Perverse sheaves learning seminar: Lecture 1: Sheaves

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1 The category of sheaves

Example 1.0.1 (Prototypes). A smooth function on an open subset of a smooth manifold restricts to a smooth function on any open subset, and a smooth function can be glued together (uniquely) from smooth functions on smaller subsets. These features mean that $C^\infty(M)$ is naturally a sheaf. Similarly, holomorphic functions on $\mathbb{C}$ or a complex manifold, or more generally sections of a vector bundle are all naturally sheaves.

1.1 Presheaves

Let X be a topological space. It will be harmless to assume it Hausdorff and locally compact throughout, although we will point out when these hypotheses are actually necessary. Then we have a category $\mathrm{Op}(X)$ of open sets of X, where the morphisms are inclusions, so that $\mathrm{Hom}(U, V) = pt.$ if $U \subset V$ and is empty otherwise.. A *presheaf* is (for us) a functor $\mathcal{F} \colon \mathrm{Op}(X)^{\mathrm{opp}} \to \mathbf{Vect}_{\mathbb{C}}$, the category of $\mathbb{C}$-vector spaces. Therefore for every $U \in \mathrm{Op}(X)$, we obtain a complex vector space $\mathcal{F}(U)$. If $V \subset U$, then we call the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ of $\mathbb{C}$-vector spaces *restriction.* An element $s \in \mathcal{F}(U)$ is called a *section of $\mathcal{F}$ over U*. Its image under the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is usually written $s\restriction_V$ or $\mathrm{res}_{UV}(s)$.

A *morphism of presheaves* is a natural transformation of functors. That is, a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism $\phi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ of $\mathbb{C}$-vector spaces for every $U \in \mathrm{Op}(X)$ such that all the squares

$$\begin{array}{ccccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) & \qquad & U \\ \downarrow & & \downarrow & & \uparrow \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) & & V \end{array}$$

commute.

Given a point $x \in X$, the neighbourhoods $U \ni x$ form a filtered subcategory of $\mathrm{Op}(X)$. This means simply that if U and V are neighbourhoods of x, so is $U \cap V$. The *stalk of $\mathcal{F}$* at x is then the (cofilitered) colimit

$$\mathcal{F}_x := \operatorname*{colim}_{U \ni x} \mathcal{F}(U).$$

Its elements are called *germs.* If $U \ni x$, then by definition there is a canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ written $s \mapsto s_x$. Concretely,

$$\mathcal{F}_x = \{(s, V) \mid V \ni x,\ s \in \mathcal{F}(V)\} / \sim$$

where $(s, V) \sim (t, U)$ if there is $W \subset U \cap V$ such that $W \ni x$ and $s\restriction_W = t\restriction_W$. Each stalk is naturally a $\mathbb{C}$-vector space for formal reasons, but we can also see this from the concrete realization. Thus under the heuristic that sheaves should be the functions on a space, the germ of a section records the behaviour of s near x.

It will be helpful to know when we can calculate stalks using only certain open sets.

Definition 1.1.1. A subset B of a partially ordered set A is *cofinal* if for every $a \in A$,there is $b \in B$ with $a \leq b$.

We will apply this with $\leq$ meaning reverse inclusion. For formal reasons, colimits taken over cofinal subsets are isomorphic.

A morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ of presheaves induces a map $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ for each $x \in X$. Concretely, it is given by

$$\phi_p(s_p) = (\phi(V)(s))_p$$

where (s, V) is a representative of s_p.

The *support* of a presheaf $\mathcal{F}$ on X is the closed subset

$$\operatorname{supp} \mathcal{F} := \overline{\{x \in X \mid \mathcal{F}_x \neq 0\}}.$$

The *support of a section* $s \in \mathcal{F}(U)$ is the closed set

$$\operatorname{supp}(s) = \{x \in U \mid s_x \neq 0\}.$$

Under the heuristic that sections are functions, this notion corresponds to the support of a function. However, in the case of sections we do not need to take the closure: If $s_x = 0$ then there is a small neighbourhood of x inside U where s is zero. Hence its germ at any other point in the small neighbourhood is zero.

We will write $\operatorname{Presh}(X)$ for the category of presheaves of $\mathbb{C}$- vector spaces on X.

1.2 Sheaves

A sheaf is a presheaf satisfying two axioms which serve to make the sections of a sheaf more "function-like" than the sections of an arbitrary presheaf.

Definition 1.2.1. A *sheaf* is a presheaf satisfying the following axioms:

1. (Identity). If $U = \bigcup_\alpha U_\alpha$, $s \in \mathcal{F}(U)$ and $s \restriction_{U_\alpha} = 0$ for all α, then $s = 0$.
2. (Gluing). If $U = \bigcup_\alpha U_\alpha$ and we are given $s_\alpha \in \mathcal{F}(U_\alpha)$ such that

$$s_\alpha \restriction_{U_\alpha \cap U_\beta} = s_\beta \restriction_{U_\alpha \cap U_\beta}$$

   for all α, β, then there is $s \in \mathcal{F}(U)$ such that $s \restriction_{U_\alpha} = s_\alpha$. If we require this s to be unique (*Strong Glueing*), axiom 2 subsumes the identity axiom.

Another way to package identity and gluing is to say that a presheaf $\mathcal{F}$ is a sheaf if the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{r_0} \prod_\alpha \mathcal{F}(U_\alpha) \xrightarrow{\quad r_1 \quad} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

$$s \longmapsto (s \restriction_{U_\alpha})_\alpha$$

$$(s_\alpha)_\alpha \longmapsto (s \restriction_{U_\alpha \cap U_\beta} (s_\alpha) - s \restriction_{U_\alpha \cap U_\beta} (s_\beta))_{\alpha,\beta}$$

is exact. Here $U = \bigcup_\alpha U_\alpha$. This sequence is of particular use when the U_α is disjoint.

We have $r_1 \circ r_0 = 0$ for any presheaf. Exactness at $\mathcal{F}(U)$ is the identity axiom. Exactness at $\prod_\alpha \mathcal{F}(U_\alpha)$ is the gluing axiom.

A *morphism of sheaves* is by definition a morphism of presheaves. We denote the category of sheaves on X by $\operatorname{Sh}(X)$.

A presheaf satisfying the identity axiom is called *separated* or *decent*.

Lemma 1.2.2. *Let $\mathcal{F}$ be a presheaf on X. The following are equivalent.*

1. *$\mathcal{F}$ is separated;*
2. *For all $U \in \operatorname{Op}(X)$ and $s \in \mathcal{F}(U)$, $s_x = 0$ for all $x \in U$ implies $s = 0$;*
3. *For all $U \in \operatorname{Op}(X)$ and $s, t \in \mathcal{F}(U)$, $s_x = t_x$ all $x \in U$ implies $s = t$.*

1.3 Sheafification

We will need sheaf versions of the basic objects of homological algebra, but the naive definitions of the image and cokernel, in particular, fail in general to be sheaves if we attempt to define them by *e.g.* $\operatorname{coker}(\phi)(U) = \operatorname{coker}(\phi(U))$. We will see this has to do with subtleties in the definition of a surjection of sheaves.

Therefore we need a way to produce sheaves from presheaves. An alternative reference for this section is Vakil's notes.

Definition 1.3.1. An element

$$(s_p)_{p\in U} \in \prod_{p\in U} \mathcal{F}_p$$

is a *compatible set of germs* if for each $p \in U$, there is an open set $U_p \subset U$ such that $U_p \ni p$ and a section s'_p such that $(s'_p)_q = s_q$ for all $q \in U_p$. We define

$$\mathcal{F}^+(U) = \{(s_p)_{p\in U} \mid (s_p)_{p\in U} \text{ is compatible}\} \subset \prod_{p\in U} \mathcal{F}_p.$$

The compatibility condition forces the strong gluing axiom.

Lemma 1.3.2. *The set of compatible sets of germs is precisely the image of the map*

$$s \mapsto (s_p)_{p\in U}.$$

Theorem 1.3.3. *Let $\mathcal{F}$ be a presheaf. Then $\mathcal{F}^+$ as defined above is a sheaf with a morphism $\iota\colon \mathcal{F} \to \mathcal{F}^+$ with the following universal property: given a morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ with $\mathcal{G}$ a sheaf, there is a unique morphism ϕ^+ such that*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\iota} & \mathcal{F}^+ \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi^+} \\ & & \mathcal{G} \end{array}$$

commutes. Moreover, ι induces an isomorphism $\mathcal{F}_p \xrightarrow{\sim} \mathcal{F}^+_p$ for all $p \in X$.

Definition 1.3.4. We call the sheaf $\mathcal{F}^+$ the *sheafification of* $\mathcal{F}$.

Sketch of proof of theorem. The sheafification is unique because it solves a universal mapping problem.

The restriction maps are

$$(s_p)_{p\in U} \longmapsto (s_p)_{p\in V}$$

$$V \hookrightarrow U.$$

Identity axiom: If $(s_p)_p \in \mathcal{F}^+(U)$ restricts to zero on a open cover $\{U_\alpha\}$ of U, then $s_p = 0$ for all p, so $(s_p)_p = 0$.

Gluing axiom: If for each α we have $(s_{\alpha,p})_{p\in U_\alpha}$, then agreeing on overlaps means $s_{\alpha,p} = s_{\beta,p}$ for all $p \in U_\alpha \cap U_\beta$, so define $(s_p)_{p\in U}$ by $s_p = s_{\alpha,p}$ if $p \in U_\alpha$. We must show this is compatible. Given $p_0 \in U$, take α such that $p_0 \in U_\alpha$. By compatibility (s_α, p) there is $U_{p_0} \subset U_\alpha$ and $s'_{p_0} \in \mathcal{F}(U_{p_0})$ representing s_{p_0}. Then if $q \in U_{p_0}$,

$$(s'_{p_0})_q = s_{\alpha,q} = s_q.$$

Thus $(s_p)_{p\in U}$ is compatible. To show that the sets of compatible germs form a $\mathbb{C}$-vector space we must show they are an abelian group with scalar multiplication. This is straightforward. If $U = \bigcup_\alpha U_\alpha$ and is such that $(s_p)\restriction_{U_\alpha} = (s_p)_{p\in U_\alpha} = 0$ for all α, then for all $p \in U$, $s_p = 0$, so that $(s_p)_{p\in U} = 0$. Therefore $\mathcal{F}^+$ satisfies the identity axiom. The gluing axiom is an exercise.

The morphism ι is given by $\iota(U)(s) = (s_p)_{p\in U}$.

If $(s_p)_{p\in U}$ is compatible, so is $(\phi_p(s_p))_{p\in U}$, so this set of germs comes from some $t \in \mathcal{G}(U)$. Define $\phi^+((s_p)_{p\in U}) = t$. □

1.4 Examples of sheaves

Example 1.4.1. There is an isomorphism of categories $\mathrm{Sh}(\mathrm{pt}) \simeq \mathbf{Vect}_{\mathbb{C}}$.

Example 1.4.2 (Constant sheaves). Let M be a vector space. Then

$$\underline{M}_X(U) = \{s\colon U \to M \mid s \text{ constant on connected components}\}$$

with restriction given by restriction of functions gives the *constant sheaf* with value M. It is the sheafification of presheaf of constant M-valued functions.

If X is connected, for all $x \in X$, we have $\underline{M}_X(X) \simeq (\underline{M}_X)_x$.

Lemma 1.4.3. *Let X be a topological space and M a $\mathbb{C}$-vector space. Then*

1. *For any connected open $U \subset X$, $\underline{M}_X(U) \simeq M$ canonically.*
2. *$V \subset U$ both connected open, then restriction is identified with identity $M \to M$.*

Proof. A locally constant function on a connected set is constant. Then 2 follows from 1. □

Example 1.4.4 (Skyscraper sheaves). Given X and a vector space M, we can define the *skyscraper sheaf* by

$$\underline{M}^x(U) = \begin{cases} M & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}.$$

Restrictions are either the identity or zero. In particular, $\underline{M}^x(U) = (\underline{M}^x)_x$. We have $\operatorname{supp} \underline{M}^x = \{x\}$, and any sheaf supported at a single point is a skyscraper sheaf.

Remark 1.4.5. If we want $\underline{M}^x$ to be supported only at x, we should ask that X be at least T_1, *i.e.* that for any points $x \neq y$ in X, there is a neighbourhood of x that does not contain y.

1.5 Morphisms of sheaves

Definition 1.5.1.

- A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if $\phi(U)$ is injective for all U;
- A morphism of sheaves is surjective if ϕ_x is surjective for all x;
- A morphism of sheaves is an *isomorphism* if it has an inverse morphism of sheaves, equivalently $\phi(U)$ is an isomorphism for all U.

This example shows a morphism of sheaves of abelian groups with a surjective, but not surjective on sections:

Example 1. Let $\mathcal{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$, and $\mathcal{O}^\times$ be the sheaf of nonzero holomorphic functions on $\mathbb{C}$. Consider the exponential map $\exp\colon \mathcal{O} \to \mathcal{O}^\times$. Recalling that every nonzero holomorphic function on a simply connected domain has a logarithm and that $\mathbb{C}$ is locally simply connected, we see that exp is surjective on stalks. However, $\exp(\mathbb{C}^\times)\colon \mathcal{O}(C^\times) \to \mathcal{O}(\mathbb{C}^\times)$ is not surjective; if $z = e^{f(z)}$ then on $\mathbb{C} \setminus [0, \infty)$ we would have $f(z) = \log(z)$, and $\log(z)$ doesn't extend to $\mathbb{C}^\times$.

Lemma 1.5.2. *The following are equivalent:*

1. *ϕ is an isomorphism of sheaves;*
2. *ϕ_x is an isomorphism of $\mathbb{C}$-vector spaces for each x;*
3. *ϕ is injective and surjective.*

Definition 1.5.3 (with exercises)**.** Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

1. The presheaf
$$U \mapsto \ker \phi(U)$$
is a sheaf. Call it $\ker \phi$. It stalks are $(\ker \phi)_p = \ker(\phi_p)$.

2. The sheafification of the presheaf
$$U \mapsto \operatorname{coker}(\phi(U))$$
is called $\operatorname{coker} \phi$, and $(\operatorname{coker} \phi)_p = \operatorname{coker}(\phi_p)$.

The fact about the stalks descends from the fact that colimits in vector spaces are exact, so commute with taking kernels and cokernels.

Proposition 1.5.4. *A sequence of sheaves*
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
(by definition, this means we have equalities of sheaves $\mathrm{im} = \ker$ *at each place) iff*
$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$
is exact for all x.

Remark 1.5.5. Note that exactness says that $\mathrm{im}\,\phi = \ker \psi$ for some morphisms of sheaves, so in this case the image presheaf is already a sheaf.

A useful description of surjectivity:

Lemma 1.5.6. *A morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ is surjective iff for all open $U \subset X$, and $s \in \mathcal{G}(U)$, there is an open cover $\{U_\alpha\}$ and sections $t_\alpha \in \mathcal{F}(U_\alpha)$ such that $s \restriction_{U_\alpha}$ is in the image of $\phi(U_\alpha)$.*

Thus surjectivity for sheaves means "local surjectivity," and the issue is we can have sections whose images glue, but which do not themselves glue.

Lemma 1.5.7. *A sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact iff it gives an exact sequence of vector spaces for all open $U \subset X$.*

Definition 1.5.8. An *abelian category* is an additive category in which

1. Every morphism has a kernel and cokernel;

2. Every monomorphism is the kernel of its cokernel;

3. Every epimorphism is the cokernel of its kernel.

We won't give the full categorical definitions of kernels and cokernels, but given a morphism $f \colon A \to B$, we will use the word "kernel" to also mean the morphism $\ker f \to A$, and "cokernel" to also mean the morphism $B \to \operatorname{coker} f$.

Theorem 1.5.9. $\mathrm{Sh}(X)$ *is abelian.*

Proof. Only have to check requirements 2 and 3. Hint to do so: Given a monomorphism $\phi \colon \mathcal{F} \to \mathcal{G}$, we want to show that $\mathcal{F}$ is isomorphic to the kernel of the map $\mathcal{G} \to \operatorname{coker} \phi$. Reduce this to the fact that ϕ_x are all monomorphisms and use that $\mathbf{Vect}_{\mathbb{C}}$ is abelian. □

2 Pullback and pushforward

The table on p. 90 is excellent.

2.1 Pullbacks

Let $f\colon X \to Y$. Then we have a functor $f^*_{pre}\colon \operatorname{Presh}(Y) \to \operatorname{Presh}(X)$ given by

$$(f^*_{pre}\mathcal{F})(U) = \operatorname*{colim}_{\substack{V \in \operatorname{Op}(Y)\\ V \supset f(U)}} \mathcal{F}(V).$$

Sheafification gives a functor $f^*\colon \operatorname{Sh}(Y) \to \operatorname{Sh}(X)$ sending $\mathcal{F} \mapsto (f^*_{pre}\mathcal{F})^+$.

Example 2.1.1. Pullback from a point is the same as the constant sheaf functor.

Definition 2.1.2. If $f\colon X \hookrightarrow Y$ is an inclusion, then $f^*_{pre}\mathcal{F}$ sheafifies to the *restriction of* $\mathcal{F}$ *to* X, written $\mathcal{F}\restriction_X$.

Stalks of the pullback are easy to calculate, which will give nice exactness properties to the pullback.

Lemma 2.1.3. *Let $f\colon X \to Y$. Then we have a natural isomorphism $(f^*\mathcal{F})_x \simeq \mathcal{F}_{f(x)}$.*

Proof. Note we can use the presheaf versions. For each open $U \ni x$ we get a family of opens $V \subset Y$ such that $V \supset f(U) \ni f(x)$, and so we get maps $\operatorname{colim}_{V\supset f(U)} \mathcal{F}(V) \to \mathcal{F}_{f(x)}$. We have morphisms $\operatorname{colim}_{V\supset f(U_2)} \mathcal{F}(V) \to \operatorname{colim}_{V\supset f(U_1)} \mathcal{F}(V)$ when $U_1 \subset U_2$, and by the univeral property of colimits, all the triangles

$$\begin{array}{ccc} \operatorname{colim}_{V\supset f(U_2)} \mathcal{F}(V) & \longrightarrow & \operatorname{colim}_{V\supset f(U_1)} \mathcal{F}(V) \\ & \searrow \quad \swarrow & \\ & M & \end{array}.$$

commute for any vector space M. Thus $\operatorname{colim}_{U\ni x} \operatorname{colim}_{\substack{V \in \operatorname{Op}(Y)\\ V \supset f(U)}} \mathcal{F}(V)$ satisfies the universal property of $\mathcal{F}_x$ so they are isomorphic. □

Lemma 2.1.4. *The pullback functor is exact.*

Proof. $$0 \longrightarrow f^*\mathcal{F} \longrightarrow f^*\mathcal{G} \longrightarrow f^*\mathcal{H} \longrightarrow 0$$

is exact iff

$$0 \longrightarrow f^*\mathcal{F}_x \longrightarrow f^*\mathcal{G}_x \longrightarrow f^*\mathcal{H}_x \longrightarrow 0$$

is exact, but this is

$$0 \longrightarrow \mathcal{F}_{f(x)} \longrightarrow \mathcal{G}_{f(x)} \longrightarrow \mathcal{H}_{f(x)} \longrightarrow 0,$$

which is exact. □

Proposition 2.1.5. *The pullback f is functorial in the sense that: $(g \circ f)^*\mathcal{F} = g^*(f^*\mathcal{F})$ if $g \circ f\colon X \to Y \to Z$ and $\mathcal{F} \in \operatorname{Sh}(Z)$.*

Proof. Exercise. As presheaves, the two colimits are over cofinal sets, and so are identified. We have to show the isomorphism holds for the sheafifications. Get a diagram

$$\begin{array}{ccc} f^*_{pre}(g^*_{pre}\mathcal{F}) & \longrightarrow & (g \circ f)^*(\mathcal{F}) \\ \downarrow & & \downarrow \\ f^*_{pre}(g^*_{pre}\mathcal{F}) & \longrightarrow & f^*(g^*\mathcal{F}) \end{array}$$

using functoriality of f^*_{pre} and universal property of sheafification. Taking stalks, horizontal arrows become isomorphisms, and left arrow is isomorphism $(g^*_{pre}\mathcal{F})_{f(x)} \to (g^*\mathcal{F})_{f(x)}$. □

2.2 Pushforward

Given $f\colon X \to Y$, we have $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ sending a sheaf $\mathcal{F}$ to the sheaf (no sheafification needed) $U \mapsto \mathcal{F}(f^{-1}(U))$.

The book writes $^\circ f_*$ for this functor, and f_* for its (right) derived functor. No point doing this yet.

Proposition 2.2.1. *f_* is left exact.*

Proof. By definition 1.5.1, left exactness can be checked on sections. □

Proposition 2.2.2. *Let $f\colon X \to Y$ be continuous with $\mathcal{F} \in \mathrm{Sh}(Y)$ and $\mathcal{G} \in \mathrm{Sh}(X)$. We have an adjunction*

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(f^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{F}, f_*\mathcal{G}).$$

Example 2.2.3. Pushing forward to a point is the same as taking global sections: let $f\colon X \to \mathrm{pt}$ be constant. Then $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(\mathrm{pt}) \simeq \mathbf{Vect}_{\mathbb{C}}$, and $f_* \simeq \Gamma(X, -)$; $f^{-1}(pt) = X$. If $M \in \mathrm{Sh}(pt) = \mathbf{Vect}_{\mathbb{C}}$ and $\mathcal{F} \in \mathrm{Sh}(X)$, the adjunction says

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\underline{M}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathbb{C}}(M, \Gamma(\mathcal{F})).$$

Example 2.2.4. If $\iota_x\colon \{x\} \hookrightarrow X$ is inclusion of a point, and $M \in \mathrm{Sh}(\{x\}) = \mathbf{Vect}_{\mathbb{C}}$, then $\iota_{x,*}M$ is a skyscraper $\underline{M}^x$ on X supported at x. To see that $\iota_{x,*}M$ is not supported outside of x, one can use the fact that the set of open neighbourhoods of any $y \neq x$ not containing x is cofinal with the set of all open neighbourhoods of y.

The adjunction now gives

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_x, M) \simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, \underline{M}^x)$$

for $x \in X$ and $\mathcal{F} \in \mathrm{Sh}(X)$.

Proposition 2.2.5. *Let $f\colon X \to Y$ be continuous. Then we have a morphism*

$$(f_*\mathcal{F})_y \to \Gamma(\mathcal{F} \restriction_{f^{-1}(y)}).$$

Proof. We have $(f_*\mathcal{F})_y = \mathrm{colim}_{U \ni y} \mathcal{F}(U)$, so $f^{-1}(U)$ is an open set containing $f^{-1}(y)$. Thus there is a morphism into $\mathrm{colim}_{V \supset f^{-1}(y)} \mathcal{F}(V)$, and this maps by sheafification to the right-hand-side. □

Example 2.2.6. This map need not be an isomorphism: Let $\Lambda \subset \mathbb{C}$ be a lattice and let $f\colon \mathbb{C} \to \mathbb{C}/\Lambda$ to be the natural map. Let $\mathcal{F}$ be the sheaf of holomorphic functions on $\mathbb{C}$. Then $(f_*\mathcal{F})_y = \bigoplus_{\lambda \in \Lambda} \mathcal{F}_{y+\lambda}$, whereas $\Gamma(\mathcal{F} \restriction_\Lambda)$ should be $\prod_{\lambda \in \Lambda} \mathcal{F}_\lambda$.

3 Local systems and monodromy

3.1 Local systems

Definition 3.1.1. A *local system* or *locally constant sheaf* is a sheaf $\mathcal{F}$ such that there is an open cover $\{U_\alpha\}$ of X with $\mathcal{F} \restriction_{U_\alpha}$ a constant sheaf.

Theorem 3.1.2. *The category* $\mathrm{Loc}(X)$ *of local systems is a full abelian subcategory of* $\mathrm{Sh}(X)$*, when X is locally connected.*

Idea of proof. It suffices to show that $\mathrm{Loc}(X)$ is closed under (co)kernels and extensions. We will show the kernel of a morphism of local systems is a local system. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism and V be open connected such that $\mathcal{F} \restriction_V$ and $\mathcal{G} \restriction_V$ are constant. Then for any open connected U, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\ker f)(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\ker f)(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \end{array}$$

□

with exact rows, where the vertical maps are restrictions. The rightmost two restrictions are isomorphisms, so the restriction $(\ker \mathcal{F})$ is an isomorphism. It follows that $(\ker f) \upharpoonright_V$ is constant.

Closure under cokernels and extensions: exercise.

Lemma 3.1.3. *Let X be connected and locally connected. The constant sheaf functor $M \mapsto \underline{M}_X$ is fully faithful. The global sections functor $\mathcal{F} \mapsto \Gamma(\mathcal{F})$ is a left inverse to the constant sheaf functor.*

Proof. X is connected, so lemma 1.4.3 says that the global sections of $\underline{M}_X$ are naturally identified with M. Recall that Γ is right adjoint to the constant sheaf functor. Therefore

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\underline{M}_X, \underline{N}_X) \simeq \mathrm{Hom}_{\mathbb{C}}(M, \Gamma(\underline{N}_X)) \simeq \mathrm{Hom}_{\mathbb{C}}(M, N).$$

So the constant sheaf functor is fully faithful, and Γ is left inverse to it (look just at rightmost isomorphism and apply the Yoneda lemma).

Any point $x \in X$ has a neighbourhood basis of connected sets, so we can compute stalks using just these sets. Because restrictions between these sets are identity maps, $M \simeq \Gamma(\underline{M}_X) \to (\underline{M}_X)_x$. □

3.2 Pushforward of local systems

Note that inclusion of a point is proper, but skyscraper sheaves are certainly not locally constant, so we cannot expect to push forward a local system and obtain a local system using just any map, even with proper pushforward.

Definition 3.2.1. A map f is a *covering map* if it is surjective and every point in Y has a neighbourhood U such that $f^{-1}(U)$ is a disjoint union of open subsets V_α with $f \upharpoonright_{V_\alpha}$ a homeomorphism (onto U).

Proposition 3.2.2. *If $f\colon X \to Y$ is a covering map and Y is locally path- connected and locally simply connected, and $\mathcal{F} \in \mathrm{Loc}(X)$, then $f_*\mathcal{F} \in \mathrm{Loc}(Y)$. Moreover, f_* is exact between these categories.*

Proof. Let $y \in Y$. We can always pick a simply connected neighbourhood $U \ni y$ whose preimage is a disjoint union $\coprod_\alpha V_\alpha$ of copies of U. By the equivalence of categories between representations of π_1 and local systems, any local system on U or V_α is constant. By exactness of the sequence defining the sheaf condition, $\mathcal{F}(f^{-1}(U)) \simeq \prod_\alpha \mathcal{F}(V_\alpha)$.

Let $\mathcal{F} \in \mathrm{Loc}(X)$. Let $M_\alpha := \mathcal{F}(V_\alpha)$ so that $\mathcal{F} \upharpoonright_{V_\alpha} \simeq \underline{M_\alpha}_{V_\alpha}$. For any connected open $U' \subset U$,

$$f_*(\mathcal{F})(U') = \mathcal{F}'(f^{-1}(U')) \simeq \prod_\alpha \mathcal{F}(V_\alpha \cap f^{-1}(U')) \simeq \prod_\alpha M_\alpha.$$

By a recognition criterion developed in the book, $f_*\mathcal{F}$ is locally constant.

We already know f_* is left exact, so it is enough to prove it preserves surjective morphisms. Let $q\colon \mathcal{F} \twoheadrightarrow \mathcal{G}$. It is enough to show f_*q is surjective on sections, for a small enough open set in Y. Let U be as in the beginning of the proof, and the V_α as above. Since $\mathcal{F} \upharpoonright_{V_\alpha}$ and $\mathcal{G} \upharpoonright_{V_\alpha}$ are locally constant and the constant sheaf functor is fully faithful, $q(V_\alpha)$ is surjective. Then we have

$$(f_*q)(U) = \prod_\alpha q(V_\alpha),$$

which is surjective. □

Remark 3.2.3. The same proof yields the same conclusion for $f_!$, expect with $\bigoplus$ instead of $\prod$.

Example 2 (Exercise 2.2.1 in the book). Let $\mathcal{Q}$ be the sheaf on $X = \mathbb{C}^\times$ with sections

$$\mathcal{Q}(U) = \left\{ f\colon U \to \mathbb{C} \,\middle|\, 2z\frac{dg}{dz} = g \right\}.$$

We claim that $\mathcal{Q}$ is locally constant, but not constant. On any open connected simply-connected set $U \subset X$, we may define a branch of the square root function, by $g(z) = e^{\frac{1}{2}\log(z)}$ for a branch log of the complex logarithm. Note $g \neq 0$ on U. We claim any section of $\mathcal{Q}(U)$ is proportional to g. Indeed,

$$\frac{d}{dz}(\frac{h}{g}) = g^{-1}\frac{dh}{dz} - g^{-2}h\frac{dg}{dz} = g^{-1}\frac{dh}{dz} - \frac{1}{z}2z\frac{dh}{dz}\frac{g}{2z} = (g^{-1} - \frac{1}{z}g)\frac{dh}{dz},$$

and $g^{-1} = \frac{1}{z}g$ is zero. This gives isomorphisms

$$\mathcal{Q}(U) \simeq \{\alpha g \mid \alpha \in \mathbb{C}\} \simeq \mathbb{C} \simeq \underline{\mathbb{C}}_X(U).$$

and the existence of g on U gives a natural isomorphism $\mathcal{Q} \restriction_U \simeq \underline{\mathbb{C}}_X \restriction_{\mathbb{C}^\times}$.

The sheaf $\mathcal{Q}$ is not constant. One way to show this is that it does not have nontrivial global sections. If g is a solution to the differential equation and γ is a loop around the origin, then

$$\int_\gamma \frac{g}{g'}\,dz = \int_\gamma \frac{1}{2z}\,dz = \pi i,$$

but this is the winding number of the path $g \circ \gamma$, so should be in $2\pi i\mathbb{Z}$.

Example 3 (Exercise 2.2.2 from the book). Let $f\colon \mathbb{C}^\times \to \mathbb{C}^\times$ be the two-sheeted cover $f\colon z \mapsto z^2$. Then we claim that $f_*\underline{\mathbb{C}}_X \simeq \underline{\mathbb{C}}_X \oplus \mathcal{Q}$. We will construct a map out of the direct product from its universal property. Define

$$\underline{\mathbb{C}}_X(U) \to f_*\underline{C}_X)(U)$$

by

$$g \mapsto g(f(z)).$$

The image function is constant on connected components because continuous images of connected sets are connected. Next define

$$\mathcal{Q}(U) \to f_*\underline{C}_X(U)$$

by

$$g \mapsto \frac{g(f(z))}{z}.$$

These images have zero derivative, hence are locally constant, because g solves the differential equation.

The induced map φ_w on stalks is an isomorphism. It is enough to show it is injective; $(f_*\underline{\mathbb{C}}_X)_w \simeq \mathbb{C}_{z_1} \oplus \mathbb{C}_{z_2}$ where $z_1^2 = z^2 = w$. Say that $\varphi_z(g_1, g_2) = 0$. This means there is $V \ni w$ and small balls $f^{-1}(V) \supset V_{z_i} \ni z_i$ such that

$$\frac{g_1(z^2)}{z} = -g_2(z^2).$$

On each V_i. On each V_i, $g_2(z^2)$ is constant, and $z_1^2 = z_2^2$, so $g_2(z^2)$ is constant on $V_1 \sqcup V_2$. But $z_1 = -z_2$, and now

$$-g_2(w) = \frac{g_1(w)}{z_1} = -\frac{g_1(w)}{z_2} = g_2(w).$$

Then $g_2 = 0$ in V whence $g_1 = 0$ in V.

3.3 Local systems and representations of π_1

The main result is:

Theorem 3.3.1. *Let X be path-connected, and locally simply connected, $x_0 \in X$. Then there is an equivalence of abelian categories*

$$\mathrm{Mon}_{x_0}\colon\ \mathrm{Loc}(X) \xrightarrow{\sim} \mathbf{Rep}(\pi_1(X, x_0)).$$

given by monodromy functor *sending a local system to its* monodromy representation at x_0.

Representations of π_1 will come from composing isomorphisms along paths. We have to know some things about covering paths.

Lemma 3.3.2. *If $\mathcal{F}$ is a local system on a path-connected space X, then $\mathcal{F}_x \simeq \mathcal{F}_y$ for all $x, y \in X$.*

Proof. Chain finitely-many isomorphisms together by covering a path with connected subsets. □

Proof of theorem. The vector space will be $\mathcal{F}_{x_0}$. Let $*$ denote usual concatenation of paths. Given a path γ, let sets U_α cover a it from $\gamma(0)$ to $\gamma(1)$ such that $\mathcal{F}\restriction_{U_\alpha}$ is constant. The $\gamma^{-1}(U_\alpha)$ cover $[0,1]$.

Lemma 3.3.3. *Let $\{V_\alpha\}_\alpha$ cover $[0,1]$. There is a sequence of real numbers a_i such that $0 = a_0 < a_1 < \cdots < a_n = 1$ and $[a_i, a_{i+1}]$ contained in only one V_α.*

Therefore we get indicies $\alpha_1, \ldots, \alpha_n$ and points $a_1, \ldots, a_n \in [0,1]$ such that $\gamma[a_i, a_{i+1}] \subset U_{\alpha_{i+1}}$. Now, $\mathcal{F}\restriction_{\gamma[a_i,a_{i+1}]}$ is a constant sheaf for each i (lemma on pullbacks of constant sheaves). We get isomorphisms:

$$\begin{array}{ccccccccc} & \Gamma(\mathcal{F}\restriction_{\gamma([a_{n-1},a_n])}) & & \Gamma(\mathcal{F}\restriction_{\gamma([a_{n-2},a_{n-1}])}) & & \cdots & & \Gamma(\mathcal{F}\restriction_{\gamma([a_0,a_1])}) & \\ \swarrow & & \searrow\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{F}_{\gamma(a_n)} & & \mathcal{F}_{\gamma(a_{n-1}} & & \mathcal{F}_{\gamma(a_{n-2})} & \cdots & \mathcal{F}_{\gamma(a_1)} & & \mathcal{F}_{\gamma(a_0)} \end{array}$$

We want to set $\rho(\gamma)$ equal to the composition, so we must show this isomorphism does not depend on the points a_i chosen. Possible to show by induction that adding a finite number of points gives the same isomorphism, so (take unions) any numbers $\{a_i\}$ give the same isomorphism.

Pasting the diagrams shows $\rho(\gamma_1 * \gamma_2) = \rho(\gamma_1) \circ \rho(\gamma_2)$.

Finally have to show ρ descends to quotient by homotopy. The key ingredient in checking this is a version of lemma 3.3.3 for rectangles.

Functoriality in $\mathcal{F}$: exercise.

Now define $\mathrm{Mon}_{x_0}(\mathcal{F})$ to be the representation with vector space $\mathcal{F}_{x_0}$ and action as above.

We will now sketch the inverse functor. Fix a choice of path α_x from x_0 to x, for each $x \in X$, and chose α_{x_0} to be the constant path. Given a representation M, set

$$Q(M)(U) := \left\{ s\colon U \to M \,\middle|\, s(\gamma(0)) = (\alpha_{\gamma(0)} * \gamma * \alpha_{\gamma(1)}^{-1}) \cdot s(\gamma(1))\ \forall \gamma\colon [0,1] \to U \right\}.$$

This is a sheaf. The open sets on which $Q(M)$ restricts to a constant sheaf are the ones coming from the locally simply connected hypothesis. □

Here is another way of seeing what kind of sheaf $Q(M)$ really is. Given M, consider $\underline{M}_{\tilde{X}}$. This is a trivial local system, but it's pushforward to X need not be. The local system $Q(M)$ should be what is obtained by pushing $\underline{M}_{\tilde{X}}$ forward along the covering map, then taking $\pi_1(X, x_0)$- invariants. Pullback sends local systems to local systems, in the nicest possible way.

Proposition 3.3.4. *$M \in \mathbf{Vect}_{\mathbb{C}}$, $f\colon X \to Y$. Then $f^*\underline{M}_Y \simeq \underline{M}_X$. Further, if $\mathcal{F}$ is a local system, so is $f^*\mathcal{F}$.*

Proof. Let $p\colon Y \to \mathrm{pt}$ be constant, and then by a previous example, $\underline{M}_Y = p^*M$ for a sheaf (vector space) on pt. By functoriality,

$$f^*\underline{M}_Y \simeq (f^*(p^*M)) \simeq (p \circ f)^*(M) \simeq \underline{M}_X.$$

□

Recall that a map $f\colon (X, x_0) \to (Y, y_0)$ of pointed spaces gives a homomorphism

$$\pi_1(f)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0),$$

yielding a restriction functor

$$\mathrm{Res}^{\pi_1(Y,y_0)}_{\pi_1(X,x_0))}\colon \mathbf{Rep}(\pi_1(Y, y_0)) \to \mathbf{Rep}(\pi_1(X, x_0)).$$

Lemma 3.3.5. *Given* $f\colon (X, x_0) \to (Y, y_0)$ *and* $\mathcal{F} \in \mathrm{Loc}(Y)$*, we have*

$$\mathrm{Mon}_{x_0}(f^*\mathcal{F}) = \mathrm{Res}^{\pi_1(Y,y_0)}_{\pi_1(X,x_0)} \mathrm{Mon}_{y_0}(\mathcal{F}).$$

Idea of proof. Recall that $(f^*\mathcal{F})_{x_0} \simeq \mathcal{F}_{y_0}$, so the vector spaces on both sides are identified. Therefore to prove the lemma one just needs to show that $\rho(\gamma)$ and $\rho(f \circ \gamma)$ can be identified, where γ is a loop based at x_0. □

Example 4. We have a two-sheeted cover $\mathbb{C}^\times \to \mathbb{C}^\times$ given by $z \mapsto z^2$. Pushing forward the trivial representation $\underline{\mathbb{C}}_{\mathbb{C}^\times}$ gives a two-dimensional representation of $\pi_1(C^\times) = \mathbb{Z}$, which decomposes into the trivial representation and sign representation.

4 Proper pushforward

A map $f\colon X \to Y$ is called *proper* if inverse images of compacts are compact. If X and Y are locally compact (Hausdorff with neighbourhood bases of precompact sets, *i.e.* whose closures are compact) then properness is equivalent to f being closed (images of closed sets are closed) and all fibres being compact. Spaces in this section, and usually when discussing pushforwards in general, are all locally compact.

The *proper pushforward* is $f_!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

$$f_!(\mathcal{F})(U) = \left\{ s \in \mathcal{F}(f^{-1}(U)) \,\middle|\, f\restriction_{\operatorname{supp} s}\colon \operatorname{supp} s \to U \text{ is proper} \right\}.$$

If f is proper then $f_! = f_*$.

The presheaf $f_!\mathcal{F}$ is actually a sheaf. We need only check the gluing axiom. Let $U = \bigcup_\alpha U_\alpha \subset Y$ so that $f^{-1}(U) = \bigcup_\alpha f^{-1}(U_\alpha) \subset X$ with $s_\alpha \in f_!\mathcal{F}(U_\alpha)$ all agreeing on overlaps. The s_α glue to a section $s \in f_*\mathcal{F}(U)$; we must show $f\restriction_{\operatorname{supp} s}$ is proper. Let $K \subset U$ be compact, so that

$$K \subset \bigcup_{\text{finite}} U_\beta.$$

Then

$$f^{-1}(K) \cap \operatorname{supp}(s) = f^{-1}(K) \cap \left(\bigcup_\alpha \operatorname{supp} s_\alpha \right) = \bigcup_{\text{finite}} (f^{-1}(K) \cap \operatorname{supp} s_\beta),$$

which is a finite union of compact sets (f is proper on each $\operatorname{supp} s_\beta$), hence is compact. This proves the claim.

Proposition 4.0.1. $f_!$ *is left-exact.*

Proof. Diagram chase using exactness of the bottom row of

$$\begin{array}{ccccccc} 0 & \longrightarrow & f_!\mathcal{F}(U) & \longrightarrow & f_!\mathcal{G}(U) & \longrightarrow & f_!\mathcal{H}(U) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f_*\mathcal{F}(U) & \longrightarrow & f_*\mathcal{G}(U) & \longrightarrow & f_*\mathcal{H}(U) \end{array}.$$

□

Let Γ_c be the functor giving global sections with compact support. Then we have isomorphisms $f_! \simeq \Gamma_c\colon \mathrm{Sh}(X) \to \mathbf{Vect}_{\mathbb{C}}$, where $f\colon X \to \mathrm{pt}$ is constant.

Proper pushforward also has good description of the stalks, under the condition X and Y be locally compact.

Proposition 4.0.2. *Let* $f\colon X \to Y$ *be a morphism of locally compact spaces, then if* $y \in Y$*,* $\mathcal{F} \in \mathrm{Sh}(X)$*, have natural isomorphism*

$$(f_!\mathcal{F})_y \xrightarrow{\sim} \Gamma_c(\mathcal{F}\restriction_{f^{-1}(y)}).$$

Proof. We have

$$f_!(\mathcal{F})_y = \operatorname*{colim}_{\substack{V \subset Y \\ V \ni y}} \left\{ s \in \mathcal{F}(f^{-1}(V)) \,\middle|\, f \restriction_{\operatorname{supp} s} \text{ is proper} \right\}$$

and

$$\Gamma_c(\mathcal{F} \restriction_{pre, f^{-1}(y)}) = \left\{ s \in \operatorname*{colim}_{\substack{U \subset X \\ X \supset f^{-1}(y)}} \mathcal{F}(U) \,\middle|\, \operatorname{supp} s \cap f^{-1}(y) \text{ is compact} \right\},$$

where $\mathcal{F} \restriction_{pre,f^{-1}(y)}$ is a presheaf which sheafifies to the restriction.

Let $s \in f_!(\mathcal{F})_y$ be represented by $s' \in \mathcal{F}(f^{-1}(V))$ with $f \restriction_{\operatorname{supp} s'}$ proper. Thus $(f \restriction_{\operatorname{supp} s'})^{-1}(y) = \operatorname{supp} s \cap f^{-1}(y)$ is compact. Thus $s' \in \Gamma_c(\mathcal{F} \restriction_{pre,f^{-1}(y)})$. Thus we get a map

$$f_!(\mathcal{F})_y \to \Gamma_c(\mathcal{F} \restriction_{pre,f^{-1}(y)}) \xrightarrow{sh} \Gamma_c(\mathcal{F} \restriction_{f^{-1}(y)}),$$

where sh is the (global component of the) sheafification morphism.

One can show this is a bijection. Injectivity is easy: If $s \mapsto s' = 0$, then there is an open set U such that $f^{-1}(y) \subset U \subset f^{-1}(V)$ with $s' \restriction_U = 0$. Thus $y \notin \operatorname{supp} s'$. As f is in particular closed when restricted to $\operatorname{supp} s'$, we have $V' := V \setminus f(\operatorname{supp} s')$ is an open neighbourhood of y. By construction, $s' \restriction_{f^{-1}(V')} = 0$, so s is the zero germ at y.

Surjectivity: exercise; see the proof of lemma 2.3.6 in the book for full details. □

Proposition 4.0.3. *Proper pushforward is functorial in the sense that:* $(g \circ f)_! \mathcal{F} \simeq g_!(f_! \mathcal{F})$ *if* $g \circ f \colon X \to Y \to Z$ *and* $\mathcal{F} \in \operatorname{Sh}(X)$.

Proof. $s \in f_!\mathcal{F}(U) \subset \mathcal{F}(f^{-1}(U))$ has a support $\operatorname{supp}_X(s) \subset f^{-1}(U)$ as a section of $\mathcal{F}$, and a (distinct) support $\operatorname{supp}_Y(s) \subset U$ as a section of $f_!\mathcal{F}$. We have $\operatorname{supp}_X(s) = \left\{ x \in f^{-1}(U) \,\middle|\, s_x \neq 0 \text{ in } \mathcal{F}(f^{-1}(U) \right\}$ and $\operatorname{supp}_Y(s) = \{ y \in U \mid s_y \neq 0 \text{ in } (f_!\mathcal{F})_y \}$. We have $f(\operatorname{supp}_X s) = \operatorname{supp}_Y s$. We have to show the containment

$$\begin{aligned} g_!(f_!(\mathcal{F}))(U) = \left\{ s \in \mathcal{F}(g^{-1}(f^{-1}(U)) \,\middle|\, f \restriction_{\operatorname{supp}_X s} \text{ and } g \restriction_{\operatorname{supp}_Y s} \text{ proper} \right\} \\ \subset (g \circ f)_!(\mathcal{F})(U) = \left\{ s \in \mathcal{F}(g^{-1}(f^{-1}(U)) \,\middle|\, (g \circ f) \restriction_{\operatorname{supp}_X s} \text{ proper} \right\}. \end{aligned}$$

is an equality. To see this containment, check directly that the composite is proper when f and g are. To prove the opposite containment: point set topology shows that

$$(g \circ f) \restriction_{\operatorname{supp}_X s} \text{ proper} \implies f \restriction_{\operatorname{supp}_X s} \text{ proper},$$

so we can view s as a section of $f_!\mathcal{F}(U)$, and $\operatorname{supp}_Y s$ is defined. Then more point set topology shows that $g \restriction_{\operatorname{supp}_Y s}$ is proper, finishing the proof. □

4.1 Proper base change

Theorem 4.1.1. *Given a Cartesian square (pullback diagram) of locally compact spaces*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

there is an isomorphism of functors $g^* \circ f_! \simeq f'_! \circ (g')^* \colon \operatorname{Sh}(X) \to \operatorname{Sh}(Y')$.

The horizontal maps are *base change morphisms*. The steps of the proof are:

1. Produce a natural map $g^* f_* \mathcal{F} \to f'_*(g')^*\mathcal{F}$;
2. Show our map restricts to a map $g^* f_! \mathcal{F} \to f'_!(g')^*\mathcal{F}$;

3. Show this restricted map is an isomorphism on stalks.

Steps 1 and 2 are the content of

Lemma 4.1.2. *Given a Cartesian square, there is a commutative square*

$$\begin{array}{ccc} g^* f_! \mathcal{F} & \longrightarrow & f'_!(g')^*\mathcal{F} \\ \downarrow & & \downarrow \\ g^* f_* \mathcal{F} & \longrightarrow & f'_*(g')^*\mathcal{F} \end{array}$$

with the vertical maps being the natural inclusions induced by $f_! \hookrightarrow f_*$ and likewise for f'.

Proof of lemma. Given $V' \subset Y'$, set

$$V = f^{-1}(g(V')) = g'((f')^{-1}(V')).$$

Then we have a cartesian square of topological spaces

$$\begin{array}{ccc} (f')^{-1}(V') & \xrightarrow{g'} & V \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ V' & \xrightarrow{g} & g(V') \end{array}$$

We will construct a commutative diagram

$$\begin{array}{ccc} (g^*_{pre} f_! \mathcal{F})(V') & \longrightarrow & (f_!(g')^*_{pre}\mathcal{F})(V') \\ \downarrow{\scriptstyle g^*_{pre}(f_! \hookrightarrow f_*)} & & \downarrow{\scriptstyle \text{inclusion}} \\ (g^*_{pre} f_* \mathcal{F})(V') & \longrightarrow & (f'_*(g')^*\mathcal{F})(V') \end{array}$$

The bottom map exists because the directed system for $g^*_{pre} f_* \mathcal{F}$ is a subset of the directed system for $f'_*(g')^*_{pre}$: by the square (4.1)

$$(g^*_{pre} f_*)\mathcal{F} = \operatorname*{colim}_{\substack{U \subset X \\ U = f^{-1}(U') \\ U' \supset g(V')}} \mathcal{F}(U).$$

and

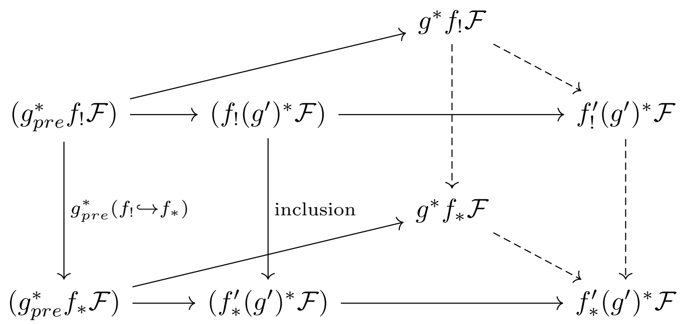
$$f'_*(g')^*_{pre}\mathcal{F} = \operatorname*{colim}_{\substack{U \subset X \\ U \supset V}} \mathcal{F}(U)$$

We must show the bottom map restricts appropriately to complete our square. Given $s \in \mathcal{F}(U)$ such that $f \restriction_{\operatorname{supp} s}$ is proper, we must show that $f' \restriction_{(g')^{-1}(\operatorname{supp} s) \cap (f')^{-1}(V')}$ is proper. This follows from the description of pullbacks in **Top**, and the fact that the outer square in

$$\begin{array}{ccccc} (g')(\operatorname{supp} s) \cap (f')^{-1}(V') \subset X' & \longrightarrow & \operatorname{supp} s \cap V \subset X & \hookrightarrow & \operatorname{supp} s \subset X \\ \downarrow & & \downarrow & & \downarrow \\ V' \subset Y' & \longrightarrow & g(V') \subset Y' & \hookrightarrow & U \subset Y \end{array}$$

is a pullback (pasting of pullbacks).

The natural transformation $f_! \to f_*$ applied to sheafification $(g')^*_{pre}\mathcal{F} \to (g')^*\mathcal{F}$ gives two squares, and sheafification again gives



The dashed square is the one we wanted. ☐

Proof of theorem. We get $(g^* f_!)\mathcal{F} \to (f'_!(g')^*)\mathcal{F}$ from the lemma, and it suffices to show it is an isomorphism on stalks. We have

$$(f_!\mathcal{F})_{g(y)} \simeq (g^* f_! \mathcal{F})_y \xrightarrow{\text{lemma}} (f'_!(g')^* \mathcal{F})_y \xrightarrow{\text{description of stalks},\sim} \Gamma_c\left([(g')^*\mathcal{F}] \restriction_{(f')^{-1}(y)}\right) \simeq \Gamma_c(\mathcal{F} \restriction) f^{-1}(y))$$

The description of stalks gives an isomorphism $(f_!\mathcal{F})_{g(y)} \xrightarrow{\sim} \Gamma_c(\mathcal{F} \restriction_{f^{-1}(y)}))$, which is actually this composite. Thus the lemma map is an isomorphism. ☐

5 Open and closed embeddings

Lemma 5.0.1. *Let $h\colon Y \hookrightarrow X$ be an inclusion of a locally closed subspace. Then if $\mathcal{F} \in \mathrm{Sh}(Y)$, $h_!\mathcal{F}$ is naturally isomorphic to the sheafification of the presheaf $h_{!,pre}(\mathcal{F})$ defined by*

$$U \mapsto \begin{cases} \mathcal{F}(U \cap Y) & \text{if } U \cap \bar{Y} \subset Y \\ 0 & \text{otherwise} \end{cases}.$$

$h_!$ is exact, with stalks

$$h_!(\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the Hausdorff assumption,

$$\mathrm{supp}(s) \hookrightarrow U \text{ is proper} \iff \mathrm{supp}(s) \text{ is closed in } U$$

so

$$h_!(\mathcal{F})(U) = \{s \in \mathcal{F}(U \cap Y) \,|\, \mathrm{supp}(s) \subset U \text{ is closed}\}$$

If $U \cap \bar{Y} \subset Y$, then $U \cap \bar{Y} = U \cap Y$ is closed in U, so supports are closed in U as well. Therefore if $U \cap \bar{Y} \subset Y$, then $h_!(\mathcal{F})(U) = \mathcal{F}(U \cap Y)$. This shows that there is a morphism of presheaves $h_{!,pre}(\mathcal{F}) \to h_!(\mathcal{F})$, hence $h^+_{!,pre}(\mathcal{F}) \to h_!(\mathcal{F})$. Can show that stalks of both are as in the statement, so this is an isomorphism. (Exercise.)

Description of stalks implies exactness. ☐

$h_!$ preserves stalks over Y and has zero stalks outside, so it is called the *extension by zero functor*.

5.1 Restriction with supports

Definition 5.1.1. Let $h\colon Y \hookrightarrow X$ be an inclusion of a locally closed subspace. *Restriction with supports* is the functor $h^!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ defined by

$$h^!(\mathcal{F})(U) = \operatorname*{colim}_{\substack{V \in \mathrm{Op}(X) \\ V \cap \bar{Y} = U}} \{s \in \mathcal{F}(V) \,|\, \mathrm{supp}(s) \subset U\}.$$

Exercise: this is a sheaf. Hint: For all $V \subset X$ open such that $V \cap \bar{Y} = U$, we actually have

$$\{s \in \mathcal{F}(V) \,|\, \mathrm{supp}(s) \subset U\} \xrightarrow{\sim} h^!(\mathcal{F})(U).$$

to show this, let $V' \subset V$ with $V' \cap \bar{Y} = U$, and do a diagram chase with

$$\begin{array}{ccccccc} 0 & \longrightarrow & \{s \in \mathcal{F}(V) \,|\, \mathrm{supp}(s) \subset U\} & \hookrightarrow & \mathcal{F}(V) & \xrightarrow{res} & \mathcal{F}(V \setminus U) \\ & & \downarrow q & & \downarrow res & & \downarrow res \\ 0 & \longrightarrow & \{s \in \mathcal{F}(V') \,|\, \mathrm{supp}(s) \subset U\} & \hookrightarrow & \mathcal{F}(V') & \xrightarrow{res} & \mathcal{F}(V' \setminus U) \end{array}$$

to show q is an isomorphism.

The set of open subsets $V \subset X$ such that $V \cap \bar{Y} = U$ is cofinal with the set of all $V \subset X$ such that $V \supset U$ (Given $V \supset U$, take V' such that $U = V' \cap \bar{Y}$. Then $V_1 := V \cap V' \subset V$ and $V_1 \cap \bar{Y} = U$.) Therefore we have a have natural injective map $h^!(\mathcal{F}) \hookrightarrow h^*(\mathcal{F})$.

If h is an open inclusion, then $h^! = h^*$: If $U' \subset U \hookrightarrow X$ is open in an open, then there is a unique minimal V, namely $V = U'$. So the support condition is automatic.

Proposition 5.1.2. *Let $h\colon Y \hookrightarrow X$ be inclusion of locally closed subspace. We have an adjunctionf*

$$\mathrm{Hom}_{Sh(X)}(h_!\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{F}, h^!\mathcal{G}).$$

Proof. Enough to construct a natural isomorphism of hom sets with $h_{!,pre}(\mathcal{F})$ in place of $h_!\mathcal{F}$. Let $U \subset Y$ be open, and let $V \subset X$ be open such that $V \cap \bar{Y} = U$. As Y is locally closed, all open subsets of Y arise this way. Note that U is closed in V. Given $f\colon h_{!,pre}\mathcal{F} \to \mathcal{G}$, consider

$$\begin{array}{ccc} h_{!,pre}\mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \\ \downarrow & & \downarrow \\ h_{!,pre}\mathcal{F}(V \setminus U) = 0 & \xrightarrow{f_{V \setminus U}} & \mathcal{G}(V \setminus U). \end{array}$$

The over-down composition is then also zero, and so f_V must land in $h^!(\mathcal{G})(U)$, yielding a map

$$\mathcal{F}(U) = h_{!,pre}\mathcal{F}(V) \to \{s \in \mathcal{G}(V) \,|\, \mathrm{supp}(s) \subset U\} \simeq h^!(\mathcal{G})(U). \tag{1}$$

Define $\phi\colon \mathrm{Hom}(h_{!,pre}\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, h^!\mathcal{G})$ by defining $\phi(f)_U$ as in (1).

If $\phi(f)$ is zero, then f_V is zero for all $V \subset X$ with $V \cap \bar{Y} \subset Y$. But for any open subset of X not of this form, $h_{!,pre}\mathcal{F}$ is itself zero. So ϕ is injective.

Sketch of surjectivity: Given $g\colon \mathcal{F} \to h^!\mathcal{G}$, define f as the composite

$$h_{!,pre}\mathcal{F}(V) = \mathcal{F}(V \cap Y) \to h^!\mathcal{G}(V \cap Y) \simeq \{s \in \mathcal{G}(V) \,|\, \mathrm{supp}(s) \subset V \cap U\} \hookrightarrow \mathcal{G}(V)$$

can check this is a morphism of sheaves and gives a section of ϕ. □

Corollary 5.1.3. *$h^!$ is left exact.*

Proof. $h^!$ has a left adjoint. □

Corollary 5.1.4. *$(h \circ j)^!\mathcal{F} \simeq j^!h^!\mathcal{F}$, if $h\colon Y \hookrightarrow X$ and $j\colon W \hookrightarrow Y$ are locally closed inclusions.*

Proof. Functoriality of proper pushforward, plus uniqueness of adjoint functors. □

Proposition 5.1.5. *Let $h\colon Y \to X$ be a locally closed inclusion. For any $\mathcal{F} \in \mathrm{Sh}(Y)$, the natural maps*

$$h^!h_*\mathcal{F} \xrightarrow{\iota} h^*h_*\mathcal{F} \to \mathcal{F} \text{ and } \mathcal{F} \to h^!h_!\mathcal{F} \xrightarrow{\iota'} h^*h_!\mathcal{F},$$

where ι and ι' are induced by the the natural transformation $h^! \hookrightarrow h^*$ and the other maps are adjunction morphisms, are all isomorphisms.

Proof. By definition we have

$$h^*_{pre} h_*(\mathcal{F})(U) = \operatorname*{colim}_{\substack{V\subset X\\ V\supset U}} h_*(\mathcal{F})(V) = \operatorname*{colim}_{\substack{V\subset X\\ V\supset U}} \mathcal{F}(V\cap Y) \xrightarrow{\phi} \mathcal{F}(U),$$

where the map ϕ is induced by the restrictions $\mathcal{F}(V\cap Y)\to\mathcal{F}(U)$. For V small enough, we have $V\cap Y = U$, so ϕ is an isomorphism. Thus $h^*h_*\mathcal{F}\to\mathcal{F}$ is an isomorphism.

From the description of stalks of $h_!\mathcal{F}$ from lemma 5.0.1, we know if $s\in h_!\mathcal{F}(U)$, then $\operatorname{supp}(s)\subset U\cap Y$. We will leave showing ι is an isomorphism as an exercise. From the definition of $h^!$, we see that $h^!h_!\mathcal{F}\to h^*h_!\mathcal{F}$ is an isomorphism. Again using lemma 5.0.1, we see that

$$\mathcal{F}\to h^!h_!\mathcal{F}\to h^*h!\mathcal{F}$$

is isomorphism, so the first map is an isomorphism, too.

ι is an isomorphism: exercise. Hint: Can factor any h into open and closed inclusions.

$h^!h_*\mathcal{F}\simeq h^*h_*\mathcal{F}$: If h is a closed inclusion, $h_*\simeq h_!$, and if h is an open inclusion, $h^*\simeq h^!$. We can factor any h as $h = j\circ i\colon Y\hookrightarrow \bar{Y}\hookrightarrow X$, and we are done modulo showing ι' is an isomorphism:

$$\begin{aligned}(j\circ i)^!(j\circ i)_*\mathcal{F} &\simeq i^!j^!(j_*i_*\mathcal{F})\\ &\simeq i^*j^!j_!i_*\mathcal{F}\\ &\simeq i^*j^*j_!i_*\mathcal{F}\\ &\simeq (j\circ i)^*j_*i_*\mathcal{F}\\ &\simeq h^*h_*\mathcal{F}.\end{aligned}$$

□

Theorem 5.1.6. *Let $i\colon Z\to X$ be a closed inclusion, and $j\colon U\to X$ be the open inclusion of the complement. Then*

$$i^*\circ j_! = 0,\ \ i^!\circ j_* = 0,\ \ j^*\circ i_* = 0.$$

Proof. Exercise. The functors are both $\operatorname{Sh}(U)\to\operatorname{Sh}(X)\to\operatorname{Sh}(Z)$. At $z\in Z$,

$$(i^*j_!\mathcal{F})_z = (j_!\mathcal{F})_{i(z)} \simeq \Gamma_c(\mathcal{F}\restriction_{j^{-1}(i(z))}) = 0$$

as $j^{-1}(i(z)) = \emptyset$.

If $W\subset Z$ is open,

$$(i^!j_*\mathcal{F})(W) = \operatorname*{colim}_{\substack{V\subset X\\ V\cap Z = W}} \{s\in\mathcal{F}(V\cap U)\,|\,\operatorname{supp}(s)\subset W\},$$

but $(V\cap U)\cap W = \emptyset$ so the colimit is zero. □

6 Tensor product and internal hom

Sheaf versions of usual constructions in $\mathbf{Vect}_{\mathbb{C}}$. Both will send local systems to local systems in the nicest possible way.

6.1 Tensor product

Sheafification of the presheaf

$$U\mapsto \mathcal{F}(U)\otimes_{\mathbb{C}}\mathcal{G}(U)$$

gives the *tensor product of sheaves* $\mathcal{F}\otimes\mathcal{G}$.

Lemma 6.1.1. *If $f\colon X\to Y$ and $\mathcal{F},\mathcal{G}\in\operatorname{Sh}(Y)$, then $f^*(\mathcal{F}\otimes\mathcal{G})\simeq f^*\mathcal{F}\otimes f^*\mathcal{G}$. In particular $(\mathcal{F}\otimes\mathcal{G})_y\simeq\mathcal{F}_y\otimes\mathcal{G}_y$.*

Proof. Stalks: Tensor product of vector spaces commutes with colimits in $\mathbf{Vect}_{\mathbb{C}}$ because it has a right adjoint $\mathrm{Hom}_{\mathbb{C}}(-,-)$. This proves the claim about stalks for presheaves, and sheafification preserves stalks.

For general pullbacks, we again swap $\otimes$ and colimits to see that $f^*_{pre}\mathcal{F} \otimes_{pre} f^*_{pre}\mathcal{G} \simeq f^*_{pre}(\mathcal{F} \otimes_{pre} \mathcal{G})$. Sheafification(s) now give a map which must be an isomorphism as it will be an isomorphism on stalks. Details: exercise. For general pullbacks, sheafification gives maps

$$f^*_{pre}\mathcal{F} \otimes_{pre} f^*_{pre}\mathcal{G} \to f^*\mathcal{F} \otimes_{pre} f^*\mathcal{G} \to f^*\mathcal{F} \otimes f^*\mathcal{G}.$$

Both are isomorphisms on stalks, which are $(f^*\mathcal{F})_x \otimes (f^*\mathcal{G})_x$. We have $(f^*_{pre}\mathcal{F} \otimes f^*_{pre}\mathcal{G})(U) \simeq (f^*_{pre}(\mathcal{F} \otimes_{pre} \mathcal{G}))(U)$, again because $\otimes$ commutes with colimits in $\mathbf{Vect}_{\mathbb{C}}$. Now we have

$$\begin{array}{ccccc} f^*_{pre}\mathcal{F} \otimes_{pre} f^*_{pre}\mathcal{G} & & \longrightarrow & & f^*\mathcal{F} \otimes f^*\mathcal{G} \\ \downarrow \sim & & & & \downarrow \phi \\ f^*_{pre}(\mathcal{F} \otimes_{pre} \mathcal{G}) & \longrightarrow & f^*_{pre}(\mathcal{F} \otimes \mathcal{G}) & \longrightarrow & f^*(\mathcal{F} \otimes \mathcal{G}) \end{array}$$

Every other arrow is an isomorphism on stalks, so ϕ must be. Here ϕ is induced by sheafification of the top left presheaf, the top map is from above step. □

Lemma 6.1.2. $- \otimes -\colon \mathrm{Sh}(X) \times \mathrm{Sh}(X) \to \mathrm{Sh}(X)$ *is exact in both variables.*

Proof. Calculate on stalks, and use that tensor product of vector spaces is exact. □

Lemma 6.1.3. *X connected and locally connected. $M, N \in \mathbf{Vect}_{\mathbb{C}}$, then*

$$\underline{M}_X \otimes \underline{N}_X \simeq \underline{M \otimes N}_X.$$

If $\mathcal{F}$ and $\mathcal{G}$ are local systems, for all $x_0 \in X$,

$$\mathrm{Mon}_{x_0}(\mathcal{F}) \otimes \mathrm{Mon}_{x_0}(\mathcal{G}) \simeq \mathrm{Mon}_{x_0}(\mathcal{F} \otimes \mathcal{G}).$$

Proof. The identity gives a map $\underline{M}_X(U) \otimes \underline{N}_X(U) \to \underline{M \otimes N}_X(U)$, so we get a map $\underline{M}_X \otimes_{pre} \underline{N}_X \simeq \underline{M \otimes N}_X$ which is an isomorphism on stalks.

If γ is a loop based at x_0, then by construction of $\rho(\gamma)$,

$$\begin{array}{ccc} \mathcal{F}_{x_0} \otimes \mathcal{G}_{x_0} & \xrightarrow{\sim} & (\mathcal{F} \otimes \mathcal{G})_{x_0} \\ \downarrow \rho_{\mathcal{F}}(\gamma) \otimes \rho_{\mathcal{G}}(\gamma) & & \downarrow \rho_{\mathcal{F} \otimes \mathcal{G}}(\gamma) \\ \mathcal{F}_{x_0} \otimes \mathcal{G}_{x_0} & \xrightarrow{\sim} & (\mathcal{F} \otimes \mathcal{G})_{x_0} \end{array}$$

commutes. □

6.2 Internal hom

The sheaf hom is given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathrm{Sh}(U)}(\mathcal{F} \restriction_U, \mathcal{G} \restriction_U).$$

Note that $\Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G})) = \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, \mathcal{F})$.

The sheaf hom is a sheaf, even if $\mathcal{F}$ is only a presheaf.

Example 6.2.1.
1. Let $X = \mathbb{C}$ and let $\mathcal{F} = \underline{\mathbb{C}}^x$ be a skyscraper sheaf supported at $x \in \mathbb{C}$. Let $\mathcal{G} = \underline{\mathbb{C}}_X$. Then $\mathcal{H}om(\underline{\mathbb{C}}^x, \underline{\mathbb{C}}_X) = 0$. Indeed, if $U \subset X$ is open then consider ϕ in $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$. There is $V \subset U$ such that $V \not\ni x$ and restriction maps $\mathcal{G}(U) \hookrightarrow \mathcal{G}(V)$ (connected components of open sets in $\mathbb{C}$ are open; assemble V by finding an open subset avoiding x for each connected component.) Then we have

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) = 0 & \xrightarrow{\phi_V} & \mathcal{G}(V), \end{array}$$

so $\phi = 0$.

2. We have $\mathcal{H}om(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}^x) = \underline{\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})^x}$. Indeed, if $U \not\ni x$, then $\mathcal{H}om(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}^x)(U) = 0$. Otherwise

$$\mathcal{H}om(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}^x)(U) = \mathrm{Hom}_{\mathrm{Sh}(U)}(\underline{\mathbb{C}}_U, \underline{\mathbb{C}}^x \upharpoonright_U) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$$

by adjunction.

The sheaf hom does not commute with taking stalks: We just saw $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x = 0$, but of course $\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_x, \mathbb{C})$ is not zero.

Internal hom is left exact in both variables.

For the covariant variable, the definition of injectivity shows preservation of injective maps, and the rest of injectivity is a short gluing argument. For the contravariant variable, have to be mindful of what a short exact sequence in the opposite category means *e.g.* epimorphisms and monomorphisms are exchanged.

Sheaf tensor and sheaf hom are an adjoint pair, left and right, respectively.

Proposition 6.2.2. *We have an adjunction*

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})).$$

Lemma 6.2.3. *Let X be path-connected and locally simply-connected. Then*

$$\underline{\mathrm{Hom}_{\mathbb{C}}(M, N)}_X \simeq \mathcal{H}om(\underline{M}_X, \underline{N}_X).$$

In particular $\mathcal{H}om(-,-)$ takes local systems to local systems. Also

$$\mathrm{Mon}_{x_0}(\mathcal{H}om(\mathcal{F}, \mathcal{G})) \simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathrm{Mon}_{x_0}(\mathcal{F}), \mathrm{Mon}_{x_0}(\mathcal{G})).$$

Proof.

$$\mathcal{H}om(\underline{M}_X, \underline{N}_X)(U) = \mathrm{Hom}_{\mathrm{Sh}(X)}(\underline{M}_X \upharpoonright_U, \underline{N}_X \upharpoonright_U) \simeq \mathrm{Hom}_{\mathrm{Sh}(U)}(\underline{M}_U, \underline{N}_U) \simeq \mathrm{Hom}_{\mathbb{C}}(M, N).$$

The last isomorphism is because the constant sheaf functor is fully faithful. Thanks to monodromy functor and adjunction, we have

$$\begin{aligned} \mathrm{Hom}_{\pi_1(X,x_0)}(M, \mathrm{Mon}_{x_0}(\mathcal{H}om(\mathcal{F}, \mathcal{G}))) &\simeq \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathrm{Mon}_{x_0}^{-1}(M) \otimes \mathcal{F}, \mathcal{G}) \\ &\simeq \mathrm{Hom}_{\pi_1(X,x_0)}(M \otimes \mathrm{Mon}_{x_0}(\mathcal{F}), \mathrm{Mon}_{x_0}(\mathcal{G})) \\ &\simeq \mathrm{Hom}_{\pi_1(X,x_0)}(M, \mathrm{Hom}_{\mathbb{C}}(\mathrm{Mon}_{x_0}(\mathcal{F}), \mathrm{Mon}_{\mathrm{x}_0}(\mathcal{G}))). \end{aligned}$$

This holds for any M, so apply Yoneda to see the claim. ∎