Representation theory, geometric Langlands duality and categorification

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October 21, 2014

Abstract

The representation theory of reductive groups, such as the group GL_n of invertible complex matrices, is an important topic, with applications to number theory, algebraic geometry, mathematical physics, and quantum topology. One way to study this representation theory is through the geometric Satake correspondence (also known as geometric Langlands duality). This correspondence relates the geometry of spaces called affine Grassmannians with the representation theory of reductive groups. This correspondence was originally developed from the viewpoint of the geometric Langlands program, but it has many other interesting applications. For example, this correspondence can be used to construct knot homology theories in the framework of categorification.

In these lectures, we will begin by explaining the representation theory of GL_n, beginning with classification of irreducible representations. We will also give a presentation of the category of representations; such a presentation is known as "skein" or "spider" theory. We will also discuss quantum GL_n and see how it can be used to define knot invariants. We will then define the affine Grassmannian for GL_n and explain the geometric Satake correspondence from the perspective of skein theory. We will conclude by explaining how these ideas, along with the theory of derived categories of coherent sheaves, can be used to construct knot homology theory.

1 Representation theory of GL_n

We will begin by reviewing the representation theory of the group GL_n of invertible complex matrices. Good references are [FH] and [GW].

1.1 Preliminaries and examples

Definition 1.1. An **representation** of GL_n is a pair (V, ρ) where V is a finite-dimensional complex vector space V and $\rho : GL_n \to GL(V)$ is a group homomorphism which is also a morphism of algebraic varieties. Saying that ρ is a morphism of algebraic varieties means the following. Given any $v \in V$ and $\alpha \in V^*$, we can define a map

$$\rho_{v,\alpha} : GL_n \to \mathbb{C}, \quad \rho_{v,\alpha}(g) = \alpha(\rho(g)(v)).$$

ρ is said to be a morphism of algebraic varieties if, for all v, α, $\rho_{v,\alpha}$ is a rational function in the entries of g. (Equivalently, if we choose a basis for V, we get a matrix $[\rho(g)]$ and we require that the entries of this matrix be rational functions in the entries of g.)

We will often drop ρ from the notation and we will write $\rho(g)(v)$ as $g \cdot v$.

Example 1.2. The simplest representation of GL_n is the "standard representation" which is the action of GL_n on $\mathbb{C}^n$ (so $V = \mathbb{C}^n$ and the map $GL_n \to GL(V)$ is the identity).

Example 1.3. More generally, we can also consider actions of GL_n on vector spaces built out of $\mathbb{C}^n$. The first examples are the actions of GL_n on the kth symmetric powers, $\mathrm{Sym}^k\mathbb{C}^n$, and kth exterior powers, $\bigwedge^k\mathbb{C}^n$. The action of GL_n on $\mathrm{Sym}^k\mathbb{C}^n$ is given by

$$g \cdot v_1 \cdots v_k = (g \cdot v_1) \cdots (g \cdot v_k)$$

and similarly for $\bigwedge^k\mathbb{C}^n$.

Exercise 1.4. Identify $\mathrm{Sym}^k\mathbb{C}^n$ with the vector space of homogeneous polynomials of degree k in n variables. Consider the case $n = 2$. Describe the action of an invertible matrix $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ on a homogeneous polynomial $p(x, y)$. Using this description, write the matrix for the action of g on $\mathrm{Sym}^n\mathbb{C}^2$.

Example 1.5. As a special case of the previous example, we may consider $V = \bigwedge^n\mathbb{C}^n$. This is a 1-dimensional vector space. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{C}^n$. Then $e_1 \wedge \cdots \wedge e_n$ is a basis for $\bigwedge^n\mathbb{C}^n$.

With respect to this basis, the action of GL_n becomes a group homomorphism $GL_n \to GL_1 = \mathbb{C}^\times$. This homomorphism is the map taking a matrix to its determinant. Hence we will refer to $\bigwedge^n\mathbb{C}^n$ as the **determinant representation**.

The representations $\bigwedge^k\mathbb{C}^n$, for $k = 1, \dots, n-1$ are usually called the **fundamental representations** of GL_n. For the purposes of this paper, we will include the determinant representation $\bigwedge^n\mathbb{C}^n$ as a fundamental representation as well.

Example 1.6. Take $n = 1$ and take $V = \mathbb{C}$. Consider the map $\rho : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $\rho(z) = \overline{z}$ (complex conjugation). This map is a group homomorphism, but it is not algebraic. So this is not a representation that we will study.

Remark 1.7. There is a close connection between the representation theory of GL_n and the smooth representation theory of the subgroup $U(n)$ of unitary matrices. If we have a representation (V, ρ) of GL_n, we can restrict it to $U(n)$ and we get a continuous representation of $U(n)$. On the other hand, if we have a continuous representation of $U(n)$ on a finite-dimensional complex vector space V, then it extends uniquely to a representation of GL_n on the the same vector space.

Similarly, there is a close connection with integral weight representations of the Lie algebra $\mathfrak{gl}_n$.

Definition 1.8. If V, W are representations of GL_n, then their **direct sum** $V \oplus W$ is naturally a representation of GL_n, with $g \cdot (v, w) := (g \cdot v, g \cdot w)$. Similarly, their **tensor product** $V \otimes W$ is also naturally a representation, with $g \cdot v \otimes w = g \cdot v \otimes g \cdot w$.

1.2 Representations of tori

Our goal now is to classify all representations of GL_n.

Let us begin with the case of $GL_1 = \mathbb{C}^\times$. A one dimensional representation of $\mathbb{C}^\times$ is given by a (algebraic) group homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$.

Proposition 1.9. *Every algebraic group homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$ is given by $z \mapsto z^n$ for some integer n.*

Proof. Suppose that $\rho : \mathbb{C}^\times \to \mathbb{C}^\times$ is an algebraic group homomorphism. Then since $\rho(z)$ is a rational function and since it is well-defined for any $z \in \mathbb{C}^\times$, we see that $\rho(z) \in \mathbb{C}[z, z^{-1}]$. The condition that $\rho(z)$ is a group homomorphism forces $\rho(z) = z^n$ for some $n \in \mathbb{Z}$. □

More generally, let (V, ρ) be a representation of $\mathbb{C}^\times$.

Definition 1.10. A vector $v \in V$ is said to have **weight** n if $\rho(z)v = z^n v$ for all $z \in \mathbb{C}^\times$. We write

$$V_n = \{v \in V : \rho(z)(v) = z^n v, \text{ for all } z \in \mathbb{C}^\times\}$$

for the subspace of vectors of weight n.

Theorem 1.11. *Let V a representation of $\mathbb{C}^\times$. Then we have a direct sum decomposition $V = \oplus_{n\in\mathbb{Z}} V_n$.*

We will not give a proof of this result. One way to prove it is to appeal to $U(1) \subset \mathbb{C}^\times$ and to construct a $U(1)$-invariant inner product on V. Another approach is to use that morphisms of algebraic groups take semisimple elements to semisimple elements.

In this theorem, we have written V as a direct sum of eigenspaces for the elements of $\mathbb{C}^\times$. This theorem captures two important facts about the representation theory of $\mathbb{C}^\times$. First, all representations are semisimple; meaning that they can be written as a direct sum of irreducible subrepresentations. Second, all irreducible representations of $\mathbb{C}^\times$ are 1-dimensional (this holds since $\mathbb{C}^\times$ is abelian).

More generally, let us consider a group $T = (\mathbb{C}^\times)^n$. Such a group is called a **torus**. As before, we begin by studying the one dimensional representations of this group. These are all of the form

$$z = (z_1, \dots, z_n) \mapsto \mu(z) := z_1^{\mu_1} \cdots z_n^{\mu_n}$$

for some sequence of integers $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$.

Definition 1.12. A homomorphism $\mu : T \to \mathbb{C}^\times$ as above is called a **weight** of T. The set of all weights $P = \mathbb{Z}^n$ is called the **weight lattice** of T.

Now let V be an arbitrary representation of T. For any weight μ, we can consider the subspace of vectors of weight μ

$$V_\mu := \{v \in V : z \cdot v = \mu(z)v\}$$

The following result generalizes Theorem 1.11.

Proposition 1.13. *We have a direct sum decomposition $V = \oplus_{\mu\in\mathbb{Z}^n} V_\mu$.*

1.3 Weight spaces of representations of GL_n

Within GL_n, there is a large torus $T = (\mathbb{C}^\times)^n$ consisting of the invertible diagonal matrices. We will study representations of GL_n by restricting them to representations of this maximal torus.

Let (V, ρ) be a representation of GL_n. Then it is also a representation of the maximal torus T and so we get a weight decomposition $V = \oplus_\mu V_\mu$. We can record some numerical information about the representation by recording the dimensions of these weight spaces.

Definition 1.14. The formal expression

$$\chi_V := \sum_{\mu \in \mathbb{Z}^n} \dim(V_\mu)\mu \in \mathbb{Z}[P].$$

is called the **character** of the representation V.

Note that if $g \in (\mathbb{C}^\times)^n$ is any diagonal element of GL_n,, then we have an equality

$$\chi_V(g) = \sum_{\mu \in \mathbb{Z}^n} \dim(V_\mu)\mu(g) = tr(\rho(g))$$

Thus the character χ_V determines the trace of the diagonalizable elements of GL_n acting on V. Since the diagonalizable elements of GL_n are dense in GL_n, the character χ_V determines the trace of all elements of GL_n acting on V. So there is really a lot of information in χ_V.

Here is the first fundamental result about the representation theory of GL_n.

Theorem 1.15. *(i) Consider the action of the symmetric group S_n on $\mathbb{Z}^n$ given by*

$$w(\mu_1, \dots, \mu_n) = (\mu_{w(1)}, \dots, \mu_{w(n)}).$$

For any representation V, we have $\dim V_\mu = \dim V_{w\mu}$ for any $w \in S_n$.

(ii) If V, W are two representations and $\chi_V = \chi_W$, then $V \cong W$.

Thus the character completely determines the representation. We will not prove this result. (One approach to (ii) is to restrict to $U(n)$ and use results about the character of compact groups.)

Exercise 1.16. Prove (i). Hint: consider the inclusion $S_n \hookrightarrow GL_n$ which takes permutations to permutation matrices.

Example 1.17. Consider the representation $\mathrm{Sym}^3\mathbb{C}^2$. It has a basis given by

$$e_1^3, e_1^2 e_2, e_1 e_2^2, e_2^3.$$

Note that if $z = \left[\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right]$, then

$$z \cdot e_1^a e_2^b = (z_1 e_1)^a (z_2 e_2)^b = z_1^a z_2^b e_1^a e_2^b$$

and hence $e_1^a e_2^b$ has weight (a, b). Hence the non-zero weight spaces of $\mathrm{Sym}^3\mathbb{C}^2$ are $(3,0), (2,1), (1,2), (0,3)$ and each weight space has dimension 1.

Example 1.18. More generally consider the representation $\mathrm{Sym}^k\mathbb{C}^n$. It has a basis given by monomials $e_1^{a_1}\dots e_n^{a_n}$ of total degree k. Such a monomial has weight $(a_1,\dots,a_n)$. Hence μ is a weight if and only if $(\mu_1+\cdots+\mu_n)=k$. and all weight spaces have dimension 1.

Exercise 1.19. Find the character of the representation $\bigwedge^k\mathbb{C}^n$.

Definition 1.20. Let $R(GL_n)$ denote the **representation ring** of GL_n. This is the abelian group generated by isomorphisms classes $[V]$ of representations of GL_n, modulo the relation $[V\oplus W]=[V]+[W]$. The multiplication in this ring is given by $[V][W]:=[V\otimes W]$.

The map $\chi: R(GL_n)\to\mathbb{Z}[P]$ given by $[V]\mapsto\chi_V$ is easily seen to be a ring homomorphism (the multiplication on $\mathbb{Z}[P]$ is defined using addition in P). Theorem 1.15 shows that the χ is injective and that its image lands in the subspace $\mathbb{Z}[P]^{S_n}$ of invariants for the symmetric group.

1.4 Irreducible representations of GL_n

Definition 1.21. A **subrepresentation** $W\subset V$ of a representation of GL_n is a subspace of $W\subset V$ which is invariant under the action of GL_n.

A representation V of GL_n is called **irreducible** if it has no subresentations (other than $0,V$).

Let V,W be representations of GL_n. A **morphism of representations** $A:V\to W$ is a linear map such that $A(g\cdot v)=g\cdot A(v)$ for all $g\in GL_n, v\in V$. The set of all morphisms from V to W is denoted $\mathrm{Hom}_{GL_n}(V,W)$.

The following basic result is called Schur's Lemma. Its proof is straightforward.

Theorem 1.22. *Suppose that V,W are irreducible representations. Then $\mathrm{Hom}_{GL_n}(V,W)$ is 1-dimensional if $V\cong W$ and is 0-dimensional otherwise.*

We also have the following semisimplicity result.

Theorem 1.23. *Every representation V can be written as a direct sum of irreducible subresentations.*

There are a number of proofs of this theorem. One approach is to construct a $U(n)$-invariant inner product on V.

Example 1.24. Consider the representation of GL_n on $\mathbb{C}^n\otimes\mathbb{C}^n$. We have a direct sum decomposition into subrepresentations $\mathbb{C}^n\otimes\mathbb{C}^n=\bigwedge^2\mathbb{C}^n\oplus\mathrm{Sym}^2\mathbb{C}^n$. Later we will see that $\bigwedge^2\mathbb{C}^n$ and $\mathrm{Sym}^2\mathbb{C}^n$ are irreducible representations.

Now we would like to describe the irreducible representations of GL_n. Before we do this, we will need a few definitions.

Definition 1.25. The **positive roots** of GL_n are the elements of $\mathbb{Z}^n$ of the form

$$\beta = (0, \dots, 0, 1, 0 \dots, 0, -1, 0, \dots, 0)$$

where the 1 occurs before the -1.

We say $\lambda \geq \mu$ if we can write

$$\lambda - \mu = \sum_{\beta} k_\beta \beta$$

for some non-negative integers k_β, where the sum varies over all positive roots.

A weight $\mu = (\mu_1, \dots, \mu_n)$ is called a **dominant weight** if $\mu_1 \geq \cdots \geq \mu_n$ or equivalently if $\langle \mu, \beta \rangle \geq 0$ for all positive roots β. We write P_+ for the set of dominant weights.

A representation V is said to have **highest weight** λ, if $V_\lambda \neq 0$ and whenever $V_\mu \neq 0$, then $\mu \leq \lambda$.

Note that there is a unique dominant weight in every orbit of S_n on the weight lattice $P = \mathbb{Z}^n$. Also note that because the set of weights of a representation is invariant under S_n, if λ is the highest weight of a representation, then λ must be dominant.

Example 1.26. Consider $\mathrm{Sym}^3\mathbb{C}^2$. As calculated in example 1.17, the weights are $(3, 0), (2, 1), (1, 2), (0, 3)$. Hence this representation has highest weight $(3, 0)$.

Here is the final main theorem concerning the representation theory of GL_n.

Theorem 1.27. *For each $\lambda \in P_+$, there exists a unique representation $V(\lambda)$ which is irreducible and of highest weight λ.*

Proof. Let us prove the existence. Since λ is dominant, we can write $\lambda = \sum_{k=1}^n m_k \omega_k$, with $m_k \geq 0$ for $k < n$.

Consider the tensor product

$$W = \bigotimes_{k=1}^{n} (\bigwedge^k \mathbb{C}^n)^{\otimes m_k}.$$

We can see that W is of highest weight λ, as the sum of the highest weights of each factor is λ.

We can write $W = W_1 \oplus \cdots \oplus W_p$ for some irreducible subrepresentations $W_1, \dots, W_p$. At least one of these must have a non-zero weight space for λ and this one will be an irreducible representation of highest weight λ.

□

Exercise 1.28. Use the injectivity of the character map $R(GL_n) \to \mathbb{Z}[P]^{S_n}$ to prove the uniqueness of $V(\lambda)$.

Example 1.29. Take $\lambda = (k, 0, \dots, 0)$. Then $V(\lambda) = \mathrm{Sym}^k\mathbb{C}^n$.

Exercise 1.30. Prove that $\bigwedge^k \mathbb{C}^n$ is irreducible and is of highest weight

$$\omega_k = (1, \ldots, 1, 0, \ldots, 0)$$

(where there are k 1s). Conclude that $\bigwedge^k \mathbb{C}^n = V(\omega_k)$.

Example 1.31. Consider $\mathfrak{sl}_3$ the vector space of 3×3 trace 0 matrices. We have an action of GL_3 on this space by conjugation. This is the irreducible representation $V(1, 0, -1)$. In particular

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has weight $(1, 0, -1)$.

The weights of this representation are

$$(1, 0, -1), (0, 1, -1), (1, -1, 0), (0, 0, 0), (0, -1, 1), (-1, 1, 0), (-1, 0, 1).$$

All of these weight spaces are 1-dimensional with the exception of the $(0, 0, 0)$ weight space which consists of the diagonal matrices and hence is 2-dimensional.

1.5 Determinant representation and polynomial representations

The determinant representation described above is the irreducible representation of highest weight $(1, \ldots, 1)$. Its dual representation has highest weight $(-1, \ldots, -1)$. In this representation, we have $\rho(g) = \det(g)^{-1}$. Tensoring with these representations moves us around the set of dominant weights in the sense that

$$V(1, \ldots, 1) \otimes V(\lambda) \cong V(\lambda_1 + 1, \ldots, \lambda_n + 1)$$
$$V(-1, \ldots, -1) \otimes V(\lambda) \cong V(\lambda_1 - 1, \ldots, \lambda_n - 1).$$

Let P_{++} be the set of dominant weights λ such that $\lambda_n \geq 0$ for all i.. We say that an irreducible representation $V(\lambda)$ is **polynomial**, if $\lambda \in P_{++}$. More generally, a representation of GL_n is called polynomial if it is the direct sum of polynomial irreducible representations.

Starting with polynomial irreducible representations, we can get to all irreducible representations by repeatedly tensoring with $V(-1, \ldots, -1)$. For this reason it is enough to consider just these polynomial irreducible representations which we will do in later sections.

2 GL_n spider/skein theory

We will now give a diagrammatic description of the category of GL_n representations.

2.1 The representation category

From Theorems 1.22, 1.23, 1.27, the category $\mathcal{R}ep(GL_n)$ of representations of GL_n is pretty easy to describe. Every representation V can be decomposed uniquely as

$$V = \bigoplus_{\lambda} V(\lambda) \otimes M(\lambda)$$

where $M(\lambda)$ is a vector space with trivial GL_n action, called the multplicity space.

If $V' = \oplus_\lambda V(\lambda) \otimes M'(\lambda)$ is another representation, then we have

$$\mathrm{Hom}_{GL_n}(V, V') = \bigoplus_{\lambda} \mathrm{Hom}(M(\lambda), M'(\lambda)) \tag{1}$$

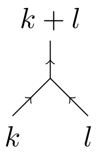
However, we can get a more interesting combinatorial structure if we work with a restricted set of representations.

Definition 2.1. $\mathcal{R}ep_f(GL_n)$ denotes the category whose objects are isomorphic to tensor products of the fundamental representations $\bigwedge^k \mathbb{C}^n$ of GL_n, for $k = 1, \dots, n$ and whose morphisms are morphisms of representations.

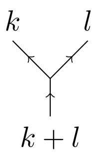
Of course, one can describe morphisms between these tensor product representation by decomposing into irreducibles and then applying (1). However, we will pursue a different approach here. We will try to describe morphisms in this category as being built from the natural maps

$$\textstyle\bigwedge^k \mathbb{C}^n \otimes \bigwedge^l \mathbb{C}^n \to \bigwedge^{k+l} \mathbb{C}^n \quad \text{and} \quad \bigwedge^{k+l} \mathbb{C}^n \to \bigwedge^k \mathbb{C}^n \otimes \bigwedge^l \mathbb{C}^n$$

which we depict diagrammatically as follows (where we read from the bottom up)



and



. (2)

It is relatively easy to show that these are indeed generators, *i.e.* that every GL_n-linear map between tensor products of fundamental representations can be written as tensor products and compositions of these maps.

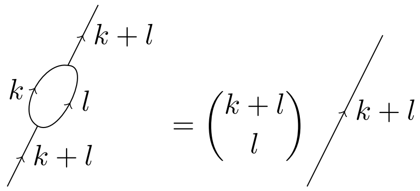
We define a diagrammatic category, the "free spider category" $\mathcal{FS}p(GL_n)$, whose objects are sequences $\underline{k} = (k_1, \dots, k_m)$ (with $k_j \in \{1, \dots, n\}$) and whose morphisms are linear combinations of "webs", trivalent graphs made up by glueing together the pieces in (2). The edges in these webs are oriented and labelled by $\{1, \dots, n\}$.

Theorem 2.2. *We have a full and dominant functor $\mathcal{FS}p(GL_n) \to \mathcal{R}ep(GL_n)$.*

We will now describe the kernel of this functor. This is a problem which was first posed by Kuperberg [Ku], who found the relations when $n = 3$.

2.2 Definition of the spider category

The spider category $\mathcal{S}p(GL_n)$ is the quotient of $\mathcal{FS}p(GL_n)$ by the following relations:



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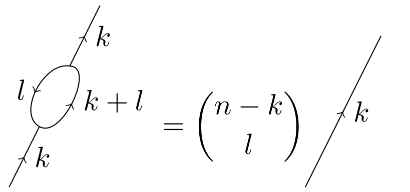

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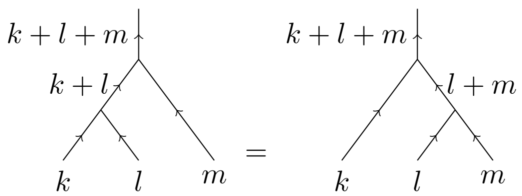

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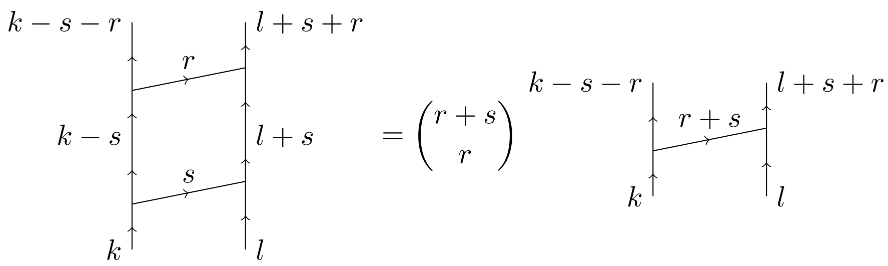

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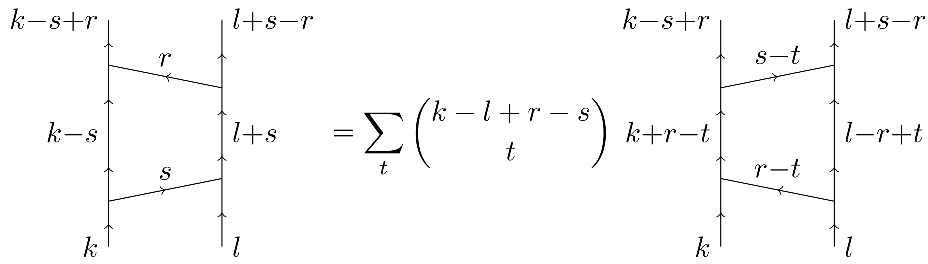


(7)

together with the mirror reflections of these.

It is fairly easy to check that all these relations hold in $\mathcal{R}ep_f(GL_n)$ and thus the functor $\mathcal{FS}p(GL_n) \to \mathcal{R}ep_f(GL_n)$ descends to a functor $\Gamma_n : \mathcal{S}p(GL_n) \to \mathcal{R}ep_f(GL_n)$.

The following result was proved with Cautis and Morrison [CKM].

Theorem 2.3. *The functor $\Gamma_n : \mathcal{S}p(GL_n) \to \mathcal{R}ep_f(GL_n)$ is an equivalence of categories.*

The idea behind the proof is to use the theory of "skew Howe duality". More precisely, we consider the representation $\bigwedge^K \mathbb{C}^n \otimes \mathbb{C}^m$ which carries commuting actions of GL_n and GL_m.

2.3 Quantum groups

The group GL_n and its representations admit **q-defomations**. More precisely, there exists a Hopf algebra $U_q(\mathfrak{gl}_n)$ over the ring $\mathbb{C}[q, q^{-1}]$. It has representations $V_q(\lambda)$ which are free modules over $\mathbb{C}[q, q^{-1}]$ which specialize to $V(\lambda)$ when $q = 1$. Everything written above about representations of GL_n carries over to $U_q(\mathfrak{gl}_n)$.

In particular, there are $U_q(\mathfrak{gl}_n)$ representations $\bigwedge_q^k \mathbb{C}_q^n$ which are free $\mathbb{C}[q, q^{-1}]$ modules of dimension $\binom{n}{k}$. The category $\mathcal{R}ep_f(U_q(\mathfrak{gl}_n))$ can be defined in a similar way, as can $FSp(U_q(\mathfrak{gl}_n))$ and $Sp(U_q(\mathfrak{gl}_n))$ except that in all the relations (3)-(7) we replace $\binom{n}{k}$ with $\begin{bmatrix} n \\ k \end{bmatrix}_q$ which is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q \cdots [1]_q}{[k]_q \cdots [1]_q [n-k]_q \cdots [1]_q}, \text{ where } [k]_q = q^{k-1} + q^{k-3} + \cdots q^{1-k}$$

The analog of Theorem 2.3 holds in this context. (More precisely, it hold after tensoring with $\mathbb{C}(q)$.)

2.4 Braiding

There is one interesting new feature about $U_q(\mathfrak{gl}_n)$ representations, called **braiding**. Suppose that V, W are representations of GL_n. Then of course we may form $V \otimes W$ or $W \otimes V$ and the map

$$\begin{aligned} \sigma_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v \end{aligned}$$

is easily seen to be an isomorphism between these two representations. Now, if V, W are $U_q(\mathfrak{gl}_n)$ representations, there is a natural way to give $V \otimes W$ the structure of a $U_q(\mathfrak{gl}_n)$ representation. However, it turns out that the map $\sigma_{V,W}$ is not a map of $U_q(\mathfrak{gl}_n)$ representations. However, there does exist a natural isomorphism $\beta_{V,W} : V \otimes W \to W \otimes V$, which is called the **braiding**. When we set $q = 1$, then β becomes σ.

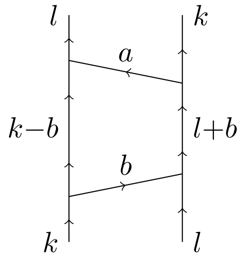
Example 2.4. Consider $n = 2$ and $V = W = \mathbb{C}[q, q^{-1}]^2$. Then with respect to the standard basis of $\mathbb{C}[q, q^{-1}]^2 \otimes \mathbb{C}[q, q^{-1}]^2$, the braiding $\beta_{\mathbb{C}[q,q^{-1}]^2, \mathbb{C}[q,q^{-1}]^2}$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In terms of the diagrammatics, there is a nice expression for the braiding of fundamental representations.

Proposition 2.5. *The braiding $\beta_{\bigwedge_q^k \mathbb{C}_q^n, \bigwedge_q^l \mathbb{C}_q^n}$ is given by the following sum of webs*

$$(-1)^{k+kl} q^{k-\frac{kl}{n}} \sum_{\substack{a,b\geq 0\\ b-a=k-l}} (-q)^{-b}$$



Now take V to be any representation. If consider $V^{\otimes n}$, then the braidings $\beta_{V,V}$ of neighbouring pairs in this tensor product can be used to generate an action of the braid group.

Definition 2.6. The **braid group** B_n is defined topologically as $\pi_1(\mathbb{C}^n \smallsetminus \Delta/S_n)$, the fundamental group of the configuration space of n points on $\mathbb{C} = \mathbb{R}^2$. It has generators $s_1, \ldots, s_{n-1}$ and relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \text{ if } |i-j| \geq 2.$$

The following important result is due to Drinfeld.

Theorem 2.7. *There is an action of B_n on $V^{\otimes n}$ where s_i acts by $I_{V^{\otimes(i-1)}} \otimes \beta_{V,V} \otimes I_{V^{\otimes(n-i-1)}}$.*

2.5 Knot invariants

The representation theory of $U_q(\mathfrak{gl}_n)$ can be used to define knot invariants. These were developed by Reshetikhin-Turaev following earlier work by Jones, Witten, and others.

Recall that a knot is an embedding of finitely many circles into $\mathbb{R}^3$. We will consider knots up to isotopy. Given a knot, we can consider projections of the knot to $\mathbb{R}^2$ to produce a knot diagram, a planar graph with vertices marked by over- or under-crossings. There is a well-known combinatorial theory of Reidemeister moves, which determine when two knot diagrams come from isotopic knots.

Now suppose that we have knot K whose components are coloured with representations of $U_q(\mathfrak{gl}_n)$. Then we can compose together the braidings which given by the crossings of a knot diagram of K, as well as coevaluation $\mathbb{C}[q, q^{-1}] \to V \otimes V^*$ and evaluation $V \otimes V^* \to \mathbb{C}[q, q^{-1}]$ maps given by the cups and caps of the diagram. Reshetikhin-Turaev [RT] showed that the resulting polynomial $P(K) \in \mathbb{C}[q, q^{-1}]$ only depends on the isotopy class of K. When $n = 2$ and we use the standard representation $\mathbb{C}_q^2$, then $P(K)$ is called the **Jones polynomial** and was earlier discovered by Jones [J].

If K is labelled by fundamental representations, then we can compute the knot polynomial $P(K)$ by replacing all crossings with webs using Proposition 2.5 and then evaluating the resulting webs using relations (3)-(7). The special case where all strands are labelled by $\mathbb{C}_q^n$ was investigated by Murakami-Ohtsuki-Yamada [MOY].

3 Geometric Satake correspondence

We will now give an exposition of the geometric Satake correspondence which is due to Lusztig [L], Ginzburg [G], and Mirkovic-Vilonen [MV].

3.1 The varieties Gr^λ

Consider the (infinite-dimensional) $\mathbb{C}$-vector space $\mathbb{C}[z]\otimes\mathbb{C}^n$. So if $\mathbb{C}^n$ has basis $e_1,\dots,e_n$, then $\mathbb{C}[z]\otimes\mathbb{C}^n$ has a basis $\{z^k e_i\}$ where $k=0,1,\dots$ and $i=1,\dots,n$. Define a linear operator

$$w : \mathbb{C}[z]\otimes\mathbb{C}^n \to \mathbb{C}[z]\otimes\mathbb{C}^n$$
$$z^k e_i \mapsto \begin{cases} z^{k-1}e_i, & \text{if } k\geq 1 \\ 0, & \text{if } k=0 \end{cases}$$

Definition 3.1. Let $\mathrm{Gr} = \{L\subset \mathbb{C}[z]\otimes\mathbb{C}^n : wL\subset L\}$ be the set of all w-invariant finite-dimensional subspaces of $\mathbb{C}[z]\otimes\mathbb{C}$. This is called the positive part of the **affine Grassmannian**.

Let $L\in\mathrm{Gr}$. We can consider the restriction of w to L. This will be a nilpotent operator and hence it will have a Jordan type. Note that since $\dim ker(w)=n$, it follows that $\dim ker(w|_L)\leq n$ which implies that $w|_L$ has at most n Jordan blocks. Hence the Jordan type of $w|_L$ can be written as $(\lambda_1,\dots,\lambda_n)$ with $\lambda_1\geq\cdots\geq\lambda_n\geq 0$ and thus can be regarded as an element of P_{++}.

We define

$$\mathrm{Gr}^\lambda := \{L\subset\mathbb{C}[z]\otimes\mathbb{C}^n : wL\subset L, \text{ and } w|_L \text{ has Jordan type } \lambda\}.$$

It is a locally closed subset in Gr and we may take its closure to obtain $\overline{\mathrm{Gr}^\lambda}$.

Proposition 3.2. $\overline{\mathrm{Gr}^\lambda}$ *is a projective variety of dimension* $(n-1)\lambda_1+(n-3)\lambda_2+\cdots+(-1-n)\lambda_n$.

It is easy to see exactly which subspaces we pick up in the closure — in the closure the Jordan type of $w|_L$ becomes more "special", i.e. more like the zero matrix. This can be expressed as follow.

Lemma 3.3.

$$\overline{\mathrm{Gr}^\lambda} = \bigcup_{\mu\in P_{++}:\mu\leq\lambda} \mathrm{Gr}^\mu$$

Example 3.4. Suppose that $\lambda=\omega_k=(1,\dots,1,0,\dots,0)$. Having Jordan type ω_k means that $w|_L$ is 0 and that $\dim L=k$. Hence we see that Gr^λ is isomorphic to the Grassmannian $G(k,n)$ of k dimensional subspaces of $ker(w)=\mathbb{C}^n$. Note that in this case, $\mathrm{Gr}^\lambda=\overline{\mathrm{Gr}^\lambda}$.

Exercise 3.5. Prove that $\overline{Gr^{(2,0)}}$ is isomorphic to the singular projective variety in $\mathbb{P}^3$ defined by the equation $x^2+yz=0$ (in other words, the projective closure of the 2-dimensional quadratic cone defined by the same equation).

It is not immediately obvious that these varieties Gr^λ are non-empty. Let us demonstrate this now. For each $\mu \in \mathbb{N}^n$, we define a subspace

$$L_\mu = \mathrm{span}_\mathbb{C}(z^k e_i : k < \mu_i) \subset \mathbb{C}[z] \otimes \mathbb{C}^n.$$

Exercise 3.6. $L_\lambda \in \mathrm{Gr}^\lambda$. More generally $L_{w\lambda} \in \mathrm{Gr}^\lambda$ for all $w \in S_n$.

Now suppose $\underline{k} = (k_1, \dots, k_m)$ is a sequence with $k_j \in \{1, \dots, n\}$. We define a flag-like variety

$$\begin{aligned}\mathrm{Gr}^{\underline{k}} := \{0 = &L_0 \subset L_1 \subset \cdots \subset L_m \subset \mathbb{C}[z] \otimes \mathbb{C}^n : \\ &wL_i \subset L_{i-1}, \dim L_i = \dim L_{i-1} + k_i\}\end{aligned}$$

Exercise 3.7. Consider the map $\mathrm{Gr}^{(k_1,\dots,k_m)} \to \mathrm{Gr}^{(k_1,\dots,k_{m-1})}$. Prove that this map is a fibre bundle with fibre $G(k_m, n)$.

The geometric Satake correspondence relates the representation $V(\lambda)$ to the variety Gr^λ and the representation $\bigwedge^{k_1} \otimes \cdots \otimes \bigwedge^{k_m}$ to the variety $\mathrm{Gr}^{\underline{k}}$. To explain this more precisely, we will need some general facts about sheaves.

3.2 Sheaf theory

Let X be a reasonable topological space.

Definition 3.8. A **sheaf** $\mathcal{F}$ on X is a choice of a vector space $\mathcal{F}(U)$ for every open set $U \subset X$ along with restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $U \subset V$. These restriction maps must compose properly. Finally, we require that whenever we have a decomposition $U = \cup_i U_i$ and sections $s_i \in \mathcal{F}(U_i)$ which agree (after restriction) in $\mathcal{F}(U_i \cap U_j)$ for all i, j, then there exists a unique $s \in \mathcal{F}(U)$ which restricts to all of the s_i.

The simplest example of a sheaf is the "constant sheaf" $\underline{\mathbb{C}}_X$, with $\underline{\mathbb{C}}_X(U) = \mathbb{C}$ for every connected open set U.

If $\mathcal{F}$ is a sheaf, then one can define the cohomology of X with coefficients in $\mathcal{F}$, denoted $H^*(X, \mathcal{F})$ using Cech cohomology. When $\mathcal{F} = \underline{\mathbb{C}}_X$, then $H^*(X, \underline{\mathbb{C}}_X)$ is the same as the singular or simplicial cohomology of X.

There is another special sheaf on X, called the intersection cohomology sheaf, denote IC_X. The cohomology $IH^*(X) := H^*(X, IC_X)$ is called the intersection homology of X and it satisfies Poincaré duality for any reasonable space X. When X is a manifold, then $IC_X \cong \underline{\mathbb{C}}_X$.

If $p : X \to Y$ is a continuous map, then we can define the push-forward sheaf[1] $p_*\mathcal{F}$ by

$$p_*\mathcal{F}(U) = H^*(p^{-1}(U), \mathcal{F})$$

[1] Actually, $p_*\mathcal{F}$ is not a sheaf, but rather a complex of sheaves (more precisely, an object in the derived category of sheaves on X). For the purposes of these lectures, the above approximate definition will be sufficient. Similarly, when we speak about morphisms of sheaves, we really mean morphisms in the derived category.

Now suppose that we have two maps $p : X \to Y$ and $p' : X' \to Y$ with X, X' manifolds. The following result of Chriss-Ginzburg [CG] will be quite useful for us, since it relates morphisms between pushforwards of constant sheaves and homology of fibre products.

Theorem 3.9. *We have an isomorphism*

$$\mathrm{Hom}(p_* \underline{\mathbb{C}}_X, p'_* \underline{\mathbb{C}}_{X'}) \cong H_m(X \times_Y X')$$

for some m.

(If the maps p, p' are semismall, then m equals the (real) dimension of $X \times_Y X'$.)

To explain the second part of the theorem, suppose that $p'' : X'' \to Y$ is a third variety with a map Y. Then we can define a **convolution product**

$$* : H_*(X \times_Y X') \otimes H_*(X' \times_Y X'') \to H_*(X \times_Y X'')$$

by the formula

$$c_1 * c_2 = (\pi_{13})_*(\pi_{12}^*(c_1) \cap \pi_{23}^*(c_2)),$$

where "$\cap$" denotes the intersection product (with support), relative to the ambient manifold $X \times X' \times X''$ and $p_{12} : X \times X' \times X'' \to X \times X'$, etc. For more details about this construction, see [CG, Sec. 2.6.15] or [F, Sec. 19.2].

The following result is also due to Chriss-Ginzburg.

Theorem 3.10. *The isomorphism of Theorem 3.9 interwines the composition of* Hom*s with the convolution product on homology.*

3.3 Geometric Satake

We are now in a position to roughly formulate the geometric Satake correspondence. Let $\mathcal{P}(\mathrm{Gr})$ denote the category of sheaves on Gr which direct sums of the sheaves $IC_{\overline{\mathrm{Gr}^\lambda}}$ and let $\mathcal{P}_f(\mathrm{Gr})$ denote the category of sheaves which are of the form $p_* \underline{\mathbb{C}}_{\mathrm{Gr}^{\underline{k}}}$. The decomposition theorem shows us that $\mathcal{P}_f(\mathrm{Gr}) \subset \mathcal{P}(\mathrm{Gr})$.

Theorem 3.11. *We have an equivalence of categories $\mathcal{R}ep(GL_n) \cong \mathcal{P}(\mathrm{Gr})$, taking $\mathcal{R}ep_f(GL_n) \cong \mathcal{P}_f(\mathrm{Gr})$. On objects, this equivalence is given by*

$$V(\lambda) \mapsto IC_{\overline{\mathrm{Gr}^\lambda}}, \quad \bigwedge^{k_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k_m} \mathbb{C}^n \mapsto p_* \underline{\mathbb{C}}_{\mathrm{Gr}^{\underline{k}}}$$

Moreover, this equivalence is compatible with the functors to vector spaces on both sides, so that

$$IH^*(\overline{\mathrm{Gr}^\lambda}) \cong V(\lambda), \text{ and } H^*(\mathrm{Gr}^{\underline{k}}) \cong \bigwedge^{k_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k_m} \mathbb{C}^n.$$

Exercise 3.12. Take $\lambda = \omega_k$. As we have seen $\overline{\mathrm{Gr}^{\omega_k}} \cong \mathrm{Gr}^{\omega_k} \cong G(k, n)$. Show that $IH_*(\overline{\mathrm{Gr}^{\omega_k}}) = H^*(G(k, n))$ has a basis labelled by k element subsets of $\{1, \ldots, n\}$. Using this construct an isomorphism $IH_*(\overline{\mathrm{Gr}^{\omega_k}}) \to V(\omega_k)$.

3.4 Weight space decomposition

As we know, representations of GL_n have a weight space decomposition $V = \oplus_\mu V_\mu$. It is interesting to examine this decomposition from the perspective of perverse sheaves.

We have an action of the torus T on $\mathbb{C}^n$ and thus on $C[z] \otimes \mathbb{C}^n$ and thus on Gr. The fixed points of this torus action on Gr are precisely the points L_μ, for $\mu \in P$.

Let us pick a generic $\mathbb{C}^\times \to \mathbb{C}^n$, for example given by the diagonal matrix with entries $(t, t^2, \dots, t^n)$. Then for each $\mu \in P$, we can consider the attracting set

$$S^\mu := \{L \in \mathrm{Gr} : \lim_{t\to\infty} t \cdot L = L_\mu\}$$

This set S^μ provides a kind of cell-decomposition of Gr, except the pieces S^μ are not finite-dimensional. They are called "semi-infinite" cells.

Nonetheless, we can use them to study the intersection homology of each $\overline{\mathrm{Gr}^\lambda}$. The following result is due to Mirkovic-Vilonen [MV].

Theorem 3.13. *For each λ, we have a decomposition*

$$IH_*(\overline{\mathrm{Gr}^\lambda}) = \bigoplus_\mu H_{\mathrm{top}}(\mathrm{Gr}^\lambda \cap S^\mu)$$

Moreover, this decomposition matches the weight space decomposition of $V(\lambda)$ under the isomorphism provided by geometric Satake.

The vector space $H_{\mathrm{top}}(\mathrm{Gr}^\lambda \cap S^\mu)$ has a basis given by the irreducible components of $\mathrm{Gr}^\lambda \cap S^\mu$. The irreducible components are called Mirkovic-Vilonen cycles.

3.5 Spiders and geometric Satake

We have previous described $\mathcal{R}ep_f(GL_n)$ using webs. It is natural to ask how this description looks like after passing to $\mathcal{P}_f(\mathrm{Gr})$ using geometric Satake. This will also allow us to understand geometric Satake on the level of morphisms.

Recall that by Theorem 3.9, for any two sequences $\underline{k} = (k_1, \dots, k_m), \underline{k'} = (k'_1, \dots, k'_{m'})$, we have

$$\mathrm{Hom}_{\mathcal{P}_f(\mathrm{Gr})}(p_*\underline{\mathbb{C}}_{\mathrm{Gr}^{\underline{k}}}, p_*\underline{\mathbb{C}}_{\mathrm{Gr}^{\underline{k'}}}) \cong H_{\mathrm{top}}(Z(\underline{k}, \underline{k'}))$$

where

$$Z(\underline{k}, \underline{k'}) = \mathrm{Gr}^{\underline{k}} \times_{\mathrm{Gr}} \mathrm{Gr}^{\underline{k'}} = \{(L_\cdot, L'_\cdot) \in \mathrm{Gr}^{\underline{k}} \times_{\mathrm{Gr}} \mathrm{Gr}^{\underline{k'}} : L_m = L'_{m'}$$

Recall that we have the equivalences of categories $\mathcal{S}p(GL_n) \to \mathcal{R}ep_f(GL_n) \to \mathcal{P}_f(\mathrm{Gr})$ and thus an isomorphism

$$\mathrm{Hom}_{\mathcal{S}p(GL_n)}(\underline{k}, \underline{l}) \to \mathrm{Hom}_{\mathcal{P}_f(\mathrm{Gr})}(p_*\underline{\mathbb{C}}_{\mathrm{Gr}^{\underline{k}}}, p_*\underline{\mathbb{C}}_{\mathrm{Gr}^{\underline{l}}}) \cong H_{\mathrm{top}}(Z(\underline{k}, \underline{k'})) \qquad (8)$$

An element of the LHS is represented by a web with bottom endpoints $\underline{k}$ and top endpoints $\underline{l}$. On the RHS, we have a basis given by the irreducible components of $Z(\underline{k}, \underline{k'})$. Thus under the geometric Satake correspondence, the web will be mapped to a linear combination of these components. In [FKK], with Fontaine and Kuperberg, we showed that for each web w, there exists a configuration space $X(w)$ in the affine Grassmannian which maps to $Z(\underline{k}, \underline{k'})$ and such that (under good circumstances) the image of the web under (8) is a linear combination of the components of $Z(\underline{k}, \underline{k'})$ with coefficients being the Euler characteristics of the generic fibres of $X(w) \to Z(\underline{k}, \underline{k'})$.

Now, let us focus on the case $\underline{k} = (k, l)$ and $\underline{k'} = (k + l)$. We can see that

$$Z((k, l), (k + l)) = \{(L_1, L_2), (L'_1) : L'_1 = L_2\}$$

is just isomorphic to a $G(k, l)$ bundle over $G(k + l, n)$.

On the other hand, $\mathrm{Hom}_{\mathcal{S}p(GL_n)}((k, l), (k+l))$ is just spanned by the web consisting of a single trivalent vertex. Under (8), this trivalent vertex is mapped to $[Z((k, l), (k+l))]$.

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