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THE IMPLEMENTATION DUALITY

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Conjugate duality relationships are pervasive in matching and implementation problems and provide much of the structure essential for characterizing stable matches and implementable allocations in models with quasilinear (or transferable) utility. In the absence of quasilinearity, a more abstract duality relationship, known as a Galois connection, takes the role of (generalized) conjugate duality. While weaker, this duality relationship still induces substantial structure. We show that this structure can be used to extend existing results for, and gain new insights into, adverse-selection principal-agent problems and two-sided matching problems without quasilinearity.

KEYWORDS: Implementation, conjugate duality, Galois connection, optimal transport, imperfectly transferable utility, principal-agent model, two-sided matching.

1. INTRODUCTION

MUCH OF THE THEORY OF MECHANISM DESIGN with quasilinear utility can be developed from a linear programming perspective, with duality-based arguments taking center stage (Vohra (2011)). The fundamental duality of linear programming also plays a central role in the theory of matching models with quasilinear (transferable) utility, from the theory's inception in Shapley and Shubik (1972) to the more recent adoption of optimal transport methods (cf. Galichon (2016)) based on the Kantorovich duality for infinite-dimensional linear programs (Villani (2009)).

In the context of matching problems with transferable utility (in their guise as optimal transportation problems), it is well understood that the linear programming duality gives rise to a second layer of duality relationships: stable outcomes in such models are composed of optimal assignments (obtained as the solution to a primal linear programming problem) together with optimal utility profiles (obtained as the solution to the dual linear programming problem), with the utility profiles being generalized conjugate duals of each other and the optimal assignment being drawn from the argmax correspondence of the maximization problems inducing this duality (Galichon (2016, Chapter 7)). Generalized conjugate duality also plays a prominent role in mechanism design with quasilinear utility, giving (for instance) rise to the characterization of implementable assignments in Rochet (1987). This is no coincidence: as Carlier (2003) has shown, testing for the imple-

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mentability of a given assignment is equivalent to checking whether the assignment solves an optimal transportation problem (cf. Galichon (2016, Section 9.6.2)).

Models based on quasilinear utility are ill-suited for mechanism design problems in which the stakes are sufficiently large to make income effects or risk aversion salient (Mirrlees (1971), Stiglitz (1977)), and are also ill-suited for matching problems in which—either because of income effects or because of the structure of the underlying bilateral relationship—utility is imperfectly transferable (Legros and Newman (2007), Chiappori and Salanié (2016), Chiappori (2017), Galichon, Kominers, and Weber (2017)).

This paper studies implementation without invoking quasilinearity. In so doing, we lose access to the linear programming duality. Nonetheless, we find that much of the conjugate duality structure and the link between matching problems and implementation problems remains.

The first part of the paper, Sections 2 and 3, introduces a pair of "implementation maps" and shows that they satisfy a duality relationship, known as a Galois connection (Birkhoff (1995, p. 124)), which is a more abstract version of the generalized conjugate duality relationship from the quasilinear case. Implementable utility profiles are abstract conjugate duals of each other, and implementable assignments are drawn from the corresponding argmax correspondence.

The second part of the paper, Sections 4 to 6, illustrates the potential application of our results by developing an "abstract duality" approach to two-sided matching problems and adverse-selection principal-agent problems.

Section 4 examines stable outcomes in two-sided matching models. We show that stable outcomes can be characterized in terms of a pair of profiles implementing each other together with the argmax correspondences associated with these profiles. We then leverage familiar existence results for matching models with a finite number of agents in order to obtain an existence result for more general models. We also derive lattice results for sets of stable utility profiles from the underlying duality structure.

Section 5 turns to adverse-selection principal-agent models. Our first finding is an existence result. The important step here is that we can formulate the principal's problem as a nonlinear pricing problem in which the principal maximizes over the set of *implementable* tariffs. We next show that, unlike the quasilinear case, the solution to the principal's problem may leave slack in the participation constraint for *every* type of agent. We explore two sufficient conditions for a solution to entail a binding participation constraint. One is a *strong implementability* condition that captures the essential implication of quasilinearity in a more general form, and the other is a private values condition on the principal's payoff. In both cases, the argument exploits the lattice structure of the set of implementable utility profiles.

Section 6 considers the special case in which a single-crossing condition holds and type spaces are one-dimensional. We show that there exists a unique stable match that is positively assortative. With our duality results in place, the proof is a straightforward generalization of the one which yields the existence of a unique solution to the optimal transport problem under supermodularity conditions. It then follows almost immediately from the parallels between matching and principal-agent models that an assignment is implementable if and only if it is increasing, just as in the quasilinear case.

2. IMPLEMENTATION

2.1. *Basic Ingredients*

The basic ingredients of our model are two sets, X and Y, and a function $\phi : X \times Y \times \mathbb{R} \to \mathbb{R}$. We offer two interpretations of these ingredients.

Matching model. X and Y are the possible types of two disjoint sets of agents that we refer to as buyers (X) and sellers (Y). The function ϕ specifies the utility frontier describing the feasible utilities that can be realized in a match between buyer type x and seller type y. That is, $u = \phi(x, y, v)$ is the maximal utility buyer type x can obtain when matched with seller type y and providing utility v to the seller. We complete the specification of a two-sided matching model in Section 4 by specifying distributions and reservation utilities for the buyer and seller types.

Principal-agent model. X is a set of possible types for an agent, Y is a set of possible decisions to be taken by the agent, and $u = \phi(x, y, v)$ is the utility of an agent of type x, who takes decision y and provides monetary transfer v to a principal. We complete the specification of an adverse-selection principal-agent model in Section 5 by specifying a utility function for the principal, her beliefs over the agent's types, and reservation utilities for the agent's types.

In the following, we will often refer to ϕ as the *generating function* as it plays the same role in our analysis as the generating function of a duality plays in Penot (2010).

ASSUMPTION 1: *The sets X and Y are compact subsets of metric spaces. The function $\phi : X \times Y \times \mathbb{R} \to \mathbb{R}$ is continuous, strictly decreasing in its third argument, and satisfies the full range condition $\phi(x, y, \mathbb{R}) = \mathbb{R}$ for all $(x, y) \in X \times Y$.*

The conditions on the generating function in Assumption 1 are satisfied if ϕ is quasilinear, that is, there exists a continuous function $f : X \times Y \to \mathbb{R}$ such that $\phi(x, y, v) = f(x, y) - v$. Our main interest is in generating functions that are not quasilinear.

In the context of the matching model, the assumption that ϕ is strictly decreasing excludes the case of *nontransferable utility* introduced in Gale and Shapley (1962), in which there is no possibility for compensatory transfers between a pair of matched agents. If the generating function is quasilinear, we have *perfectly transferable utility* as considered in Shapley and Shubik (1972), with Assumption 1 also allowing for *imperfectly transferable utility* as in Demange and Gale (1985).[1] Legros and Newman (2007, Section 5), Nöldeke and Samuelson (2015, Section 2), and Galichon, Kominers, and Weber (2017, Section 3) presented economic examples giving rise to non-quasilinear generating functions in matching models. In the principal-agent model, strict monotonicity of ϕ in its third argument squares with the interpretation of v as a monetary transfer, while going beyond the case in which the agent's utility function is quasilinear in the monetary transfer v by allowing for income effects. The importance of doing so in models of optimal nonlinear pricing has been emphasized in Wilson (1993, Chapter 7).

The essential implication of the full range condition in Assumption 1 is that (for example) for any agent type x and decisions y and $\tilde{y}$, there are transfers under which the agent prefers decision y, as well as transfers under which the agent prefers decision $\tilde{y}$. Demange and Gale (1985, Section 3) discussed the full range condition in the context of the matching model. In the principal-agent model, the condition ensures that the taxation principle is applicable without taking recourse to tariffs specifying infinite transfers (cf. Remark 1). All of our analysis goes through if A and B are open intervals in $\mathbb{R}$ and the generating function $\phi : X \times Y \times A \to B$ satisfies the counterpart to Assumption 1 with $\phi(x, y, A) = B$.

[1] Our terms for the case distinction between perfectly transferable, imperfectly transferable, and nontransferable follow (for example) Chade, Eeckhout, and Smith (2017) and Nöldeke and Samuelson (2015). Other authors (e.g., Legros and Newman (2007)) use the term nontransferable utility whenever utility is not perfectly transferable.

2.2. *The Inverse Generating Function*

Assumption 1 ensures that for all $x \in X$, $y \in Y$, and $u \in \mathbb{R}$, there is a unique value $v \in \mathbb{R}$ satisfying $u = \phi(x, y, v)$, so that the *inverse generating function* $\psi : Y \times X \times \mathbb{R} \to \mathbb{R}$ specified as the solution to

$$u = \phi\big(x, y, \psi(y, x, u)\big) \tag{1}$$

is well-defined and satisfies the "reverse" inverse relationship

$$v = \psi\big(y, x, \phi(x, y, v)\big). \tag{2}$$

The inverse generating function inherits the properties of the generating function stated in Assumption 1: ψ is continuous, strictly decreasing in its third argument, and satisfies $\psi(y, x, \mathbb{R}) = \mathbb{R}$ for all $(y, x) \in Y \times X$. (The straightforward verification is in Appendix B.1 of the Supplemental Material (Nöldeke and Samuelson (2018)).) Throughout the following, we freely make use of the compactness of X and Y and the properties of the generating function ϕ and its inverse ψ without explicitly referring to Assumption 1 or the argument in Appendix B.1.

In the context of the matching model, the interpretation of ψ is analogous to the one given for ϕ: the utility $v = \psi(y, x, u)$ is the maximal utility a seller type y can obtain when matched with a buyer type x and providing utility u to the buyer.[2] In the principal-agent model, ψ identifies the largest transfer an agent of type x can pay for the decision y while obtaining utility level u. In either context, as indicated by (1)–(2), the inverse generating function contains the same information about preferences as the generating function.

2.3. *Profiles, Assignments, and Implementability*

Let $\mathbf{B}(X)$ denote the set of bounded functions from X to $\mathbb{R}$ and let $\mathbf{B}(Y)$ denote the set of bounded functions from Y to $\mathbb{R}$. We refer to $\mathbf{u} \in \mathbf{B}(X)$ and $\mathbf{v} \in \mathbf{B}(Y)$ as *profiles*. We endow $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ with the supremum norm, denoted by $\| \cdot \|$ in both cases, making them complete metric spaces for the induced metric. We order $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ with the pointwise partial order inherited from the standard order $\geq$ on $\mathbb{R}$. For simplicity, we also denote these pointwise partial orders on $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ by $\geq$. The join $\mathbf{u} \vee \mathbf{u}'$ and meet $\mathbf{u} \wedge \mathbf{u}'$ are respectively the pointwise maximum and minimum of the profiles $\mathbf{u}$ and $\mathbf{u}'$. With these operations, the sets $\mathbf{B}(X)$ and $\mathbf{B}(Y)$) are conditionally complete lattices.[3]

Let Y^X denote the set of functions from X to Y and let X^Y be the set of functions from Y to X. Any function $\mathbf{y} \in Y^X$ and any function $\mathbf{x} \in X^Y$ will be referred to as an *assignment*.

We say that $(\mathbf{u}, \mathbf{y}) \in \mathbf{B}(X) \times Y^X$ is *implementable* if there exists a profile $\mathbf{v} \in \mathbf{B}(Y)$ that *implements* $(\mathbf{u}, \mathbf{y})$, meaning that the conditions

$$\mathbf{y}(x) \in \mathbf{Y}_{\mathbf{v}}(x) := \operatorname*{argmax}_{y \in Y} \phi\big(x, y, \mathbf{v}(y)\big), \tag{3}$$

[2]Observe that, in the definition of ψ, the order of the first two arguments has been exchanged, so that in the matching model for both ϕ and ψ the first argument gives the type of the agent whose maximal utility is specified and the second argument gives the type of his or her partner. In the quasilinear case, we have $\psi(y, x, u) = g(y, x) - u$, where $g(y, x) = f(x, y)$ holds for all $(x, y) \in X \times Y$.

[3]A lattice is conditionally complete if every nonempty subset that is bounded has both an infimum and a supremum. Here and throughout the following, we simply refer to a set of profiles in $\mathbf{B}(X)$ or $\mathbf{B}(Y)$ as being bounded without distinguishing between boundedness in order and boundedness in norm, as these two notions are equivalent in our setting.

$$\mathbf{u}(x) = \max_{y \in Y} \phi\big(x, y, \mathbf{v}(y)\big) \tag{4}$$

hold for all $x \in X$ (which, obviously, implies that the argmax correspondence $\mathbf{Y}_\mathbf{v} : X \rightrightarrows Y$ defined in (3) is nonempty-valued). Similarly, $(\mathbf{v}, \mathbf{x}) \in \mathbf{B}(Y) \times X^Y$ is implementable if there exists a profile $\mathbf{u} \in \mathbf{B}(X)$ implementing $(\mathbf{v}, \mathbf{x})$, meaning that for all $y \in Y$,

$$\mathbf{x}(y) \in \mathbf{X}_\mathbf{u}(y) := \operatorname*{argmax}_{x \in X} \psi\big(y, x, \mathbf{u}(x)\big), \tag{5}$$

$$\mathbf{v}(y) = \max_{x \in X} \psi\big(y, x, \mathbf{u}(x)\big). \tag{6}$$

We also say that a profile $\mathbf{v}$ *implements* the profile $\mathbf{u}$ (assignment $\mathbf{y}$) if there exists $\mathbf{y}$ (there exists $\mathbf{u}$) such that $\mathbf{v}$ implements $(\mathbf{u}, \mathbf{y})$. We use the analogous terms for a profile $\mathbf{u}$ implementing the profile $\mathbf{v}$ and assignment $\mathbf{x}$. Profiles and assignments are said to be *implementable* if there exists a profile implementing them. We let $\mathbf{I}(X)$ and $\mathbf{I}(Y)$ denote the sets of implementable profiles, so that (for example) $\mathbf{I}(X) = \{\mathbf{u} \in \mathbf{B}(X) | \exists \mathbf{v} \in \mathbf{B}(Y) \text{ s.t. (4) holds}\}$.

In the matching model, $\mathbf{u}$ is a utility profile for buyers, whereas $\mathbf{v}$ is a utility profile for sellers. An assignment $\mathbf{y}$ specifies, for each buyer type x, a seller type $y = \mathbf{y}(x)$ with whom x matches; the interpretation of an assignment $\mathbf{x}$ is analogous.[4] The utility profile $\mathbf{v}$ implements $(\mathbf{u}, \mathbf{y})$ if every buyer type x finds it optimal to select seller type $\mathbf{y}(x)$ as a partner and by doing so obtains the utility $\mathbf{u}(x)$, given that sellers have to be provided with the utility profile $\mathbf{v}$. The interpretation of conditions (5)–(6) is analogous.

In the principal-agent model, $\mathbf{u}$ specifies a utility level for each agent type, whereas an assignment $\mathbf{y}$ specifies a decision for each agent type. The profile $\mathbf{v}$ is a nonlinear tariff offered by the principal to the agent, with $\mathbf{v}(y)$ specifying the transfer to the principal at which any type of agent can purchase decision y. Such a tariff implements the pair $(\mathbf{u}, \mathbf{y})$ if all agent types find it optimal to choose the decisions specified in $\mathbf{y}$ when faced with the tariff $\mathbf{v}$, and $\mathbf{u}$ is the resulting rent function. We may think of a type assignment $\mathbf{x}$ as specifying, for each decision y, an agent type $\mathbf{x}(y)$ to whom the principal wants to sell decision y, as in Nöldeke and Samuelson (2007). Though the interpretation of a rent function $\mathbf{u}$ implementing a pair $(\mathbf{v}, \mathbf{x})$ is less obvious in the principal-agent model, Section 5 shows that the notion of an implementable tariff can nonetheless be helpful.

REMARK 1—Implementability and Direct Mechanisms: In defining implementability, we have taken a nonlinear pricing (rather than a direct mechanism) approach and, in addition, have required the profiles $\mathbf{u}$ and $\mathbf{v}$ to be both bounded. The taxation principle (e.g., Guesnerie and Laffont (1984), Rochet (1985)) is applicable in our setting and ensures that there is no loss of generality in using a nonlinear pricing approach when studying principal-agent models. What is less obvious is that the restriction to bounded profiles is innocent, but this follows from Assumption 1.[5] Appendix B.2 of the Supplemental Material verifies this claim.

[4]Note that the definition of an assignment does not incorporate any notion of feasibility (e.g., an assignment $\mathbf{x}$ could specify that all types of the seller match with the same type of buyer). In the matching context, an assignment is sometimes referred to as a pre-matching (Adachi (2000)) or a semi-matching (Lawler (2001)).

[5]In the absence of the full range condition from Assumption 1, this conclusion may fail. To see this, it suffices to consider a direct mechanism in which type x obtains utility u from choosing y, but there exists y' such that $\lim_{v \to \infty} \phi(x, y', v) > u$. Then, no matter what transfer $\mathbf{v}(y') \in \mathbb{R}$ is specified, type x will prefer to choose y' rather than y.

2.4. *Strongly Implementable Assignments*

We say that a profile $\mathbf{u} \in \mathbf{B}(X)$ satisfies the initial condition $(x_0, u_0) \in X \times \mathbb{R}$ if $\mathbf{u}(x_0) = u_0$ holds and say that an assignment $\mathbf{y} \in Y^X$ is *strongly implementable* if, for all initial conditions (x_0, u_0), there exists $\mathbf{u}$ such that $(\mathbf{u}, \mathbf{y})$ is implementable and $\mathbf{u}$ satisfies the initial condition. Similarly, a profile $\mathbf{v} \in \mathbf{B}(Y)$ satisfies the initial condition $(y_0, v_0) \in Y \times \mathbb{R}$ if $\mathbf{v}(y_0) = v_0$ holds and an assignment $\mathbf{x} \in X^Y$ is strongly implementable if, for all initial conditions (y_0, v_0), there exists $\mathbf{v}$ such that $(\mathbf{v}, \mathbf{x})$ is implementable and $\mathbf{v}$ satisfies the initial condition. An assignment is thus strongly implementable if it can be implemented while pegging the utility level of an arbitrary agent at an arbitrary level.

With a quasilinear generating function, every implementable assignment is strongly implementable, so that the distinction between these two concepts is moot. This follows from the translational invariance of the incentive constraints under quasilinearity: $\mathbf{u}(x) = f(x, \mathbf{y}(x)) - \mathbf{v}(\mathbf{y}(x)) = \max_{y \in Y}[f(x, y) - \mathbf{v}(y)]$ implies $\mathbf{u}(x) - t = f(x, \mathbf{y}(x)) - (\mathbf{v}(\mathbf{y}(x)) + t) = \max_{y \in Y}[f(x, y) - (\mathbf{v}(y) + t)]$ for all $x \in X$ and $t \in \mathbb{R}$, so that by choosing the constant t appropriately, a tariff $\mathbf{v}$ implementing an assignment $\mathbf{y}$ can be adjusted to satisfy any given initial condition while continuing to implement $\mathbf{y}$ (with an analogous argument applying to implementable $\mathbf{x} \in X^Y$).

In general, the implementability of an assignment does not imply its strong implementability. This causes some salient differences between the quasilinear and the general case. For example, if every implementable profile is strongly implementable, then—just as in the quasilinear case—the participation constraint must be binding for some type of agent in a solution to the principal-agent model (Proposition 10), whereas this property may fail otherwise (see the example in Appendix C.2 of the Supplemental Material). Remark 2 and Section 6.2 identify circumstances in which all implementable profiles are strongly implementable, ensuring that an important structural property of the quasilinear case is preserved, even though the generating function is not quasilinear.

REMARK 2—A Sufficient Condition for Strong Implementability: Appendix B.3 of the Supplemental Material shows that every implementable assignment is strongly implementable if the generating function satisfies

$$\big[\phi(x, y, v) - \phi\big(x, y', v'\big)\big] = \big[\phi(x, y, \hat{v}) - \phi\big(x, y', \hat{v}'\big)\big]$$

$$\Longrightarrow \tag{7}$$

$$\big[\phi\big(x', y, v\big) - \phi\big(x', y', v'\big)\big] = \big[\phi\big(x', y, \hat{v}\big) - \phi\big(x', y', \hat{v}'\big)\big]$$

for any x, x', y, and y' and any v, v', $\hat{v}$, and $\hat{v}'$.

Condition (7) imposes a restriction *across* types, demanding that whatever change in tariff is required to preserve all utility differences for one type will also preserve all utility differences for any other type. Condition (7) holds, of course, if the characteristic function is quasilinear. More generally, it is satisfied if the characteristic function takes the form $\phi(x, y, v) = f(x, y) - h(y, v)$.

We note that, in the context of the principal-agent model, condition (7) embodies *no* restriction on the preferences of a single agent type x over (y, v) pairs beyond the weak regularity properties from Assumption 1, and hence allows arbitrary income effects. This is in contrast to the quasilinear case, which implies the absence of income effects.

3. DUALITY

In this section, we characterize implementable profiles and assignments. Section 3.1 introduces a pair of functions between sets of profiles that we refer to as implementation maps, and shows that these maps are a Galois connection between the sets of profiles $\mathbf{B}(X)$ and $\mathbf{B}(Y)$. Equivalently, these maps are dualities that are dual to each other. Section 3.2 uses the structure of the implementation maps to characterize implementable profiles. Building on these results, Section 3.3 characterizes implementable assignments and Section 3.4 establishes some key properties of sets of implementable profiles.

3.1. *Implementation Maps*

Consider any profile $\mathbf{v} \in \mathbf{B}(Y)$. As X and Y are compact and ϕ is continuous, setting $\mathbf{u}(x) = \sup_{y\in Y} \phi(x, y, \mathbf{v}(y))$ for all $x \in X$ results in a bounded profile $\mathbf{u} \in \mathbf{B}(X)$. Together with a similar argument for $\mathbf{v}(y) = \sup_{x\in X} \psi(y, x, \mathbf{u}(x))$, this ensures that the implementation maps $\Phi : \mathbf{B}(Y) \to \mathbf{B}(X)$ and $\Psi : \mathbf{B}(X) \to \mathbf{B}(Y)$ obtained by setting

$$\Phi\mathbf{v}(x) = \sup_{y\in Y} \phi\big(x, y, \mathbf{v}(y)\big) \quad \forall x \in X, \tag{8}$$

$$\Psi\mathbf{u}(y) = \sup_{x\in X} \psi\big(y, x, \mathbf{u}(x)\big) \quad \forall y \in Y \tag{9}$$

are well-defined. Appendix B.4 of the Supplemental Material proves that these maps are also reasonably well-behaved:

LEMMA 1: *Let Assumption 1 hold. The implementation maps $\Phi : \mathbf{B}(Y) \to \mathbf{B}(X)$ and $\Psi : \mathbf{B}(X) \to \mathbf{B}(Y)$ are continuous and map bounded sets into bounded sets.*

We next show that Φ and Ψ are a Galois connection (Birkhoff (1995, p. 124)) between the sets $\mathbf{B}(X)$ and $\mathbf{B}(Y)$. That is,

$$\mathbf{u} \geq \Phi\mathbf{v} \iff \mathbf{v} \geq \Psi\mathbf{u} \tag{10}$$

holds for all $\mathbf{u} \in \mathbf{B}(X)$ and $\mathbf{v} \in \mathbf{B}(Y)$.[6] Equivalently, the implementation maps are dualities that are dual to each other, where a duality is a map between two partially ordered sets with the property that, for any subset of the domain which has an infimum, the image of the infimum of that set is the supremum of its image (Penot (2010, Definition 1)), and the implementation maps are dual to each other if

$$\Phi\mathbf{v} = \inf\{\mathbf{u}|\mathbf{v} \geq \Psi\mathbf{u}\} \quad \text{and} \quad \Psi\mathbf{u} = \inf\{\mathbf{v}|\mathbf{u} \geq \Phi\mathbf{v}\}$$

holds for all $\mathbf{u} \in \mathbf{B}(X)$ and $\mathbf{v} \in \mathbf{B}(Y)$.[7]

PROPOSITION 1: *Let Assumption 1 hold. The implementation maps Φ and Ψ are a Galois connection or, equivalently, are dualities that are dual to each other.*

[6] There is an alternative definition of a Galois connection in which the second inequality in (10) is reversed (Davey and Priestley (2002, Chapter 7)).

[7] Singer (1997, Definition 5.1) defined a duality as a map between *complete lattices* with the property that the image of the infimum of any set is the supremum of the image of that set. Penot's definition provides the obvious generalization to the situation under consideration here in which $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ are lattices, but are not complete. The notion of maps dual to each other is similarly adapted from Singer (1997, Definition 5.2).

PROOF: To obtain (10) and hence the claim that Φ and Ψ are a Galois connection, observe:

$$\begin{aligned}
\mathbf{u} \geq \Phi\mathbf{v} \quad &\iff \quad \mathbf{u}(x) \geq \sup_{y\in Y} \phi\big(x, y, \mathbf{v}(y)\big) \quad \text{for all } x \in X \\
&\iff \quad \mathbf{u}(x) \geq \phi\big(x, y, \mathbf{v}(y)\big) \quad \text{for all } x \in X \text{ and } y \in Y \\
&\iff \quad \psi\big(y, x, \mathbf{u}(x)\big) \leq \mathbf{v}(y) \quad \text{for all } x \in X \text{ and } y \in Y \\
&\iff \quad \mathbf{v}(y) \geq \sup_{x\in X} \psi\big(y, x, \mathbf{u}(x)\big) \quad \text{for all } y \in Y \\
&\iff \quad \mathbf{v} \geq \Psi\mathbf{u},
\end{aligned}$$

where the first equivalence holds by the definition of $\Phi\mathbf{v}$ in (8), the second is from the definition of the supremum, the third uses (2) and that the inverse generating function ψ is strictly decreasing in its third argument, the fourth is by the definition of the supremum, and the fifth holds by the definition of $\Psi\mathbf{u}$ in (9).

The result that Φ and Ψ are a Galois connection if and only if they are dualities that are dual to each other is standard for maps between complete lattices (Singer (1997, Theorem 5.4)). Appendix B.5 of the Supplemental Material contains a proof, building on Corollary 1 below, adapted to our setting in which the lattices $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ are not complete. *Q.E.D.*

To interpret the result that Φ and Ψ are a Galois connection, consider the matching context. Suppose we have a pair of profiles $\mathbf{u}$ and $\mathbf{v}$ such that each buyer $x \in X$ is content to obtain $\mathbf{u}(x)$ rather than matching with any seller $y \in Y$ and providing that seller with utility $\mathbf{v}(y)$, that is, the inequality $\mathbf{u} \geq \Phi\mathbf{v}$ holds. It is then intuitive that every seller $y \in Y$ would similarly weakly prefer to obtain utility $\mathbf{v}(y)$ to matching with any buyer $x \in X$ who insists on receiving utility $\mathbf{u}(x)$, that is, the inequality $\mathbf{v} \geq \Psi\mathbf{u}$ holds. Reversing the roles of buyers and sellers in this explanation motivates the other direction of the equivalence in (10).

The statements in the following corollary are standard implications of the fact that Φ and Ψ are a Galois connection. Our terms for these follow Davey and Priestley (2002, p. 159); for completeness, Appendix B.5 of the Supplemental Material provides a proof.

COROLLARY 1: *Let Assumption 1 hold. The implementation maps Φ and Ψ*
(1) *satisfy the cancellation rule, that is, for all $\mathbf{u} \in \mathbf{B}(X)$ and $\mathbf{v} \in \mathbf{B}(Y)$,*

$$\mathbf{v} \geq \Psi\Phi\mathbf{v} \quad \textit{and} \quad \mathbf{u} \geq \Phi\Psi\mathbf{u}; \tag{11}$$

(2) *are order reversing, that is, for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{B}(X)$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{B}(Y)$,*

$$\mathbf{v}_1 \geq \mathbf{v}_2 \Longrightarrow \Phi\mathbf{v}_1 \leq \Phi\mathbf{v}_2 \quad \textit{and} \quad \mathbf{u}_1 \geq \mathbf{u}_2 \Longrightarrow \Psi\mathbf{u}_1 \leq \Psi\mathbf{u}_2; \tag{12}$$

(3) *satisfy the semi-inverse rule, that is, for all $\mathbf{u} \in \mathbf{B}(X)$ and $\mathbf{v} \in \mathbf{B}(Y)$,*

$$\Psi\Phi\Psi\mathbf{u} = \Psi\mathbf{u} \quad \textit{and} \quad \Phi\Psi\Phi\mathbf{v} = \Phi\mathbf{v}. \tag{13}$$

To provide some interpretation for (11)–(13), we focus on the first statement in each case and consider the principal-agent model. The order-reversal property (Corollary 1(2)) asserts that all agent types are better off when the prices specified by the tariff are low rather than high. Intuitively, the tariff $\Psi\Phi\mathbf{v}$ appearing in the cancellation rule (Corollary 1(1)) specifies, for each decision $y \in Y$, the highest payment such that some agent type x can achieve the same utility from choosing y as from maximizing against the tariff $\mathbf{v}$ (i.e., $\Phi\mathbf{v}(x)$), thereby making $\Psi\Phi\mathbf{v}$ an "envelope tariff."[8] The assertion of the cancellation rule then is that the envelope tariff $\Psi\Phi\mathbf{v}$ obtained from the tariff $\mathbf{v}$ specifies payments no higher than the original tariff $\mathbf{v}$. Finally, the semi-inverse rule (Corollary 1(3)) indicates that the inequality from the cancellation rule turns into an equality when the original tariff $\mathbf{v}$ is given by $\Psi\mathbf{u}$, and hence specifies the highest payments for which, for any decision y, some agent type x can achieve utility $\mathbf{u}(x)$ by choosing decision y.

REMARK 3—Quasilinearity and Generalized Conjugate Duality: In the quasilinear case, the definitions of the implementation maps in (8) and (9) reduce to

$$\Phi\mathbf{v}(x) = \sup_{y\in Y}\big[f(x, y) - \mathbf{v}(y)\big],$$

$$\Psi\mathbf{u}(y) = \sup_{x\in X}\big[g(y, x) - \mathbf{u}(x)\big],$$

where $g(y, x) = f(x, y)$ holds for all $(x, y) \in X \times Y$ (cf. footnote 2). In this case, $\Phi\mathbf{v}$ is a familiar object, namely the f-conjugate of $\mathbf{v}$, and $\Psi\mathbf{u}$ is the g-conjugate of $\mathbf{u}$ (cf. Ekeland (2010, Section 3.2)). The properties noted in Corollary 1 generalize corresponding properties from the theory of (generalized) conjugate duality. Indeed, the cancellation property (Corollary 1(1)) corresponds to the statement that the biconjugate of any function is smaller than the function itself and the semi-inverse rule (Corollary 1(3)) corresponds to the statement that a conjugate function is its own biconjugate. These are well-known implications of conjugate duality (cf. Ekeland (2010, Section 3.4)). Martinez-Legaz and Singer (1990, 1995) offered additional illustrations of how results for abstract dualities specialize to familiar results from conjugate duality when the generating function is quasilinear.

3.2. *Implementable Profiles*

Comparing the implementation condition (4) and the definition of the implementation map Φ in (8), it is clear that $\mathbf{v} \in \mathbf{B}(Y)$ implements $\mathbf{u} \in \mathbf{B}(X)$ if and only if $\mathbf{u} = \Phi\mathbf{v}$ holds and, in addition, the suprema in (8) are attained for all $x \in X$, that is, the argmax correspondence $\mathbf{Y}_\mathbf{v}$ defined in (3) is nonempty-valued. Consequently, the set of implementable profiles $\mathbf{I}(X)$ is contained in the image $\Phi\mathbf{B}(Y)$ of the implementation map Φ. Similarly, $\mathbf{I}(Y) \subseteq \Psi\mathbf{B}(X)$ holds.

The following proposition shows that the reverse set inclusions also hold. Hence, the images of the implementation maps are precisely the sets of implementable profiles. In the course of proving this result, it is straightforward to also show that every implementable profile is continuous.[9] Let $\mathbf{C}(X) \subseteq \mathbf{B}(X)$ denote the set of continuous (and hence necessarily bounded, since X is compact) functions from X to $\mathbb{R}$, with $\mathbf{C}(Y)$ analogous. Appendix A.1 shows the following:

[8]In convex analysis, the counterpart of $\Psi\Phi\mathbf{v}$ is referred to as the convex envelope of $\mathbf{v}$, and is the greatest convex minorant of $\mathbf{v}$ (Galichon (2016, Proposition D.12)). An analogous property holds here. First, from the cancellation property, $\Psi\Phi\mathbf{v}$ is a minorant of $\mathbf{v}$. Second, consider $\mathbf{u}$ satisfying $\Psi\mathbf{u} \le \mathbf{v}$. Applying the order reversal property twice yields $\Psi\Phi\Psi\mathbf{u} \le \Psi\Phi\mathbf{v}$ and therefore, from the semi-inverse rule $\Psi\mathbf{u} \le \Psi\Phi\mathbf{v}$.

[9]Weibull (1989) has obtained related results in an optimal taxation model with one-dimensional types and decisions.

PROPOSITION 2: *Let Assumption 1 hold. A profile is implementable if and only if it is in the image of the relevant implementation map. Further, every implementable profile is continuous. That is*,

$$\mathbf{I}(X) = \Phi\mathbf{B}(Y) \subseteq \mathbf{C}(X) \quad \textit{and} \quad \mathbf{I}(Y) = \Psi\mathbf{B}(X) \subseteq \mathbf{C}(Y). \tag{14}$$

The first step in the proof of Proposition 2 shows that every lower semicontinuous profile implements its image under the relevant implementation map and that this image is continuous. The proof is then completed by showing that the image of any profile under the relevant implementation map is the same as the image of its lower semicontinuous hull.

As a direct implication of Berge's maximum theorem, the continuity of implementable profiles and of the generating function ensures that the argmax correspondences associated with implementable profiles are well-behaved. In particular, as the argmax correspondences are nonempty-valued, implementable profiles implement their images under the relevant implementation map:

COROLLARY 2: *Let Assumption 1 hold. If* $\mathbf{v} \in \mathbf{I}(Y)$, *then the argmax correspondence* $\mathbf{Y}_{\mathbf{v}}$ *is nonempty-valued and compact-valued and upper hemicontinuous and* $\mathbf{v}$ *implements* $\Phi\mathbf{v}$. *Analogously, if* $\mathbf{u} \in \mathbf{I}(X)$, *then the argmax correspondence* $\mathbf{X}_{\mathbf{u}}$ *is nonempty-valued and compact-valued and upper hemicontinuous and* $\mathbf{u}$ *implements* $\Psi\mathbf{u}$.

Combining Proposition 2 with the semi-inverse rule from Corollary 1(3) yields a characterization of implementable profiles:

PROPOSITION 3: *Let Assumption 1 hold.*
(1) $\mathbf{u} \in \mathbf{B}(X)$ *is implementable if and only if* $\mathbf{u} = \Phi\Psi\mathbf{u}$.
(2) $\mathbf{v} \in \mathbf{B}(Y)$ *is implementable if and only if* $\mathbf{v} = \Psi\Phi\mathbf{v}$.

PROOF: We prove Proposition 3(1); Proposition 3(2) is analogous.

If $\mathbf{u} = \Phi\Psi\mathbf{u}$, then obviously $\mathbf{u} \in \Phi\mathbf{B}(Y)$ and hence (by Proposition 2) $\mathbf{u} \in \mathbf{I}(X)$. Conversely, if $\mathbf{u}$ is implementable, then there exists $\mathbf{v} \in \mathbf{B}(Y)$ such that $\mathbf{u} = \Phi\mathbf{v}$, and hence (by Corollary 1(3)) we have $\mathbf{u} = \Phi\Psi\mathbf{u}$. *Q.E.D.*

For any Galois connection, the counterparts to the fixed point conditions $\mathbf{u} = \Phi\Psi\mathbf{u}$ and $\mathbf{v} = \Psi\Phi\mathbf{v}$ characterize the images of the constituent maps (Singer (1997, Corollary 5.6)). Proposition 2 allows us to strengthen this result from a characterization of the images of the implementation maps (which we are not interested in as such) to a characterization of implementable profiles.

The following is a straightforward implication of Corollary 2 and Proposition 3:

COROLLARY 3: *Let Assumption 1 hold.*
(1) *Suppose the profile* $\mathbf{u} \in \mathbf{B}(X)$ *is implementable. Then* $\mathbf{u}$ *implements and is implemented by* $\mathbf{v} = \Psi\mathbf{u}$. *Further,* $\Psi\mathbf{u}$ *is the only profile in* $\mathbf{I}(Y)$ *implementing* $\mathbf{u}$.
(2) *Suppose the profile* $\mathbf{v} \in \mathbf{B}(Y)$ *is implementable. Then* $\mathbf{v}$ *implements and is implemented by* $\mathbf{u} = \Phi\mathbf{v}$. *Further,* $\Phi\mathbf{v}$ *is the only profile in* $\mathbf{I}(X)$ *implementing* $\mathbf{v}$.

PROOF: We prove Corollary 3(1); Corollary 3(2) is analogous.

Let $\mathbf{u} \in \mathbf{I}(X)$. By Corollary 2, $\mathbf{u}$ implements $\mathbf{v} = \Psi\mathbf{u}$. Hence, $\mathbf{v}$ is implementable and, by Corollary 2, in turn implements $\Phi\mathbf{v}$, which by Proposition 3(1) is identical to $\mathbf{u}$. Hence, $\mathbf{u}$ not only implements $\mathbf{v} = \Psi\mathbf{u}$ but is also implemented by it.

Suppose $\mathbf{u} = \Phi\mathbf{v}$ holds for some implementable profiles $\mathbf{u}$ and $\mathbf{v}$. Applying the implementation map Ψ to both sides of this equality yields $\Psi\mathbf{u} = \Psi\Phi\mathbf{v}$. As $\mathbf{v}$ is implementable, we also have $\Psi\Phi\mathbf{v} = \mathbf{v}$ from Proposition 3(2). Combining the two preceding equalities implies $\mathbf{v} = \Psi\mathbf{u}$, so that $\Psi\mathbf{u}$ is the only implementable profile implementing $\mathbf{u}$. *Q.E.D.*

Corollary 3 indicates that

$$\mathbf{u} = \Phi\mathbf{v} \quad \Longleftrightarrow \quad \mathbf{v} = \Psi\mathbf{u} \tag{15}$$

holds for all implementable profiles $\mathbf{u} \in \mathbf{I}(X)$ and $\mathbf{v} \in \mathbf{I}(Y)$ with these profiles implementing each other if and only if the equivalent statements in (15) hold. In particular, the continuous implementation maps Φ and Ψ are inverse bijections between the sets of implementable profiles $\mathbf{I}(X)$ and $\mathbf{I}(Y)$ and thus (since they are continuous, by Lemma 1) homeomorphisms between these sets. Figure 1 illustrates these observations in the context provided by Proposition 2.

Corollary 3 shows that all implementable profiles can be implemented by implementable profiles. The following result shows that implementable profiles also suffice to implement all implementable assignments. The straightforward proof in Appendix B.6 of the Supplemental Material relies on the cancellation property (Corollary 1(1)).

COROLLARY 4: *Let Assumption 1 hold.*
(1) *If* $(\mathbf{u}, \mathbf{y}) \in \mathbf{B}(X) \times Y^X$ *is implementable, then* $\mathbf{y}$ *is implemented by* $\Phi\mathbf{u}$.
(2) *If* $(\mathbf{v}, \mathbf{x}) \in \mathbf{B}(Y) \times X^Y$ *is implementable, then* $\mathbf{x}$ *is implemented by* $\Phi\mathbf{v}$.

REMARK 4—Implementable Profiles in the Quasilinear Case: Following up on Remark 3, we note that in the quasilinear case, Proposition 3 is the statement that a profile is implementable if and only if it is its own generalized biconjugate (Ekeland (2010, Corollary 12)). Taken together, Corollaries 3(1) and 4(1) indicate for the quasilinear case that a profile-assignment pair $(\mathbf{u}, \mathbf{y})$ is implementable if and only if it is implemented by

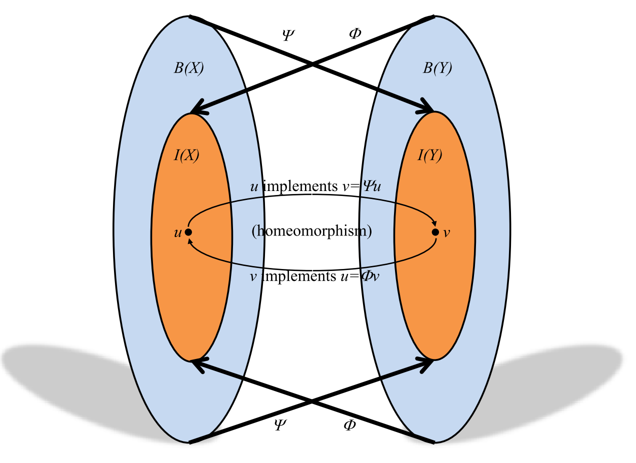

FIGURE 1.—Illustration of the implementation maps. The implementation map Φ maps the set of bounded profiles $\mathbf{B}(Y)$ onto the set of implementable profiles $\mathbf{I}(X)$ (and Ψ maps the set of bounded profiles $\mathbf{B}(X)$ onto the set of implementable profiles $\mathbf{I}(Y)$). The maps Φ and Ψ are continuous inverse bijections on the sets of implementable profiles $\mathbf{I}(X)$ and $\mathbf{I}(Y)$ with profiles $\mathbf{u}$ and $\mathbf{v}$ in these sets satisfying $\mathbf{u} = \Phi\mathbf{v} \Longleftrightarrow \mathbf{v} = \Psi\mathbf{u}$ and implementing each other.

the generalized conjugate of $\mathbf{u}$. As discussed in Basov (2006, p. 136 and p. 142), the latter result is the essence of the implementability criterion for the quasilinear case provided by Carlier (2002, Proposition 1).

3.3. *Implementable Assignments*

Given any pair of profiles $(\mathbf{u}, \mathbf{v})$, let

$$\begin{aligned}\Gamma_{\mathbf{u},\mathbf{v}} &= \{(x, y) \in X \times Y | \mathbf{u}(x) = \phi(x, y, \mathbf{v}(y))\} \\ &= \{(x, y) \in X \times Y | \mathbf{v}(y) = \psi(y, x, \mathbf{u}(x))\},\end{aligned} \tag{16}$$

where the second equality holds by definition of the inverse generating function ψ. If $\boldsymbol{v}$ implements $\boldsymbol{u}$, then $\Gamma_{\boldsymbol{u},\boldsymbol{v}}$ coincides with the graph of the argmax correspondence $\boldsymbol{Y}_{\boldsymbol{v}}$ defined in (3), i.e., if $\boldsymbol{v}$ implements $\boldsymbol{u}$ then $y \in \boldsymbol{Y}_{\boldsymbol{v}}(x)$ is equivalent to $\boldsymbol{u}(x) = \boldsymbol{\phi}(x, y, \boldsymbol{v}(y))$. Similarly, if $\mathbf{u}$ implements $\mathbf{v}$, the equality in the second line indicates that $\Gamma_{\mathbf{u},\mathbf{v}}$ coincides with the graph of the argmax correspondence $\mathbf{X}_{\mathbf{u}}$ defined in (5). For the special case in which the profiles $\mathbf{u}$ and $\mathbf{v}$ implement each other, the graphs of both $\mathbf{X}_{\mathbf{u}}$ and $\mathbf{Y}_{\mathbf{v}}$ thus coincide with $\Gamma_{\mathbf{u},\mathbf{v}}$. This proves the following:

LEMMA 2: *Let Assumption 1 hold and suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *implement each other. The argmax correspondences* $\mathbf{X}_{\mathbf{u}}$ *and* $\mathbf{Y}_{\mathbf{v}}$ *are inverses and their graphs coincide with* $\Gamma_{\mathbf{u},\mathbf{v}}$, *that is, they satisfy*

$$\hat{x} \in \mathbf{X}_{\mathbf{u}}(\hat{y}) \iff \hat{y} \in \mathbf{Y}_{\mathbf{v}}(\hat{x}) \iff (\hat{x}, \hat{y}) \in \Gamma_{\mathbf{u},\mathbf{v}}. \tag{17}$$

Lemma 2 indicates that the inverse relationship (15) between profiles that implement each other extends to the argmax correspondences associated with these two profiles.[10] Making use of Corollaries 3 and 4, this observation leads to the following characterization of implementable assignments.

PROPOSITION 4: *Let Assumption 1 hold.*

(1) *An assignment* $\mathbf{y} \in Y^X$ *is implementable if and only if there exist profiles* $\mathbf{u} \in \mathbf{B}(X)$ *and* $\mathbf{v} \in \mathbf{B}(Y)$ *that implement each other with* $\Gamma_{\mathbf{u},\mathbf{v}}$ *containing the graph of* $\mathbf{y}$, *that is,*

$$(x, \mathbf{y}(x)) \in \Gamma_{\mathbf{u},\mathbf{v}} \quad \textit{for all } x \in X.$$

(2) *An assignment* $\mathbf{x} \in X^Y$ *is implementable if and only if there exist profiles* $\mathbf{u} \in \mathbf{B}(X)$ *and* $\mathbf{v} \in \mathbf{B}(Y)$ *that implement each other with* $\Gamma_{\mathbf{u},\mathbf{v}}$ *containing the graph of* $\mathbf{x}$, *that is,*

$$(\mathbf{x}(y), y) \in \Gamma_{\mathbf{u},\mathbf{v}} \quad \textit{for all } y \in Y.$$

PROOF: We prove Proposition 4(1); Proposition 4(2) is analogous. First, suppose the profiles $\mathbf{u}$ and $\mathbf{v}$ implement each other and let $\mathbf{y} \in Y^X$ satisfy $(x, \mathbf{y}(x)) \in \Gamma_{\mathbf{u},\mathbf{v}}$ for all $x \in X$. Then it follows from (17) in Lemma 2 that for all $x \in X$, we have $\mathbf{y}(x) \in \mathbf{Y}_{\mathbf{v}}(x)$. Hence $\mathbf{v}$ implements $\mathbf{y}$ (cf. (3)) and $\mathbf{y}$ is therefore implementable. Conversely, suppose that $\mathbf{y} \in Y^X$ is implementable, so that there exists $\mathbf{u}$ such that $(\mathbf{u}, \mathbf{y})$ is implementable. Let $\mathbf{v} =$

[10]The counterpart of Lemma 2 in the quasilinear case is the following: if $\mathbf{u}$ and $\mathbf{v}$ are each other's conjugates, then the graphs of both of their subdifferentials coincide with the set of points for which equality holds in the Fenchel inequality (cf. Ekeland (2010, Corollary 13)).

$\Psi\mathbf{u}$. Then, from Corollary 3(1), $\mathbf{u}$ and $\mathbf{v}$ implement each other, and from Corollary 4(1), $\mathbf{v}$ implements $(\mathbf{u},\mathbf{y})$. From (3), we then have that for all $x \in X$, $\mathbf{y}(x) \in \mathbf{Y}_{\mathbf{v}}$. Using Lemma 2, it then follows that for all $x \in X$, we have $(x, \mathbf{y}(x)) \in \Gamma_{\mathbf{u},\mathbf{v}}$, finishing the proof. *Q.E.D.*

REMARK 5—Implementable Assignments and Strong Implementability: In the quasi-linear case, an assignment is implementable if and only if it is cyclically monotone (Rochet (1987, Theorem 1)). Importantly, and in contrast to the characterization result in Proposition 4, cyclical monotonicity is a condition on assignments that does not involve any profiles and therefore can be verified directly.[11] In general, the existence of implementable assignments that are not strongly implementable precludes any hope to verify the implementability of an assignment without considering the associated profiles. On the other hand, if it is known that all implementable assignments are strongly implementable, a sharper characterization of implementable assignments might be possible. Section 6 provides an illustration.

REMARK 6—Another Characterization of Implementable Profiles: Proposition 4 characterizes implementable assignments in terms of the argmax correspondences $\mathbf{X}_{\mathbf{u}}$ and $\mathbf{Y}_{\mathbf{v}}$. Implementable profiles can be characterized in an analogous way. Appendix B.7 of the Supplemental Material shows:

$$\mathbf{u} \in \mathbf{I}(X) \iff X_{\mathbf{u}} \text{ is nonempty-valued and onto,}$$
$$\mathbf{v} \in \mathbf{I}(Y) \iff Y_{\mathbf{v}} \text{ is nonempty-valued and onto.}$$

3.4. *Sets of Implementable Profiles*

We use $\mathcal{U}_{\mathbf{y}}$ to denote the subset of implementable profiles $\mathbf{I}(X)$ for which $(\mathbf{u},\mathbf{y})$ is implementable and define $\mathcal{V}_{\mathbf{x}}$ analogously:

$$\mathcal{U}_{\mathbf{y}} = \{\mathbf{u} \in \mathbf{I}(X) : (\mathbf{u},\mathbf{y}) \text{ is implementable}\},$$
$$\mathcal{V}_{\mathbf{x}} = \{\mathbf{v} \in \mathbf{I}(Y) : (\mathbf{v},\mathbf{x}) \text{ is implementable}\}.$$

We will sometimes refer to these sets as the set of profiles compatible with $\mathbf{y}$, resp. with $\mathbf{x}$.

3.4.1. *Metric Structure*

The following corollary establishes properties of sets of implementable profiles that play a key role throughout our study of matching and principal-agent models.

COROLLARY 5: *Let Assumption 1 hold. Then*,
(1) $\mathbf{I}(X)$ *is closed and so is* $\mathcal{U}_{\mathbf{y}}$ *for all* $\mathbf{y} \in Y^X$.
(2) *If* $\mathcal{U} \subset \mathbf{I}(X)$ *is bounded, then it is equicontinuous.*
(3) *If* $\mathcal{U} \subset \mathbf{I}(X)$ *is closed and bounded, then it is compact.*
Analogously, $\mathbf{I}(Y)$ *and* $\mathcal{V}_{\mathbf{x}}$ *are closed, if* $\mathcal{V} \subset \mathbf{I}(Y)$ *is bounded, then it is equicontinuous, and if it is closed and bounded, then it is compact.*

[11] In essence, Rochet's proof of his Theorem 1 shows how to construct $\mathbf{u}$ and $\mathbf{v}$ satisfying the sufficient conditions in Proposition 4 if the assignment is cyclical monotone, and also shows that doing so is impossible if cyclical monotonicity fails.

Appendix B.8 of the Supplemental Material contains the proof. First, we invoke Corollary 3 to show that for any converging sequence of profiles in (for example) $\mathbf{I}(X)$, there exists a converging sequence of profiles in $\mathbf{I}(Y)$ that implement the former sequence. It then follows from the continuity of the implementation map Φ (Lemma 1) that the limit of the latter sequence implements the limit of the former sequence, allowing us to conclude that $\mathbf{I}(X)$ is closed. An analogous argument shows that $\mathcal{U}_\mathbf{y}$ is closed. Next, we use Lemma 1 and Corollary 3 to show that any bounded set $\mathcal{U} \subset I(X)$ is implemented by a bounded set $\mathcal{V}$ of profiles (namely, the image of the set $\mathcal{U}$ under the implementation map Ψ). This ensures that the continuous function ϕ (Proposition 2) is uniformly continuous on the relevant domain. An application of the incentive constraints then gives equicontinuity. Finally, Corollary 5(3) follows from Corollary 5(2) by applying the Arzela–Ascoli theorem.

3.4.2. *Order Structure*

As the implementation maps are dualities (Proposition 1), the sets of implementable profiles are join semi-sublattices of the lattices of profiles: If, say, $\mathbf{v}_1$ implements $\mathbf{u}_1$ and $\mathbf{v}_2$ implements $\mathbf{u}_2$, then we have $\mathbf{u}_1 = \Phi\mathbf{v}_1$ and $\mathbf{u}_2 = \Phi\mathbf{v}_2$. Because Φ is a duality, $\Phi(\mathbf{v}_1 \wedge \mathbf{v}_2) = \mathbf{u}_1 \vee \mathbf{u}_2$ follows immediately. Proposition 2 ensures that $\mathbf{u}_1 \vee \mathbf{u}_2$ is not only in the image of the implementation map Φ but is indeed implementable.

Even when the generating function is quasilinear, the meet of two implementable profiles may not be implementable. In such a case, the sets of implementable profiles are not sublattices of the lattices of profiles. Appendix C.1 of the Supplemental Material provides a simple example illustrating this.

There are, however, interesting subsets of implementable profiles that are sublattices. The most prominent example are the sets of stable profiles in a matching model that we will investigate in Section 4. Here we give two preliminary results that consider the sets of implementable profiles compatible with a given assignment.

LEMMA 3: *Let Assumption 1 hold. The set $\mathcal{U}_\mathbf{y}$ is a sublattice of $\mathbf{B}(X)$ for all implementable $\mathbf{y} \in Y^X$ and the set $\mathcal{V}_\mathbf{x}$ is a sublattice of $\mathbf{B}(Y)$ for all implementable $\mathbf{x} \in X^Y$.*

The proof of Lemma 3, which considers the set $\mathcal{U}_\mathbf{y}$ (the other case is analogous) is in Appendix B.9 of the Supplemental Material. The essence of the argument is that if, say, $y \in Y$ is optimal for an agent type $x \in X$ when faced with the tariff $\mathbf{v}_1$ and also optimal when faced with the tariff $\mathbf{v}_2$, then y remains an optimal choice for x both when faced with the tariff $\mathbf{v}_1 \wedge \mathbf{v}_2$ and when faced with the tariff $\mathbf{v}_1 \vee \mathbf{v}_2$. Consequently, when $\mathbf{v}_1$ implements $(\mathbf{u}_1, \mathbf{y})$ and $\mathbf{v}_2$ implements $(\mathbf{u}_2, \mathbf{y})$, then $\mathbf{v}_1 \wedge \mathbf{v}_2$ implements $(\mathbf{u}_1 \vee \mathbf{u}_2, \mathbf{y})$ and $\mathbf{v}_1 \vee \mathbf{v}_2$ implements $(\mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{y})$.

Next, we consider sets of profiles that are compatible with a given implementable assignment and in addition satisfy a participation constraint. For example, consider the set $\{\mathbf{u} \in \mathcal{U}_\mathbf{y} | \mathbf{u} \geq \underline{\mathbf{u}}\}$ for some continuous profile $\underline{\mathbf{u}}$. As the intersection of the sublattice $\mathcal{U}_\mathbf{y}$ (Lemma 3) and the sublattice of profiles satisfying $\mathbf{u} \geq \underline{\mathbf{u}}$, this set is also a sublattice. In the quasilinear case, it is not difficult to see that this sublattice (i) is nonempty and (ii) has a minimum element, say $\mathbf{u}^*$, for which the participation constraint is binding, that is, $\mathbf{u}^*(x) = \underline{\mathbf{u}}(x)$ holds for some $x \in X$. The proof of the following result (in Appendix A.2) shows that these two additional properties do not require quasilinearity but hold under the weaker condition that the assignment under consideration is strongly implementable.

LEMMA 4: *Let Assumption 1 hold and let* $\underline{\mathbf{u}} \in \mathbf{C}(X)$ *and* $\underline{\mathbf{v}} \in \mathbf{C}(Y)$.

(1) *If* $\mathbf{y} \in Y^X$ *is strongly implementable, then the sublattice* $\{\mathbf{u} \in \mathcal{U}_\mathbf{y} | \mathbf{u} \geq \underline{\mathbf{u}}\}$ *has a minimum element* $\mathbf{u}^*$ *and this minimum element satisfies* $\mathbf{u}^*(x) = \underline{\mathbf{u}}(x)$ *for some* $x \in X$.

(2) *If* $\mathbf{x} \in X^Y$ *is strongly implementable, then the sublattice* $\{\mathbf{v} \in \mathcal{V}_\mathbf{x} | \mathbf{v} \geq \underline{\mathbf{v}}\}$ *has a minimum element* $\mathbf{v}^*$ *and this minimum element satisfies* $\mathbf{v}^*(y) = \underline{\mathbf{v}}(y)$ *for some* $y \in Y$.

The main difficulty in establishing Lemma 4(1) (the other case is analogous) is to exclude the possibility that the minimum element $\mathbf{u}^*$ is strictly greater than $\underline{\mathbf{u}}$ for all $x \in X$. We resolve this difficulty by exploiting the lattice structure observed in Lemma 3 and the assumption of strong implementability to construct an increasing sequence of profiles in $\mathcal{U}_\mathbf{y}$ that satisfy $\mathbf{u}(x) = \underline{\mathbf{u}}(x)$ for some $x \in X$ (but may violate the participation constraint) and then show (using Corollary 5) that this sequence has a limit that satisfies the participation constraint for all $x \in X$ and satisfies it with equality for some $x \in X$.[12]

4. STABILITY IN MATCHING MODELS

This section applies the results from Section 3 to study stable outcomes in two-sided matching models. Section 4.1 introduces the matching model and defines the stability notions—stable outcomes and pairwise stable outcomes—that we consider. The notion of a pairwise stable outcome, which abstracts from participation constraints, is important because such outcomes can be characterized in terms of a pair of profiles implementing each other together with the argmax correspondences associated with these profiles. Section 4.2 develops this link. Section 4.3 then exploits it to show how familiar results for the existence of stable outcomes in matching models with a finite number of agents can be combined with our duality results to obtain, via a limiting argument, the existence of stable outcomes in matching models with an infinity of types. The role of the implementation duality in this argument is analogous to the role of (generalized) conjugate duality in McCann's proof (McCann (1995)) of the Kantorovich duality for optimal transport problems (see also Villani (2009, Chapter 5)).[13]

The main result in Section 4.4 is Proposition 8, which establishes that the sets of stable profiles are complete sublattices of the sets of profiles, thereby generalizing a corresponding result for matching models with a finite number of agents (Demange and Gale (1985)).

4.1. *The Matching Model*

To obtain a matching model, we add to our basic ingredients (X, Y, ϕ) a pair of finite nonzero Borel measures μ on X and ν on Y, describing the distribution of agent types on

[12]In the quasilinear case, a much simpler argument will do: Suppose $\mathbf{u}^*(x) > \underline{\mathbf{u}}(x)$ holds for all $x \in X$. As $\underline{\mathbf{u}}$ has been assumed to be continuous, $\mathbf{u}^*$ is continuous by Proposition 2, and X is compact, there then exists $\epsilon > 0$ such that $\mathbf{u}^*(x) - \epsilon \geq \underline{\mathbf{u}}(x)$ holds for all $x \in X$. In the quasilinear case, the profile given by $\mathbf{u}^*(x) - \epsilon$ is an element of $\mathcal{U}_\mathbf{y}$, contradicting the minimality of $\mathbf{u}^*$. Appendix C.2 of the Supplementary Appendix provides an example featuring a profile $\boldsymbol{y}$ that is not strongly implementable and for which $\boldsymbol{u}^*(x)$ is strictly greater than $\underline{\boldsymbol{u}}(x)$ for all $x \in X$.

[13]Previously, Gretsky, Ostroy, and Zame (1992) have used tools from optimal transport to establish existence of stable outcomes in matching models with perfectly transferable (quasilinear) utility. Kaneko and Wooders (1986, 1996) established an existence result for a class of infinite cooperative games which includes matching models with both perfectly and imperfectly transferable utility as special cases, but to do so, resorted to a notion of approximate feasibility. In work contemporaneous to ours, Greinecker and Kah (2018) obtained the existence of stable outcomes for a broad class of matching problems (including problems with nontransferable utility) with an infinity of types, using tools quite different from the ones we employ.

each side of the market, and a pair of continuous reservation utility profiles $\underline{\mathbf{u}} : X \to \mathbb{R}$ and $\underline{\mathbf{v}} : Y \to \mathbb{R}$, describing the utilities agents achieve when remaining unmatched. A matching model is then a collection $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$.

4.1.1. *Matches and Outcomes*

We follow the optimal transportation literature (Villani (2009), Galichon (2016)) and Gretsky, Ostroy, and Zame (1992) in using a measure λ on $X \times Y$ to describe who is matched with whom and who remains unmatched. Formally, a match for a matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ is a Borel measure λ on $X \times Y$ satisfying the conditions

$$\lambda_X(\tilde{X}) := \lambda(\tilde{X} \times Y) \leq \mu(\tilde{X}), \tag{18}$$

$$\lambda_Y(\tilde{Y}) := \lambda(X \times \tilde{Y}) \leq \nu(\tilde{Y}), \tag{19}$$

for all measurable $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$. We interpret $\lambda(\tilde{X} \times \tilde{Y})$ as identifying the mass of buyers from $\tilde{X}$ who are matched with sellers from $\tilde{Y}$. Condition (18) indicates that the mass of buyers with types in $\tilde{X}$, given by the marginal measure $\lambda_X(\tilde{X})$, who are matched to *some* seller cannot exceed the mass of these buyers, with mass $\mu(\tilde{X}) - \lambda_X(\tilde{X}) \geq 0$ of the agents in the set $\tilde{X}$ remaining unmatched. The interpretation of condition (19) is analogous.

An outcome is a triple $(\lambda, \mathbf{u}, \mathbf{v})$ consisting of a match λ and a pair of utility profiles $\mathbf{u} \in \mathbf{B}(X)$ and $\mathbf{v} \in \mathbf{B}(Y)$ satisfying the (dual) feasibility conditions

$$\mathbf{u}(x) = \phi\big(x, y, \mathbf{v}(y)\big) \quad \text{and} \quad \mathbf{v}(y) = \psi\big(y, x, \mathbf{u}(x)\big) \quad \forall (x, y) \in \operatorname{supp}(\lambda) \tag{20}$$

for matched agents and the feasibility conditions

$$\mathbf{u}(x) = \underline{\mathbf{u}}(x) \quad \forall x \in \operatorname{supp}(\mu - \lambda_X), \tag{21}$$

$$\mathbf{v}(y) = \underline{\mathbf{v}}(y) \quad \forall y \in \operatorname{supp}(\nu - \lambda_Y), \tag{22}$$

for unmatched agents.[14] These feasibility conditions require that matched pairs receive utilities that can be generated in their matches and unmatched agents obtain their reservation utilities. Observe that we require feasibility for all types in the supports of μ and ν. This is in contrast to the approximate feasibility notion employed in Kaneko and Wooders (1986, 1996).

4.1.2. *Stable Outcomes*

An outcome for a matching model is *stable* if it satisfies the participation constraints

$$\mathbf{u}(x) \geq \underline{\mathbf{u}}(x) \quad \forall x \in \operatorname{supp}(\nu), \tag{23}$$

$$\mathbf{v}(y) \geq \underline{\mathbf{v}}(y) \quad \forall y \in \operatorname{supp}(\mu), \tag{24}$$

and the (dual) incentive constraints

$$\mathbf{u}(x) \geq \phi\big(x, y, \mathbf{v}(y)\big) \quad \text{and} \quad \mathbf{v}(y) \geq \psi\big(y, x, \mathbf{u}(x)\big) \quad \forall (x, y) \in \operatorname{supp}(\nu) \times \operatorname{supp}(\mu). \tag{25}$$

[14] By specifying an outcome in terms of utility profiles, we are imposing the equal treatment property that all agents of the same type receive the same utility level. Greinecker and Kah (2018) demonstrated that this is an innocent simplification under Assumption 1. Similarly, by requiring the equalities in (20), we are imposing efficiency within each match rather than obtaining this as an implication of stability.

A match or profile will be called stable if it is part of a stable outcome.

The stability conditions require that, as indicated by (23)–(24), no matched agent in the support of one of the type distributions would rather be unmatched, and, as indicated by (25), no pair of agents in the supports of the type distributions can achieve strictly higher utilities by matching with each other than by sticking to the outcome under consideration.

Conditions (20)–(25) impose no constraints whatsoever on types that do not appear in the supports of the type distributions. Further, (25) does not preclude the possibility that some type x in the support of μ might prefer to match with a type outside of the support of ν (and vice versa). In essence, we are thus treating types that lie outside the supports of the type distributions as being nonexistent in the definition of stable outcomes. We could exclude such types from the model by assuming that μ and ν have full support, but retaining them allows us to consider *finite-support matching models*.

The matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ has *finite support* if there exists $(x_1, \dots, x_m) \in X^m$ and $(y_1, \dots, y_n) \in Y^n$ such that the measures μ and ν on X and Y satisfy

$$\mu(\tilde{X}) = \sum_{i=1}^{m} \delta_{x_i}(\tilde{X}) \quad \text{and} \quad \nu(\tilde{Y}) = \sum_{j=1}^{n} \delta_{y_i}(\tilde{Y})$$

for all measurable $\tilde{X} \subseteq X$ and measurable $\tilde{Y} \subseteq Y$, where m and n are natural numbers and δ_x (and similarly δ_y) is the Dirac measure on X assigning mass 1 to x.

The import of such models for our analysis is that they can be interpreted as *matching models with a finite number of agents*, with known results about stable outcomes carrying over from matching models with a finite number of agents to finite-support matching models. In particular, every finite-support matching model satisfying Assumption 1 has a stable outcome. See Appendix B.10 of the Supplemental Material for details.

4.1.3. *Pairwise Stable Outcomes in Balanced Matching Models*

We say that a matching model is balanced if $\mu(X) = \nu(Y)$ holds, so that the masses of buyers and sellers are identical. A match λ for a balanced matching model is full if the inequalities in (18) and (19) hold as equalities,

$$\lambda(\tilde{X} \times Y) = \mu(\tilde{X}), \tag{26}$$

$$\lambda(X \times \tilde{Y}) = \nu(\tilde{Y}), \tag{27}$$

for all measurable $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$, indicating that there are no unmatched agents. An outcome $(\lambda, \mathbf{u}, \mathbf{v})$ for a balanced matching model is full if it features a full match. For any full match, the feasibility conditions (21) and (22) are vacuous (because $\text{supp}(\mu - \lambda_X) = \text{supp}(\nu - \lambda_Y) = \emptyset$), so that an outcome is full if and only if it satisfies (20), (26), and (27). In line with our definition of profiles $\mathbf{u}$ or $\mathbf{v}$ satisfying an initial condition (cf. Section 2.4), we say that a full outcome $(\lambda, \mathbf{u}, \mathbf{v})$ for a balanced matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ satisfies initial condition $(x_0, u_0) \in X \times \mathbb{R}$ if $\mathbf{u}(x_0) = u_0$, and satisfies initial condition $(y_0, v_0) \in Y \times \mathbb{R}$ if $\mathbf{v}(y_0) = v_0$.

A full outcome is *pairwise stable* if it satisfies the incentive constraints (25). A pairwise stable outcome is stable if and only if it also satisfies the participation constraints (23) and (24). Note that full matches and full outcomes exist only for balanced matching models and that whenever we call an outcome, match, or profile pairwise stable, it is implied that it is part of a full outcome.

Our definition of a full match for a balanced matching model is identical to the definition of a transportation (or transference) plan in the literature on optimal transport. This allows us to borrow results from this literature when analyzing full matches and full outcomes. For instance, it is well known that (under our maintained compactness assumption on X and Y) the set of full matches is compact in the topology of weak convergence of measures (cf. Villani (2009, p. 45)).

4.1.4. *Deterministic Matches*

In many economic applications, it is natural to focus on full matches that can be described in terms of assignments, thereby identifying, for all agent types on one side of the matching market, a unique type on the other side with whom they are matched. This is captured by the notion of a deterministic match—corresponding to the notion of a deterministic coupling or transport map in the optimal transportation literature (Villani (2009, p. 6))—defined in the following.[15]

We say that a measure λ on the set $X \times Y$ is deterministic and denote it by $\lambda_{\mathbf{y}}$ if there exists a measurable assignment $\mathbf{y}$ such that

$$\lambda(\tilde{X} \times \tilde{Y}) = \mu\big(\{x \in \tilde{X} | \mathbf{y}(x) \in \tilde{Y}\}\big) \tag{28}$$

for measurable $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$. If such a deterministic measure λ is a full match in the balanced matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$, then it is a deterministic match.

If $\lambda_{\mathbf{y}}$ is a deterministic match, then the assignment $\mathbf{y}$ must be measure preserving (and hence necessarily measurable), that is, $\nu(\tilde{Y}) = \mu(\mathbf{y}^{-1}(\tilde{Y}))$ must hold for all measurable $\tilde{Y} \subseteq Y$.

In general, pairwise stable deterministic matches do not exist in balanced matching models, even when the generating function is quasilinear and the existence of measure-preserving assignments is assured (e.g., when μ is atomless).[16]

4.2. *Connecting Implementability and Pairwise Stability*

With a quasilinear generating function $\phi(x, y, v) = f(x, y) - v$, a full match is pairwise stable if and only if it maximizes the surplus $\int_{X \times Y} f(x, y)\, d\lambda(x, y)$ over the set of full matches. Standard results from the optimal transport literature then imply that a full match λ is pairwise stable if and only if its support is contained in $\Gamma_{\mathbf{u},\mathbf{v}}$ for a pair of profiles $(\mathbf{u}, \mathbf{v})$ implementing each other, and that, for such a pair of profiles, the full outcome $(\lambda, \mathbf{u}, \mathbf{v})$ is a pairwise stable outcome (cf. Galichon (2016, Chapters 6 and 7)). These results carry over to our case:

PROPOSITION 5: *Let Assumptions 1 hold and let the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be balanced.*

[15] We focus on assignments $\mathbf{y} \in Y^X$ with all our definitions and observations carrying over to assignments $\mathbf{x} \in X^Y$ in the obvious way.

[16] Villani (2009, Example 4.9) provided a simple example for an optimal transport problem (with both μ and ν atomless) which has no deterministic solution. This example is easily modified to demonstrate the nonexistence of pairwise stable deterministic matches. See also Gretsky, Ostroy, and Zame (1992) for an extended discussion of related existence questions in the context of a two-sided matching model and an argument which, when transferred to our setting, suggests that it is possible to interpret any of the full matches we consider as measure-preserving bijections between suitably enlarged measure spaces. Greinecker and Kah (2018) pursued such a construction.

(1) *If* λ *is a full match, then* $(\lambda, \mathbf{u}, \mathbf{v})$ *is a full outcome if and only if* $\text{supp}(\lambda) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$.

(2) *If* $(\lambda, \mathbf{u}, \mathbf{v})$ *is a full outcome and* (i) $\mathbf{u}$ *implements* $\mathbf{v}$ *or* (ii) $\mathbf{v}$ *implements* $\mathbf{u}$, *then* $(\lambda, \mathbf{u}, \mathbf{v})$ *is pairwise stable*.

(3) *If* $(\lambda, \mathbf{u}, \mathbf{v})$ *is a pairwise stable outcome, then there exist profiles* $\tilde{\mathbf{u}}$ *and* $\tilde{\mathbf{v}}$ *with the properties that* (i) $\tilde{\mathbf{u}}(x) = \mathbf{u}(x)$ *on the support of* μ *and* $\tilde{\mathbf{v}}(y) = \mathbf{v}(y)$ *on the support of* ν, (ii) $(\lambda, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ *is a pairwise stable outcome for* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$, *and* (iii) $\tilde{\mathbf{u}}$ *and* $\tilde{\mathbf{v}}$ *implement each other.*

PROOF:

(1) If λ is a full match, then (20) is necessary and sufficient for $(\lambda, \mathbf{u}, \mathbf{v})$ to be a full outcome. By definition of $\Gamma_{\mathbf{u},\mathbf{v}}$ (see (16)), condition (20) holds if and only if $\text{supp}(\lambda) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$.

(2) If $(\lambda, \mathbf{u}, \mathbf{v})$ is a full outcome, then (20), (26), and (27) hold. Therefore, (25), which holds if $\mathbf{v}$ implements $\mathbf{u}$ or $\mathbf{v}$ implements $\mathbf{u}$, is sufficient for $(\lambda, \mathbf{u}, \mathbf{v})$ to be pairwise stable.

(3) See Appendix B.11 of the Supplemental Material. *Q.E.D.*

If the type measures μ and ν both have full support, Proposition 5(3) reduces to the statement that the profiles $\mathbf{u}$ and $\mathbf{v}$ in every pairwise stable outcome $(\lambda, \mathbf{u}, \mathbf{v})$ implement each other (which in this case is immediate from (20) and (25)). Otherwise, Proposition 5(3) indicates that the profiles $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ in any pairwise stable outcome can be adjusted outside the supports of μ and ν in such a way that the suitably adjusted profiles implement each other. In either case, in conjunction with the first two parts of the proposition, we obtain the conclusion that a full match λ is pairwise stable if and only if it satisfies $\text{supp}(\lambda) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$ for a pair of profiles implementing each other.

For a deterministic match $\lambda_{\mathbf{y}}$ with implementable $\mathbf{y}$, it is not difficult to show (using Proposition 4) that $\text{supp}(\lambda_{\mathbf{y}}) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$ holds for profiles $\mathbf{u}$ and $\mathbf{v}$ implementing each other, so that Proposition 5 implies that $\lambda_{\mathbf{y}}$ is a pairwise stable match. Obtaining a converse statement involves dealing with some technical complications, arising out of the fact that $\text{supp}(\lambda_{\mathbf{y}}) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$ does not necessarily imply that the graph of $\mathbf{y}$ is contained in $\Gamma_{\mathbf{u},\mathbf{v}}$. We tackle these complications in Appendix B.12 of the Supplemental Material, thereby proving the following:

LEMMA 5: *Let Assumption 1 hold, let the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be balanced, and let* λ *be a deterministic match. Then* λ *is pairwise stable if and only if there exists an implementable* $\mathbf{y} \in Y^X$ *such that* $\lambda = \lambda_{\mathbf{y}}$ *holds.*

4.3. *Existence of (Pairwise) Stable Outcomes*

We begin by exploiting our duality results to establish the existence of pairwise stable outcomes in balanced matching models satisfying arbitrary initial conditions. Appendix A.3 proves the following:

PROPOSITION 6: *Let Assumption 1 hold and let the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be balanced. Then, for every initial condition* (y_0, v_0) *(and similarly for every initial condition* (x_0, u_0)*), there exists a pairwise stable outcome* $(\lambda, \mathbf{u}, \mathbf{v})$ *satisfying* $\mathbf{v}(y_0) = v_0$ *in which* $\mathbf{u}$ *and* $\mathbf{v}$ *implement each other.*

The proof of Proposition 6 begins by considering balanced finite-support matching models with at most n types of buyers and at most n types of sellers. We exploit Lemma 3 in Demange and Gale (1985) to show that such a finite-support matching model has a pairwise stable outcome $(\lambda_n, \mathbf{u}_n, \mathbf{v}_n)$ satisfying the given initial condition. In addition, Proposition 5(3) ensures that we can take the profiles $(\mathbf{u}_n, \mathbf{v}_n)$ to implement each other. We next construct a sequence of such finite-support balanced matching models for which the associated measures μ_n and ν_n converge weakly to the target measures μ and ν. Prokhorov's theorem implies that the sequence of measures $(\lambda_n)_{n=1}^{\infty}$ has a subsequence converging weakly to a full match λ^*. Using the fact that the initial condition holds along the sequences to show that the sequences of profiles $(\mathbf{u}_n)_{n=1}^{\infty}$ and $(\mathbf{v}_n)_{n=1}^{\infty}$ are bounded, it becomes a straightforward consequence of our duality results that these sequences have subsequences converging to profiles $\mathbf{u}^*$ and $\mathbf{v}^*$ implementing each other and that, further, $(\lambda^*, \mathbf{u}^*, \mathbf{v}^*)$ is a pairwise stable outcome for $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ satisfying the initial condition. This gives us the desired result.

To go from the existence result for pairwise stable outcomes in balanced matching models in Proposition 6 to an existence result for stable outcomes in any matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ satisfying Assumption 1, we consider an augmented matching model. As in a similar construction in Chiappori, McCann, and Nesheim (2010), in this augmented model the type spaces differ from X and Y by the addition of dummy types x_0 and y_0 on each side of the market. Adding the dummy types x_0 and y_0 transforms the original matching model into a balanced matching model in which (i) being unmatched in the original model corresponds to being matched with a dummy agent in the augmented matching model, (ii) for an appropriate choice of initial condition, a pairwise stable outcome in the augmented model corresponds to a stable outcome in the original model, and (iii) Assumption 1 holds for the augmented model. Given these properties of the augmented matching model, Proposition 6 implies the existence of a stable outcome for the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$. The proof of the following result, in Appendix B.13 of the Supplemental Material, shows how to construct an augmented matching model with the requisite properties.

COROLLARY 6: *Let Assumption 1 hold. There exists a stable outcome* $(\lambda, \mathbf{u}, \mathbf{v})$ *for the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$.

4.4. *Lattice Structure of (Pairwise) Stable Profiles*

The main result of this section is Proposition 8, which establishes that the sets of stable profiles are complete sublattices of the sets of bounded profiles. As in Section 4.3, we first establish lattice results for pairwise stable outcomes. These lattice results for pairwise stable outcomes will also be of independent use when we turn to the principal-agent model.

The following assumption simplifies the exposition by ensuring (from Proposition 5(3)) that in every pairwise stable outcome $(\lambda, \mathbf{u}, \mathbf{v})$, the profiles $\mathbf{u}$ and $\mathbf{v}$ implement each other.[17]

ASSUMPTION 2: *The type measures* μ *and* ν *have full support*.

4.4.1. *The Lattice of Pairwise Stable Profiles*

Let

$$\mathbb{U} = \{\mathbf{u} \in \mathbf{B}(X) | (\lambda, \mathbf{u}, \mathbf{v}) \text{ is pairwise stable for some full match } \lambda \text{ and } \mathbf{v} \in \mathbf{B}(Y)\},$$

[17]Without Assumption 2, the argument would require an additional step, adjusting a pair of profiles $(\mathbf{u}, \mathbf{v})$ outside the supports of μ and ν to ensure they implement each other, as in the proof of Proposition 5(3).

$$\mathbb{V} = \{\mathbf{v} \in \mathbf{B}(Y) | (\lambda, \mathbf{u}, \mathbf{v}) \text{ is pairwise stable for some full match } \lambda \text{ and } \mathbf{u} \in \mathbf{B}(X)\}$$

denote the sets of pairwise stable profiles in a balanced matching model. From Proposition 6, the sets $\mathbb{U}$ and $\mathbb{V}$ are nonempty if Assumption 1 holds. The following result shows that they are also closed sublattices (of $\mathbf{B}(X)$, resp. of $\mathbf{B}(Y)$).

PROPOSITION 7: *Let Assumptions 1 and 2 hold and let the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be balanced. The sets* $\mathbb{U}$ *and* $\mathbb{V}$ *of pairwise stable profiles are closed sublattices.*

Appendix A.4 contains the proof. The idea behind the proof that $\mathbb{U}$ and $\mathbb{V}$ are sublattices is the same as the one behind the Decomposition Lemma in Demange and Gale (1985, Lemma 1): Given two pairwise stable outcomes $(\lambda_1, \mathbf{u}_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{u}_2, \mathbf{v}_2)$, we show that both X and Y can be partitioned into two sets each, say X into X_1 and X_2 and Y into Y_1 and Y_2, such that both λ_1 and λ_2 match buyer types from X_1 with seller types in Y_1 and buyer types in X_2 with seller types in Y_2. Further, when faced with $\mathbf{v}_1 \wedge \mathbf{v}_2$, all buyers in X_1 prefer to be matched as under λ_1, whereas the reverse preference holds for buyers in X_2. Constructing a measure λ_3 on $X \times Y$ by matching the types in X_1 and Y_1 as under λ_1 and the types in X_2 and Y_2 as under λ_2 then yields a pairwise stable outcome $(\lambda_3, \mathbf{u}_1 \vee \mathbf{u}_2, \mathbf{v}_1 \wedge \mathbf{v}_2)$. An analogous argument establishes the existence of a full match λ_4 such that $(\lambda_4, \mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{v}_1 \vee \mathbf{v}_2)$ is a pairwise stable outcome. The existence of the pairwise stable outcomes $(\lambda_3, \mathbf{u}_1 \vee \mathbf{u}_2, \mathbf{v}_1 \wedge \mathbf{v}_2)$ and $(\lambda_4, \mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{v}_1 \vee \mathbf{v}_2)$ implies that both $\mathbb{U}$ and $\mathbb{V}$ are sublattices. The closedness claim in the statement of the proposition follows from the same arguments we have used in the proof of Proposition 6 to establish that the limit of the pairwise stable outcomes in the approximating finite-support matching models considered there is pairwise stable.

The proof of Proposition 7 would be much simpler if we could assume that all pairs $(\mathbf{u}_1, \mathbf{v}_1)$ and $(\mathbf{u}_2, \mathbf{v}_2)$ of stable profiles are compatible with the *same* stable match λ.[18] In that case, an argument analogous to that of Lemma 3 would yield that $\mathbb{U}$ and $\mathbb{V}$ are sublattices. However, as illustrated by Roth and Sotomayor (1990, Example 9.6, p. 225) and Quint (1994, Example 6.1, p. 612), this is generally not the case if the generating function is not quasilinear.

Recall that Lemma 4 in Section 3.4.2 has established that the set of profiles $\mathcal{U}_{\mathbf{y}}$ compatible with a given strongly implementable assignment $\mathbf{y}$ satisfying a participation constraint has a minimum element in which the participation constraint is binding for some type. The only properties of $\mathcal{U}_{\mathbf{y}}$ used in the proof of Lemma 4 were that the set $\mathcal{U}_{\mathbf{y}}$ is a closed (Corollary 5(1)) sublattice (Lemma 3) of implementable profiles containing a profile for every possible initial condition (by strong implementability). The set of pairwise stable profiles $\mathbb{U}$ satisfies the same properties: it is a closed (Proposition 6) sublattice (Proposition 7) of implementable profiles (Proposition 5(2)) with the set $\{u \in \mathbb{U} | \mathbf{u}(x) = \underline{\mathbf{u}}(x)\}$ nonempty for all $x \in X$ (Proposition 6). Therefore, the following counterpart to Lemma 4 holds for the sets of pairwise stable profiles (with the proof being identical):

COROLLARY 7: *Let Assumptions 1 and 2 hold and let* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be a balanced matching model. Then the set of pairwise stable buyer profiles satisfying the participation*

[18]This is trivially true if there is a unique stable match, as is the case under a strict single-crossing condition (Proposition 12 in Section 6). It is also true with a quasilinear generating function, as with transferable utility all stable profiles are compatible with the same stable match; see Roth and Sotomayor, (1990 Corollary 8.7, p. 207) for finite matching models and Gretsky, Ostroy, and Zame (1999), who also used this fact to establish a counterpart to our Proposition 8 below (Gretsky, Ostroy, and Zame (1999, Proposition 5)), for a model with an infinity of types.

constraint $\mathbf{u}(x) \geq \underline{\mathbf{u}}(x)$ *for all* $x \in X$ *has a minimum element* $\mathbf{u}^*$ *satisfying* $\mathbf{u}^*(x) = \underline{\mathbf{u}}(x)$ *for some* $x \in X$. *Similarly, the set of pairwise stable seller profiles satisfying the participation constraints* $\mathbf{v}(y) \geq \underline{\mathbf{v}}(y)$ *for all* $y \in Y$ *has a minimum element* $\mathbf{v}^*$ *satisfying* $\mathbf{v}^*(y) = \underline{\mathbf{v}}(y)$ *for some* $y \in Y$.

4.4.2. *The Lattice of Stable Profiles*

The connection between pairwise stability in balanced matching models and stability in arbitrary matching models underlying the proof of Corollary 6 in Section 4.3 allows us to extend our results about the lattice structure of pairwise stable profiles to results about the lattice structure of stable profiles.

First, we use Proposition 7 to show that the sets of stable buyer and seller profiles are complete sublattices. Appendix B.14 of the Supplemental Material proves the following:

PROPOSITION 8: *Let Assumptions 1 and 2 hold. The sets of stable seller profiles and stable buyer profiles of the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *are complete sublattices.*

Second, we use Corollary 7 to establish a counterpart to Lemma 3 in Demange and Gale (1985), asserting that, in a balanced matching model, *both* the minimum buyer stable profile $\boldsymbol{u}^*$ and the minimum seller stable profile $\boldsymbol{v}^*$ feature binding participation constraints.[19]

COROLLARY 8: *Let Assumptions 1 and 2 hold and let* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be a balanced matching model. Then the minimum stable buyer profile* $\mathbf{u}^*$ *satisfies* $\mathbf{u}^*(x) = \underline{\mathbf{u}}(x)$ *for some* $x \in X$ *and the minimum stable seller profile* $\mathbf{v}^*$ *satisfies* $\mathbf{v}^*(y) = \underline{\mathbf{v}}(y)$ *for some* $y \in Y$.

PROOF: The claim is immediate from the feasibility conditions (21)–(22) unless all stable outcomes are fully matched. We therefore suppose this to be the case. The set of stable outcomes then coincides with the set of pairwise stable outcomes $(\lambda, \mathbf{u}, \mathbf{v})$, satisfying the participation constraints $\mathbf{u} \geq \underline{\mathbf{u}}$ and $\mathbf{v} \geq \underline{\mathbf{v}}$. Recalling that, for any pairwise stable outcome $(\lambda, \mathbf{u}, \mathbf{v})$, the profiles $\mathbf{u}$ and $\mathbf{v}$ implement each other (Assumption 2 and Proposition 5(3)), the result then follows from Corollary 7, provided that the profiles $\mathbf{u}^*$ and $\mathbf{v}^*$ appearing in the statement of that corollary satisfy $\Psi\mathbf{u}^* \geq \underline{\mathbf{v}}$ and $\Phi\mathbf{v}^* \geq \underline{\mathbf{u}}$. Because the implementation maps are order reserving, these conditions must be satisfied (as otherwise the set of stable profiles would be empty). *Q.E.D.*

5. OPTIMAL OUTCOMES IN PRINCIPAL-AGENT MODELS

This section applies our characterization of implementable profiles and assignments to adverse-selection principal-agent models. Section 5.1 formulates the principal's problem as choosing a measure λ on $X \times Y$, as well as a rent function $\mathbf{u}$ and a tariff $\mathbf{v}$, subject to incentive and participation constraints. This formulation allows us to interpret triples

[19]In an unbalanced matching model (satisfying $\mu(X) \neq \nu(Y)$), it is trivially the case that in every outcome there are unmatched agents on the "long side" of the market. By the feasibility conditions (21)–(22), such unmatched agents receive their reservation utility, so that either the minimum buyer stable profile $\mathbf{u}^*$ or the minimum seller stable profile $\mathbf{v}^*$ features a binding participation constraint. In particular, if $\mu(X) > \nu(Y)$, then there exists $x \in X$ satisfying $\mathbf{u}^*(x) = \underline{\mathbf{u}}(x)$ and, similarly, if $\mu(X) < \nu(Y)$, then there exists $y \in Y$ satisfying $\mathbf{v}^*(y) = \underline{\mathbf{v}}(y)$. Note the existence of $\mathbf{u}^*$ and $\mathbf{v}^*$ is ensured because the sets of stable profiles are complete sublattices (Proposition 8).

$(\lambda, \mathbf{u}, \mathbf{v})$ satisfying the incentive constraints in the principal's problem as pairwise stable outcomes in a balanced matching model.

Section 5.2 reformulates the principal's problem as a nonlinear pricing problem in which the principal maximizes over a set of tariffs, and then uses this reformulation to establish that the principal's problem has a solution. Moreover, it has a solution in which the measure λ chosen by the principal is deterministic and thus corresponds to the choice of an optimal assignment. Our duality results play a central role in this existence argument, with Corollaries 3(1) and 4(1) ensuring that we can model the principal as choosing an *implementable* tariff, and Corollary 5 ensuring that the resulting feasible set is compact.[20]

In general, the agent's participation constraint may fail to bind in a solution to the principal's problem.[21] Section 5.3 shows that this cannot happen if every implementable profile is strongly implementable or if the principal's utility function exhibits private values. The first result is consistent with our view of strong implementability as a useful generalization of quasilinearity, while the second makes essential use of the connections to the matching model.

5.1. *The Principal-Agent Model*

To obtain a principal-agent model, we add to our basic ingredients (X, Y, ϕ) a function $\pi : X \times Y \times \mathbb{R} \to \mathbb{R}$ describing the principal's utility of receiving payment v from agent type x who takes decision y, a finite Borel measure μ on the set X describing the distribution of agent types, and a continuous profile $\underline{\mathbf{u}} : X \to \mathbb{R}$ describing the agent's reservation utilities. A principal-agent model is then a collection $(X, Y, \phi, \mu, \pi, \underline{\mathbf{u}})$.

ASSUMPTION 3: *The function π is continuous, strictly increasing in its third argument, and satisfies $\pi(x, y, \mathbb{R}) = \mathbb{R}$ for all $(x, y) \in X \times Y$. The type measure μ has full support.*

Let $\mathbb{M}$ be the set of Borel measures on $X \times Y$ whose marginal distribution on the set X equals μ. We formulate the principal's problem as choosing a triple $(\lambda, \mathbf{u}, \mathbf{v})$ consisting of a measure $\lambda \in \mathbb{M}$, a utility profile $\mathbf{u} \in \mathbf{B}(X)$, and a tariff $\mathbf{v} \in \mathbf{B}(Y)$ to maximize

$$\int_X \int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda(x, y) \tag{29}$$

subject to the feasibility constraints

$$\mathbf{v} \text{ implements } \mathbf{u}$$

[20]Obtaining compactness of the feasible set (and the requisite continuity properties of the principal's objective function) is the main difficulty in the existence proofs in Kahn (1993), Carlier (2001), and Carlier (2002), who considered special cases of the principal-agent model in which the agent's utility function is quasilinear. Using the structure resulting from the imposition of a single-crossing condition when X and Y are intervals, Jullien (2000) provided a straightforward existence argument which uses Helly's selection theorem in lieu of compactness arguments. Working without quasilinearity, the existence proofs in Page (1991, 1992, 1997) and Balder (1996) impose compactness as an assumption on the set of feasible contracts. Allowing for stochastic contracts, Kadan, Reny, and Swinkels (2017) obtained a very general existence result for principal-agent models with both adverse selection and moral hazard using tools rather different from the ones we employ. We explain in Appendices D.2 and D.3 of the Supplemental Material how our approach can be extended to allow for stochastic contracts and moral hazard.

[21]Appendix C.1 of the Supplemental Material provides an example.

$$\text{supp}(\lambda) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$$
$$\mathbf{u} \geq \underline{\mathbf{u}}.$$

If λ is a deterministic measure $\lambda_{\mathbf{y}}$ (cf. (28)), then the first two constraints in this maximization problem are the standard incentive constraints, requiring that (i) $\mathbf{u}$ is the rent function that results when each agent type maximizes against the tariff $\mathbf{v}$ and (ii) all agent types x are assigned to one of their optimal decisions $\mathbf{y}(x) \in Y_{\mathbf{v}}(x)$. Intuitively, for measures $\lambda \in \mathbb{M}$ that are not deterministic, the second of these conditions is weakened to allow the principal to randomize over the set of decisions that are optimal for the agent.

The principal's expected utility in (29) is well-defined for any feasible $(\lambda, \mathbf{u}, \mathbf{v})$: Because $\text{supp}(\lambda) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$, we have $\mathbf{v}(y) = \psi(y, x, \mathbf{u}(x))$ for all $(x, y) \in \text{supp}(\lambda)$, and hence

$$\int_X \int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda(x, y) = \int_X \int_Y \pi\big(x, y, \psi\big(y, x, \mathbf{u}(x)\big)\big)\, d\lambda(x, y), \tag{30}$$

where the latter integral is well-defined because π, ψ, and the implementable profile $\mathbf{u}$ are continuous (the last of these by Proposition 2). A useful implication is that the principal's payoff can be written in terms of only the measure λ and rent function $\mathbf{u}$, implying that any two feasible outcomes $(\lambda, \mathbf{u}, \mathbf{v})$ and $(\lambda, \mathbf{u}, \tilde{\mathbf{v}})$ give the same payoff to the principal.

REMARK 7—Pairwise Stability and Feasibility in the Principal's Problem: Consider a triple $(\lambda, \mathbf{u}, \mathbf{v})$ that satisfies the incentive constraints in the principal's problem, that is, $\mathbf{v}$ implements $\mathbf{u}$ and $\text{supp}(\lambda) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$. Define the measure ν on Y by setting $\nu(\tilde{Y}) = \lambda_Y(\tilde{Y})$ for all measurable $\tilde{Y} \subset Y$ and specify an arbitrary continuous reservation utility profile $\underline{\mathbf{v}}$. Then λ is a full match for the balanced matching problem $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$. Further, it is immediate from Proposition 5 that $(\lambda, \mathbf{u}, \mathbf{v})$ is pairwise stable in this balanced matching problem. Vice versa, if $(\lambda, \mathbf{u}, \mathbf{v})$ is pairwise stable for a matching problem $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ in which μ has full support, then $(\lambda, \mathbf{u}, \mathbf{v})$ satisfies the incentive constraints in any principal-agent model $(X, Y, \phi, \mu, \pi, \underline{\mathbf{u}})$ in which π has the properties from Assumption 3. See Carlier (2003, Theorem 2) and, more recently, Dworczak and Zhang (2017) for related observations in the quasilinear case.

5.2. *Existence of a Solution to the Principal's Problem*

To obtain our existence result, we begin by transforming the principal's problem into a nonlinear pricing problem over the set of implementable tariffs $\mathbf{v} \in \mathbf{I}(Y)$. Towards this end, define the function $F : \mathbf{I}(Y) \times \mathbb{M} \to \mathbb{R}$ by

$$F(\mathbf{v}, \lambda) = \int_X \int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda(x, y) \tag{31}$$

and define the correspondence $G : \mathbf{I}(Y) \to \mathbb{M}$ by

$$G(\mathbf{v}) = \big\{\lambda \in \mathbb{M} : \text{supp}(\lambda) \subseteq \Gamma_{\Phi\mathbf{v},\mathbf{v}}\big\}. \tag{32}$$

Also, for $\mathbf{v} \in \mathbf{I}(Y)$, let

$$\Pi(\mathbf{v}) = \max_{\lambda \in G(\mathbf{v})} F(\mathbf{v}, \lambda). \tag{33}$$

Observe that $F(\mathbf{v}, \lambda)$ is nothing but the objective function of the principal's problem specified in (29). The heuristic interpretation of (33) therefore is that $\Pi(\mathbf{v})$ specifies the maximal payoff the principal can obtain by probabilistically assigning agents to decisions that are optimal for them when facing the implementable tariff $\mathbf{v}$ (i.e., by choosing $\lambda \in G(\mathbf{v})$). Appendix B.15 of the Supplemental Material shows that this problem has a solution for every implementable tariff, so that the function $\Pi : \mathbf{I}(Y) \to \mathbb{R}$ is well-defined. Further, it shows the following:

LEMMA 6: *Let Assumptions 1 and 3 hold. The function $\Pi : \mathbf{I}(Y) \to \mathbb{R}$ is upper semicontinuous. If $\mathbf{v}^*$ solves*

$$\max_{\{\mathbf{v}\in\mathbf{I}(Y):\mathbf{v}\leq\Psi\underline{\mathbf{u}}\}} \Pi(\mathbf{v}), \tag{34}$$

then there exists $\lambda^* \in G(\mathbf{v}^*)$ such that the triple $(\lambda^*, \Phi\mathbf{v}^*, \mathbf{v}^*)$ solves the principal's problem.

The first step in the proof of Lemma 6 uses Corollaries 3(1) and 4(1) to show that replacing an arbitrary tariff $\mathbf{v}$ in a feasible triple $(\lambda, \mathbf{u}, \mathbf{v})$ with the implementable tariff $\Psi\mathbf{u}$ results in another feasible triple. Doing so leaves the principal's expected payoff unchanged (cf. (30)). This allows us to reduce the principal's problem to the choice of an implementable tariff $\mathbf{v}$ and an associated measure $\lambda \in G(\mathbf{v})$, with the agent's utility profile given by the rent function $\mathbf{u} = \Phi\mathbf{v}$. The continuity of implementable profiles $\mathbf{v}$ (Proposition 2) and the compactness of the set of measures $\mathbb{M}$ (by Prokhorov's theorem) then ensure that the function F and the correspondence G are sufficiently well-behaved to imply the upper semicontinuity of the function Π. Maximizing this function subject to the constraint that the associated rent function $\Phi\mathbf{v}$ satisfies the participation constraints $\Phi\mathbf{v} \geq \underline{\mathbf{u}}$, which we rewrite as $\mathbf{v} \leq \Psi\underline{\mathbf{u}}$, then yields an optimal tariff $\mathbf{v}^*$ that, together with the associated measure λ^* and induced rent function $\mathbf{u}^* = \Phi\mathbf{v}^*$, solves the principal's problem.

To show the existence of a solution to the principal's problem, it remains to show that the nonlinear pricing problem (34) in the statement of Lemma 6 has a solution. To do so, we begin by observing that the feasible set of the nonlinear pricing problem is bounded above by $\Psi\underline{\mathbf{u}}$. While there is no corresponding lower bound in the formulation of the nonlinear pricing problem, it is intuitive that a suitable lower bound can be imposed without impinging on the value of the principal's maximization problem. We can thereby restrict the choice set in the nonlinear pricing problem to a closed and bounded set of tariffs. Moreover, and crucially, the maximization in (34) is over a set of implementable profiles, and we have established in Corollary 5(3) that closed and bounded sets of implementable profiles are compact. As Π is upper semicontinuous (Lemma 6), an application of Weierstrass's extreme value theorem then yields the existence of a solution to the nonlinear pricing problem. Appendix A.5 shows, in addition, that the measure in the associated solution to the principal's problem can be "purified" to obtain a solution to the principal's problem featuring a deterministic match:

PROPOSITION 9: *Let Assumptions 1 and 3 hold. Then there exists a solution $(\lambda, \mathbf{u}, \mathbf{v})$ to the principal's problem in which $\mathbf{u}$ and $\mathbf{v}$ implement each other and λ is deterministic.*

5.3. *Is the Participation Constraint Binding?*

As the principal must respect the agent's participation constraint when choosing an optimal tariff, we have $\mathbf{u} \geq \underline{\mathbf{u}}$ in any solution $(\lambda, \mathbf{u}, \mathbf{v})$ to the principal's problem. Here we ask whether the agent's participation constraint must be binding in the sense that there exists some $x \in X$ satisfying $\mathbf{u}(x) = \underline{\mathbf{u}}(x)$.[22]

If all implementable assignments $\mathbf{y}$ are strongly implementable, then the answer is straightforward from the lattice result in Lemma 4. Appendix A.6 shows the following:

PROPOSITION 10: *Let Assumptions 1 and 3 hold. If every implementable assignment* $\mathbf{y}$ *is strongly implementable, then the participation constraint is binding in any solution to the principal's problem.*

In the absence of strong implementability, the conclusion of Proposition 10 may fail. Appendix C.2 of the Supplemental Material provides an example illustrating this. In this example, it is optimal for the principal to implement an assignment that is not strongly implementable and to leave strictly positive rents to all agent types.

The example in Appendix C.2 features common values in the sense that the principal cares directly about which type of the agent obtains which decision. Our next result demonstrates that no such example can be constructed if the principal-agent model has private values, that is, the principal's payoff function π does not depend on x and can thus be rewritten as $\hat{\pi} : Y \times \mathbb{R} \to \mathbb{R}$:

PROPOSITION 11: *Let Assumptions 1 and 3 hold and let the principal-agent model have private values. Then, in any solution to the principal's problem, the participation constraint is binding for some type of agent.*

Appendix A.7 contains the proof. The key idea is that any $(\lambda, \mathbf{u}, \mathbf{v})$ which is feasible in the principal's problem corresponds to a pairwise stable outcome satisfying the participation constraint $\mathbf{u} \geq \underline{\mathbf{u}}$ in a suitably constructed balanced matching model (cf. Remark 7). We can then apply the result in Corollary 8 to obtain a minimum (in the set of buyer profiles) pairwise stable outcome, in which the participation constraint binds for some type of buyer. The principal can implement this outcome, which features the same induced distribution ν over decisions as the one that we started from. The private-values assumption ensures that this leads to a strictly higher payoff for the principal than any feasible outcome in which the participation constraint is not binding.

6. SINGLE CROSSING

For uni-dimensional principal-agent models in which the agent's utility function is quasilinear, assuming the agent's willingness to pay to be strictly supermodular leads to a sharp characterization of implementable assignments: an assignment is implementable (and therefore strongly implementable, Section 2.4) if and only if it is increasing (Rochet (1987); also see Vohra (2011, Theorem 4.2.5)). Similarly, for uni-dimensional matching models with perfectly transferable utility, assuming that the surplus function is strictly supermodular implies that all stable full matches feature positive assortative matching (Becker (1973)).

[22]Throughout the following discussion, we impose Assumption 3 and, therefore, suppose that the principal's utility is strictly increasing in the transfer received from the agent. As noted in Guesnerie and Laffont (1984), there is no reason to suppose that the participation constraint should be binding if this assumption fails.

In this section, we show that these results carry over to our setting with imperfectly transferable utility. The only change required is to replace the assumption of strict supermodularity with the assumption that the generating function satisfies a strict single-crossing condition.[23]

ASSUMPTION 4: *The sets X and Y are compact intervals in $\mathbb{R}$. The generating function ϕ satisfies the strict single-crossing condition*:

$$\phi(x_1, y_2, v_2) \geq \phi(x_1, y_1, v_1) \quad \Longrightarrow \quad \phi(x_2, y_2, v_2) > \phi(x_2, y_1, v_1) \tag{35}$$

for all $x_1 < x_2 \in X$, $y_1 < y_2 \in Y$, and $v_1, v_2 \in \mathbb{R}$.

A quasilinear generating function $\phi(x, y, v) = f(x, y) - v$ satisfies the strict single-crossing condition if and only if $f(x, y)$ is strictly supermodular.[24]

We begin by considering matching models satisfying Assumption 4 and then show how the results obtained for this case can be leveraged into a corresponding result for implementable assignments. Our results generalize previous results for principal-agent models without quasilinear preferences by Guesnerie and Laffont (1984) and for matching models with imperfectly transferable utility by Legros and Newman (2007). The former imposed a smoothness condition on the generating function and restricted attention to piecewise continuously differentiable assignments. The latter considered a model with a finite number of agents and showed that their generalized increasing differences condition, which is equivalent to our strict single-crossing condition, ensures that stable matches are positive assortative.

6.1. *Positive Assortative Matching*

We consider balanced matching models $(X, \mathcal{Y}, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ satisfying Assumptions 1 and 4. Given that X and Y are compact intervals in the reals, it will be convenient to identify the measures μ, ν, and λ with their distribution functions, denoted by F_μ, G_ν, and H_λ. Let λ^* be the unique full match satisfying

$$H_{\lambda^*}(x, y) = \min\{F_\mu(x), G_\nu(y)\} \quad \text{for all } (x, y) \in X \times Y. \tag{36}$$

Following Galichon (2016, Chapter 4), we refer to λ^* as the positive assortative match.

When both F_μ and G_ν are continuous and strictly increasing, the positive assortative match is obtained by matching each agent with his or her uniquely determined counterpart on the other side who has the same "rank" in the type distribution (as determined by the quantile functions F^{-1} and G^{-1}). Note that, in general, the positive assortative match need not be deterministic but will be so when μ is atomless (Galichon (2016, Lemma 4.2)). This provides us with the condition in the following proposition ensuring that the pairwise stable match is not only unique but also deterministic.

[23] We could equivalently define strict single crossing in terms of the inverse generating function ψ.

[24] Under quasilinearity, the strict single-crossing condition (35) becomes

$$f(x_1, y_2) - f(x_1, y_1) \geq v_2 - v_1 \quad \Longrightarrow \quad f(x_2, y_2) - f(x_2, y_1) > v_2 - v_1.$$

This is obviously implied by the strict supermodularity condition $f(x_2, y_2) - f(x_2, y_1) > f(x_1, y_2) - f(x_1, y_1)$, while choosing $v_2 - v_1 = f(x_1, y_2) - f(x_1, y_1)$ ensures that strict single crossing implies supermodularity.

PROPOSITION 12: *Let Assumptions 1 and 4 hold and the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be balanced. Then the positive assortative match* λ^* *is the unique pairwise stable match for all initial conditions* (x_0, u_0). *Further, if* μ *is absolutely continuous with respect to Lebesgue measure, then* λ^* *is deterministic.*

PROOF: Proposition 6 ensures that there exists a pairwise stable outcome $(\lambda, \mathbf{u}, \mathbf{v})$ with $\mathbf{u}$ and $\mathbf{v}$ implementing each other and satisfying $\mathbf{u}(x_0) = u_0$.

Suppose $\Gamma_{\mathbf{u},\mathbf{v}}$ is comonotonic, that is, for (x_1, y_1) and $(x_2, y_2) \in \Gamma_{\mathbf{u},\mathbf{v}}$ we have $x_2 > x_1 \Longrightarrow y_2 \geq y_1$ and $y_2 > y_1 \Longrightarrow x_2 \geq x_1$. Proposition 5(1) then implies that $\text{supp}(\lambda)$ is comonotonic. From Theorem 3 in Dhaene, Denuit, Goovaerts, Kaas, and Vyncke (2002), λ then satisfies (36) and therefore is the positive assortative match λ^*. If μ is absolutely continuous with respect to Lebesgue measure, then F_μ is continuous and λ^* is deterministic (Galichon (2016, Lemma 4.2)).

It remains to verify that the strict single-crossing condition (35) in Assumption 4 implies that $\Gamma_{\mathbf{u},\mathbf{v}}$ is comonotonic. It suffices to show that there does not exist $(x_1, y_1), (x_2, y_2) \in \Gamma_{\mathbf{u},\mathbf{v}}$ satisfying $x_2 > x_1$ and $y_1 > y_2$. To show this, observe that (because $\mathbf{v}$ implements $\mathbf{u}$) from Lemma 2, we have that $(x_1, y_1), (x_2, y_2) \in \Gamma_{\mathbf{u},\mathbf{v}}$ implies

$$\phi\big(x_1, y_1, \mathbf{v}(y_1)\big) \geq \phi\big(x_1, y_2, \mathbf{v}(y_2)\big),$$

$$\phi\big(x_2, y_2, \mathbf{v}(y_2)\big) \geq \phi\big(x_2, y_1, \mathbf{v}(y_1)\big).$$

With $x_2 > x_1$ and $y_1 > y_2$, the first of these inequalities and (35) imply $\phi(x_2, y_1, \mathbf{v}(y_1)) > \phi(x_2, y_2, \mathbf{v}(y_2))$, contradicting the second inequality. *Q.E.D.*

Extending Proposition 12 to show that the unique pairwise stable match λ^* is also the unique stable match requires the existence of a pairwise stable outcome $(\lambda^*, \mathbf{u}, \mathbf{v})$ satisfying the participation constraints $\mathbf{u} \geq \underline{\mathbf{u}}$ and $\mathbf{v} \geq \underline{\mathbf{v}}$. This is not guaranteed. For an extreme counterexample, it may be that there is no pair of agents capable of generating individually rational payoffs (i.e., $\underline{\mathbf{u}}(x) > \phi(x, y, \underline{\mathbf{v}}(y))$ holds for all (x, y)), obviously implying that, in the unique stable outcome, all agents are unmatched. Suppose, however, that for all $(x, y) \in X \times Y$, we have

$$\underline{\mathbf{u}}(x) < \phi\big(x, y, \underline{\mathbf{v}}(y)\big), \tag{37}$$

and consider a stable outcome $(\lambda, \mathbf{u}, \mathbf{v})$. If there were unmatched types in Y (i.e., $\text{supp}(\nu - \lambda_Y) \neq \emptyset$), then we could conclude from (22) that there exists $\hat{y} \in \text{supp}(\nu)$ such that $\mathbf{v}(\hat{y}) = \underline{\mathbf{v}}(\hat{y})$ holds. Using (25) and (37), this implies $\mathbf{u}(x) > \underline{\mathbf{u}}(x)$ for all $x \in \text{supp}(\mu)$, which in turn implies (from (21)) that there exist no unmatched types in X (i.e., $\text{supp}(\mu - \lambda_X) = \emptyset$). As in a balanced match there are no matches featuring a strictly positive measure of unmatched agents on one side but not on the other, we may thus conclude that λ is a full match. As every stable outcome featuring a full match is also pairwise stable, Proposition 12 then implies the following:

COROLLARY 9: *Let Assumptions 1 and 4 hold, let the matching model* $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ *be balanced, and let (37) hold. Then the positive assortative match* λ^* *is the unique stable match.*

Similar arguments, though with more tedious notation, show that if Assumptions 1 and 4 hold, then in any stable match, all *matched* agents are matched positive assortatively.

6.2. *Increasing Assignments*

It is a familiar result that implementable assignments must be increasing if a strict single-crossing condition holds (e.g., Fudenberg and Tirole (1991, Theorem 7.2)). Therefore, the main challenge in proving the following result is to show that every increasing assignment can be implemented with any initial condition. To obtain this, we build on Proposition 12 to show that, for every increasing assignment, the deterministic measure associated with it can arise as the unique pairwise stable match in a suitably defined matching model.

PROPOSITION 13: *Let Assumptions 1 and 4 hold. Then an assignment* $\mathbf{y}$ *is implementable if and only if it is increasing. In addition, every implementable assignment is strongly implementable.*

PROOF: Suppose the assignment $\mathbf{y}$ is implementable. Then there exist $\mathbf{u}$ and $\mathbf{v}$ implementing each other such that $(x, \mathbf{y}(x)) \in \Gamma_{\mathbf{u},\mathbf{v}}$ holds for all $x \in X$ (Proposition 4(1)). Because $\Gamma_{\mathbf{u},\mathbf{v}}$ is comonotonic (cf. the proof of Proposition 12), this implies that $\mathbf{y}$ is increasing.

Fix an increasing assignment $\mathbf{y}$ and an initial condition (x_0, u_0). We construct a balanced matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$: Let μ be Lebesgue measure on the Borel sets of X, and let ν be the pushforward of μ through $\mathbf{y}$ (which is well-defined because an increasing function $\mathbf{y}$ is measurable). The reservation utilities $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$ will play no role, and so we can take $\underline{\mathbf{u}} \equiv 0 \equiv \underline{\mathbf{v}}$.

Let λ^* denote the positive assortative match for the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$. From Proposition 12, λ^* is deterministic. The construction of ν ensures $\lambda^* = \lambda_{\mathbf{y}}$. Applying Proposition 12, we obtain that there exists $(\mathbf{u}, \mathbf{v})$ such that $(\lambda_{\mathbf{y}}, \mathbf{u}, \mathbf{v})$ is a pairwise stable outcome with $\mathbf{u}(x_0) = u_0$. From Proposition 6, we may take $\mathbf{u}$ and $\mathbf{v}$ to implement each other.

We complete the argument by showing that $\mathbf{v}$ implements $(\mathbf{u}, \mathbf{y})$. It suffices to show that, for every $x \in X$, $(x, \mathbf{y}(x)) \in \Gamma_{\mathbf{u},\mathbf{v}}$ (Proposition 4). From Proposition 5(1), we have $\text{supp}(\lambda_{\mathbf{y}}) \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$. Fix a value $x \in X$. If $\mathbf{y}$ is continuous at x, then we immediately have $(x, \mathbf{y}(x)) \in \text{supp}(\lambda_{\mathbf{y}})$ (since otherwise $\lambda_{\mathbf{y}}(\tilde{X} \times Y) = 0$ for some neighborhood $\tilde{X}$ of x, a contradiction). If $\mathbf{y}$ is not continuous at x, then the increasing function $\mathbf{y}$ must take an upward jump at x, and we have $(x, \mathbf{y}(x)) \in [\lim_{\tilde{x} \nearrow x} \mathbf{y}(x), \lim_{\tilde{x} \searrow x} \mathbf{y}(x)] \subseteq \Gamma_{\mathbf{u},\mathbf{v}}$. The final inclusion follows from the facts that, for each $y' \in [\lim_{\tilde{x} \nearrow x} \mathbf{y}(x), \lim_{\tilde{x} \searrow x} \mathbf{y}(x)]$, there exists $x' \in X$ such that $(x', y') \in \Gamma_{\mathbf{u},\mathbf{v}}$ (because, from Lemma 2, $\Gamma_{\mathbf{u},\mathbf{v}}$ coincides with the graph of the argmax-correspondence $\mathbf{Y}_{\mathbf{v}}$, which is nonempty-valued) and that $\Gamma_{\mathbf{u},\mathbf{v}}$ is comonotonic (cf. the proof of Proposition 12), which implies $x' = x$. *Q.E.D.*

Recall from Section 2.4 that, in the absence of quasilinearity, an assignment may be implementable without being strongly implementable. Proposition 13 shows that strict single crossing precludes this possibility. It follows that strict single crossing is a sufficient condition for the participation constraint to bind in any solution to the principal-agent model (Proposition 10).

REMARK 8—Single Crossing Versus Strict Single Crossing: Say that the generating function satisfies the single-crossing condition if the final inequality in (35) is weak. Under this weaker condition, there may be (pairwise) stable matches that are different from the positive assortative match λ^* and non-increasing assignments may be implementable (as can be easily seen by considering the trivial quasilinear example in which the generating function is given by $\phi(x, y, v) = -v$). However, under otherwise identical assumptions, it remains true that (i) in a balanced matching model, the positive assortative match λ^* is pairwise stable for all initial conditions, (ii) every balanced matching model satisfying condition (37) has a stable outcome featuring the match λ^*, and (iii) every increasing assignment $\mathbf{y}$ is strongly implementable. Proving this is more tedious under single crossing than under strict single crossing, as an extra step is required in the proof of Proposition 12 to show that the support of λ^* is contained in $\Gamma_{\mathbf{u},\mathbf{v}}$ for every pairwise stable outcome $(\lambda, \mathbf{u}, \mathbf{v})$ with $\mathbf{u}$ and $\mathbf{v}$ implementing each other.

7. DISCUSSION

We have introduced and studied a duality relationship that provides a characterization of implementable profiles and assignments suitable for adverse-selection principal-agent models and two-sided matching models. This has allowed us to extend results previously developed for the quasilinear case, and to clarify the logic behind these results.

Throughout our analysis, we have eschewed smoothness assumptions, as these play no role for the duality structure and are not required for the existence and characterization results pursued here. However, much of the power of conjugate duality stems from the inherent smoothness properties of convex functions, and many of the more useful implications of generalized conjugate duality for the quasilinear case—ranging from the familiar integral representation of implementable utility profiles (e.g., Myerson (1979)) to results asserting the uniqueness and determinateness of stable matchings (e.g., Chiappori, McCann, and Nesheim (2010))—require smoothness conditions. Adding such conditions to our Assumption 1 opens the possibility to investigate questions that go beyond those addressed in this paper. For instance, McCann and Zhang (2017) used the implementation duality to show how the conditions from Figalli, Kim, and McCann (2011), under which the principal's problem can be reduced to a convex maximization program, can be extended to the non-quasilinear case.

A number of extensions suggest themselves. First, much is known about the structure of the set of stable outcomes in matching models with a finite number of agents (Roth and Sotomayor (1990, Chapter 9)), including connectedness and comparative static properties, that one might want to extend to our setting. Second, Appendix D.1 of the Supplemental Material extends the principal-agent model to allow exclusion. For much the same reasons that the participation constraint may not bind in a solution to the principal's problem (Section 5.3), we find that the principal may prefer to pay agents for nonparticipation. Third, Appendices D.2 and D.3 of the Supplemental Material explain how our analysis can be extended to include stochastic contracts and moral hazard in the principal-agent model. In the course of these extensions, we note that our compactness assumption on Y is sometimes restrictive because it is natural to allow for unbounded Y. Similarly, the assumption that the type space X is compact is violated in some applications in finance (such as Glosten (1989)) in which normally distributed types are considered.[25]

The implementation relationships studied here also appear in economic contexts different from the ones we have considered, with possible applications ranging from the characterization of hedonic pricing equilibria (cf. Chiappori, McCann, and Nesheim (2010 in the quasilinear case)) to the development of new econometric techniques for discrete-choice random-utility models (Bonnet, Galichon, and Shum (2017)). Finally, while Galois connections have played little role in economic theory so far, their appearance in the study of

[25]In the quasilinear case, Bardsley (2017) provided an illuminating duality-based analysis of principal-agent models that avoids compactness assumptions.

information aggregation (under the guise of a residual mapping) in Chambers and Miller (2011) and in the study of preference aggregation (Monjardet (1978, 2007)) suggest that further applications may be found in other areas.

APPENDIX A

A.1. *Proof of Proposition 2*

It is immediate from the definitions that $\mathbf{I}(X) \subseteq \Phi\mathbf{B}(Y)$. Hence, to establish the first statement in (14), we need to show that the image $\Phi\mathbf{v}$ of any profile $\mathbf{v} \in \mathbf{B}(Y)$ is implementable and continuous. The remaining statement in (14) follows by an analogous argument.

Given any profile $\mathbf{v} \in \mathbf{B}(Y)$, let $s_\mathbf{v} = \sup_{y\in Y} \mathbf{v}(y)$ denote its supremum and $i_\mathbf{v} = \inf_{y\in Y} \mathbf{v}(y)$ its infimum. These are finite because $\mathbf{v}$ is bounded. Let $E_\mathbf{v} = \{(y, v) \in Y \times \mathbb{R} | v \geq \mathbf{v}(y)\}$ denote the epigraph of $\mathbf{v}$, and let $Z_\mathbf{v} = \{(y, v) \in Y \times \mathbb{R} | s_\mathbf{v} \geq v \geq \mathbf{v}(y)\}$. Observe that the set $Z_\mathbf{v} \subset E_\mathbf{v}$ is bounded, contains the graph of $\mathbf{v}$, and is contained in $[i_\mathbf{v}, s_\mathbf{v}] \times Y$, which is a compact set (because Y is compact).

We now proceed in two steps.

Step 1: Consider $\mathbf{v} \in \mathbf{B}(Y)$ that is lower semicontinuous. Then its epigraph $E_\mathbf{v}$ is closed and so is $Z_\mathbf{v}$. As $Z_\mathbf{v}$ is contained in the compact set $[i_\mathbf{v}, s_\mathbf{v}] \times Y$, it follows that $Z_\mathbf{v}$ is compact. As the generating function ϕ is continuous, a solution to the problem

$$\max_{(y,v)\in Z_\mathbf{v}} \phi(x, y, v) \tag{A.1}$$

thus exists for all $x \in X$ by Weierstrass's extreme value theorem. As ϕ is continuous and $Z_\mathbf{v}$ is compact, it follows from Berge's maximum theorem (Ok (2007, p. 306)) that the profile $\mathbf{u} \in \mathbf{B}(X)$ defined by $\mathbf{u}(x) = \max_{(y,v)\in Z_\mathbf{v}} \phi(x, y, v)$ for all $x \in X$ is continuous.

As the graph of $\mathbf{v}$ is contained in $Z_\mathbf{v}$, and ϕ is strictly decreasing in its third argument, any solution to (A.1) lies on the graph of $\mathbf{v}$, implying that for every $x \in X$, there exists $\mathbf{y}(x) \in Y$ such that

$$\max_{(y,v)\in Z_\mathbf{v}} \phi(x, y, v) = \phi\big(x, \mathbf{y}(x), \mathbf{v}\big(\mathbf{y}(x)\big)\big)$$

holds. This ensures that the suprema in the definition of $\Phi\mathbf{v}$ are attained and that $\mathbf{v}$ implements $\Phi\mathbf{v} = \mathbf{u}$.

Step 2: It remains to consider the case in which $\mathbf{v} \in \mathbf{B}(Y)$ is not lower semicontinuous. Let $\bar{\mathbf{v}}$ be the lower semicontinuous hull of $\mathbf{v}$, that is, the greatest element of the family of lower semicontinuous functions from Y to $\mathbb{R}$ majorized by $\mathbf{v}$. (The existence of $\bar{\mathbf{v}}$ is assured, cf. Penot (2013, Proposition 1.21).) As $\mathbf{v}$ is bounded, so is $\bar{\mathbf{v}}$, that is, we have $\bar{\mathbf{v}} \in \mathbf{B}(Y)$. From the previous step, the profile $\bar{\mathbf{v}}$ implements $\Phi\bar{\mathbf{v}}$, which is continuous. It remains to show that $\Phi\bar{\mathbf{v}} = \Phi\mathbf{v}$ holds. Because the epigraph $E_{\bar{\mathbf{v}}}$ of $\bar{\mathbf{v}}$ is the closure of the epigraph $E_\mathbf{v}$ of $\mathbf{v}$ (Penot (2013, Proposition 1.21)), we have that $Z_{\bar{\mathbf{v}}}$ is the closure of $Z_\mathbf{v}$. Therefore,

$$\sup_{(y,v)\in Z_\mathbf{v}} \phi(x, y, v) = \max_{(y,v)\in Z_{\bar{\mathbf{v}}}} \phi(x, y, v),$$

and thus (because ϕ is decreasing in its third argument) we have $\sup_{y\in Y} \phi(x, y, \mathbf{v}(y)) = \max_{y\in Y} \phi(x, y, \bar{\mathbf{v}}(y))$ for all $x \in X$, which is the desired result.

A.2. *Proof of Lemma 4*

We prove Lemma 4(1); the proof for Lemma 4(2) is analogous.

Let $\mathfrak{U} \subset \mathbf{I}(X)$ be a closed sublattice of $\mathbf{B}(X)$ for which

$$U_x = \{\mathbf{u} \in \mathfrak{U} | \mathbf{u}(x) = \underline{\mathbf{u}}(x)\}$$

is nonempty for all $x \in X$. For the current proof, the important observation is that if $\mathbf{y} \in Y^X$ is strongly implementable, then one such set is $\mathcal{U}_\mathbf{y}$, which is a subset of $\mathbf{I}(X)$ (by definition), closed (Corollary 5(1)), and, by Lemma 3, a sublattice of $\mathbf{B}(X)$, with the strong implementability of $\mathbf{y}$ ensuring that $\{\mathbf{u} \in \mathcal{U}_\mathbf{y} | \mathbf{u}(x) = \underline{\mathbf{u}}(x)\}$ is nonempty for all $x \in X$.

Let

$$S = \{\mathbf{u} \in \mathfrak{U} | \mathbf{u} \geq \underline{\mathbf{u}}\}.$$

We proceed in two steps. The first step establishes that there exists $\hat{\mathbf{u}} \in S$ satisfying $\hat{\mathbf{u}}(x) = \underline{\mathbf{u}}(x)$ for some $x \in X$. The second step then completes the argument by showing that S has a minimum element.

Step 1: Pick an arbitrary $x_0 \in X$ and $\mathbf{u}_0 \in U_{x_0}$. We construct a sequence $(x_n)_{n=1}^\infty$ in X and an associated sequence $(\mathbf{u}_n)_{n=1}^\infty$ of profiles in $\mathfrak{U}$, satisfying $\mathbf{u}_n \in U_{x_n}$ for all n, by the following recursion: Given $(x_{n-1}, \mathbf{u}_{n-1})$ with $\mathbf{u}_{n-1} \in U_{x_{n-1}}$, let $x_n \in \arg\min_{x \in X}[\mathbf{u}_{n-1}(x) - \underline{\mathbf{u}}(x)]$. Because both $\mathbf{u}_{n-1}$ (as an implementable profile, Proposition 2) and $\underline{\mathbf{u}}$ (by assumption) are continuous and X is compact, such an x_n exists. Pick any $\hat{\mathbf{u}}_n \in U_{x_n}$. Define $\mathbf{u}_n = \mathbf{u}_{n-1} \vee \hat{\mathbf{u}}_n$. Because $\mathfrak{U}$ is a sublattice, we then have $\mathbf{u}_n \in \mathfrak{U}$. Because $\mathbf{u}_{n-1} \in U_{x_{n-1}}$ implies $\min_{x \in X}[\mathbf{u}_{n-1}(x) - \underline{\mathbf{u}}(x)] \leq 0$, we further have $\mathbf{u}_n(x_n) = \underline{\mathbf{u}}(x_n)$, implying $\mathbf{u}_n \in U_{x_n}$.

The sequence $(\mathbf{u}_n)_{n=1}^\infty$ is increasing by construction. It is also bounded above.[26] Therefore, it is bounded and thus equicontinuous (Corollary 5(2)). Hence, $(\mathbf{u}_n)_{n=1}^\infty$, which is a sequence in the closed set $\mathfrak{U}$, has a limit point $\hat{\mathbf{u}} \in \mathfrak{U}$. Because $\hat{\mathbf{u}} \in \mathfrak{U} \subset \mathbf{I}(X)$ is implementable, it is continuous (Proposition 2).

Because X is compact, the sequence $(x_n)_{n=1}^\infty$ has a converging subsequence, denoted by x_{n_k}, with limit $x^* \in X$. As $(\mathbf{u}_n)_{n=1}^\infty$ is a sequence of continuous functions converging uniformly to the continuous function $\hat{\mathbf{u}}$, we have

$$\lim_{k \to \infty} \mathbf{u}_{n_k}(x_{n_k}) = \hat{\mathbf{u}}(x^*), \tag{A.2}$$

$$\lim_{k \to \infty} \mathbf{u}_{n_{k-1}}(x_{n_k}) = \hat{\mathbf{u}}(x^*). \tag{A.3}$$

As $\mathbf{u}_n(x_n) = \underline{\mathbf{u}}(x_n)$ holds for all n and $\underline{\mathbf{u}}$ is continuous, (A.2) implies

$$\hat{\mathbf{u}}(x^*) = \underline{\mathbf{u}}(x^*). \tag{A.4}$$

By construction of the sequence $(x_n)_{n=1}^\infty$, we have

$$\mathbf{u}_{n-1}(x) - \underline{\mathbf{u}}(x) \geq \mathbf{u}_{n-1}(x_n) - \underline{\mathbf{u}}(x_n)$$

[26] By continuity of ψ and $\underline{\mathbf{u}}$, the profile $\underset{\sim}{\mathbf{y}} \in \mathbf{B}(Y)$ given by $\underset{\sim}{\mathbf{y}}(y) = \min_{x \in X} \psi(y, x, \underline{\mathbf{u}}(x))$ for all $y \in Y$ is well-defined. For any profile $\mathbf{v} \in \mathbf{B}(Y)$ satisfying $\mathbf{v}(\hat{y}) < \underset{\sim}{\mathbf{y}}(\hat{y})$ for some $\hat{y} \in Y$, we have $\phi(x, \hat{y}, \mathbf{v}(\hat{y})) > \underline{\mathbf{u}}(x)$ for all $x \in X$ by construction. For such $\mathbf{v}$, $\mathbf{u} = \Phi\mathbf{v}$ thus satisfies $\mathbf{u}(x) > \underline{\mathbf{u}}(x)$ for all $x \in X$, implying that $\mathbf{u}$ is not in $\bigcup_{x \in X} U_x$. By the order reversal property of the implementation map Φ, it follows that $\overline{\mathbf{u}} = \Phi\underset{\sim}{\mathbf{y}}$ is an upper bound for $\bigcup_{x \in X} U_x$ and therefore an upper bound for $(\mathbf{u}_n)_{n=1}^\infty$.

for all $x \in X$ and $n \geq 1$. Taking limits for the sequence n_k, we thus obtain

$$\hat{\mathbf{u}}(x) - \underline{\mathbf{u}}(x) \geq \hat{\mathbf{u}}(x^*) - \underline{\mathbf{u}}(x^*)$$

for all $x \in X$, where we have used the continuity of $\underline{\mathbf{u}}$ and (A.3) to obtain the right side of the inequality. Taking account of (A.4), this implies

$$\hat{\mathbf{u}}(x) \geq \underline{\mathbf{u}}(x) \tag{A.5}$$

for all $x \in X$. Combining (A.4) and (A.5), we have established the desired result.

Step 2: As S contains $\hat{\mathbf{u}}$ satisfying $\hat{\mathbf{u}}(x) = \underline{\mathbf{u}}(x)$ for some $x \in X$, it is immediate that a minimum element $\mathbf{u}^*$ of S must satisfy $\mathbf{u}^*(x) = \underline{\mathbf{u}}(x)$ for the same $x \in X$. It remains to show that such a minimum element exists.

Given any $\bar{\mathbf{u}} \in S$, let $S_{\bar{\mathbf{u}}} = \{\mathbf{u} \in \mathfrak{U}\} | \bar{\mathbf{u}} \geq \mathbf{u} \geq \underline{\mathbf{u}}\}$. The set $S_{\bar{\mathbf{u}}}$ contains $\bar{\mathbf{u}}$ and hence is nonempty. Further, it is bounded. As the intersection of two closed sets, the set $S_{\bar{\mathbf{u}}}$ is closed, and as an intersection of two sublattices of $\mathbf{B}(X)$, it is a sublattice. With the set $S_{\bar{\mathbf{u}}}$ being a closed and bounded subset of $\mathbf{I}(X)$, it is compact (Corollary 5(3)) and thus a complete sublattice of $\mathbf{B}(X)$.[27] The complete sublattice $S_{\bar{\mathbf{u}}}$ has a minimum element $\mathbf{u}^*$, which clearly is also the minimum element of S.

A.3. *Proof of Proposition 6*

Let $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ be a balanced matching problem satisfying Assumption 1. Since this matching model is balanced, nothing is lost (and some convenience is gained) by taking μ and ν to be probability measures, which we hereafter maintain.

Let $(x_1, \dots, x_n) \in X^n$ and $(y_1, \dots, y_n) \in Y^n$ satisfy $y_1 = y_0$, where $y_0 \in Y$ is the agent appearing as part of the initial condition (y_0, v_0) in the statement of the proposition. Define a measure μ_n on X by

$$\mu_n(\tilde{X}) = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}(\tilde{X}) \tag{A.6}$$

for measurable $\tilde{X} \subseteq X$, and define the measure ν_n on Y similarly by

$$\nu_n(\tilde{Y}) = \frac{1}{n} \sum_{k=1}^{n} \delta_{y_k}(\tilde{Y}) \tag{A.7}$$

for all measurable $\tilde{Y} \subseteq Y$.

The proof of the following lemma, which builds on Lemma 3 in Demange and Gale (1985) (for matching models with a finite number of agents), is in Appendix B.16 of the Supplemental Material.

LEMMA 7: *Let Assumption 1 hold. The matching model $(X, Y, \phi, \mu_n, \nu_n, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ has a pairwise stable outcome $(\lambda_n, \mathbf{u}_n, \mathbf{v}_n)$ with profiles $\mathbf{u}_n$ and $\mathbf{v}_n$ that implement each other and that satisfy $\mathbf{v}_n(y_0) = v_0$.*

[27]The set $S_{\bar{\mathbf{u}}}$ is compact in the norm topology. A lattice is complete if and only if it is compact in the interval topology (Birkhoff (1995, p. 250, Theorem 20)). Compactness in the norm topology implies compactness in the interval topology, as any set open under the latter is also open under the former.

Let $(x_n)_{n_1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences in X and Y with $y_1 = y_0$ and such that the probability measures μ_n and ν_n defined in (A.6)–(A.7) converge weakly to μ, respectively ν. The existence of such sequences is assured: for example, if all but x_1 and y_1 are obtained by taking sequences of independent random draws from the probability measures μ and ν, then with probability 1 we obtain sequences of measures μ_n and ν_n that converge weakly (as $n \to \infty$) to the measures μ and ν (Villani (2009, p. 64)). For each n, the matching model $(X, Y, \phi, \mu_n, \nu_n, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ has a pairwise stable outcome $(\lambda_n, \mathbf{u}_n, \mathbf{v}_n)$ satisfying the properties in the statement of Lemma 7. Let $(\lambda_n, \mathbf{u}_n, \mathbf{v}_n)_{n=1}^{\infty}$ be a sequence of such outcomes. The following lemma establishes that this sequence has a limit point, which is the pairwise stable outcome we seek.

LEMMA 8: *Let Assumption 1 hold. The sequence $(\lambda_n, \mathbf{u}_n, \mathbf{v}_n)_{n=1}^{\infty}$ has a subsequence converging (weakly in the case of the measures λ_n, and in norm for the profiles) to a pairwise stable outcome $(\lambda^*, \mathbf{u}^*, \mathbf{v}^*)$ of the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ that satisfies $\mathbf{v}^*(y_0) = v_0$.*

PROOF OF LEMMA 8: Because each of the probability measures λ_n is defined on the compact (and hence separable) metric space $X \times Y$, the collection $\{\lambda_n\}_{n=1}^{\infty}$ is tight, and Prokhorov's theorem (Shiryaev (1996, p. 318)) ensures that there is a subsequence of $(\lambda_n)_{n=1}^{\infty}$ converging weakly to a probability measure λ^* on $X \times Y$. Further, as each λ_n is a full match, so is λ^*, that is, conditions (26)–(27) are preserved in the limit (Villani (2009, p. 64)). For convenience of notation, we assume that the sequence $(\lambda_n)_{n=1}^{\infty}$ itself converges to λ^*.

We show below that the sequences $(\mathbf{u}_n)_{n=1}^{\infty}$ and $(\mathbf{v}_n)_{n=1}^{\infty}$ are bounded.

Because $\{\mathbf{u}_n\}_{n=1}^{\infty}$ and $\{\mathbf{v}_n\}_{n=1}^{\infty}$ are sets of implementable profiles, Corollary 5(2) then ensures that both of these sets are equicontinuous and the Ascoli theorem (Kelley (1955, p. 233)) ensures that they have compact closures, and hence $(\mathbf{u}_n, \mathbf{v}_n)_{n=1}^{\infty}$ has a subsequence (which, for notational convenience, we take to be the sequence itself) converging to some limit $(\mathbf{u}^*, \mathbf{v}^*)$. As the sets of implementable profiles are closed (Corollary 5(1)), it follows that $\mathbf{u}^*$ and $\mathbf{v}^*$ are implementable. Further, the arguments in the proof of Corollary 5(1) show that $(\mathbf{u}^*, \mathbf{v}^*)$ implement each other. As $\mathbf{v}_n(y_0) = v_0$ holds for all n, we obtain $\mathbf{v}^*(y_0) = v_0$. In light of Proposition 5, the desired result then follows provided that $\operatorname{supp}(\lambda^*) \subseteq \Gamma_{\mathbf{u}^*,\mathbf{v}^*}$ holds, that is, we need to establish

$$\mathbf{u}^*(x) = \boldsymbol{\phi}\big(x, y, \mathbf{v}^*(y)\big)$$

for all $(x, y) \in \operatorname{supp}(\lambda^*)$. The weak convergence of the sequence $(\lambda_n)_{n=1}^{\infty}$ to λ^* ensures that for every (x, y) in the support of λ^*, there is a sequence $(x_n, y_n)_{n=1}^{\infty}$, with each (x_n, y_n) in the support of λ_n, converging to (x, y). For each n and each $(x_n, y_n) \in \operatorname{supp}(\lambda_n)$, we have

$$\mathbf{u}_n(x_n) = \phi\big(x_n, y_n, \mathbf{v}_n(y_n)\big).$$

The convergence of the equicontinuous sequences $(\mathbf{u}_n)_{n=1}^{\infty}$ and $(\mathbf{v}_n)_{n=1}^{\infty}$ of continuous profiles to the continuous profiles $(\mathbf{u}^*, \mathbf{v}^*)$ then gives the result.

It remains to establish boundedness of the sequences $(\mathbf{u}_n)_{n=1}^{\infty}$ and $(\mathbf{v}_n)_{n=1}^{\infty}$. To do so, we first recall that in the pairwise stable outcome $(\lambda_n, \mathbf{u}_n, \mathbf{v}_n)$ of the nth matching model, the profiles $\mathbf{u}_n$ and $\mathbf{v}_n$ implement each other and (because $y_1 = y_0$) satisfy $\mathbf{v}_n(y_1) = v_0$. Hence, for each x and n, we have $\mathbf{u}_n(x) \geq \phi(x, y_1, v_0)$, providing a lower bound for $(\mathbf{u}_n)_{n=1}^{\infty}$. Similarly, we note that some buyer x is matched with seller y_1. The ability of any seller to match with buyer x puts a lower bound on $\mathbf{v}_n$. We cannot be sure which buyer is involved in such a match, but we know that the buyer in question receives utility $\phi(x, y_1, v_0)$, and so we have

$$\mathbf{v}_n(y) \geq \min_{x\in X} \psi\big(y, x, \phi(x, y_1, v_0)\big),$$

providing a lower bound for $(\mathbf{v}_n)_{n=1}^{\infty}$. By the order reversal property of the implementation maps (Corollary 1(2)), the lower bound on $(\mathbf{u}_n)_{n=1}^{\infty}$ provides an upper bound on $(\mathbf{v}_n)_{n=1}^{\infty}$ and the lower bound on $(\mathbf{v}_n)_{n=1}^{\infty}$ provides an upper bound on $(\mathbf{u}_n)_{n=1}^{\infty}$. Hence, the sequences $(\mathbf{u}_n)_{n=1}^{\infty}$ and $(\mathbf{v}_n)_{n=1}^{\infty}$ are bounded, finishing the proof. *Q.E.D.*

This completes the proof of Proposition 6.

A.4. *Proof of Proposition 7*

Let $(\lambda_1, \mathbf{u}_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{u}_2, \mathbf{v}_2)$ be pairwise stable outcomes. Because the type measures μ and ν have full support (Assumption 2), Proposition 5(3) then implies that $\mathbf{u}_1$ and $\mathbf{v}_1$ as well as $\mathbf{u}_2$ and $\mathbf{v}_2$ implement each other.

To show that $\mathbb{U}$ and $\mathbb{V}$ are sublattices of $\mathbf{B}(X)$ and $\mathbf{B}(Y)$, it suffices to show that there exist full matches λ_3 and λ_4 such that $(\lambda_3, \mathbf{u}_1 \vee \mathbf{u}_2, \mathbf{v}_1 \wedge \mathbf{v}_2)$ and $(\lambda_4, \mathbf{u}_1 \wedge \mathbf{u}_2, \mathbf{v}_1 \vee \mathbf{v}_2)$ are pairwise stable outcomes. The conditions for the pairwise stability of these two outcomes differ from each other only by a reversal of the role of the buyer profiles and the seller profiles, so that we may focus on the first of these, namely the existence of a full match λ_3 such that $(\lambda_3, \mathbf{u}_1 \vee \mathbf{u}_2, \mathbf{v}_1 \wedge \mathbf{v}_2)$ is a pairwise stable outcome.

Because $\mathbf{v}_1$ implements $\mathbf{u}_1$ and $\mathbf{v}_2$ implements $\mathbf{u}_2$, it is immediate from the fact that the implementation maps are dualities (Proposition 1) that $\mathbf{v}_1 \wedge \mathbf{v}_2$ implements $\mathbf{u}_1 \vee \mathbf{u}_2$ (cf. the discussion at the beginning of Section 3.4.2). Hence, from Propositions 5(1) and 5(2), it suffices to construct a full match λ_3 with $\operatorname{supp}(\lambda_3) \subseteq \Gamma_{\mathbf{u}_1\vee\mathbf{u}_2,\mathbf{v}_1\wedge\mathbf{v}_2}$ to obtain the desired pairwise stable outcome $(\lambda_3, \mathbf{u}_1 \vee \mathbf{u}_2, \mathbf{v}_1 \wedge \mathbf{v}_2)$.

To simplify notation throughout the following, let $\mathbf{u}_3 = \mathbf{u}_1 \vee \mathbf{u}_2$ and $\mathbf{v}_3 = \mathbf{v}_1 \wedge \mathbf{v}_2$. Using this notation, we may rewrite the condition $\operatorname{supp}(\lambda_3) \subseteq \Gamma_{\mathbf{u}_1\vee\mathbf{u}_2,\mathbf{v}_1\wedge\mathbf{v}_2}$ as

$$(x, y) \in \operatorname{supp}(\lambda_3) \Longrightarrow \mathbf{u}_3(x) = \phi\big(x, y, \mathbf{v}_3(y)\big). \tag{A.8}$$

Our task is to construct a full match λ_3 satisfying (A.8). To do so, we define

$$Y_1 = \{y \in Y : \mathbf{v}_1(y) < \mathbf{v}_2(y)\} \quad \text{and} \quad X_1 = \{x \in X : \mathbf{Y}_{\mathbf{v}_2}(x) \cap Y_1 \neq \emptyset\}.$$

Let $X_2 = X \setminus X_1$ and $Y_2 = Y \setminus Y_1$ denote the complements of X_1 and Y_1.

*S*tep 1: The sets X_1, X_2, Y_1, and Y_2 are measurable.

That $Y_1 \subseteq Y$ is measurable is immediate from the continuity of the implementable assignments $\mathbf{v}_1$ and $\mathbf{v}_2$ (Proposition 2), which ensures that Y_1 is open in Y. The argmax correspondence $\mathbf{Y}_{\mathbf{v}_2}$ has a closed graph (Corollary 2) and hence is weakly measurable (Aliprantis and Border (2006, Theorem 18.20 and Lemma 18.2)). Hence, the pre-image of the open set Y_1 under $\mathbf{Y}_{\mathbf{v}_2}$, namely X_1, is measurable. As the complements of measurable sets, X_2 and Y_2 are also measurable.

*S*tep 2: The measures λ_1 and λ_2 are both concentrated on $(X_1 \times Y_1) \cup (X_2 \times Y_2)$.

Recall that $\mathbf{v}_2$ and $\mathbf{u}_2$ implement each other. By definition of X_1 and Lemma 2, we thus have that $\Gamma_{\mathbf{u}_2,\mathbf{v}_2}$ and $X_2 \times Y_1$ do not intersect each other. Because $\operatorname{supp}(\lambda_2)$ is contained in $\Gamma_{\mathbf{u}_2,\mathbf{v}_2}$ (Proposition 5(1)), it follows that the support of λ_2 does not intersect $X_2 \times Y_1$ so that

$$\lambda_2(X_2 \times Y_1) = 0 \tag{A.9}$$

holds. Because λ_2 is a full match, (A.9) implies $\lambda_2(X_1 \times Y_1) = \nu(Y_1)$. Consequently, we have

$$\mu(X_1) \geq \lambda_2(X_1 \times Y_1) = \nu(Y_1), \tag{A.10}$$

where the inequality obtains because λ_2 is a match.

Next, we have

$$\lambda_1(X_1 \times Y_2) = 0. \tag{A.11}$$

To establish this, consider any $x' \in X_1$. By definition of X_1, there exists $y' \in Y_1$ such that $\mathbf{u}_2(x') = \phi(x', y', \mathbf{v}_2(y')) \geq \phi(x', y, \mathbf{v}_2(y))$, with the inequality holding for all $y \in Y$. As $\mathbf{v}_1(y') < \mathbf{v}_2(y')$ holds (because $y' \in Y_1$) and $\mathbf{v}_1$ implements $\mathbf{u}_1$, we obtain

$$\mathbf{u}_1(x') \geq \phi\bigl(x', y', \mathbf{v}_1(y')\bigr) > \phi\bigl(x', y', \mathbf{v}_2(y')\bigr) \geq \phi\bigl(x', y, \mathbf{v}_2(y)\bigr)$$

for all $y \in Y$. As $\mathbf{v}_1(y) \geq \mathbf{v}_2(y)$ holds for all $y \in Y_2$, this implies $\mathbf{u}_1(x') > \phi(x', y, \mathbf{v}_1(y))$ for all $y \in Y_2$. As $(\lambda_1, \mathbf{u}_1, \mathbf{v}_1)$ is an outcome, this implies that there does not exist $(x, y) \in X_1 \times Y_2$ contained in the support of λ_1, establishing (A.11).

Because λ_1 is a match, we have $\nu(Y_1) \geq \lambda_1(X_1 \times Y_1)$. Condition (A.11) implies $\lambda_1(X_1 \times Y_1) = \mu(X_1)$, and hence we have

$$\nu(Y_1) \geq \lambda_1(X_1 \times Y_1) = \mu(X_1). \tag{A.12}$$

Combining (A.10) and (A.12) yields

$$\lambda_1(X_1 \times Y_1) = \lambda_2(X_1 \times Y_1) = \mu(X_1) = \nu(Y_1).$$

Because λ_1 and λ_2 are matches, this in turn implies $\lambda_1(X_2 \times Y_1) = 0$ and $\lambda_2(X_1 \times Y_2) = 0$, finishing the argument for this step.

Step 3: Completion of the proof that $\mathbb{U}$ and $\mathbb{V}$ are sublattices.

By Step 1, setting

$$\lambda_3(\tilde{X} \times \tilde{Y}) = \lambda_1\bigl((\tilde{X} \cap X_1) \times (\tilde{Y} \cap Y_1)\bigr) + \lambda_2\bigl((\tilde{X} \cap X_2) \times (\tilde{Y} \cap Y_2)\bigr)$$

for all measurable $\tilde{Y} \subseteq Y$ and $\tilde{X} \subseteq X$ defines a measure on $X \times Y$. By Step 2, λ_3 is a full match. It remains to show (A.8). To obtain this, we show first that $\mathbf{u}_3(x) = \phi(x, y, \mathbf{v}_3(y))$ holds on a subset of $X \times Y$ on which λ_3 is concentrated and then use a continuity argument to extend the result to the support of λ_3.

By construction, λ_3 is concentrated on $(X_1 \times Y_1) \cup (X_2 \times Y_2)$. It is therefore also concentrated on the union of $\operatorname{supp}(\lambda_3) \cap (X_1 \times Y_1)$ and $\operatorname{supp}(\lambda_3) \cap (\mathcal{X} \times Y_2)$, where $\mathcal{X}$ is any measurable subset of X_2 satisfying $\lambda_3(\mathcal{X} \times Y_2) = \lambda_3(X_2 \times Y_2)$.

Consider $(x', y') \in \operatorname{supp}(\lambda_3) \cap (X_1 \times Y_1)$. By construction of λ_3, we then have $(x', y') \in \operatorname{supp}(\lambda_1)$, implying $\mathbf{u}_1(x') = \phi(x', y', \mathbf{v}_1(y'))$. As $y' \in Y_1$, we have $\mathbf{v}_1(y') = \mathbf{v}_3(y')$. As $x' \in X_1$, the argument that we have used to establish (A.11) in Step 2 yields $\mathbf{u}_1(x') > \mathbf{u}_2(x')$ and thus $\mathbf{u}_3(x') = \mathbf{u}_1(x')$, establishing (A.8) for the case under consideration.

Let

$$\mathcal{X} = \bigl\{x \in X_2 | \mathbf{Y}_{\mathbf{v}_1}(x) \cap Y_2 \neq \emptyset\bigr\}.$$

We show $\lambda_3(\mathcal{X} \times Y_2) = \lambda_3(X_2 \times Y_2)$ and then consider $(x', y') \in \operatorname{supp}(\lambda_3) \cap (\mathcal{X} \times Y_2)$.

An argument akin to the one used in Step 1 of the proof shows that $\mathcal{X}$ is measurable.[28] By definition of $\mathcal{X}$, $(x, y) \in (X_2 \setminus \mathcal{X}) \times Y_2$ implies $(x, y) \notin \text{supp}(\lambda_1)$, so that $\lambda_1((X_2 \setminus \mathcal{X}) \times Y_2) = 0$ holds. Because λ_1 is a full match, this in turn implies $\lambda_1((X_2 \setminus \mathcal{X}) \times Y_1) = \mu(X_2 \setminus \mathcal{X})$ with $\lambda_1(X_2 \times Y_1) = 0$ (cf. Step 2 of the proof) then implying $\mu(X_2 \setminus \mathcal{X}) = 0$, yielding $\mu(\mathcal{X}) = \mu(X_2)$. As $\lambda_3(\mathcal{X} \times Y_2) = \mu(\mathcal{X})$ and $\lambda_3(X_2 \times Y_2) = \mu(X_2)$ holds, this establishes the requisite property $\lambda_3(\mathcal{X} \times Y_2) = \lambda_3(X_2 \times Y_2)$.

By construction of λ_3, we then have $(x', y') \in \text{supp}(\lambda_2)$, implying $\mathbf{u}_2(x') = \phi(x', y', \mathbf{v}_2(y'))$. As $y' \in Y_2$, we have $\mathbf{v}_3(y') = \mathbf{v}_2(y')$, so that it remains to establish $\mathbf{u}_2(x') \geq \mathbf{u}_1(x')$ to obtain (A.8) for the case under consideration. Suppose to the contrary that $\mathbf{u}_1(x') > \mathbf{u}_2(x')$ holds. As $\mathbf{v}_2(y) \leq \mathbf{v}_1(y)$ holds on Y_2, this implies $\mathbf{u}_1(x') > \phi(x', y, \mathbf{v}_1(y))$ for all $y \in Y_2$, which contradicts $x' \in \mathcal{X}$.

Finally, consider any $(x', y') \in \text{supp}(\lambda_3)$. As λ_3 is concentrated on the union of $\text{supp}(\lambda_3) \cap (X_1 \times Y_1)$ and $\text{supp}(\lambda_3) \cap (\mathcal{X} \times Y_2)$, there exists a sequence $(x_n, y_n)_{n=1}^{\infty}$ in this union which converges to (x', y'). As shown above, $\mathbf{u}_3(x_n) = \phi(x_n, y_n, \mathbf{v}_3(y_n))$ holds for all n in this sequence. As ϕ, $\mathbf{v}_3$, and $\mathbf{u}_3$ are all continuous, the convergence of $(x_n, y_n)_{n=1}^{\infty}$ to (x', y') implies $\mathbf{u}_3(x') = \phi(x', y', \mathbf{v}_3(y'))$, which is the desired result. Hence, $\mathbb{U}$ and $\mathbb{V}$ are sublattices of $\boldsymbol{B}(X)$ and $\boldsymbol{B}(Y)$.

It remains to show that the set of pairwise stable outcomes for the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ is closed. Let $(\lambda_k, \mathbf{u}_k, \mathbf{v}_k)_{k=1}^{\infty}$ be a sequence of pairwise stable outcomes for the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ converging to $(\lambda^*, \mathbf{u}^*, \mathbf{v}^*)$. Using the assumption that $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ have full support, Proposition 5 implies that $(\mathbf{u}_k, \mathbf{v}_k)$ implement each other for all k. The same arguments as in the proof of Lemma 8 (in Appendix A.3) then imply that $(\lambda^*, \mathbf{u}^*, \mathbf{v}^*)$ is a pairwise stable outcome for $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$.

A.5. *Proof of Proposition 9*

We proceed in two steps, first establishing the existence of a solution $\mathbf{v}$ to the nonlinear pricing problem (34) and then showing that, in the associated solution $(\lambda, \Phi\mathbf{v}, \mathbf{v})$ to the principal's problem, the measure λ can be taken to be deterministic.

Step 1: We first show that we can restrict attention to a bounded set of tariffs. To simplify notation, let $\overline{\mathbf{v}} = \Psi\underline{\mathbf{u}}$ denote the upper bound for the feasible set in the nonlinear pricing problem. By Proposition 2, we have $\overline{\mathbf{v}} \in \mathbf{I}(Y)$, so that $\Pi(\overline{\mathbf{v}})$ is well-defined. To obtain a lower bound, let $v^\dagger \in \mathbb{R}$ be such that, for all $(x, y) \in X \times Y$,

$$\pi(x, y, v^\dagger) < \Pi(\overline{\mathbf{v}}). \tag{A.13}$$

The existence of such a $v^\dagger$ is ensured because π satisfies the full range condition in Assumption 3 and X and Y are compact. By Assumption 1, there also exists $\underline{v} \in \mathbb{R}$ such that, for all (x, y) in $X \times Y$ and $v \leq \underline{v}$, we have

$$\phi(x, y, v) > \max_{\hat{y} \in Y} \phi(x, \hat{y}, v^\dagger). \tag{A.14}$$

Inequality (A.14) ensures that for any tariff $\mathbf{v} \in \mathbf{I}(Y)$ with the property that $\mathbf{v}(y) \leq \underline{v}$ holds for some $y \in Y$, we have that $(\hat{x}, \hat{y}) \in \Gamma_{\Phi\mathbf{v},\mathbf{v}}$ implies $\mathbf{v}(\hat{y}) < v^\dagger$. From (A.13), this ensures that $F(\mathbf{v}, \lambda) < \Pi(\overline{\mathbf{v}})$ holds for all $\lambda \in G(\mathbf{v})$, implying that $\Pi(\mathbf{v}) < \Pi(\overline{\mathbf{v}})$ holds for any such

[28]As the complement of the open set Y_1, the set Y_2 is closed with Theorem 17.20 in Aliprantis and Border (2006) then ensuring that $\{x \in X | \mathbf{Y}_{\mathbf{v}_1}(x) \cap Y_2 \neq \emptyset\}$ is measurable. As the intersection of this set with the measurable set X_2, the set $\mathcal{X}$ is measurable.

tariff. Hence, $\Pi(\mathbf{v}) \geq \Pi(\bar{\mathbf{v}})$ implies $\mathbf{v}(y) \geq \underline{v}$ for all $y \in Y$ and there thus exists a tariff $\underline{\mathbf{v}} \in \mathbf{I}(Y)$ such that $\Pi(\mathbf{v}) \geq \Pi(\bar{\mathbf{v}})$ implies $\mathbf{v} \geq \underline{\mathbf{v}}$.

Clearly, we have $\underline{\mathbf{v}} \leq \bar{\mathbf{v}}$. Thus, the order interval $[\underline{\mathbf{v}}, \bar{\mathbf{v}}] = \{\mathbf{v} \in \mathbf{B}(Y) | \underline{\mathbf{v}} \leq \mathbf{v} \leq \bar{\mathbf{v}}\}$ is a nonempty, closed, and bounded subset of $\mathbf{B}(Y)$. As $\mathbf{I}(Y)$ is also closed (Corollary 5(1)), it follows that $\mathcal{V} = [\underline{\mathbf{v}}, \bar{\mathbf{v}}] \cap \mathbf{I}(Y)$ is a closed and bounded subset of $\mathbf{I}(Y)$. By Corollary 5(3), $\mathcal{V}$ is therefore compact. As $\bar{\mathbf{v}}$ is an element of both $\mathcal{V}$ and $\mathbf{I}(Y)$, this set is also nonempty. As Π is upper semicontinuous (Lemma 6), Weierstrass's extreme value theorem for upper semicontinuous functions (Ok (2007, p. 234)) then implies that the problem

$$\max_{\{\mathbf{v} \in \mathbf{I}(Y): \underline{\mathbf{v}} \leq \mathbf{v} \leq \bar{\mathbf{v}}\}} \Pi(\mathbf{v})$$

has a solution $\mathbf{v}^*$. We obviously have $\Pi(\mathbf{v}^*) \geq \Pi(\bar{\mathbf{v}})$ and hence $\Pi(\mathbf{v}^*) \geq \Pi(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{I}(Y)$ satisfying $\mathbf{v} \leq \bar{\mathbf{v}} = \Psi\underline{\mathbf{u}}$, ensuring that $\mathbf{v}^*$ solves the nonlinear pricing problem (34).

Step 2: Let $(\lambda, \mathbf{u}, \mathbf{v})$ be feasible in the principal's problem with $\mathbf{v} \in \mathbf{I}(Y)$. We first observe that $\max_{y \in \mathbf{Y}_\mathbf{v}(x)} \pi(x, y, \mathbf{v}(y))$ is a measurable function of x and that there exists a measurable assignment $\mathbf{y}^*$ solving this maximization problem for all x. This follows from Aliprantis and Border (2006, Theorem 18.19) upon observing that (i) the function $(x, y) \to \pi(x, y, \mathbf{v}(y))$ is continuous on its domain $X \times Y$ (from Proposition 2 and Assumption 3) and thus a Caratheodory function and (ii) the properties of the correspondence $\mathbf{Y}_\mathbf{v}$ noted in Corollary 2 imply that this correspondence has a closed graph, ensuring that it is weakly measurable (Aliprantis and Border (2006, Theorem 18.20 and Lemma 18.2)).

We can then write

$$\begin{aligned} F(\mathbf{v}, \lambda) &= \int_X \int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda(x, y) \\ &= \int_X \left(\int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda(y|x) \right) d\mu(x) \\ &\leq \int_X \max_{y \in \mathbf{Y}_\mathbf{v}(x)} \pi\big(x, y, \mathbf{v}(y)\big)\, d\mu(x) \\ &= \int_X \pi\big(x, \mathbf{y}^*(x), \mathbf{v}\big(\mathbf{y}^*(x)\big)\big)\, d\mu(x) \\ &= F(\mathbf{v}, \lambda_{\mathbf{y}^*}), \end{aligned}$$

where the equality in the second line follows from the disintegration theorem (Chang and Pollard (1997, Theorem 1)), with $\lambda(\cdot|x)$ being the disintegration measure on $\{x\} \times Y$ for each $x \in X$. The inequality holds because the support of $\lambda(\cdot|x)$ is contained in $\mathbf{Y}_\mathbf{v}(x)$ for μ-almost all $x \in X$. The equality on the penultimate line is by definition of $\mathbf{y}^*$. As $(\lambda_{\mathbf{y}^*}, \mathbf{u}, \mathbf{v})$ is feasible in the principal's problem and this problem has a solution, the inequality $F(\mathbf{v}, \lambda) \leq F(\mathbf{v}, \lambda_{\mathbf{y}^*})$ implies that the principal's problem has a deterministic solution.

A.6. *Proof of Proposition 10*

Suppose $(\lambda, \mathbf{u}, \mathbf{v})$ solves the principal's problem with $\mathbf{u}(x) > \underline{\mathbf{u}}(x)$ for all $x \in X$. From Proposition 9, there exists a deterministic match $\lambda_\mathbf{y}$, such that $(\lambda_\mathbf{y}, \mathbf{u}, \mathbf{v})$ is also a solution to the principal's problem. By the same argument as the one proving Lemma 5, we can take $\mathbf{y}$ to be implementable and therefore (by assumption) to be strongly implementable. From Lemma 4, there thus exists a profile $\mathbf{u}^*$ such that $(\mathbf{u}^*, \mathbf{y})$ is implementable, $\mathbf{u} \geq \mathbf{u}^* \geq \underline{\mathbf{u}}$ holds, and there exists $x \in X$ such that $\mathbf{u}(x) > \mathbf{u}^*(x)$ for some $x \in X$. As both $\mathbf{u}$ and $\mathbf{u}^*$ are implementable (and therefore continuous by Proposition 2), the set $\mathcal{X} = \{x \in X | u(x) > u^*(x)\}$ is measurable. Because μ has full support, we have $\mu(\mathcal{X}) > 0$.

Now, let $\mathbf{v}^* = \Psi \mathbf{u}^*$. Then $\mathbf{v}^*$ implements $(\mathbf{u}^*, \mathbf{y})$ (Corollaries 3(1) and 4(1)) and the triple $(\lambda_{\mathbf{y}}, \mathbf{u}^*, \mathbf{v}^*)$ is therefore feasible in the principal's problem. We also have that the principal obtains a strictly higher expected payoff from $(\lambda_{\mathbf{y}}, \mathbf{u}^*, \mathbf{v}^*)$ than from $(\lambda_{\mathbf{y}}, \mathbf{u}, \mathbf{v})$, contradicting the optimality of $(\lambda_{\mathbf{y}}, \mathbf{u}, \mathbf{v})$:

$$\begin{aligned}
&\int_X \int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda_{\mathbf{y}}(x, y) \\
&\quad = \int_X \int_Y \pi\big(x, y, \psi\big(y, x, \mathbf{u}(x)\big)\big)\, d\lambda_{\mathbf{y}}(x, y) \\
&\quad < \int_X \int_Y \pi\big(x, y, \psi\big(y, x, \mathbf{u}^*(x)\big)\big)\, d\lambda_{\mathbf{y}}(x, y) \\
&\quad = \int_X \int_Y \pi\big(x, y, \mathbf{v}^*(y)\big)\, d\lambda_{\mathbf{y}}(x, y),
\end{aligned}$$

where the equalities follow as in (30) and the strict inequality holds because $\mu(\mathcal{X}) > 0$, ψ is strictly decreasing in its third argument, and π is strictly increasing in its third argument.

A.7. *Proof of Proposition 11*

Suppose $(\lambda, \mathbf{u}, \mathbf{v})$ is a solution to the principal's problem with $\mathbf{u}(x) > \underline{\mathbf{u}}(x)$ for all $x \in X$. Then, as we have noted in Remark 7, $(\lambda, \mathbf{u}, \mathbf{v})$ is a pairwise stable outcome of the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$, where ν is the marginal measure λ_Y of λ on Y and $\underline{\mathbf{v}} : Y \to \mathbb{R}$ is an arbitrary continuous function. Let $\mathcal{Y}$ be the support of ν. It exposes the logic of the argument most clearly by first proceeding under the assumption that $\mathcal{Y} = Y$.

The assumption $\mathcal{Y} = Y$ ensures that the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ satisfies Assumption 2, so that this matching model has a pairwise stable outcome $(\hat{\lambda}, \hat{\mathbf{u}}, \hat{\mathbf{v}})$ satisfying $\mathbf{u} \geq \hat{\mathbf{u}} \geq \underline{\mathbf{u}}$, with the first inequality holding strictly for some $x \in X$ (Corollary 8). Because $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ implement each other (Proposition 5(3)) and the implementation maps are order-reversing inverse bijections (cf. (15)), we thus obtain $\mathbf{v} \leq \hat{\mathbf{v}}$ with strict inequality for some $y \in Y$. From the continuity of the two profiles $\mathbf{v}$ and $\hat{\mathbf{v}}$ (Proposition 2) and the assumption that ν has full support, we thus obtain

$$\nu\big(\{y : \mathbf{v}(y) < \hat{\mathbf{v}}(y)\}\big) > 0. \tag{A.15}$$

We can now write

$$\begin{aligned}
F(\mathbf{v}, \lambda) &= \int_X \int_Y \pi\big(x, y, \mathbf{v}(y)\big)\, d\lambda(x, y) \\
&= \int_Y \hat{\pi}\big(y, \mathbf{v}(y)\big)\, d\nu(y) \\
&< \int_Y \hat{\pi}\big(y, \hat{\mathbf{v}}(y)\big)\, d\nu(y)
\end{aligned}$$

$$= \int_X \int_Y \pi(x, y, \hat{\mathbf{v}}(y))\, d\hat{\lambda}(x, y)$$
$$= F(\hat{\mathbf{v}}, \hat{\lambda}),$$

where the two inner equalities are from the private-values assumption and the inequality follows from (A.15) because $\hat{\pi}$ is strictly increasing in its second argument (Assumption 3). We thus obtain $F(\mathbf{v}, \lambda) < F(\hat{\mathbf{v}}, \hat{\lambda})$. As $(\hat{\lambda}, \hat{\mathbf{u}}, \hat{\mathbf{v}})$ is feasible in the principal's problem, this contradicts the optimality of $(\lambda, \mathbf{u}, \mathbf{v})$.

If $\mathcal{Y}$ is a strict subset of Y, then the above argument is not directly applicable because the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ violates the full support condition in Assumption 2. It is, however, straightforward to establish a "restriction lemma" (similar in spirit to the extension result of Proposition 5(3)) showing that if $(\lambda, \mathbf{u}, \mathbf{v})$ is a pairwise stable outcome of the matching model $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$, then $(\lambda, \mathbf{u}, \mathbf{v})$ can be restricted to give a pairwise stable outcome of the matching model derived from $(X, Y, \phi, \mu, \nu, \underline{\mathbf{u}}, \underline{\mathbf{v}})$ by restricting the sets X and Y to the supports $\mathcal{X}$ and $\mathcal{Y}$ of μ and ν. This latter model satisfies Assumption 2, allowing us to repeat the argument above (and in particular to apply Corollary 8). The conclusion of this argument is that the principal can secure a higher payoff than under $(\boldsymbol{\lambda}, \mathbf{u}, \mathbf{v})$, even if restricted to assigning only decisions in $\mathcal{Y}$ to the agents.

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