1. Exercises from 3.2

In this tutorial we'll study different ways of representing smooth curves. There are three principle ways to do this. For curves in $\mathbb{R}^2$, they are as follows:

(1) Explicitly, as the graph of a function $y = f(x)$
(2) Explicitly by a parameterization $t \mapsto (f_1(t), f_2(t))$
(3) Implicitly, by the vanishing of a function $S = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$

The implicit function theorem implies the *local* equivalence of these statements.

We say that a curve is *smooth* if every point has a neighbourhood on which the curve is the graph of a differentiable function. There are two obvious ways a curve can fail to be smooth: (1) It can intersect itself, or (2) it can have a cusp.

Example of a smooth curve: Let $S = \{(x, y) \mid F(x, y) = y - x^2 = 0\}$. We can also think of S as the graph of the map $f : x \mapsto x^2$, or alternatively, as the image of the curve $\gamma(t) : (-\infty, \infty) \to \mathbb{R}^2$, $\gamma(t) = (t, t^2)$. This curve is smooth almost by definition, since it is the map of $y = f(x) = x^2$, a differentiable map.

Example of a non-smooth curve: Let $S = \{(x, y) \mid x^3 - y^2 = 0\}$, then we can define S piecewise as a curve by:

$$\gamma(t) = (t^2, t^3)\ t \in (-\infty, \infty)$$

Alternatively, we can think of S as the graph of the function $f(y) = y^{2/3}$. Notice though that this is not differentiable at the origin, since $f'(y) = (2/3)y^{-2/3}$ is not defined at $y = 0$. This shows that S is not a smooth curve.

Problem 1. *Let $F(x, y) = xy(x+y-1)$, and set $S = \{(x, y) \mid F(x, y) = 0\}$. Sketch S. Is S smooth? Near which points is S the graph of a function $y = f(x)$, or $x = f(y)$?*

- $F(x, y) = 0$ if and only if $x = 0$, or $y = 0$, or $y = 1 - x$.
- (Draw S).
- Thm. 3.11 says that if $a \in S$ and $\nabla F(a) \neq 0$, then S is the graph of a C^1 function in a neighbourhood of a. Taking the contrapositive, if we want to find possible points where the curve S is not smooth, then we should look for points in S such that $\nabla F(a) = 0$.

$$\nabla F = \begin{pmatrix} y(2x + y - 1 + xy) \\ x(2y + x - 1 + xy) \end{pmatrix}$$

- Case 1: $x = 0$ and $y = 0$.
- Case 2: $y = 0$ and $x \neq 0$, then $2y + x - 1 + xy = x - 1 = 0$ implies $x = 1$.
- Case 3: $x = 0$ and $y \neq 0$, then $2x + y - 1 + xy = y - 1 = 0$ implies $y = 1$.
- Case 4: $x \neq 0$ and $y \neq 0$, then $2y + x - 1 + xy = 0$ and $2x + y - 1 + xy = 0$. Subtracting the second from the first gives $y = x$. Now we need $x^2 + 3x - 1 = 0$, which can be solved to give $y_0 = x_0 = (-3 \pm \sqrt{13})/2$. However, $F(x_0, y_0) \neq 0$ so this point is not in S.
- Near each of the points where $\nabla F = 0$, S is a union of two lines; therefore S could not be the graph of a single-valued function near any of these points.
- We have found the points of S such that $\nabla F = 0$, so by thm. 3.11 we know that S can be represented by the graph of a function near every point except $(0, 0)$, $(0, 1)$, and $(1, 0)$.

Problem 2. *Let $\gamma(t) = (t^3 - 1, t^3 + 1)$. Is $\gamma(t)$ a smooth curve? Sketch the curve. Examine S near any points where $\gamma'(t) = 0$.*

- If we take $x = \gamma_1(t)$, $y = \gamma_2(t)$, then $x - y + 2 = 0$.
- Define $F(x, y) = x - y + 2$, then $\nabla F(x, y) = (1, -1) \neq 0$ so the curve $\gamma(t)$ must be smooth

- (Sketch the plane)
- Notice that $\gamma'(t) = (3t^2, 3t^2)$ which has a zero at $t = 0$, however, the curve is still smooth at the point $(-1, 1)$.