Perverse Sheaves Learning Seminar: Derived Categories and its Applications to Sheaves

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1 Derived Categories

Unless otherwise stated, let $\mathscr{A}$ be an abelian category.

Definition 1.1. Let $Ch(\mathscr{A})$ be the category of chain complexes in $\mathscr{A}$. Objects in this category are chain complexes $A^\bullet$, which is a sequence of objects and morphisms in $\mathscr{A}$ of the form

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots$$

satisfying $d^i \circ d^{i-1} = 0$ for every $i \in \mathbb{Z}$.

A morphism $f : A^\bullet \to B^\bullet$ between two complexes is a collection of morphisms $f = (f^i : A^i \to B^i)_{i\in\mathbb{Z}}$ in $\mathscr{A}$ such that $f^{i+1} \circ d_A^i = d_B^i \circ f^i$ for every $i \in \mathbb{Z}$.

Definition 1.2. A chain complex $A^\bullet$ is said to be **bounded above** if there is an integer N such that $A^i = 0$ for all $i > N$. Similarly, $A^\bullet$ is said to be **bounded below** if there is an integer N such that $A^i = 0$ for all $i < N$. $A^\bullet$ is said to be **bounded** if it is bounded above and bounded below.

Let $Ch^-(\mathscr{A})$ (resp. $Ch^+(\mathscr{A})$, $Ch^b(\mathscr{A})$) denote the full subcategory of $Ch(\mathscr{A})$ consisting of bounded-above (resp. bounded-below, bounded) complexes.

Let $Ch^\circ(\mathscr{A})$ denote any of the four categories above. For a complex $A^\bullet$, let $[1] : Ch(\mathscr{A}) \to Ch(\mathscr{A})$ denote the shift functor where $A[1]^i = A^{i-1}$.

Definition 1.3. A **quasi-isomorphism** in $Ch(\mathscr{A})$ is a chain map $f : A^\bullet \to B^\bullet$ such that the induced maps $H^n(f) : H^n(A) \to H^n(B)$ are isomorphisms for all n.

The derived category for $\mathscr{A}$ can be thought of as a category obtained from $Ch(\mathscr{A})$ by having quasi-isomorphisms be actual isomorphisms. To do this, we localize (= invert) quasi-isomorphisms.

Definition 1.4. Let $\mathscr{A}$ be an additive category and let $\mathscr{S}$ be a class of morphisms in $\mathscr{A}$ closed under composition. Let $\mathscr{A}_\mathscr{S}$ be an additive category and let $L : \mathscr{A} \to \mathscr{A}_\mathscr{S}$ be an additive functor. We say $(\mathscr{A}_\mathscr{S}, L)$ is obtained by **localizing** $\mathscr{A}$ at $\mathscr{S}$ if $\mathscr{A}'$ is an additive category and $F : \mathscr{A} \to \mathscr{A}'$ is an additive functor that sends all morphisms of $\mathscr{S}$ to isomorphisms, then there exists a unique functor $\overline{F} : \mathscr{A}_\mathscr{S} \to \mathscr{A}'$ and a unique isomorphism $\epsilon : \overline{F} \circ L \xrightarrow{\sim} F$.

This is similar to the construction of localizing a ring. However, like in the case of localizing a ring, localizations may not always exist or be nice. Proposition I.6.3 of [?] state that a localization $\mathscr{A}_\mathscr{S}$ exists for $\mathscr{A}$ if $\mathscr{S}$ satisfies the following conditions:

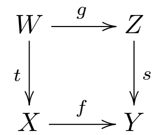
L0 For every $X \in \mathscr{A}$, we have $id_X \in \mathscr{S}$.

L1 Given morphisms $f : X \to Y$ and $s : Z \to Y$ with $s \in \mathscr{S}$, there is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}$$

with $t \in \mathscr{S}$.

L2 Given morphisms $g : W \to Z$ and $t : W \to X$ with $t \in \mathscr{S}$, there is a commutative diagram



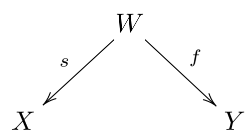
with $s \in \mathscr{S}$.

L3 Given morphisms $f, g : X \to Y$, the following are equivalent:

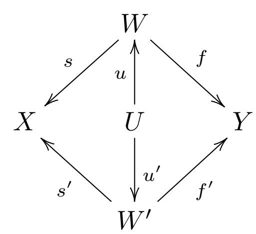
- There is a morphism $t : Y \to Y'$ with $t \in \mathscr{S}$ such that $t \circ f = t \circ g$.
- There is a morphism $s : X' \to X$ with $s \in \mathscr{S}$ such that $f \circ s = g \circ s$.

The objects of $\mathscr{A}_{\mathscr{S}}$ are the same as $\mathscr{A}$, but the morphisms are "roofs".

Definition 1.5. Let $\mathscr{S}$ be a class of morphisms closed under composition. For $X, Y \in \mathscr{A}$, a **roof diagram** from X to Y is a diagram of morphisms

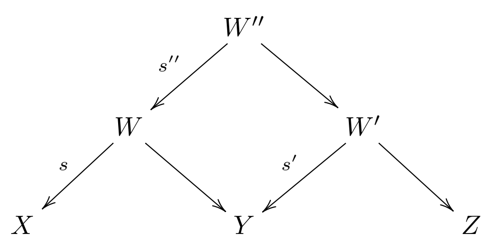


with $s \in \mathscr{S}$. Two roof diagrams $X \overset{s}{\leftarrow} W \overset{f}{\rightarrow} Y$ and $X \overset{s'}{\leftarrow} W' \overset{f'}{\rightarrow} Y$ are equivalent if there is a commutative diagram in $\mathscr{A}$



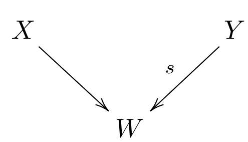
with $u \in \mathscr{S}$. Note that this means the compositions $U \to W \to X$ and $U \to W' \to X$ are homotopy equivalent, so since the first is a quasi-isomorphism, so is the second.

If $\mathscr{S}$ satisfies L0-L3, one can identify $Hom_{\mathscr{A}_{\mathscr{S}}}(X, Y)$ with equivalence classes of roof diagrams, where composition of roof diagrams $X \overset{s}{\leftarrow} W \to Y$ and $Y \overset{s'}{\leftarrow} W' \to Z$ is a commutative diagram $X \leftarrow W' \to Z$ of the form



with $s'' \in \mathscr{S}$. The existence of such a diagram follows from L1.

Remark 1.6. Basement diagrams can also be used instead of roof diagrams to describe $Hom_{\mathscr{A}_{\mathscr{S}}}(X, Y)$. These are diagrams of the form



where $s \in \mathscr{S}$.

Unfortunately, quasi-isomorphisms in $Ch(\mathscr{A})$ do not satisfy these conditions. To remedy this, instead of working with $Ch(\mathscr{A})$, we work with the homotopy category $K(\mathscr{A})$.

Definition 1.7. The **homotopy category** of $\mathscr{A}$, denoted by $K(\mathscr{A})$, is the category whose objects are those of $Ch(\mathscr{A})$, but whose morphisms are homotopy classes of chain maps. That is, $Hom_{K(\mathscr{A})}(A^\bullet, B^\bullet) := Hom_{Ch(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$, where for two morphisms $f, g : A^\bullet \to B^\bullet$ in $Ch(\mathscr{A})$, we say $f \sim g$ if there exists a collection of morphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

As in the case of $Ch(A)$, let $K^-(\mathscr{A})$ (resp. $K^+(\mathscr{A})$, $K^b(\mathscr{A})$) denote the full subcategory of $K(\mathscr{A})$ of bounded-above (resp. bounded-below, bounded) complexes.

Let $K^\circ(\mathscr{A})$ denote any of the four homotopy categories.

Remark 1.8. One can show that $K^\circ(\mathscr{A})$ is equivalent to $Ch^\circ(\mathscr{A})$ localized at chain homotopies.

Proposition 1.9. *In $K^\circ(\mathscr{A})$ the class of quasi-isomorphisms satisfies L1-L3.*

Proof. See Section I.6 of *[?]*. □

Definition 1.10. The **derived category** (resp. bounded-above derived category, bounded-below derived category, bounded derived category) of $\mathscr{A}$, denoted $D(\mathscr{A})$ (resp. $D^-(\mathscr{A}), D^+(\mathscr{A}), D^b(\mathscr{A})$) is the category obtained from $K(\mathscr{A})$ (resp. $K^-(\mathscr{A}), K^+(\mathscr{A}), K^b(\mathscr{A})$) by localizing at the quasi-isomorphisms.

Let $D^\circ(\mathscr{A})$ denote any of the four derived categories.

Remark 1.11. For an object $A \in \mathscr{A}$, we can view A as a chain complex $A^\bullet$ where $A^0 = A$ and $A^i = 0$ for $i \neq 0$. This allows us to embed $\mathscr{A}$ into $D^b(\mathscr{A})$ as a full subcategory.

We will now proceed to the important notion of distinguished triangles.

Definition 1.12. Let $f : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ be a chain map. The **chain-map cone (mapping cone)** of f, denoted by *cone*(f), is the chain complex given by

$$cone(f)^i = A^{i+1} \oplus B^i$$

with differential $d^i : cone(f)^i \to cone(f)^{i+1}$ given by

$$d^i = \begin{bmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{bmatrix}.$$

The inclusion maps $B^i \to cone(f)^i$ and projection maps $cone(f)^i \to A^{i+1}$ give chain maps

$$i_2 : B \to cone(f) \quad \text{and} \quad p_1 : cone(f) \to A[1]$$

Exercise 1.13. Show that the composition $B^\bullet \to cone(f) \to A^\bullet[1]$ is zero and the composition $A^\bullet \to B^\bullet \to cone(f)$ is homotopic to the zero map.

Definition 1.14. A diagram

$$A_1 \to A_2 \to A_3 \to A_1[1]$$

in $K^\circ(\mathscr{A})$ (resp. $D^\circ(\mathscr{A})$) is called a **distinguished triangle** if it is isomorphic in $K^\circ(\mathscr{A})$ (resp. $D^\circ(\mathscr{A})$) to a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{i_2} cone(f) \xrightarrow{p_1} A[1]$$

for some chain map f.

An additive category with a shift functor (automorphism) and distinguished triangles (collection of diagrams) satisfying some natural axioms is called a triangulated category. The homotopy category $K^\circ(\mathscr{A})$ and the derived category $D^\circ(\mathscr{A})$ are natural examples.

Remark 1.15. If we have a distinguished triangle $X \to Y \to Z \to X[1]$, then it gives us a long exact sequence in cohomology

$$\cdots \longrightarrow H^k(X) \longrightarrow H^k(Y) \longrightarrow H^k(Z) \longrightarrow H^{k+1}(X) \longrightarrow \cdots.$$

2 Derived Functors

Definition 2.1. Let $\mathscr{T}$ and $\mathscr{T}'$ be triangulated categories (e.g. $D^b(\mathscr{A})$ and $D^b(\mathscr{A}')$). A **triangulated functor** is an additive functor $F : \mathscr{T} \to \mathscr{T}'$ with a natural isomorphism

$$F(X[1]) \cong F(X)[1]$$

such that for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{T}$,

$$F(X) \to F(Y) \to F(Z) \to F(X)[1]$$

is a distinguished triangle in $\mathscr{T}'$.

Lemma 2.2. *If $F : \mathscr{A} \to \mathscr{B}$ is an additive functor of additive categories, the induced functor $F : K^\circ(\mathscr{A}) \to K^\circ(\mathscr{B})$ is a triangulated functor. If in addition F is an exact functor of abelian categories, the induced functor $F : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ is a triangulated functor.*

Proof. Easy exercise. □

Recall that a complex $A^\bullet$ in $Ch^\circ(\mathscr{A})$ or $K^\circ(\mathscr{A})$ is called **acyclic** $H^i(A^\bullet) = 0$ for all i. If we have a functor F that is not exact, the image of an acyclic complex may not be acyclic, or it may not send quasi-isomorphisms to quasi-isomorphisms.

In the case of an exact functor F, we obtain a triangulated functor $\overline{F} : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ and a natural isomorphism $\theta : L_{\mathscr{B}} \circ F \xrightarrow{\sim} \overline{F} \circ L_{\mathscr{A}}$ where $L_{\mathscr{A}} : K^\circ(\mathscr{A}) \to D^\circ(\mathscr{A})$ is the localization functor. Then in the case where F is not exact, the next best thing is to have a natural transformation in one direction.

Definition 2.3. Let $F : K^\circ(\mathscr{A}) \to K^\circ(\mathscr{B})$ be a triangulated functor. A **right derived functor** of F is a triangulated functor $RF : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ with a natural transformation

$$\epsilon : L_{\mathscr{B}} \circ F \to RF \circ L_{\mathscr{A}}$$

that is universal in the following sense: if $G : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ is another triangulated functor with a natural transformation $\phi : L_{\mathscr{B}} \circ F \to G \circ L_{\mathscr{A}}$, then there exists a unique functor morphism $\tilde{\phi} : RF \to G$ such that $\phi = (\tilde{\phi} L_{\mathscr{A}}) \circ \epsilon$, where $\tilde{\phi} L_{\mathscr{A}} : RF \circ L_{\mathscr{A}} \to G \circ L_{\mathscr{A}}$.

Similarly, a **left derived functor** of F is a triangulated functor $LF : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ together with a natural transformation

$$\eta : LF \circ L_{\mathscr{A}} \to L_{\mathscr{B}} \circ F$$

that is universal in the following sense: if $G : D^\circ(\mathscr{A}) \to D^\circ(\mathscr{B})$ is another triangulated functor with a natural transformation $\phi : G \circ L_{\mathscr{A}} \to L_{\mathscr{B}} \circ F$, then there exists a unique functor morphism $\tilde{\phi} : G \to LF$ such that $\phi = \eta \circ (\tilde{\phi} L_{\mathscr{A}})$ where $\tilde{\phi} L_{\mathscr{A}} : G \circ L_{\mathscr{A}} \to LF \circ L_{\mathscr{A}}$.

Definition 2.4. Let $\mathscr{A}$ be an abelian category and $\mathscr{Q} \subset \mathscr{A}$ a full subcategory. $\mathscr{Q}$ is said to be **large enough on the right** if for any object $A \in \mathscr{A}$, there is an injective map $A \to X$ with $X \in \mathscr{Q}$.

Similarly, $\mathscr{Q}$ is said to be **large enough on the left** if for any object $A \in \mathscr{A}$, there is a surjective map $X \to A$ with $X \in \mathscr{Q}$.

Definition 2.5. Let $\mathscr{Q} \subset \mathscr{A}$ be a full subcategory.

1. Given $A \in Ch^\circ(\mathscr{A})$, a **right $\mathscr{Q}$-resolution** of A is a quasi-isomorphism $q : A \to Q$ such that $Q \in Ch^\circ(\mathscr{Q})$. For $A \in Ch^+(\mathscr{A})$, such a right resolution is said to be **strict** if $A \in Ch(\mathscr{A})^{\geq n}$ and $Q \in Ch(\mathscr{A})^{\geq n}$ for a fixed n.

2. Given $A \in Ch^\circ(\mathscr{A})$, a **left $\mathscr{Q}$-resolution** of A is a quasi-isomorphism $q : Q \to A$ such that $Q \in Ch^\circ(\mathscr{Q})$. For $A \in Ch^-(\mathscr{A})$, such a right resolution is said to be **strict** if $A \in Ch(\mathscr{A})^{\leq n}$ and $Q \in Ch(\mathscr{A})^{\leq n}$ for a fixed n.

Example 2.6. Consider the case of $A \in \mathscr{A}$ as a sequence $A^\bullet$ where $A^i = 0$ for $i \neq 0$ and $A^0 = A$. Then a strict right $\mathscr{Q}$-resolution $Q^\bullet$ of A is the same as giving an exact sequence

$$0 \longrightarrow A^0 \xrightarrow{q} Q^0 \xrightarrow{d_Q^0} Q^1 \xrightarrow{d_Q^1} \cdots$$

The map $q : A^0 \to Q^0$ is called the **augmentation map**.

Proposition 2.7. *Let $\mathscr{A}$ be an abelian category and let $\mathscr{Q} \subset \mathscr{A}$ be a full subcategory.*

1. *If $\mathscr{Q}$ is large enough on the right, then every object in $Ch^+(\mathscr{A})$ admits a strict right $\mathscr{Q}$-resolution.*

2. *If $\mathscr{Q}$ is large enough on the left, then every object in $Ch^-(\mathscr{A})$ admits a strict left $\mathscr{Q}$-resolution.*

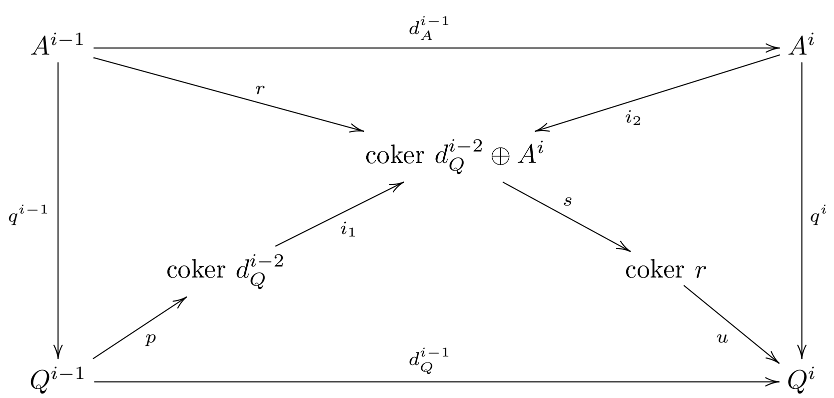
Proof. 1) Exercise. Hint: Take an injection $q^0 : A^0 \to Q^0$ where $Q^0 \in \mathscr{Q}$. To construct Q^1, let $r : A^0 \to Q^0 \oplus A^1$ be given by $r = \begin{bmatrix} q^0 \\ -d_A^0 \end{bmatrix}$. Then choose an injection coker $r \to Q^1$ with $Q^1 \in \mathscr{Q}$. The map $q^1 : A^1 \to Q^1$ is the composition

$$A^1 \hookrightarrow Q^0 \oplus A^1 \twoheadrightarrow \text{coker } r \to Q^1$$

and the differential $d_Q^0 : Q^0 \to Q^1$ is the composition

$$Q^0 \hookrightarrow Q^0 \oplus A^1 \twoheadrightarrow \text{coker } r \to Q^1.$$

[WLOG, assume $A \in Ch^+(\mathscr{A})^{\geq 0}$. We want to construct a quasi-isomorphism $q = (q^i) : A^\bullet \to Q^\bullet$ where $Q^\bullet \in Ch^+(\mathscr{Q})^{\geq 0}$. As $\mathscr{Q}$ is large enough, we have an injection $q^0 : A^0 \to Q^0$. Suppose we have already constructed $Q^\bullet$ and maps $q^\bullet$ up to the ith step. Let $p : Q^{i-1} \twoheadrightarrow \text{coker } d_Q^{i-2}$ be the quotient map. Let $r : A^{i-1} \to \text{coker } d_Q^{i-2} \oplus A^i$ be the map given by $r = \begin{bmatrix} pq^{i-1} \\ -d_A^{i-1} \end{bmatrix}$. Let $s : \text{coker } d_Q^{i-2} \oplus A^i \twoheadrightarrow \text{coker } r$ be the quotient. Choose an injection $u : \text{coker } r \to Q^i$ with $Q^i \in \mathscr{Q}$. Define $d_Q^{i-1} = u \circ s \circ i_1 \circ p$ and $q^i = u \circ s \circ i_2$ as in the diagram:



Then check that $Q^\bullet$ is a complex and q is a chain map and quasi-isomorphism.] ☐

Definition 2.8. Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor of abelian categories. A full subcategory $\mathscr{Q} \subset \mathscr{A}$ is said to be a **right adapted class** for F if it satisfies the following conditions:

1. The class $\mathscr{Q}$ is large enough on the right.

2. If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence with $X', X \in \mathscr{Q}$, then $X'' \in \mathscr{Q}$.

3. For any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathscr{Q}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Similarly, for a right exact functor $F : \mathscr{A} \to \mathscr{B}$, a full subcategory $\mathscr{Q} \subset \mathscr{A}$ is said to be a **left adapted class** for F if it satisfies the following conditions:

1. The class $\mathscr{Q}$ is large enough on the left.
2. If $0 \to X' \to X \to X'' \to 0$ is a short exact sequence with $X, X'' \in \mathscr{Q}$, then $X' \in \mathscr{Q}$.
3. For any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathscr{Q}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

Example 2.9. Let A be an algebra and A-mod be the category of A-modules. Then the full subcategory of projective modules $\mathcal{P}$ is large enough on the left and the full subcategory of injective modules $\mathcal{I}$ is large enough on the right.

Let M be an A-module. Then $Hom(M, -)$ is left exact with $\mathcal{I}$ as a right adapted class and $M \otimes -$ (or equivalently $- \otimes M$) is right exact with $\mathcal{P}$ as a left adapted class.

Lemma 2.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

1. *Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor, and let $\mathscr{Q}$ be a right adapted class for F. If $Q \in Ch^+(\mathscr{Q})$ is acyclic, then $F(Q)$ is acyclic. If $f : X \to Y$ is a quasi-isomorphism in $Ch^+(\mathscr{Q})$, then $F(f)$ is a quasi-isomorphism.*
2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor, and let $\mathscr{Q}$ be a left adapted class for F. If $Q \in Ch^-(\mathscr{Q})$ is acyclic, then $F(Q)$ is acyclic. If $f : X \to Y$ is a quasi-isomorphism in $Ch^-(\mathscr{Q})$, then $F(f)$ is a quasi-isomorphism.*

Proof. Suppose F is left exact and let $Q \in Ch^+(\mathscr{Q})$ be acyclic. Let $K^i = \text{im } d^{i-1} = \ker d^i$. Any left exact functor preserves kernels, then $F(K^i) \cong \ker F(d^i)$. Using induction, suppose im $F(d^{i-2}) = F(K^{i-1})$ and $K^{i-1} \in \mathscr{Q}$. We have a short exact sequence

$$\eta : 0 \to K^{i-1} \to Q^i \stackrel{d^{i-1}}{\to} K^i \to 0.$$

As $\mathscr{Q}$ is an adapted class, we have $K^i \in \mathscr{Q}$ and $F(\eta)$ is an exact sequence, so im $F(d^{i-1}) \cong F(K^i)$. Thus $F(Q)$ is acyclic.

Suppose $f : X \to Y$ is a quasi-isomorphism. Extend f to a distinguished triangle $X \stackrel{f}{\to} Y \to K \to$ in $K^+(\mathscr{Q})$. Note that f is a quasi-isomorphism if and only if K is acyclic, so apply the above result (apply cohomology to the triangle to get a long exact sequence of cohomology). □

Theorem 2.11. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.*

1. *If $F : \mathscr{A} \to \mathscr{B}$ is a left exact functor that admits a right adapted class, then it admits a right derived functor $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$.*
2. *If $F : \mathscr{A} \to \mathscr{B}$ is a right exact functor that admits a left adapted class, then it admits a left derived functor $LF : D^-(\mathscr{A}) \to D^-(\mathscr{B})$.*

We will describe what the functor RF does on objects and morphisms. For $X \in Ch^+(\mathscr{A})$, choose a right $\mathscr{Q}$-resolution $q_X : X \to Q_X$. Define

$$RF(X) = F(Q_X).$$

Let $f : X \to Y$ be a morphism. As q_X is a quasi-isomorphism, we can form $\tilde{f} = q_Y \circ f \circ q_X^{-1} : Q_X \to Q_Y$. As a basement, $\tilde{f}$ can be represented by the diagram

$$Q_X \xrightarrow{h} W \xleftarrow{s} Q_Y$$

where s is a quasi-isomorphism. Then $q_W \circ s : Q_Y \to Q_W$ is a quasi-isomorphism so $F(q_W \circ s)$ is a quasi-isomorphism. Define $RF(f) : RF(X) \to RF(Y)$ to be the basement diagram

$$RF(X) = F(Q_X) \xrightarrow{F(q_W \circ h)} F(Q_W) \xleftarrow{F(q_W \circ s)} RF(Y) = F(Q_Y)$$

Define the natural transformation $\epsilon : L_{\mathscr{B}} \circ F \to RF \circ L_{\mathscr{A}}$ where for $X \in K^+(\mathscr{A})$, let ϵ_X be the map

$$L_{\mathscr{B}}(F(X)) \xrightarrow{L_{\mathscr{B}}(F(q_X))} L_{\mathscr{B}}(F(Q_X)) = RF(L_{\mathscr{A}}(X)).$$

Exercise 2.12. Check the above is well-defined. In particular, check that the definition does not depend on which $\mathscr{Q}$-resolution is taken and does not depend on which basement diagram is taken.

Proposition 2.13. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be left exact functors. Suppose that F and G have right adapted classes $\mathscr{Q} \subset \mathscr{A}$ and $\mathscr{S} \subset \mathscr{B}$, respectively, such that $F(\mathscr{Q}) \subset \mathscr{S}$. Then there is a canonical isomorphism $R(G \circ F) \xrightarrow{\sim} RG \circ RF$. Similarly for right exact functors.*

Proof. Exercise. ☐

3 Sheaves

As the category of sheaves of $\mathbb{C}$-vector spaces on X, $Sh(X)$, is abelian, we can form its derived category $D^\circ(X) := D^\circ Sh(X)$.

Proposition 3.1. *$Sh(X)$ has enough injectives.*

Proof. For M a $\mathbb{C}$-vector space, as shown in Example 2.2.4 of Stefan's talk, we have $Hom_{\mathbb{C}}(\mathcal{G}_x, M) \cong Hom_{Sh(X)}(\mathcal{G}, \underline{M}^x)$ natural in $\mathcal{G}$, where $\underline{M}^x$ is the skyscraper sheaf at x. As all vector spaces are injective objects, then $Hom_{\mathbb{C}}(-, M)$ is an exact functor so $Hom_{Sh(X)}(-, \underline{M}^x)$ is also exact. Thus $\underline{M}^x$ is an injective sheaf. Using the universal property of the product, $\prod_{x \in X}(\underline{M}^x)$ is also an injective sheaf.

Let $\mathcal{F}$ be a sheaf. There is a sheaf map $\varphi : \mathcal{F} \to \underline{(\mathcal{F}_x)}^x$ with $\varphi_x : \mathcal{F}_x \to \mathcal{F}_x$ the identity. By the universal property of the product, we obtain an injective sheaf map $\theta : \mathcal{F} \to \prod_{x \in X} \underline{(\mathcal{F}_x)}^x$. ☐

By the proposition, all left exact functors have derived functors. However, $Sh(X)$ may not have enough projectives.

As the pullback functor is exact, for $f : X \to Y$, let $f^* : D^\circ(Y) \to D^\circ(X)$ denoted the induced functor. Since it is exact, we have $(g \circ f)^* \mathcal{F} \cong f^* g^* \mathcal{F}$ for $\mathcal{F} \in D^\circ(X)$, by Proposition 2.1.5 of Stefan's talk, and Proposition 2.13.

As the push-forward $^\circ f_*$ is left exact, it has a derived functor denoted by $f_* : D^+(X) \to D^+(Y)$.

Proposition 3.2. *The push-forward functor $^\circ f_*$ sends injectives to injectives.*

Proof. Exercise. Use the fact that $^\circ f_*$ is a right adjoint to f^* (Proposition 2.2.2 of Stefan's talk) and f^* is exact. ☐

Corollary 3.3. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then for $\mathcal{F} \in D^+(X)$, we have $g_* f_* \mathcal{F} \cong (g \circ f)_* \mathcal{F}$.*

Definition 3.4. Let $A \in K^-(\mathscr{A})$ and $B \in K^+(\mathscr{A})$. Their **Hom chain-complex**, denoted $chHom(A, B)$ is the chain complex in $(\mathrm{Vect}_{\mathbb{C}})$ whose terms are

$$chHom(A, B)^n = \bigoplus_{j-i=n} Hom(A^i, B^j)$$

and differential given by

$$d(f) = d_B \circ f + (-1)^{j-i+1} f \circ d_A$$

for $f \in Hom(A^i, B^j)$.

As $Sh(X)$ has enough injectives, we can form the **derived Hom functor** (in the second variable) $RHom : D^-(X)^{op} \times D^+(X) \to D^+(\mathrm{Vect}_{\mathbb{C}})$.

Proposition 3.5. *For $A \in D^-(X)$ and $B \in D^+(X)$, there is a natural isomorphism*

$$Hom_{D(X)}(A, B) \cong H^0(RHom(A, B)).$$

Theorem 3.6. *Let $f : X \to Y$ be a continuous map. For $\mathcal{F} \in D^+(Y)$ and $\mathcal{G} \in D^+(X)$, there are natural isomorphisms*

$$RHom_{D^+(X)}(f^*\mathcal{F}, \mathcal{G}) \cong RHom_{D^+(Y)}(\mathcal{F}, f_*\mathcal{G})$$
$$Hom_{D^+(X)}(f^*\mathcal{F}, \mathcal{G}) \cong Hom_{D^+(Y)}(\mathcal{F}, f_*\mathcal{G})$$

Proof. Replace $\mathcal{G}$ by an injective resolution. The first claim reduces to the claim that there is a natural isomorphism $chHom(f^*\mathcal{F}, \mathcal{G}) \cong chHom(\mathcal{F}, ^\circ f_*\mathcal{G})$, which follows from the fact that f^* is adjoint to $^\circ f_*$ in the abelian case. The second claims follows from fact that the 0th cohomology of $RHom$ is Hom. □

Remark 3.7. Let $X, Y \in Sh(X)$. For $n \in \mathbb{Z}$, the nth **Ext group** of X and Y, denoted by $Ext^n_{Sh(X)}(X, Y)$ or $Ext^n(X, Y)$, is given by

$$Ext^n(X, Y) := Hom_{D(X)}(X, Y[n]) = H^n(RHom(X, Y)).$$