MAT244 Homework 5

due: November 25, 2022

1 (5 points). Which first-order homogeneous system of linear differential equations has fundamental solutions

$$u(t) = \begin{pmatrix} t \\ 1 \end{pmatrix} \quad \text{and} \quad v(t) = \begin{pmatrix} t^2 \\ 3t \end{pmatrix}?$$

2 (5 points). Let A and B be matrices whose entries are differentiable functions of the same real variable. Suppose the number of columns in A is equal to the number of rows in B. Show that

$$(AB)' = A'B + AB'.$$

3 (10 points). Consider the matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(a) Show that each matrix is singular.

(b) For each matrix, what are the algebraic and geometric multiplicities of 0?

(c) For each matrix A, find the general solution to

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

(d) Draw all three phase portraits. What do you notice?

4 (12 points). Find the general solution of

$$\begin{aligned} tx' &= x - 2y \\ ty' &= 3x + y \end{aligned} \qquad (t > 0)$$

by guessing solutions of the form vt^p for some number p and vector v. What is the Wronskian?

5 (10 points). Consider the family of systems

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & -1 \\ \alpha & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

Classify all solutions by their asymptotic behaviour in terms of α, with reference to stability and the nature of equilibria.