MAT244 Homework 2

due: October 7, 2022

1 (5 points)**.** Consider the equation $y'' = -y$. Let s be a solution with $s(0) = 0$ and $s'(0) = 1$, and let $c = s'$. Without referring to trigonometry, show that $s^2 + c^2 = 1$.

2 (5 points)**.** Recall that the determinant of a 3-by-3 matrix is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg.$$

(a) If $a, b, c, d, e, f, g, h, i$ are differentiable functions of the same variable, show that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}' = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix}.$$

(b) Suppose u, v, and w satisfy the equation $y''' = py'' + qy' + ry$ and let $x = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$. Using the above, or otherwise, show that $x' = px$.

3 (5 points)**.** For which p and q is $x^p y^q$ an integrating factor for the equation

$$y^2 dx + (xy - x^3) dy = 0?$$

4 (10 points)**.** Consider the equation

$$(x^2 + y^2 + y)dx - xdy = 0. \tag{1}$$

(a) Show that (1) is not exact, but becomes exact upon dividing by $x^2 + y^2$.

(b) Solve (1) implicitly. That is, find U such that the level curves $U(x, y) = c$ are solutions to (1).

(c) Solve (1) explicitly. That is, isolate y by rearranging your answer in (b).

(d) Check that your solution in (c) satisfies $xy' = x^2 + y^2 + y$.

5 (12 points)**.** Torricelli's principle says that the speed at which fluid exits a punctured vessel is proportional to the square root of the distance between the surface and the hole. Suppose a hemispherical bowl is filled to the brim with 21 L of Five Alive™ and then pierced at the bottom by a 1 cm thick rusty nail. Let V and y be the volume and depth, respectively, of the juice in the bowl at time t.

(a) Find a formula for dV/dy using slices.

(b) Find a formula for dV/dt using Torricelli's principle.

(c) Using (a) and (b), write a first-order differential equation for y in terms of t.

(d) Assuming the constant of proportionality is 4.4 $\mathrm{m}^{1/2}\mathrm{s}^{-1}$, when will the juice run out? Round your answer to the nearest minute.