2)$. If we are in the interesting case where z's surplus function is decreasing at y_1, the right-hand side is less than $w'(y_1) f_y(\tilde{z}, y_2)$. Putting these together, $f_y(z, y_2) < f_y(\tilde{z}, y_2)$. Now if we can prove that $f_y(\tilde{z}, y_2) \leq w'(y_2)$, we will have shown that z's surplus function is decreasing at y_2, completing the proof.

For this last step, we use the SCP. Diamond and Stiglitz (1974) showed (Theorem 3, equivalence of (ii) and (iii)) that a gambler has a higher coefficient of absolute risk aversion if and only if he requires a higher risk premium for *any gamble*. Formally, let utility h depend on a prize x and preference parameter y. Then the coefficient of absolute risk aversion $-h_{xx}/h_x$ is smoothly falling in y if and only if the certainty equivalent of any gamble is smoothly rising in y. We extend this result by assuming that utility is just once differentiable in income—but that log marginal utility $\log h_x$ is supermodular in (x, y). As such, the Arrow-Pratt risk aversion measure need no longer exist, yet alone its derivative in y. We can find no way of patching their proof, as it includes repeated integration by parts of a third derivative. Our proof in Appendix C explicitly exploits the supermodular structure, our main focus.

LEMMA 6 (A SCP FOR GAMBLES): *Consider $h: [0, 1]^2 \mapsto \mathbb{R}$, differentiable, with h_x either positive and log-supermodular, or negative and log-submodular. For all $y_1 \in [0, 1]$ and probability densities with support $M \subseteq [0, 1]$, there is a unique certainty equivalent $\tilde{z}$: $h(\tilde{z}, y_1) \equiv E_{x \in M} h(x, y_1)$. For all $y' \geq y_1$, $h(\tilde{z}, y') \leq E_{x \in M} h(x, y')$. If $h_x > 0$ is log-submodular or $h_x < 0$ is log-supermodular, then $h(\tilde{z}, y') \geq E_{x \in M} h(x, y')$.*

In our case, $w'(y_1)$ is proportional to the certainty equivalent $E_x f_y(x, y_1)$, according to (8), and is also equal to $f_y(\bar{z}, y_1)$ by definition. Similarly, $w'(y_2)$ is proportional to the certainty equivalent $E_x f_y(x, y_2)$. Thus A3-Sup, log-supermodularity of f_{xy}, ensures $w'(y_2) \geq f_y(\bar{z}, y_2)$.

Our formal proof of Lemma 5 tightens this argument and uses a slightly different characterization of quasiconcavity. A well-behaved *non*-quasiconcave function σ has a local minimum x. The characterization in the next lemma (proof appendicized) focuses on a critical property of points y_1 and y_2 to the left and right of x.

LEMMA 7: *Assume a continuous and a.e. differentiable map* $\sigma : [0,1] \mapsto \mathbb{R}$ *obeys*: $\sigma(y_1) < \sigma(y_2)$ *for* $0 < y_1 < y_2$ *implies* $\sigma'(y_1) \geq 0$ *when defined. Then* σ *is quasiconcave.*

Now we have the tools to prove that the match surplus function is quasiconcave.

PROOF OF LEMMA 5: Since f is continuous and differentiable by A0, and w is continuous and a.e. differentiable by Lemma 1, surplus s is continuous and a.e. differentiable. We will use Lemma 7 to prove that an arbitrary $s(z, y)$ is quasiconcave in y under the supermodularity assumptions. We omit the symmetric submodularity proof.

Fix $0 < y_1 < 1$. If $w'(y_1)$ is defined, we may assume without loss of generality that $w'(y_1)$ is nonnegative. For if $w'(y_1) < 0$, we could instead work in the world with a reversed type ordering, recalling the discussion after Proposition 5. Letting $\hat{f}(x, y) \equiv f(1-x, 1-y)$ denote the production function and $\hat{L}(x) \equiv 1 - L(1-x)$ denote the type distribution in such a world, there would be a SE with value function $\hat{w}(1-y) \equiv w(y)$, and in particular $\hat{w}'(1-y_1) = -w'(y_1) > 0$. We could instead proceed with the analysis of type $1-y_1$, proving that $\hat{s}(1-z, 1-y_1)$ is quasiconcave in its second argument. By construction, this would establish quasiconcavity of $s(z, y)$, as desired.

Next, fix $y_2 \in (y_1, 1)$ and z. We must prove that if $s(z, y_1) < s(z, y_2)$, then $s_y(z, y_1) \geq 0$ when defined. If $w'(y_1)$ is undefined, then so is $s_y(z, y_1)$ and we are done. Otherwise, implicitly define $\bar{z}$ so $f_y(\bar{z}, y_1)$ is the expected value of the gamble $f_y(x, y_1)$:

$$(10) \qquad f_y(\bar{z}, y_1) \equiv \frac{\int_{\mathscr{M}(y_1)} f_y(x, y_1) u(x)\, dx}{\int_{\mathscr{M}(y_1)} u(x)\, dx}.$$

- STEP 1: $s_y(z, y_1) \geq 0$ FOR ALL $z \geq \bar{z}$.

$$0 \leq w'(y_1) = \frac{\theta \int_{\mathscr{M}(y_1)} f_y(x, y_1) u(x)\, dx}{1 + \theta \int_{\mathscr{M}(y_1)} u(x)\, dx} = \frac{f_y(\bar{z}, y_1)\theta \int_{\mathscr{M}(y_1)} u(x)\, dx}{1 + \theta \int_{\mathscr{M}(y_1)} u(x)\, dx}$$
$$\leq f_y(\bar{z}, y_1)$$

by (8) and (10), so $s_y(\bar{z}, y_1) \geq 0$. By A1-Sup, $s_y(z, y_1) > s_y(\bar{z}, y_1) \geq 0$ for all $z > \bar{z}$.

• STEP 2: $w(y_1) < w(y_2)$ AND $f(z,y_1) < f(z,y_2)$ WHENEVER $s(z,y_1) < s(z,y_2)$ AND $z < \bar{z}$. At y_2 with $s(z,y_1) \geq s(z,y_2)$, there is nothing to verify. Otherwise:

$$(11) \qquad f(z,y_2) - f(z,y_1) > w(y_2) - w(y_1) \geq \frac{\theta \int_{\mathcal{X}(y_1)} (f(x,y_2) - f(x,y_1)) u(x)\, dx}{1 + \theta \int_{\mathcal{X}(y_1)} u(x)\, dx}$$

where the second inequality applies equation (6) and inequality (7).

By A1-Sup and A3-Sup, $h = f_y$ meets Lemma 6's conditions. So for all $y' > y_1$,

$$\frac{\int_{\mathcal{X}(y_1)} f_y(x,y') u(x)\, dx}{\int_{\mathcal{X}(y_1)} u(x)\, dx} \geq f_y(\bar{z}, y')$$

by (10). For $y_2 > y_1$, integrate this over $y' \in [y_1, y_2]$, and use A1-Sup with $\bar{z} > z$:

$$(12) \qquad \frac{\int_{\mathcal{X}(y_1)} (f(x,y_2) - f(x,y_1)) u(x)\, dx}{\int_{\mathcal{X}(y_1)} u(x)\, dx} \geq f(\bar{z},y_2) - f(\bar{z},y_1) > f(z,y_2) - f(z,y_1).$$

By transitivity, the last term in (11) is smaller than the first term in (12):

$$\frac{\int_{\mathcal{X}(y_1)} (f(x,y_2) - f(x,y_1)) u(x)\, dx}{\int_{\mathcal{X}(y_1)} u(x)\, dx} > \frac{\theta \int_{\mathcal{X}(y_1)} (f(x,y_2) - f(x,y_1)) u(x)\, dx}{1 + \theta \int_{\mathcal{X}(y_1)} u(x)\, dx}.$$

Hence, $\int_{\mathcal{X}(y_1)} (f(x,y_2) - f(x,y_1)) u(x)\, dx > 0$. Using this, (11) implies $w(y_2) > w(y_1)$ and $f(z,y_2) > f(z,y_1)$. Note that (12) then implies $f(\bar{z},y_2) > f(\bar{z},y_1)$.

STEP 3: $s_y(z,y_1) \geq 0$ WHENEVER $s(z,y_1) < s(z,y_2)$ AND $z < \bar{z}$. Divide (8) by (11), and then (10) by (12), verifying that the relevant terms are positive:

$$(13) \qquad \frac{w'(y_1)}{w(y_2) - w(y_1)} \leq \frac{\int_{\mathcal{X}(y_1)} f_y(x,y_1) u(x)\, dx}{\int_{\mathcal{X}(y_1)} (f(x,y_2) - f(x,y_1)) u(x)\, dx} \leq \frac{f_y(\bar{z},y_1)}{f(\bar{z},y_2) - f(\bar{z},y_1)}.$$

Integrating A2-Sup implies that for $y_1 < y_2$ and $z < \bar{z}$,

$$f_y(\bar{z},y_1)(f(z,y_2) - f(z,y_1)) \leq f_y(z,y_1)(f(\bar{z},y_2) - f(\bar{z},y_1)).$$

Divide by $f(z,y_2) - f(z,y_1) > 0$ and $f(\bar{z},y_2) - f(\bar{z},y_1) > 0$, and combine with (13):

$$\frac{w'(y_1)}{w(y_2) - w(y_1)} \leq \frac{f_y(z,y_1)}{f(z,y_2) - f(z,y_1)}.$$

As $0 < w(y_2) - w(y_1) < f(z, y_2) - f(z, y_1)$, multiplication yields $w'(y_1) \leq f_y(z, y_1)$ or $s_y(z, y_1) \geq 0$, completing the proof. *Q.E.D.*

One can prove that A1-Sup and A2-Sup preclude all but holes in the center of acceptance sets, a pattern we are unable to produce. Instead, we offer an example showing that A3 is necessary for surplus quasiconcavity, and therefore critical to any general proof of matching set convexity. Consider the production function $f(x, y) = 1000 - (xy)^3 + 9(xy)^2 + 3(x + y)$, obeying A1-Sup, A2-Sup, and A3-Sub (but not A3-Sup). Let $\rho = 50000r$, $\delta = r$, and $L(x) = x$ on $[0, 1]$. Since a partner's type has a negligible effect on output, not surprisingly all matches are acceptable. In Appendix C, we show that this model is analytically solvable: $w(x) = 495.1144 + 2.9472(x + x^2) - 0.2456x^3$. Thus, the surplus function of $x \in [0.560, 0.562]$ is not quasiconcave. For example, $s(0.560, y)$ is minimized at $y = 0.913$.

6. CONCLUSION

This paper has pushed the assortative matching insights into a plausible search setting. We have generalized PAM and NAM for this frictional environment, and have identified the three supermodularity (submodularity) assumptions under which matching sets are convex and, with additional boundary conditions, increasing (decreasing). We have also developed a general existence theorem for search models with endogenous type distributions, and general matching sets.

We have investigated this model because it can be fully solved. In proving existence of a SE, we made assumptions on the search technology to avoid significant complications; however, our descriptive theory in Section 5 applies for any *anonymous* search technology, where the rate that searchers meet is independent of their types. This includes, for example, a linear search technology, in which the meeting rate is independent of the measure of unmatched agents.

Likewise, we have assumed that a steady-state is maintained through deaths of existing agents. We could alternatively posit an inflow of entrants, and assume that all matches are permanent. While our assortative matching and convexity conclusions do not depend on this modeling choice, our existence proof does.

This model may be generalized in several important ways. For example, so as not to double the notation, we have assumed one class of agents. All our results extend to a model with workers and firms, or men and women. Our definition of assortative matching generalizes to multiple dimensions, while convex coordinate slices (biconvexity) is the scale-independent extension of convexity in one dimension.

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APPENDICES: OMITTED PROOFS

A. VALUE FUNCTION PROPERTIES

PART 1 OF LEMMA 1: VALUE INEQUALITY. If (7) were violated, (6) would yield either $y \in \mathscr{M}(x)$, $y \notin M$, and $f(x,y) - w(x) - w(y) < 0$; or $y \in M$, $y \notin \mathscr{M}(x)$, and $f(x,y) - w(x) - w(y) > 0$. Either possibility contradicts (5). *Q.E.D.*

PART 2 OF LEMMA 1: LIPSCHITZ AND THUS CONTINUITY. Since f_x is continuous on $[0,1]^2$, $\kappa \equiv \max_{x,y} |f_x(x,y)|$ is well-defined. Also, by (6) and (7), for all $x_1 < x_2$,

$$\theta \int_{\mathscr{M}(x_2)} (f(x_2,y) - f(x_1,y) - w(x_2) + w(x_1))u(y)\,dy$$

$$\geq w(x_2) - w(x_1) \geq \theta \int_{\mathscr{M}(x_1)} (f(x_2,y) - f(x_1,y) - w(x_2) + w(x_1))u(y)\,dy.$$

Solving each inequality for $w(x_2) - w(x_1)$ and using $|f(x_2,y) - f(x_1,y)| \leq \kappa(x_2 - x_1)$,

$$\frac{\kappa(x_2 - x_1)\cdot\theta\int_{\mathscr{M}(x_2)} u(y)\,dy}{1 + \theta\int_{\mathscr{M}(x_2)} u(y)\,dy} \geq w(x_2) - w(x_1) \geq \frac{-\kappa(x_2 - x_1)\cdot\theta\int_{\mathscr{M}(x_1)} u(y)\,dy}{1 + \theta\int_{\mathscr{M}(x_1)} u(y)\,dy}.$$

So w is Lipschitz, $|w(x_2) - w(x_1)|/(x_2 - x_1) < \kappa$, and thus continuous. *Q.E.D.*

PART 3 OF LEMMA 1: DIFFERENTIABILITY. Let $D(B,C)$ be the *Hausdorff* distance between sets B and C: namely, $D(B,C) = \inf\{d \mid \forall (b,c) \in (B,C), \exists (b',c') \in (C,B)$, with $|b - b'| < d$ and $|c - c'| < d\}$. Call a correspondence M *continuous* at x if for all $\varepsilon > 0$, there is a neighborhood N_x of x, such that $D(M(x), M(x')) < \varepsilon$ if $x' \in N_x$.

- STEP 1: $\mathscr{M}$ IS CONTINUOUS AT A.E. x. First, $\mathscr{M}$ is nonempty-valued: if $\mathscr{M}(x) = \varnothing$ for some x, then $w(x) = 0$ by (6). Thus $f(x,x) - 2w(x) \geq 0$ by A0, and (5) implies $x \in \mathscr{M}(x)$, a contradiction. Next, $\mathscr{M}$ is u.h.c.: Take any sequence $(x_n, y_n) \to (x,y)$, with $y_n \in \mathscr{M}(x_n)$ for all n. Then $s(x_n, y_n) \geq 0$ for all n, and so $s(x,y) \geq 0$ as well, since s is continuous by A0 and Part 2 of this Lemma. Thus, $y \in \mathscr{M}(x)$, establishing u.h.c. Fixing $x_n = x$ for all n, $\mathscr{M}(x)$ is closed too. Also, $\mathscr{M}(x) \subseteq [0,1]$ is bounded, whence $\mathscr{M}$ is compact-valued.

We call x an *ε-continuity point* of a correspondence M, and say x belongs to $\mathscr{C}_M(\varepsilon)$, if for all x' sufficiently close to x, $D(M(x), M(x')) < \varepsilon$. Since $\mathscr{M}$ is u.h.c. and compact valued, Theorem I-B-III-4 in Hildenbrand (1974) implies that (given our nonatomic density u) for all $\varepsilon > 0$, a.e. x is an ε-continuity point of $\mathscr{M}$. Then for all $n = 1, 2, \ldots$, a.e. $x \in \mathscr{C}_M(1/n)$, and the countable intersection $\bigcap_n \mathscr{C}_M(1/n)$ contains a.e. x as well. That is, for a.e. x, for all n, if x' is sufficiently close to x, $D(\mathscr{M}(x), \mathscr{M}(x')) < 1/n$. So $\mathscr{M}$ is a.e. continuous.

- STEP 2: DECOMPOSITION OF THE VALUE FUNCTION'S SLOPE. Take any sequence $x_n \to x$. Add and subtract $\theta\int_{\mathscr{M}(x)} (f(x_n, y) - w(x_n) - w(y))u(y)\,dy$ from $w(x_n) - w(x)$ for each n, and divide through by $x_n - x$:

$$\frac{w(x_n) - w(x)}{x_n - x} = \theta\left(\int_{\mathscr{M}(x_n) \setminus \mathscr{M}(x)} \frac{f(x_n,y) - w(x_n) - w(y)}{x_n - x} u(y)\,dy \right. \tag{14}$$
$$\left. + \int_{\mathscr{M}(x)} \frac{(f(x_n,y) - f(x,y)) - (w(x_n) - w(x))}{x_n - x} u(y)\,dy\right)$$

where $f_{A-B} \equiv f_A - f_B$. The first term in brackets vanishes if $\mathcal{M}$ is continuous at x, because continuity of f and w and condition (5) imply surplus vanishes at changes in $\mathcal{M}(x)$. The remaining terms tend to the desired expression for $w'(x)$ at a.e. x. *Q.E.D.*

B. EQUILIBRIUM EXISTENCE

CONTINUITY OF $w \mapsto \alpha(w)$: PROOF OF LEMMA 3.

• STEP 1: SURPLUS FUNCTION IS RARELY CONSTANT IN ONE VARIABLE. Define $Z_s(x) = \{y : s(x,y) = 0\}$ and $Z_s = \{(x,y) : s(x,y) = 0\}$, and let μ be Lebesgue measure on [0, 1]. We claim that under A1-Sup or A1-Sub, $\mu(Z_s(x)) = 0$ for a.e. x. Let $x \neq x'$ and $y \neq y'$, with $s(x,y) = s(x',y) = s(x,y') = 0$. Under A1-Sup or A1-Sub, $f(x,y) + f(x',y') \neq f(x',y) + f(x,y')$, from which $s(x',y') \neq 0$ follows. Thus, $Z_s(x) \cap Z_s(x')$ contains at most one point whenever $x \neq x'$.

Assume $\mu(Z_s(x)) > 0$ for an uncountable number of x.[10] Then for some k, there are infinitely many $\langle x_n \rangle$ with $\mu(Z_s(x_n)) > 1/k$, whereupon $\sum_{n=1}^{\infty} \mu(Z_s(x_n)) = \infty$. Since $x_{ij} = Z_s(x_i) \cap Z_s(x_j)$ contains at most one point, $N = \bigcup_{i,j=1}^{\infty} x_{ij}$ is countable, and so $\mu(N) = 0$. Also, $Z_s(x_i) \setminus N$ and $Z_s(x_j) \setminus N$ are disjoint for all $i \neq j$. Thus

$$1 \geq \mu\left(\bigcup_{n=1}^{\infty} Z_s(x_n) \setminus N \right) = \sum_{n=1}^{\infty} \mu(Z_s(x_n) \setminus N) = \sum_{n=1}^{\infty} \mu(Z_s(x_n)) = \infty.$$

Given this contradiction, there are only countably many x with $\mu(Z_s(x)) > 0$. So $\mu(Z_s(x)) = 0$ for a.e. x, and by Fubini's theorem $(\mu \times \mu)(Z_s) = 0$.

• STEP 2: CLOSE SURPLUS FUNCTIONS RARELY DIFFER IN SIGN. As $\eta \to 0$, the set $\Sigma_s(\eta) = \{(x,y)$ with $|s(x,y)| \in [0,\eta]\}$ shrinks monotonically to $\bigcap_{k=1}^{\infty} \Sigma_s(1/k) = s^{-1}(0) = Z_s$. By the countable intersection property of measures,

$$\lim_{\eta \to 0} (\mu \times \mu)(\Sigma_s(\eta)) = \lim_{k \to \infty} (\mu \times \mu)(\Sigma_s(1/k)) = (\mu \times \mu)\left(\bigcap_{k=1}^{\infty} \Sigma_s(1/k) \right)$$
$$= (\mu \times \mu)(Z_s) = 0.$$

Finally, let w_1 and w_2 be value functions, with $\|w_1(x) - w_2(x)\| \leq \eta/2$, and α_1, α_2 corresponding match indicator functions. If $s_1(x,y) \equiv f(x,y) - w_1(x) - w_1(y) > \eta$, then $s_2(x,y) \equiv f(x,y) - w_2(x) - w_2(y) > 0$, and so $\alpha_1(x,y) = \alpha_2(x,y) = 1$, while if $s_1(x,y) < -\eta$, then $s_2(x,y) < 0$, and $\alpha_1(x,y) = \alpha_2(x,y) = 0$. Consequently, $\{(x,y) | \alpha_1(x,y) \neq \alpha_2(x,y)\} \subseteq \Sigma_{s_1}(\eta)$, whose Lebesgue measure vanishes as $\eta \to 0$. This implies the desired continuity $\lim_{\|w_1 - w_2\| \to 0} \|\alpha_1 - \alpha_2\|_{\mathcal{L}^1} = 0$.

CONTINUITY OF $\alpha \mapsto u(\alpha)$: FINISHING THE PROOF OF LEMMA 4. Normalize $\delta = 1$.

For fixed α_0, let u_0 be the associated unique unmatched density: i.e., $l(x) - u_0(x) = \rho u_0(x) \int_0^1 \alpha_0(x,y) u_0(y)\, dy$. Defining $G(\alpha, u) = u(x)(1 + \rho \int \alpha(x,y) u(y)\, dy) - l(x)$, we see that $u = u_\alpha$ solves (1) given α iff $G(\alpha, u_\alpha) = 0$. As proven, u_α is unique.

• STEP 1: AN INVERTIBLE STEADY-STATE DERIVATIVE OPERATOR. The derivative $G_u(\alpha, u)$ is a bounded linear operator on $\mathcal{L}^2[0,1]$, defined as the following limit:

$$G_u(\alpha, u)(g) = \lim_{t \to 0} (G(\alpha, u + tg) - G(\alpha, u))/t = g\left(1 + \rho \int_0^1 \alpha u\right) + \rho u \int_0^1 \alpha g$$

that is also clearly continuous in u. Write $G_u(\alpha, u) \equiv I + \rho H$. Since α is symmetric ($\alpha(x,y) = \alpha(y,x)$), the operator H is self-adjoint, and positive-definite on the space $\mathcal{L}^2([0,1], 1/u)$ of

[10] Toby Gifford (Math Dept., Washington U. in St. Louis) tightened the logic of this paragraph.

functions square-integrable with respect to density $1/u$, because[11]

$$2\langle g, Hg\rangle = 2\int_0^1 g(x)Hg(x)/u(x)\,dx$$

$$= 2\int_0^1\int_0^1 (g(x)^2u(y)/u(x) + g(x)g(y))\alpha(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 [g(x)^2u(y)/u(x) + 2g(x)g(y) + g(y)^2u(x)/u(y)]\alpha(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 \Big[g(x)\sqrt{u(y)/u(x)} + g(y)\sqrt{u(x)/u(y)}\Big]^2 \alpha(x,y)\,dx\,dy \geq 0.$$

Since u is *boundedly* positive (given $\rho < \infty$, $l \geq \underline{l} > 0$) and finite ($u \leq \bar{l} < \infty$), $\mathscr{L}^2([0,1], 1/u) = \mathscr{L}^2[0,1]$. So H is a self-adjoint and positive definite linear operator in $\mathscr{L}^2[0,1]$, and its spectrum is thus real and nonnegative: the spectrum of $G_u(\alpha, u)$ is then contained in $[1,\infty)$, and excludes 0. Hence, $G_u(\alpha, u)$ is invertible in $\mathscr{L}^2[0,1]$.

- Step 2: G is Lipschitz in α. First, we have the subtraction:

$$G(\alpha,u)(x) - G(\beta,u)(x) = \rho u(x)\int_0^1 [\alpha(x,y) - \beta(x,y)]u(y)\,dy$$

$$\Rightarrow \int_0^1 [G(\alpha,u)(x) - G(\beta,u)(x)]^2\,dx \leq \rho^2\bar{l}^4\int_0^1\left(\int_0^1[\alpha(x,y) - \beta(x,y)]\,dy\right)^2 dx$$

$$\leq \rho^2\bar{l}^4\int_0^1\int_0^1 [\alpha(x,y) - \beta(x,y)]^2\,dy\,dx$$

since $u(x) \leq \bar{l}$. Then $\|G(\alpha,u) - G(\beta,u)\| \leq \rho\bar{l}^2\|\alpha - \beta\|$, both norms in $\mathscr{L}^2$.

- Step 3: Continuity of $\alpha \mapsto u_\alpha$. Since $G(\alpha, u)$ has Lipschitz constant $L = \rho\bar{l}^2 > 0$ in α, we have $\|G(\alpha', u_\alpha)\| = \|G(\alpha', u_\alpha) - G(\alpha, u_\alpha)\| < L\|a - \alpha'\|$, and so

$$\|G(\alpha', u_{\alpha'}) + J(u_\alpha - u_{\alpha'})\| = \|G(\alpha', u_{\alpha'} + u_\alpha - u_{\alpha'})\| = \|G(\alpha', u_\alpha)\| < L\|\alpha - \alpha'\|$$

for some $J_t = G_u(\alpha', tu_\alpha + (1-t)u_{\alpha'})$ and $t \in [0,1]$ by the mean value theorem, given continuity of G_u in u. By Step 1, J_t is an invertible linear operator in $\mathscr{L}^2[0,1]$ with spectrum contained in $[1,\infty)$. So its inverse K_t exists, and $\|K_t\| \leq 1$ for all t. Hence,

$$\|u_\alpha - u_{\alpha'}\| = \|K_tJ_t(u_\alpha - u_{\alpha'})\| \leq \|K_t\|\,\|J_t(u_\alpha - u_{\alpha'})\| < L\|\alpha - \alpha'\|$$

since $\|K_tv\| \leq \|K_t\|\,\|v\| \leq \|v\|$ by the norm inequality and $\|K_t\| \leq 1$. So when α is near α' in $\mathscr{L}^2([0,1]^2)$, u_α is close to $u_{\alpha'}$ in $\mathscr{L}^2[0,1]$. Finally, continuity holds for both norms in $\mathscr{L}^1$: The Cauchy-Schwarz inequality implies $\|u_\alpha - u_{\alpha'}\|_{\mathscr{L}^1} \leq \|u_\alpha - u_{\alpha'}\|_{\mathscr{L}^2}$; and $\|\alpha - \alpha'\|_{\mathscr{L}^2} \leq \sqrt{\|\alpha - \alpha'\|_{\mathscr{L}^1}}$ holds in our restricted domain, as $|\alpha - \alpha'| \leq 1$.

Continuity of the Operator T: Finishing the Proof of Proposition 1.

For all $w^1, w^2 \in \mathscr{W}$ and $x \in [0,1]$, the triangle inequality implies $|Tw^2(x) - Tw^1(x)| \leq D_1(x) + D_2(x)$, where

$$D_i(x) \equiv \left|\frac{\theta\int_0^1 \max\langle f(x,y) - w^i(y), w^i(x)\rangle u^{w^1}(y)\,dy}{1 + \theta\bar{u}^{w^1}} - \frac{\theta\int_0^1 \max\langle f(x,y) - w^2(y), w^2(y), w^2(x)\rangle u^{w^i}(y)\,dy}{1 + \theta\bar{u}^{w^i}}\right|.$$

[11] We thank Robert Israel (UBC Math Dept.) for remarking on this fact.

We will show that D_1 and D_2 converge to zero when w^1 converges to w^2. First, $\sup_x D_1(x) \leq \sup_z |w^1(z) - w^2(z)|$, since

$$D_1(x) \leq \frac{\theta \int_0^1 |\max\langle f(x,y) - w^2(y), w^2(x)\rangle - \max\langle f(x,y) - w^1(y), w^1(x)\rangle| u^{w^1}(y)\, dy}{1 + \theta \bar{u}^{w^1}}$$

$$\leq \frac{\theta \int_0^1 |\max\langle w^1(y) - w^2(y), w^2(x) - w^1(x)\rangle| u^{w^1}(y)\, dy}{1 + \theta \bar{u}^{w^1}} \leq \sup_z |w^1(z) - w^2(z)|.$$

Second, we can bound $D_2(x)$ through a series of algebraic manipulations.

$$D_2(x) \leq \int_0^1 \max\langle f(x,y) - w^2(y), w^2(x)\rangle \left| \frac{\theta u^{w^2}(y)}{1 + \theta \bar{u}^{w^2}} - \frac{\theta u^{w^1}(y)}{1 + \theta \bar{u}^{w^1}} \right| dy$$

$$\leq \sup_y f(x,y) \int_0^1 \left| \frac{\theta u^{w^2}(y)}{1 + \theta \bar{u}^{w^2}} - \frac{\theta u^{w^1}(y)}{1 + \theta \bar{u}^{w^1}} \right| dy$$

$$\leq \sup_{z,y} f(z,y) \int_0^1 \left| \frac{\theta u^{w^2}(y)}{1 + \theta \bar{u}^{w^2}} - \frac{\theta u^{w^1}(y)}{1 + \theta \bar{u}^{w^2}} \right| + \left| \frac{\theta u^{w^1}(y)}{1 + \theta \bar{u}^{w^2}} - \frac{\theta u^{w^1}(y)}{1 + \theta \bar{u}^{w^1}} \right| dy$$

$$\leq \theta \sup_{z,y} f(z,y) \left(\int_0^1 |u^{w^2}(y) - u^{w^1}(y)|\, dy + |\bar{u}^{w^2} - \bar{u}^{w^1}| \right).$$

According to Lemmas 3 and 4, $\int_0^1 |u^{w^2}(y) - u^{w^1}(y)|\, dy$ converges to zero when w^2 converges to w^1. An immediate implication is that the difference between the mass of unmatched agents, $|\bar{u}^{w^2} - \bar{u}^{w^1}|$, must converge to zero as well.

C. DESCRIPTIVE THEORY PROOFS

MATCHING SET BOUND FUNCTIONS: PROOF OF PROPOSITION 3.

- STEP 1: MONOTONIC BOUND FUNCTIONS ⇒ ASSORTATIVE MATCHING. As the two cases are analogous, we prove that nondecreasing bounds a, b imply PAM. Choose $x_1 < x_2$ and $y_1 < y_2$ with $y_1 \in \mathcal{M}(x_2)$ and $y_2 \in \mathcal{M}(x_1)$. Then $b(x_2) \geq b(x_1) \geq y_2$, where the first inequality follows from b's monotonicity, and the second from the fact that $y_2 \in \mathcal{M}(x_1)$. Also, $y_2 > y_1 \geq a(x_2)$, where the second inequality follows from $y_1 \in \mathcal{M}(x_2)$. In summary, $b(x_2) \geq y_2 > a(x_2)$, and since matching sets are closed and convex, $y_2 \in \mathcal{M}(x_2)$. Similarly, a nondecreasing lower bound function a ensures that $y_1 \in \mathcal{M}(x_1)$, whence matching is positively assortative.
- STEP 2: ASSORTATIVE MATCHING ⇒ MONOTONIC BOUND FUNCTIONS. Proposition 2 proved that assortative matching implies convex matching sets. To avoid tedious repetition among four similar cases, we simply prove that b is nondecreasing with PAM. If not, $b(x_1) > b(x_2)$ for some pair $x_1 < x_2$. Then since $b(x_1) \in \mathcal{M}(x_1)$ and $b(x_2) \in \mathcal{M}(x_2)$ (matching sets being closed), PAM implies $b(x_1) \in \mathcal{M}(x_2)$, contradicting that $b(x_2)$ is the upper bound of x_2's matching set. *Q.E.D.*

AN INTERMEDIATE VALUE THEOREM

CLAIM 1: *Let the correspondence $\mathcal{M}: [0,1] \rightrightarrows [0,1]$ be upper hemicontinuous (u.h.c.), and convex- and nonempty-valued. Take $y_0 < y_2$, $z_0 \in \mathcal{M}(y_0)$, and $z_2 \in \mathcal{M}(y_2)$. If $z_0 < z_2$, then for all $z_1 \in [z_0, z_2]$, there exists $y_1 \in [y_0, y_2]$ with $z_1 \in \mathcal{M}(y_1)$.*

PROOF: Define $y_1 = \sup\{y \in [y_0, y_2] | \mathcal{M}(y) \cap [0, z_1] \neq \emptyset\}$, the largest point whose image includes points less than z_1. Since $\mathcal{M}$ is u.h.c., $\mathcal{M}(y_1) \cap [0, z_1] \neq \emptyset$ as well. If $y_1 = y_2$, then convexity of $\mathcal{M}(y_2)$ implies $z_1 \in \mathcal{M}(y_2)$. Otherwise, take a convergent sequence of points $y^n \to y_1$ with $y_2 \geq y^n > y_1$ for all n. Associate with this another sequence of points $\{z^n\}$ with $z^n \in \mathcal{M}(y_n)$ for all n. By the construction of y_1, $z^n \in (z_1, 1]$ for all n. Since $\mathcal{M}$ is u.h.c., there is a convergent subsequence of $\{z^n\}$ whose limit point $z \in [z_1, 1]$ is in $\mathcal{M}(y_1)$. We have shown that $\mathcal{M}(y_1)$ includes points weakly greater than and weakly less than z_1. Since it is convex, $z_1 \in \mathcal{M}(y_1)$. *Q.E.D.*

SINGLE CROSSING PROPERTY: PROOF OF LEMMA 6. We need a preliminary result:

CLAIM 2 (DENSITY-FREE INTEGRAL COMPARISON): *Take* $M \subseteq [0,1]$ *and* $u: M \mapsto \mathbb{R}_+$. *Let* $\phi, \psi: [0,1] \mapsto \mathbb{R}$ *be increasing (decreasing) functions with a nondecreasing (nonincreasing) ratio* ϕ/ψ. *If* $\int_M \psi(x)u(x)\,dx = 0$, *then* $\int_M \phi(x)u(x)\,dx \geq 0$.

PROOF: Assume without loss of generality that ϕ and ψ are increasing, as in Figure 5. If $M \neq [0,1]$, extend u to $[0,1]$ by defining $u(x) \equiv 0$ for all $x \notin M$. Define $I(x) \equiv \int_0^x \psi(x')u(x')\,dx'$. Clearly $I(0) = 0$, while $I(1) = 0$ by assumption. Then I is quasiconvex, as ψ is increasing, and so $I(x) \leq 0$ for all $x \in [0,1]$.

Substituting with I and the nondecreasing quotient $q(x) \equiv \phi(x)/\psi(x)$ yields

$$\int_M \phi(x)u(x)\,dx \equiv \int_0^1 q(x)I'(x)\,dx = q(1)I(1) - q(0)I(0) - \int_0^1 I(x)\,dq(x)$$

where we have integrated by parts. The first two terms in the last expression are zero, and the final term nonnegative, as $I(\cdot) \leq 0$, $dq(\cdot) \geq 0$. So $\int_M \phi(x)u(x)\,dx \geq 0$. *Q.E.D.*

PROOF OF LEMMA 6: We just establish the supermodular case. First, $\bar{z}$ is uniquely defined, as $h(\bar{z}, y_1)$ is strictly increasing in $\bar{z}$. Next, integrating $h_x(x_2, y')h_x(x_1, y_1) \geq h_x(x_2, y_1)h_x(x_1, y')$ over $x_1 \in [\bar{z}, x_2]$ and then $x_2 \in [x_1, \bar{z}]$, we discover that

$$h_x(x, y')(h(x, y_1) - h(\bar{z}, y_1)) \geq h_x(x, y_1)(h(x, y') - h(\bar{z}, y'))$$

for all $y' \geq y_1$, and for all $x > \bar{z}$ *and* crucially also all $x < \bar{z}$. This is equivalent to

$$\frac{\partial}{\partial x}\frac{h(x, y') - h(\bar{z}, y')}{h(x, y_1) - h(\bar{z}, y_1)} \geq 0.$$

Let $\phi(x) \equiv h(x, y') - h(\bar{z}, y')$ and $\psi(x) \equiv h(x, y_1) - h(\bar{z}, y_1)$, increasing. Since ϕ/ψ is nondecreasing and $E_{x \in M}\psi(x) = 0$ by construction, $E_{x \in M}\phi(x) \geq 0$ by Claim 2. *Q.E.D.*

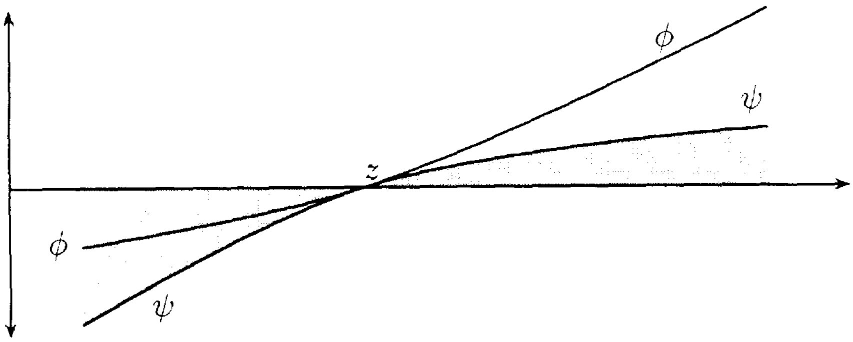

FIGURE 5.—Illustration of Claim 2. This illustrates the increasing case. As $\int_M \psi(x)u(x)\,dx = 0$, a z exists such that $\psi(x) \gtrless 0$ for all $x \gtrless z$. And since ϕ/ψ is nondecreasing, ϕ must equal zero at the same point. Then the ratio condition implies that, since the positive and negative areas of ψ balance, the positive area of ϕ must outweigh the negative area.

PROOF OF LEMMA 7: We will prove the contrapositive: If σ is not quasiconcave, there exist $0 < y_1 < y_2$ such that $\sigma'(y_1) < 0$ and $\sigma(y_1) < \sigma(y_2)$. Since σ is not quasiconcave, there are three points $y_0 < \hat{y}_1 < y_2$ with $\sigma(\hat{y}_1) < \min\langle \sigma(y_0), \sigma(y_2) \rangle$.

First take the case where $\sigma(y) < \sigma(y_2)$ for all $y \in [y_0, \hat{y}_1]$. Then as $\int_{y_0}^{\hat{y}_1} \sigma'(z)\,dz = \sigma(\hat{y}_1) - \sigma(y_0) < 0$, there exists $y_1 \in (y_0, \hat{y}_1)$ with $\sigma'(y_1) < 0$. By construction, $\sigma(y_1) < \sigma(y_2)$ as well, the desired counterexample.

Alternatively, if $\max_{y \in [y_0, \hat{y}_1]} \sigma(y) \geq \sigma(y_2)$, define $\hat{y}_0 = \sup\{y < \hat{y}_1 | \sigma(y) \geq \sigma(y_2)\}$. Note $\hat{y}_0 < \hat{y}_1$ because σ is continuous and $\sigma(\hat{y}_1) < \sigma(y_2)$. Then we can proceed as before, with $\hat{y}_0$ playing the role of y_0, to prove that there is $y_1 \in (\hat{y}_0, \hat{y}_1)$ with $\sigma'(y_1) < 0$ and, by construction, $\sigma(y_1) < \sigma(y_2)$.

Q.E.D.

ANALYTICAL SOLUTION FOR THE VALUE FUNCTION: If all matches are acceptable, it is possible to solve analytically for the value function. First, equation (1) implies the unmatched density function satisfies $u(x) \equiv \bar{u}l(x)$ for all x, where $\bar{u}$ is the aggregate unemployment rate, the larger root of the quadratic equation $\delta(1 - \bar{u}) = \rho \bar{u}^2$:

$$\bar{u} = \frac{-\delta + \sqrt{\delta^2 + 4\delta\rho}}{2\rho}.$$

Next, (8) yields an analytical solution for $w'(x)$ for all x. Finally, to pin down the *level* of the value function, use the implicit value equation (6) for type 0:

$$w(0) = \theta \int_0^1 \left(f(0, y) - w(0) - w(0) - \int_0^y w'(z)\,dz \right) u(y)\,dy$$

where we use $w(y) \equiv w(0) + \int_0^y w'(z)\,dz$. This can be solved explicitly for $w(0)$.

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