MAT244 Homework 3

due: October 21, 2022

1 (10 points)**.** Consider the equation

$$x^2y'' + 3xy' + y = 0.$$

(a) What is the largest interval I containing $x = 1$ on which a solution is guaranteed to exist?

(b) Find all numbers p such that $y_1 = x^p$ is a solution on I.

(c) Find a solution y_2 satisfying $y_2(1) = 0$ and $y_2'(1) = 1$.

(d) What is the Wronskian of y_1 and y_2?

2 (10 points)**.** Let p and q be continuous functions on an interval I, and let u and v be a pair of fundamental solutions to the second-order linear homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

(a) Show that u and v have no common roots.

(b) Let $a < b$ be two roots of u. Show that v has a root in (a, b). [*Hint*: Suppose not, and apply the Mean Value Theorem to u/v.]

(c) If v has 3 roots, how many roots can u have?

3 (10 points)**.** Let C^∞ be the vector space of all smooth (i.e., infinitely differentiable) real-valued functions on $\mathbb{R}$. Define $L : C^\infty \to C^\infty$ by $L[\phi](x) = x\phi'(x)$. Show that L is a linear map and find its eigenvalues. [*Hint*: Don't forget to check for smoothness!]

4 (10 points)**.** A ball of mass m is attached to the end of a rusty spring, drawn back, and released from rest. Assume the restoring force is proportional to the stretch of the spring and the damping force is proportional to the speed of the ball. Let x be the position of the ball (relative to equilibrium) at time t.

(a) Show that $mx'' = -kx - \gamma x'$ for some positive constants k and γ.

(b) What are the roots of the characteristic equation?

(c) Suppose $\gamma < 2\sqrt{mk}$. What is the solution with $x(0) = 1$ and $x'(0) = 0$?

(d) Show that the successive maxima form a geometric progression.