MAT244 Homework 1

due: September 23, 2022

1 (5 points)**.** Solutions to DEs come in all sorts of shapes and sizes. Show that if f is a continuous function, then

$$y = \sin x \int_0^x f(t) \cos t \, dt - \cos x \int_0^x f(t) \sin t \, dt$$

satisfies $y'' + y = f(x)$.

2 (5 points)**.** Recall that the determinant of a 2-by-2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(a) If a, b, c, d are differentiable functions of the same variable, show that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}' = \begin{vmatrix} a' & b' \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix}.$$

(b) Suppose u and v satisfy the equation $y'' = py' + qy$ and let $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$. Using the above, or otherwise, show that $w' = pw$.

3 (10 points)**.** Let f be a positive continuous function and let F be an antiderivative of $1/f$.

(a) Explain why F is invertible.

(b) Let F^{-1} be the inverse function of F and let c be an arbitrary constant. Show that $y = F^{-1}(x + c)$ satisfies $y' = f(y)$.

(c) Using the above, or otherwise, solve the following initial value problems:

(i) $y' = 1/y$ with $y(0) = 1$.

(ii) $y' = y + 1$ with $y(0) = 2$.

(iii) $y' = e^y$ with $y(0) = 3$.

4 (10 points)**.** Let N be the number of atoms in a radioactive substance. By the universal law of radioactive decay, N decreases over time at a rate proportional to N.

(a) Write a first-order linear ODE for N, denoting the (positive!) constant of proportionality by λ.

(b) Solve your ODE in terms of $N_0 = N(0)$.

(c) The half-life of a radioactive substance is the time it takes for half of it to decay. What is the product of the half-life $t_{1/2}$ and the decay constant λ?

(d) 0.0117% of all naturally occurring potassium is radioactive with a half-life of 1.25 billion years. Given that a standard banana contains 358 mg of potassium, if you kept one in your freezer til the sun burns out (about 5 billion years from now), how much potassium would be lost? Answer in µg.

5 (12 points)**.** A spherical raindrop evaporates at a rate proportional to its surface area.

(a) Write a first-order nonlinear ODE for the volume V of the raindrop as a function of time.

(b) Solve your ODE in terms of $V(0) = V_0$.

(c) Check that your answer to part (b) satisfies your equation in part (a).

(d) How does your solution behave as $t \to \infty$? Is this realistic?